A generalized proportional Caputo fractional model of multi-agent linear dynamic systems via impulsive control protocol

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ABSTRACT

This paper deals with multi-agent systems that, due to using the generalized proportional Caputo fractional derivative, possess memories. The information exchange between agents does not occur continuously but only at fixed given update times, and the lower limit of the fractional derivative changes according to the update times. Two types of multi-agent systems are studied, namely systems without a leader and systems with a leader. For a generalized proportional Caputo fractional model of multi-agent linear dynamic systems, sufficient conditions for exponential stability via impulsive control are obtained. In the case of the presence of a leader in the multi-agent system, we derive sufficient conditions for the leader-following consensus via impulsive control based on the leader’s influence. Simulation results are provided to verify the essential role of the generalized proportional Caputo fractional derivative and impulsive control in realizing the consensus of multi-agent systems.

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1. Introduction

Due to the rapid development of embedded systems and communication technology, multi-agent systems have drawn much attention from researchers in science and engineering applications. A multi-agent system is a group of usually autonomous agents that can cooperate with each other in order to accomplish given tasks that a single agent cannot. The problem of synchronization of multi-agent systems has attracted considerable attention in the last decades (see, e.g., [1–6] and survey papers [7,8]). Broadly speaking, synchronization of networked multi-agents mean that by using communication networks and local controllers, the agents should be steered towards a common trajectory. It is obvious that this problem has meaningful applications in multiple areas, such as formation control of mobile robots, target tracking, spacecraft formation flying, and so on [9–11].

It is well known that many natural phenomena in complex environments cannot be accurately explained by using the framework of integer-order dynamics. One of the possible ways to deal with this problem is by expanding

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the existing integer-order models to fractional-order models, which allow for more realistic modeling having much higher freedom to fit possible experimental data, as well as allowing the description of memories and hereditary effects of various materials and processes. In the last thirty years, research concerning applications of fractional-order dynamics has made profound and significant progress. Examples include the time-fractional damage model for hyperelastic body (e.g., to mimic the abdominal aortic aneurysm phenomena [12]), fractional-order models of long memory processes (e.g., the steelmaking process [13]), the fractional-order model of learning [14], fractional-order models in viscoelasticity [15], and economics [16, 17]. In particular, fractional calculus was also introduced into the modeling of multi-agent systems. To the best of our knowledge, the first paper devoted to this subject was [18].

2. Preliminaries

We start by recalling definitions of the generalized proportional fractional operators. Let $a, b \in \mathbb{R}$ with $b \leq \infty$ (if $b = \infty$, then the interval is half open), and $\rho \in (0, 1]$ be a fixed parameter.

**Definition 2.1** (See [34]). Let $u : [a, b] \rightarrow \mathbb{R}$ and $\alpha \geq 0$. The generalized proportional fractional integral of a function $u$ is defined by

$$
(c)^{\alpha, \rho} u(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} e^{\frac{\rho}{\alpha}s^{-\alpha}} u(s) ds, \quad t \in (a, b]
$$

(2.1)

as long as this integral is well defined.

**Definition 2.2** (See [34]). Let $u : [a, b] \rightarrow \mathbb{R}$ and $\alpha \in (0, 1)$. The generalized proportional Caputo fractional derivative of a function $u$ is defined by

$$
(c)_{a}^{\alpha, \rho} D^{1, \rho} u(t) = \left((c)^{1-\alpha, \rho} (D^{1, \rho} u)\right) (t) = \frac{1}{\rho^{1-\alpha}} \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} e^{\frac{\rho}{\alpha}s^{-\alpha}} \frac{D^{1, \rho} u(s)}{(t-s)^\rho} ds, \quad t \in (a, b],
$$

(2.2)

as long as this integral is well defined, where $D^{1, \rho} u = (1-\rho)u + \rho u'$.

Definitions 2.1 and 2.2 can be generalized componentwisely for $u \in C([a, b], \mathbb{R}^n)$. 

Remark 2.3. If \( \rho = 1 \), then the generalized proportional Caputo fractional derivative is reduced to the classical Caputo fractional derivative.

Now we cite some important results involving generalized proportional fractional operators, which are useful for our further computations.

Lemma 2.4 (See [34, Theorem 5.3]). For \( \rho \in (0, 1) \) and \( \alpha \in (0, 1) \), we have

\[
\left( aD^{\alpha}_a \left( \frac{C}{a} D^{\alpha}_a u \right) \right)(t) = u(t) - u(a) e^{\frac{\alpha}{\rho}(t-a)}.
\]  

(2.3)

Lemma 2.5 (See [34, Proposition 3.7]). For \( \rho \in (0, 1) \), \( \alpha \in (0, 1) \), and \( \beta > 0 \), we have

\[
\left( aD^{\alpha}_a u \right)(t) = \frac{\Gamma(\beta)}{\rho^\alpha \Gamma(\beta + \alpha)}(t-a)\alpha u(t), \quad \text{where} \quad u(t) = e^{\frac{\alpha}{\rho}(t-a)} \beta^{-1}.
\]  

(2.4)

Remark 2.6. Note that, if \( \rho \in (0, 1) \), then the generalized proportional Caputo fractional derivative of a constant is not zero.

Lemma 2.7 (See [34, Remark 3.2]). For \( \rho \in (0, 1) \) and \( \alpha \in (0, 1) \), we have

\[
\left( C \right) D^{\alpha}_a u(t) = 0, \quad \text{where} \quad u(t) = e^{\frac{\alpha}{\rho}(t-a)}, \quad t > a.
\]  

(2.5)

We will use the explicit form of the solution to the initial value problem for the scalar linear generalized proportional Caputo fractional differential equation, which is given in [34, Example 5.7] (with necessary slight corrections).

Lemma 2.8. A solution to the scalar linear generalized proportional Caputo fractional initial value problem

\[
\left( C \right) D^{\alpha}_a u(t) = \lambda u(t), \quad u(a) = u_0, \quad \alpha \in (0, 1), \quad \rho \in (0, 1),
\]  

is given by

\[
u(t) = u_0 e^{\frac{\alpha}{\rho}(t-a)} E_{\alpha} \left( \lambda \left( \frac{t-a}{\rho} \right) \right),
\]

where \( E_\alpha \) is the Mittag-Leffler function of one parameter.

For a vector \( x \in \mathbb{R}^n \), we denote by \( \| x \| \) its Euclidean norm. Let \( A^T \) be the transpose of a matrix \( A \). For a matrix \( A = \{ a_{ij} \}_{i,j=1}^n \in \mathbb{R}^{n \times n} \), we use the spectral norm

\[
\| A \|_2 = \sqrt{\max_{1 \leq \lambda_i \leq \| A \|}}
\]

where \( \lambda_i \) are the eigenvalues of \( A^T A \). Then we have

\[
\| A \|_2 \leq \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2, \quad \| e^A \|_2 \leq e^{\| A \|_2}, \quad \| A x \| \leq \| A \|_2 \| x \|.
\]

Definition 2.9 (See [39]). Let \( A \) be an arbitrary square matrix. The Mittag-Leffler matrix function with one parameter \( \alpha \) is defined as

\[
E_{\alpha}(Az) = \sum_{k=0}^{\infty} \frac{A^k z^k}{\Gamma(k\alpha + 1)}, \quad \alpha > 0.
\]

Let us recall some properties of the Mittag-Leffler function.

Proposition 2.10 (See [40, Theorem 1.2]). For every \( \alpha \in (0, 1) \), \( t \mapsto \frac{t^\alpha}{\alpha} - E_{\alpha}(t^\alpha) \) is completely monotonic.

Corollary 2.11. Let \( \alpha \in (0, 1) \). Then

(i) \( 0 < \frac{t^\alpha}{\alpha} - E_{\alpha}(t^\alpha) \leq \frac{1}{\alpha} - 1, \ t \geq 0; \)
(ii) \( \alpha \leq 1 - \alpha e^{-t} \left( \frac{1}{\alpha} - 1 \right) \leq \alpha e^{-t} E_{\alpha}(t^\alpha), \ t \geq 0; \)
(iii) \( \lim_{t \to 0} \left( \frac{t^\alpha}{\alpha} - E_{\alpha}(t^\alpha) \right) = 0, \ t \geq 0; \)
(iv) \( E_{\alpha}(t^\alpha) < \frac{t^\alpha}{\alpha}, \ t \geq 0. \)
Proposition 2.12. For a matrix \( A \in \mathbb{R}^{n \times n} \) and \( \alpha \in (0, 1) \), we have the inequality
\[
\|E_\alpha(A)\|_2 \leq E_\alpha(\|A\|_2).
\]

Example 2.13. Let
\[
A = \begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}.
\]
Using the codes provided by Roberto Garrappa [41], we obtain
\[
E_{0.8}(A) = \begin{bmatrix} 68.7603 & -88.6273 \\ -29.5424 & 39.2179 \end{bmatrix}.
\]
Since the maximum eigenvalue of \( A^T A \) is 0.5 \((15 + \sqrt{221})\), it follows that
\[
\|A\|_2 = \sqrt{0.5(15 + \sqrt{221})}.
\]
Thus,
\[
\|E_{0.8}(A)\|_2 = \sqrt{14993.2} = 122.447 < 281.754 = E_{0.8}(\|A\|_2) = 281.754.
\]
The following lemma is used to derive our main results.

Lemma 2.14. A solution to the initial value problem for the system of linear generalized proportional Caputo fractional differential equations
\[
\left( \frac{D^{\alpha, \rho}}{\alpha, \rho} U \right)(t) = AU(t), \quad U(a) = U_0, \quad \alpha \in (0, 1), \quad \rho \in (0, 1),
\]
where \( U_0 \in \mathbb{R}^n \) and \( A \) is an \( n \times n \) dimensional matrix, is given by
\[
U(t) = e^{\frac{\rho}{\alpha}(t-a)}U_0 \left( A \left( \frac{t-a}{\rho} \right)^\alpha \right) U_0,
\]
where \( U_0 \) is the one-parametric Mittag-Leffler function.

Proof. We use the Picard iterative process to derive the series solution to (2.6). Applying the operator \( \frac{D^{\alpha, \rho}}{\alpha, \rho} \) to both sides of the equation \( \left( \frac{D^{\alpha, \rho}}{\alpha, \rho} U \right)(t) = AU(t) \) and using the initial condition, by Lemma 2.4, we obtain
\[
U(t) = U_0 e^{\frac{\rho}{\alpha}(t-a)} + A \left( e^{\frac{\rho}{\alpha}(t-a)} \right) U_0.
\]
Define \( \Phi_0 : [a, \infty) \to \mathbb{R}^n \) by \( \Phi_0(t) = U(a)e^{\frac{\rho}{\alpha}(t-a)} \). For \( k \in \mathbb{N} \), by the recurrence formula, we calculate the \( k \)th approximate solution \( \Phi_k : [a, \infty) \to \mathbb{R}^n \):
\[
\Phi_k(t) = U_0 e^{\frac{\rho}{\alpha}(t-a)} + A \left( e^{\frac{\rho}{\alpha}(t-a)} \Phi_{k-1} \right)(t).
\]
Then, from the recurrence formula, by using Lemma 2.5 with \( \beta = 1, \alpha + 1, 2\alpha + 1 \ldots \), and
\[
E_\alpha(At^\alpha) = I + \frac{At^\alpha}{\Gamma(\alpha + 1)} + \frac{A^2 t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{A^3 t^{3\alpha}}{\Gamma(3\alpha + 1)} + \ldots,
\]
we obtain
\[
\Phi_1(t) = U_0 e^{\frac{\rho}{\alpha}(t-a)} + A \left( e^{\frac{\rho}{\alpha}(t-a)} \Phi_0 \right)(t)
\]
\[
\quad = U_0 e^{\frac{\rho}{\alpha}(t-a)} + AU_0 e^{\frac{\rho}{\alpha}(t-a)}(t-a)^\alpha \frac{1}{\rho^\alpha \Gamma(1 + \alpha)} U_0,
\]
\[
\quad = e^{\frac{\rho}{\alpha}(t-a)} \left( I + \frac{A(t-a)^\alpha}{\rho^\alpha \Gamma(1 + \alpha)} \right) U_0,
\]
\[
\Phi_2(t) = U_0 e^{\frac{\rho}{\alpha}(t-a)} + A \left( e^{\frac{\rho}{\alpha}(t-a)} \Phi_1 \right)(t)
\]
\[
\quad = e^{\frac{\rho}{\alpha}(t-a)} \left( I + \frac{A(t-a)^\alpha}{\rho^\alpha \Gamma(1 + \alpha)} + A^2 (t-a)^{2\alpha} \frac{1}{\rho^{2\alpha} \Gamma(1 + 2\alpha)} \right) U_0,
\]
\[
\Phi_3(t) = U_0 e^{\frac{\rho}{\alpha}(t-a)} + A \left( e^{\frac{\rho}{\alpha}(t-a)} \Phi_2 \right)(t)
\]
\[
\quad = e^{\frac{\rho}{\alpha}(t-a)} \left( I + \frac{A(t-a)^\alpha}{\rho^\alpha \Gamma(1 + \alpha)} + A^2 (t-a)^{2\alpha} \frac{1}{\rho^{2\alpha} \Gamma(1 + 2\alpha)} + A^3 (t-a)^{3\alpha} \frac{1}{\rho^{3\alpha} \Gamma(1 + 3\alpha)} \right) U_0.
\]
Taking the limit as $k \to \infty$ for $\Phi_k(t)$ (componentwise), we derive the series expression for a solution

$$
U(t) = e^{\frac{\rho}{\mu}(t-a)} \sum_{k=0}^{\infty} \frac{A^k(t-a)^{k\alpha}}{\rho^k \Gamma(1+k\alpha)} U_0 = e^{\frac{\rho}{\mu}(t-a)} A \left( \frac{t-a}{\rho} \right)^a U_0,
$$

completing the proof. \( \square \)

**Remark 2.15.** For the particular case $\rho = 1$, the result of Lemma 2.14 reduces to the one for systems of Caputo fractional differential equations [42].

### 3. Statement of the problem

Let $t_0$ be a given fixed initial time (usually $t_0 \geq 0$), and the sequence of points $\{\xi_k\}_{k \in \mathbb{N}}$ be such that $t_0 = \xi_0 < \xi_1 < \xi_2$, $k \in \mathbb{N}$, with $\lim_{k \to \infty} \xi_k = \infty$. In the literature, there are several different interpretations of solutions of the system of fractional differential equations with impulses (see, for example, [43,44]). Here, in our model, we consider the changeable lower limit of the generalized proportional Caputo fractional derivative at any updated time $\xi_k$, $k \in \mathbb{N}$. From now on, $\chi(\xi_k + 0) = \lim_{t \to \xi_k+0} \chi(t)$, $\chi(\xi_k - 0) = \lim_{t \to \xi_k-0} \chi(t)$, $k \in \mathbb{N}$.

#### 3.1. Multi-agent system without a leader

We consider the multi-agent system that consists of $N \in \mathbb{N} \setminus \{1\}$ agents. Each agent has its own scalar variable $x_i$, $i = 1, 2, \ldots, N$, and it has its own initial condition $x_i(t_0) = x_i^0$. Naturally, agents exchange information among them. Since continuous communication links among agents are hard to achieve in practice, we analyze the case where the information exchange among agents occurs only at update times, i.e., the controller updates of each agent occur at times $\xi_k$. The agent $i$ will suddenly update its state variable according to the state variables of itself and its neighbors at the instants $\xi_k$. Thus, the control input is called an impulsive control protocol. For any $i = 1, 2, \ldots, N$ and $k \in \mathbb{N}$, we consider the set

$$
\mathcal{N}_i(\xi_k) = \{ j = 1, 2, \ldots, N : j \neq i \text{ and the state variable } x_j(t) \text{ is available to agent } i \text{ at time } t = \xi_k \}.
$$

**Remark 3.1.** The set $\mathcal{N}_i(\xi_k)$ consists of the numbers of all agents which could influence the agent $i$ at the update time $\xi_k$.

Thus, the control input of agent $i$ at the time $\xi_k$, $k \in \mathbb{N}$, based on the information it receives from its neighboring agents, is designed by

$$
u_i(\xi_k) = \sum_{j \in \mathcal{N}_i(\xi_k)} a_{i,j,k} (x_j(\xi_k) - x_i(\xi_k)), \quad k \in \mathbb{N}, \quad (3.1)
$$

where the weights $a_{i,j,k} \in \mathbb{R}$ are entries of the weighted connectivity matrix

$$
A_k = \begin{bmatrix}
0 & a_{1,2,k} & a_{1,3,k} & \cdots & a_{1,N,k} \\
a_{2,1,k} & 0 & a_{2,3,k} & \cdots & a_{2,N,k} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
a_{N,1,k} & a_{N,2,k} & a_{N,3,k} & \cdots & 0
\end{bmatrix}
$$

and $a_{i,j,k} = 0$ iff $j \notin \mathcal{N}_i(\xi_k)$. Between two update times $\xi_k$ and $\xi_{k+1}$, any agent $i$ has information only about its own state. More precisely, the dynamics of agent $i$ is described by

$$
\left( \begin{array}{c}
x_i \\
is_i \end{array} \right) (t) = b_i x_i(t), \quad t \in (\xi_k, \xi_{k+1}],
$$

where $b_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$.

**Remark 3.2.** We do not assume the weights $a_{i,j,k}$ and coefficients $b_i$ to be positive.

At each time $\xi_k$, agent $i$ updates its state variable according to the impulsive control protocol defined by (3.1), i.e.,

$$
x_i(\xi_k + 0) = \nu_i(\xi_k), \quad i = 1, 2, \ldots, N, \quad k \in \mathbb{N}.
$$

The model described above can be written as a system of differential equations with impulses at times $\xi_k$ and generalized proportional Caputo fractional derivatives

$$
\left( \begin{array}{c}
x_i \\
is_i \end{array} \right) (t) = b_i x_i(t), \quad i = 1, 2, \ldots, N, \quad t \in (\xi_k, \xi_{k+1}], \quad k \in \mathbb{N}_0,
$$

$$
x_i(\xi_k + 0) = \sum_{j \in \mathcal{N}_i(\xi_k)} a_{i,j,k} (x_j(\xi_k) - x_i(\xi_k)), \quad k \in \mathbb{N},
$$

$$
x_i(t_0) = x_i^0, \quad i = 1, 2, \ldots, N.
$$

We refer to model (3.2) as a generalized proportional Caputo fractional model of multi-agent linear dynamic system via impulsive control protocol.
3.2. Multi-agent system with a leader

We consider the multi-agent system with fixed topology that consists of $N$ agents and a leader with state variables $x_i(t), i = 1, 2, \ldots, N$ and $x_0(t)$ at time $t$, respectively. The agents and the leader exchange information among themselves and there is an impulsive control protocol. Between two update times $\xi_k$ and $\xi_{k+1}$, the dynamics of any agent $i$ is based only on interaction between itself and other agents; the leader has no interactions with other agents. More precisely, the dynamics is described by

$$
\left( \frac{C}{C} \frac{d^\alpha}{dt^\alpha} x_i \right)(t) = \sum_{j=1}^{N} \xi_{ij}(x_i(t) - x_j(t)), \quad t \in (\xi_k, \xi_{k+1}]
$$

$$
\left( \frac{C}{C} \frac{d^\alpha}{dt^\alpha} x_0 \right)(t) = 0, \quad t \in (\xi_k, \xi_{k+1}], \quad k \in \mathbb{N}_0,
$$

where the weights $\xi_{ij} \geq 0$ are such that $\xi_{ij} = 0$ if the agent $j$ does not influence agent $i$. At each update time $\xi_k$, the leader interacts with some of the agents instantaneously, i.e., the agent $i$ updates its state variable according to the impulsive control protocol, and the state of the leader is continuous, namely

$$x_i(\xi_k + 0) = x_i(\xi_k) + u_i(\xi_k), \quad i = 1, 2, \ldots, N, \quad k \in \mathbb{N},$$

where the control input of agent $i$ at the time $\xi_k$, $k \in \mathbb{N}$, based on the interaction between the agent and the leader, is designed by

$$u_i(\xi_k) = \mu_{i,k}(x_i(\xi_k) - x_0(\xi_k)), \quad k \in \mathbb{N}.$$  

The model described above can be written as a system of differential equations with impulses at times $\xi_k$ and generalized proportional Caputo fractional derivatives

$$
\left( \frac{C}{C} \frac{d^\alpha}{dt^\alpha} x_i \right)(t) = \sum_{j=1}^{N} \xi_{ij}(x_i(t) - x_j(t)), \quad t \in (\xi_k, \xi_{k+1}].
$$

$$
\left( \frac{C}{C} \frac{d^\alpha}{dt^\alpha} x_0 \right)(t) = 0, \quad t \in (\xi_k, \xi_{k+1}], \quad k \in \mathbb{N}_0,
$$

$$x_i(\xi_k + 0) = x_i(\xi_k) + \mu_{i,k}(x_i(\xi_k) - x_0(\xi_k)), \quad i = 1, 2, \ldots, N.$$

$$x_0(\xi_k + 0) = x_0(\xi_k), \quad k \in \mathbb{N},$$

$$x_i(t_0) = x_i^0, \quad i = 1, 2, \ldots, N, \quad x_0(t_0) = x_0^0.$$  

Remark 3.3. From Lemma 2.7 with $a = \xi_k$, it follows that the state of the leader is

$$x_0(t) = x_0(\xi_k + 0) e^{\frac{\rho}{\Gamma(1-\alpha)} (t-\xi_k)} = x_0(\xi_k) e^{\frac{\rho}{\Gamma(1-\alpha)} (t-\xi_k)} \quad \text{on} \quad (\xi_k, \xi_{k+1}], \quad k \in \mathbb{N}_0.$$  

Inductively, from the impulsive condition $x_0(\xi_k + 0) = x_0(\xi_k)$, we get that the leader has the state $x_0(t) = x_0^0 e^{\frac{\rho}{\Gamma(1-\alpha)} (t-\xi_0^0)}$ for $t \geq t_0$.

Denote $z_i(t) = x_i(t) - x_0(t) = x_i(t) - x_0^0 e^{\frac{\rho}{\Gamma(1-\alpha)} (t-\xi_0^0)}, \quad i = 1, 2, \ldots, N$. Then, problem (3.3) can be written in the form of a generalized proportional Caputo fractional differential equation with impulses

$$
\left( \frac{C}{C} \frac{d^\alpha}{dt^\alpha} z_i \right)(t) = \sum_{j=1}^{N} \xi_{ij}(z_i(t) - z_j(t)), \quad t \in (\xi_k, \xi_{k+1}], \quad k \in \mathbb{N}_0,
$$

$$z_i(\xi_k + 0) = (1 + \mu_{i,k}) z_i(\xi_k), \quad i = 1, 2, \ldots, N, \quad k \in \mathbb{N},$$

$$z_i(t_0) = x_i^0 - x_0^0, \quad i = 1, 2, \ldots, N.$$  

Equivalently, (3.4) can be written in matrix form

$$
\left( \frac{C}{C} \frac{d^\alpha}{dt^\alpha} Z \right)(t) = LZ(t), \quad t \in (\xi_k, \xi_{k+1}], \quad k \in \mathbb{N}_0,
$$

$$Z(\xi_k + 0) = P_k Z(\xi_k), \quad k \in \mathbb{N},$$

$$Z(t_0) = Z_0,$$

where $P_k = \operatorname{diag}(1 + \mu_{1,k}, 1 + \mu_{2,k}, \ldots, 1 + \mu_{N,k})$.

$$L = \begin{bmatrix}
\sum_{j=1}^{N} \xi_{ij} & -\xi_{1,2} & \cdots & -\xi_{1,N} \\
-\xi_{2,1} & \sum_{j=1, j \neq 2}^{N} \xi_{2,j} & \cdots & -\xi_{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
-\xi_{N,1} & -\xi_{N,2} & \cdots & \sum_{j=1, j \neq N}^{N} \xi_{N,j}
\end{bmatrix},$$

and $Z_0 = (x_1^0 - x_0^0, x_2^0 - x_0^0, \ldots, x_N^0 - x_0^0)$.  

4. Main results

This section presents our main results. In particular, we prove sufficient conditions for exponential stability in the case of a multi-agent system without a leader (Section 4.1) and sufficient conditions for the leader following consensus in a multi-agent system with a leader (Section 4.2).

4.1. Multi-agent system without a leader

**Definition 4.1.** We say that the generalized proportional Caputo fractional model of multi-agent linear dynamic systems via impulsive control protocol (3.2) is exponentially stable if there exist positive numbers \( \lambda, M \) such that, for any \( x^0 \in \mathbb{R}^N \), the inequality \( \| x(t) \| \leq M e^{-\lambda (t-t_0)} \| x^0 \| \) holds for all \( t \geq t_0 \).

**Remark 4.2.** The exponential stability of the generalized proportional Caputo fractional model of multi-agent linear dynamic systems via impulsive control protocol (3.2) implies asymptotic stability, i.e., \( \lim_{t \to \infty} \| x(t) \| = 0 \).

Before moving on, we need the following result.

**Lemma 4.3.** Let \( \alpha \in (0, 1) \), \( \rho \in (0, 1) \), \( b \in \mathbb{R} \), and \( \eta_2 > \eta_1 \geq 0 \). Then,

\[
E_\alpha \left( b \left( \frac{\eta_2 - \eta_1}{\rho} \right)^\alpha \right) < e^{\frac{e^{\rho-\sqrt{\rho-1}} \eta_2 - \eta_1}{\alpha}}.
\]

**Proof.** For \( b \leq 0 \), we have

\[
e^{\frac{e^{\rho-\sqrt{\rho-1}} \eta_2 - \eta_1}{\alpha}} \leq e^{\frac{e^{\rho-\sqrt{\rho-1}} \eta_2 - \eta_1}{\alpha}} < \frac{e^{\frac{e^{\rho-\sqrt{\rho-1}} \eta_2 - \eta_1}{\alpha}}}{\alpha}.
\]

For \( b > 0 \), we have

\[
e^{\frac{e^{\rho-\sqrt{\rho-1}} \eta_2 - \eta_1}{\alpha}} = E_\alpha \left( \left[ \frac{e^{\rho-\sqrt{\rho-1}} \eta_2 - \eta_1}{\rho} \right]^\alpha \right).
\]

Thus, by Corollary 2.11 point (iv), it follows that

\[
e^{\frac{e^{\rho-\sqrt{\rho-1}} \eta_2 - \eta_1}{\alpha}} < \frac{e^{\frac{e^{\rho-\sqrt{\rho-1}} \eta_2 - \eta_1}{\alpha}}}{\alpha}
\]

holds, completing the proof. \( \square \)

**Theorem 4.4.** Let \( \alpha \in (0, 1) \) and \( \rho \in (0, 1) \). If \( B = \max_{i=1,2,\ldots,N} | b_i | < (1 - \rho)^\alpha \) and there exists a positive number \( K \leq \alpha \) such that

\[
\left| \sum_{j=1}^{N} a_{i,j} \right| + \sum_{j=1}^{N} |a_{i,j,k}| < K,
\]

then the generalized proportional Caputo fractional model of multi-agent linear dynamic systems via impulsive control protocol (3.2) is exponentially stable.

**Proof.** Let \( t \in [\xi_0, \xi_1] \). According to Lemma 2.8 with \( a = \xi_0, u_0 = x_0^0 \), and \( \lambda = b_i \), the solution to (3.2) is given by

\[
x_i(t) = x_i^0 e^{\frac{e^{\rho-\sqrt{\rho-1}}(t-\xi_0)}{\alpha}} E_\alpha \left( b_i \left( \frac{t - \xi_0}{\rho} \right)^\alpha \right), \quad t \in [\xi_0, \xi_1].
\]

Hence

\[
| x_i(t) | \leq | x_i^0 | e^{\frac{e^{\rho-\sqrt{\rho-1}}(t-\xi_0)}{\alpha}} E_\alpha \left( b_i \left( \frac{t - \xi_0}{\rho} \right)^\alpha \right), \quad t \in [\xi_0, \xi_1].
\]

Applying Lemma 4.3 with \( \eta_2 = \xi_1, \eta_1 = \xi_0 \) and \( b = b_i \), we obtain

\[
| x_i(\xi_1) | \leq | x_i^0 | e^{\frac{e^{\rho-\sqrt{\rho-1}}(\xi_1-\xi_0)}{\alpha}} E_\alpha \left( b_i \left( \frac{\xi_1 - \xi_0}{\rho} \right)^\alpha \right)
\]

\[
\leq | x_i^0 | \frac{e^{\frac{e^{\rho-\sqrt{\rho-1}}(\xi_1-\xi_0)}{\alpha}}}{\alpha}, \quad i = 1, 2, \ldots, N.
\]
Let \( t \in (\xi_1, \xi_2) \). Again, according to Lemma 2.8 with \( a = \xi_1, u_0 = x_i(\xi_1 + 0), \) and \( \lambda = b_i, \) the solution to (3.2) is given by
\[
  x_i(t) = x_i(\xi_1 + 0)e^{\frac{\rho - 1}{\alpha} (t - \xi_1)E_a \left( b_i \left( t - \frac{\xi_1}{\rho} \right)^\alpha \right)}, \quad t \in (\xi_1, \xi_2), \quad i = 1, 2, \ldots, N.
\]

An application of Lemma 4.3 enables us to write
\[
  |x_i(t)| = |x_i(\xi_1 + 0)|e^{\frac{\rho - 1}{\alpha} (t - \xi_1)E_a \left( b_i \left( t - \frac{\xi_1}{\rho} \right)^\alpha \right)}
  \leq |x_i(\xi_1 + 0)|e^{\left( \rho + \sqrt{|b_i| - 1} \right) t - \xi_1 \frac{1 - \rho}{\rho}}, \quad t \in (\xi_1, \xi_2).
\]

(4.2)

On account of (4.1), at the update time \( \xi_1 \), we have
\[
  |x_i(\xi_1 + 0)| = |u_i(\xi_1)| = |x_i(\xi_1)| \sum_{j=1}^{N} a_{i,j,1} - \sum_{j=1}^{N} a_{i,j,1}x_j(\xi_1) \leq |x_0^i| \frac{e^{\left( \rho + \sqrt{|b_i| - 1} \right) \frac{1 - \rho}{\rho}}}{\alpha} \sum_{j=1}^{N} |a_{i,j,1}| + \sum_{j=1}^{N} |a_{i,j,1}| |x_j^i| \frac{e^{\left( \rho + \sqrt{|b_i| - 1} \right) \frac{1 - \rho}{\rho}}}{\alpha} \leq \|x_0^i\| \frac{K}{\alpha^2} e^{\left( \rho + \sqrt{|b_i| - 1} \right) \frac{1 - \rho}{\rho}}.
\]

(4.3)

Using (4.3) in (4.2) implies
\[
  |x_i(t)| \leq \|x_0^i\| \frac{K}{\alpha^2} e^{\left( \rho + \sqrt{|b_i| - 1} \right) \frac{1 - \rho}{\rho}}
\]

for any \( t \in (\xi_1, \xi_2) \) and all \( i = 1, 2, \ldots, N. \) By induction with respect to intervals, we obtain
\[
  |x_i(t)| \leq \|x_0^i\| \frac{K^i}{\alpha^i} e^{\left( \rho + \sqrt{|b_i| - 1} \right) \frac{1 - \rho}{\rho}} \leq \|x_0^i\| \frac{K^k}{\alpha^{k+1}} e^{\left( \rho + \sqrt{|b_i| - 1} \right) \frac{1 - \rho}{\rho}}
\]

for any \( t \in (\xi_k, \xi_{k+1}), \) \( k \in \mathbb{N}_0, \) and all \( i = 1, 2, \ldots, N. \) Therefore, the solution to system (3.2) is exponentially stable with \( M = \frac{1}{\alpha} \) and \( \lambda = \frac{\rho - 1 - \sqrt{|b_i| - 1}}{\rho} > 0. \)

Remark 4.5. It is worth pointing out that the above approach does not allow obtaining of sufficient conditions for exponential stability of (3.2) in the case of Caputo derivative.

4.2. Multi-agent system with a leader

We begin with deriving an explicit form of a solution to linear impulsive system (3.5).

**Lemma 4.6.** The exact solution to system (3.5) is
\[
  Y(t) = e^{\frac{\rho - 1}{\alpha} (t-t_0)E_a \left( L \left( \frac{t - \xi_k}{\rho} \right)^\alpha \right)} \left( \prod_{i=0}^{k-1} \left( P_{k-i}E_a \left( L \left( \frac{\xi_k - i - 1}{\rho} \right)^\alpha \right) \right) \right) Z_0, \quad t \in (\xi_k, \xi_{k+1}), \quad k \in \mathbb{N}_0.
\]

(4.4)

**Proof.** The proof follows by induction. Let \( t \in [t_0, \xi_1]. \) By Lemma 2.14 with \( a = t_0, A = L, U_0 = Z_0, \) we obtain
\[
  Z(t) = e^{\frac{\rho - 1}{\alpha} (t-t_0)E_a \left( L \left( \frac{t - t_0}{\rho} \right)^\alpha \right)} Z_0.
\]

Therefore,
\[
  Z(\xi_1 - 0) = e^{\frac{\rho - 1}{\alpha} (\xi_1-t_0)E_a \left( L \left( \frac{\xi_1 - t_0}{\rho} \right)^\alpha \right)} Z_0.
\]
Let \( t \in (\xi_1, \xi_2] \). Applying Lemma 2.14 with \( a = \xi_1 \), \( A = L \), \( U_0 = Z(\xi_1 + 0) \) gives

\[
Z(t) = e^{\frac{\rho}{\rho} (t - \xi_1) E_\alpha \left( L \left( \frac{t - \xi_1}{\rho} \right)^a \right)} Z(\xi_1 + 0)
\]

\[
= e^{\frac{\rho}{\rho} (t - \xi_1) E_\alpha \left( L \left( \frac{t - \xi_1}{\rho} \right)^a \right)} P_1 E_\alpha \left( L \left( \frac{\xi_1 - t_0}{\rho} \right)^a \right) Z_0.
\]

For \( t \in (\xi_2, \xi_3] \), we use Lemma 2.14 with \( a = \xi_2 \), \( A = L \), \( U_0 = Z(\xi_2 + 0) \) and get

\[
Z(t) = e^{\frac{\rho}{\rho} (t - \xi_2) E_\alpha \left( L \left( \frac{t - \xi_2}{\rho} \right)^a \right)} P_2 E_\alpha \left( L \left( \frac{\xi_2 - \xi_1}{\rho} \right)^a \right) P_1 E_\alpha \left( L \left( \frac{\xi_1 - t_0}{\rho} \right)^a \right) Z_0.
\]

Repeated application of Lemma 2.14 yields (4.4).

**Definition 4.7.** We say that for the generalized proportional Caputo fractional model of multi-agent linear dynamic systems via impulsive control protocol, (3.3) achieves the leader following consensus if \( \lim_{t \to \infty} |x_i(t) - x_0(t)| = 0 \) for all \( i = 1, 2, \ldots, N \).

**Remark 4.8.** The important point to note here is that the leader following consensus in the generalized proportional Caputo fractional model of multi-agent linear dynamic systems via impulsive control protocol (3.3) is equivalent to the asymptotic stability of the system of generalized proportional Caputo type fractional differential equations with impulses (3.4).

**Theorem 4.9.** Let \( \alpha \in (0, 1) \) and \( \rho \in (0, 1) \). If there exist numbers \( q, \beta > 0 \) such that the inequalities

\[
\max_{1 \leq i \leq n} |1 + \mu_{i,k}| E_{\alpha} \left( \| L \|_2 \left( \frac{\beta}{\rho} \right)^a \right) \leq q < 1 \quad \text{and} \quad 0 < \xi_k - \xi_{k-1} \leq \beta < \infty, \quad k \in \mathbb{N}_0,
\]

hold, then the generalized proportional Caputo fractional model of multi-agent linear dynamic systems via impulsive control protocol (3.3) achieves the leader following consensus.

**Proof.** Let us denote by \( Z \) the solution to (3.4) (or equivalently, to (3.5)). Since

\[
P_k^T P_k = \text{diag}(1 + \mu_{1,k})^2, (1 + \mu_{2,k})^2, \ldots, (1 + \mu_{N,k})^2),
\]

we have \( \lambda_i = (1 + \mu_{i,k})^2 \) and

\[
\| P_k \|_2 = \sqrt{\max_{1 \leq i \leq n} (1 + \mu_{i,k})^2} = \max_{1 \leq i \leq n} |1 + \mu_{i,k}|.
\]

Hence,

\[
\| P_k E_\alpha \left( L \left( \frac{\xi_k - \xi_{k-1}}{\rho} \right)^a \right) \|_2 \leq \max_{1 \leq i \leq n} |1 + \mu_{i,k}| E_\alpha \left( \| L \|_2 \left( \frac{\beta}{\rho} \right)^a \right) \leq q < 1.
\]

Applying Lemma 4.6 and Proposition 2.12, we obtain

\[
\| Z(t) \| \leq e^{\frac{\rho}{\rho} (t - t_0) E_\alpha \left( L \left( \frac{t - \xi_k}{\rho} \right)^a \right)} \left( \prod_{i=0}^{k-1} \left( \| P_{k-i} E_\alpha \left( L \left( \frac{\xi_{k-i} - \xi_{k-i-1}}{\rho} \right)^a \right) \|_2 \right) \right) \| Z_0 \|
\]

\[
\leq e^{\frac{\rho}{\rho} (t - t_0) E_\alpha \left( \| L \|_2 \left( \frac{t - \xi_k}{\rho} \right)^a \right)} \| Z_0 \|
\]

\[
\leq e^{\frac{\rho}{\rho} (t - t_0) E_\alpha \left( \| L \|_2 \left( \frac{\beta}{\rho} \right)^a \right)} \| Z_0 \|,
\]

for \( t \in (\xi_k, \xi_{k+1}] \) and all \( k \in \mathbb{N}_0 \). Therefore, the generalized proportional Caputo type fractional differential equation with impulses (3.4) is exponentially stable with

\[
M = E_\alpha \left( \| L \|_2 \left( \frac{\beta}{\rho} \right)^a \right) \quad \text{and} \quad \lambda = \frac{1 - \rho}{\rho} > 0,
\]

and this, by Remark 4.8, is the desired conclusion. \( \square \)

**5. Applications**

In this section, numerical examples are presented to verify the effectiveness of the proposed impulsive control protocol for generalized proportional Caputo fractional multi-agent linear dynamical systems.
Example 5.1. Let us consider a generalized proportional Caputo fractional model of multi-agent linear dynamic system via impulsive control (see (3.2)) with
\[
N = 5, \ t_0 = \xi_0 = 0, \ \xi_{k+1} = \xi_k + 1, \ k \in \mathbb{N}_0, \ \alpha = 0.65,
\]
i.e.,
\[
\begin{align*}
\left( C_{\xi_k}^{D0.65,\rho} x_1 \right)(t) &= 0.86x_1(t), \quad \left( C_{\xi_k}^{D0.65,\rho} x_2 \right)(t) = 0.78x_2(t), \\
\left( C_{\xi_k}^{D0.65,\rho} x_3 \right)(t) &= -0.1x_3(t), \quad \left( C_{\xi_k}^{D0.65,\rho} x_4 \right)(t) = -0.65x_4(t), \\
\left( C_{\xi_k}^{D0.65,\rho} x_5 \right)(t) &= 0.85x_5(t), \quad t \in (\xi_k, \xi_{k+1}), \ k \in \mathbb{N}_0.
\end{align*}
\]
\[
x_1(\xi_k + 0) = -0.01(x_1(\xi_k) - x_2(\xi_k)) + 0.04(x_1(\xi_k) - x_3(\xi_k)) \\
-0.08(x_1(\xi_k) - x_4(\xi_k)) + 0.25(x_1(\xi_k) - x_5(\xi_k)),
\]
\[
x_2(\xi_k + 0) = -0.01(x_1(\xi_k) - x_2(\xi_k)) + 0.2(x_2(\xi_k) - x_3(\xi_k)) \\
+0.09(x_2(\xi_k) - x_4(\xi_k)) - 0.08(x_2(\xi_k) - x_5(\xi_k)),
\]
\[
x_3(\xi_k + 0) = 0.09(x_3(\xi_k) - x_1(\xi_k)) - 0.08(x_3(\xi_k) - x_2(\xi_k)) \\
-0.23(x_3(\xi_k) - x_4(\xi_k)) - 0.01(x_3(\xi_k) - x_5(\xi_k)),
\]
\[
x_4(\xi_k + 0) = -0.01(x_4(\xi_k) - x_1(\xi_k)) + 0.09(x_4(\xi_k) - x_2(\xi_k)) \\
-0.08(x_4(\xi_k) - x_3(\xi_k)) + 0.23(x_4(\xi_k) - x_5(\xi_k)),
\]
\[
x_5(\xi_k + 0) = 0.2(x_5(\xi_k) - x_1(\xi_k)) - 0.04(x_5(\xi_k) - x_2(\xi_k)) \\
+0.08(x_5(\xi_k) - x_3(\xi_k)) - 0.09(x_5(\xi_k) - x_4(\xi_k)), \quad k \in \mathbb{N},
\]
\[
x_i(0) = 1, \quad i = 1, 2, 3, 4, 5.
\]
Observe that the weighted connectivity matrix is the same at any updated times, and it is given by
\[
A = \begin{bmatrix}
0 & 0.01 & -0.04 & 0.08 & -0.25 \\
0.01 & 0 & -0.2 & -0.09 & 0.08 \\
-0.09 & 0.08 & 0 & -0.23 & 0.01 \\
0.01 & -0.09 & 0.08 & 0 & -0.23 \\
-0.2 & 0.04 & -0.08 & 0.09 & 0
\end{bmatrix}.
\]
First, we analyze the case when \( \rho = 0.2 \). Then,
\[
B = 0.86 < (1 - 0.2)^{0.65} \approx 0.864985
\]
and
\[
\begin{align*}
\sum_{j=1}^{5} |a_{1,j,k}| + \sum_{j=1}^{5} |a_{2,j,k}| &= 0.2 + 0.38 = 0.58 \leq 0.65, \\
\sum_{j=1}^{5} |a_{3,j,k}| + \sum_{j=1}^{5} |a_{4,j,k}| &= 0.2 + 0.38 = 0.58 < 0.65, \\
\sum_{j=1}^{5} |a_{5,j,k}| + \sum_{j=1}^{5} |a_{6,j,k}| &= 0.23 + 0.41 = 0.64 < 0.65. \\
\end{align*}
\]
Therefore, the conditions of Theorem 4.4 are satisfied, and the considered generalized proportional Caputo fractional model of multi-agent linear dynamic systems via impulsive control protocol is exponentially stable with
\[
\lambda = \frac{1 - 0.2}{0.2 - 0.65^{0.65}} \approx 0.106109.
\]
The graphs of \( |x_i(t)| \) are drawn in Fig. 5.1. It can be seen they are bounded above by \( \frac{\sqrt{7}}{0.65} e^{-0.106109t} \). In the second case, we put \( \rho = 0.9 \). Then the condition
\[
B = 0.85 < (1 - 0.9)^{0.65} \approx 0.223872
\]
is not satisfied. From Fig. 5.2, we can see that the system is not exponentially stable.

**Example 5.2.** Let us consider a generalized proportional Caputo fractional model of multi-agent linear dynamic system without impulsive control protocol with

\[ N = 5, \quad t_0 = 0, \quad \alpha = 0.65, \quad \rho = 0.9, \]

i.e.,

\[
\begin{align*}
\left( C^{0.65,0.2}_t D_{t_0}^{0.65,0.2} x_1 \right)(t) &= 0.86x_1(t), \\
\left( C^{0.65,0.2}_t D_{t_0}^{0.65,0.2} x_2 \right)(t) &= 0.78x_2(t), \\
\left( C^{0.65,0.2}_t D_{t_0}^{0.65,0.2} x_3 \right)(t) &= -0.1x_3(t), \\
\left( C^{0.65,0.2}_t D_{t_0}^{0.65,0.2} x_4 \right)(t) &= -0.65x_4(t), \\
\left( C^{0.65,0.2}_t D_{t_0}^{0.65,0.2} x_5 \right)(t) &= 0.85x_5(t), \\
\end{align*}
\]

\[(5.2)\]

By Lemma 2.8, the solution of (5.2) is given by

\[ x_i(t) = x_i^0 e^{-0.2t} E_{0,3} \left( b_i \left( \frac{t}{0.2} \right)^{0.3} \right) = x_i^0 e^{-4t} E_{0,3}(b_i(5t)^{0.3}), \quad i = 1, 2, 3, 4, 5, \]

where \( b_1 = 0.86, \quad b_2 = 0.78, \quad b_3 = -0.1, \quad b_4 = -0.65, \quad b_5 = 0.85 \). The graphs of \(|x_i(t)|, \ i = 1, 2, 3, 4, 5\), are shown in Fig. 5.3. It can be seen that all components of the solution approach zero.

**Example 5.3.** In order to show the importance of the type of derivative appearing in the model, we consider system (5.1), in which the generalized proportional derivatives are replaced by the integer-order ones, i.e., the first-order ordinary derivative is applied in the model instead of a fractional one. The graphs of all components \(|x_i(t)|, \ i = 1, 2, 3, 4, 5\), of the corresponding solution are shown in Fig. 5.4.
Example 5.5. Let us consider the following multi-agent system with a leader in models.

From Figs. 5.1-5.4, we may conclude that the type of the applied derivative definitely has an influence on the behavior of the studied multi-agent system. What follows is the necessity of the application of various types of derivatives in models.

**Remark 5.4.** From Figs. 5.1-5.4, we may conclude that the type of the applied derivative definitely has an influence on the behavior of the studied multi-agent system. What follows is the necessity of the application of various types of derivatives in models.

**Example 5.5.** Let us consider the following multi-agent system with a leader

\[
\begin{align*}
\left( \frac{D_t^{0.8,0.6}}{C_0} x_1 \right)(t) &= 0.9(x_1(t) - x_2(t)), \\
\left( \frac{D_t^{0.8,0.6}}{C_0} x_2 \right)(t) &= 0.1(x_2(t) - x_1(t)) + 0.1(x_2(t) - x_3(t)), \\
\left( \frac{D_t^{0.8,0.6}}{C_0} x_3 \right)(t) &= 0.5(x_3(t) - x_2(t)) + 0.3(x_3(t) - x_4(t)), \\
\left( \frac{D_t^{0.8,0.6}}{C_0} x_4 \right)(t) &= 0.1(x_4(t) - x_3(t)), \\
\left( \frac{D_t^{0.8,0.6}}{C_0} x_0 \right)(t) &= 0, & t \in \{ \xi_k, \xi_{k+1} \}, & k \in \mathbb{N}, \\
x_1(\xi_k + 0) &= x_1(\xi_k) - 0.7(x_1(\xi_k) - x_0(\xi_k)), \\
x_2(\xi_k + 0) &= x_2(\xi_k) - 1.3(x_2(\xi_k) - x_0(\xi_k)), \\
x_3(\xi_k + 0) &= x_3(\xi_k) - 0.7(x_3(\xi_k) - x_0(\xi_k)), \\
x_4(\xi_k + 0) &= x_4(\xi_k) - 1.3(x_4(\xi_k) - x_0(\xi_k)), & k \in \mathbb{N}, \\
x_i(0) &= x_i^0, & i = 0, 1, 2, 3, 4.
\end{align*}
\]

System (5.3) is of type (3.3) with $N = 4$ agents, $t_0 = \xi_0 = 0$, $\xi_{k+1} = \xi_k + 0.5$, $k \in \mathbb{N}$, $\rho = 0.6$, and $\alpha = 0.8$. In this case, we have $\beta = 0.5$, $P_k = \text{diag}(0.3, -0.3, 0.3, -0.3)$, and

\[
L = \begin{bmatrix}
0.9 & -0.9 & 0 & 0 \\
-0.1 & 0.2 & -0.1 & 0 \\
0 & -0.5 & 0.8 & -0.3 \\
0 & 0 & -0.1 & 0.1
\end{bmatrix}.
\]
Fig. 5.5. Errors of state-tracking $|x_i(t) - x_0(t)|$, $i = 1, 2, 3, 4$, for the solution to (5.3).

Fig. 5.6. Errors of state-tracking $|x_i(t) - e^{-\frac{2t}{3}}|$, $i = 1, 2, 3, 4$, for the solution to (5.4).

Hence $x_0(t) = e^{\frac{0.6}{0.8(t+1)}}$, $\|L\|_2 = 1.0466$, and

$$0.3E_{0.8} \left( 0.5 \left( 0.6^{0.8} \right) \right) = 0.873455 = q < 1.$$  

According to Theorem 4.9 for the generalized proportional Caputo fractional model of multi-agent linear dynamic systems via impulsive control protocol, (5.3) achieves a leader following consensus, i.e.,

$$\lim_{t \to \infty} |x_i(t) - x_0(t)| = \lim_{t \to \infty} |x_i(t) - e^{-\frac{2t}{3}}| = 0,$$

for $i = 1, 2, 3, 4$ (see Fig. 5.5). Now, let us consider the case without the impulsive interaction of the leader, i.e., the system without impulsive control protocol

$$\begin{align*}
\left( D^{0.8,0.6}_{0.8} x_1 \right)(t) &= 0.9(x_1(t) - x_2(t)), \\
\left( D^{0.8,0.6}_{0.8} x_2 \right)(t) &= 0.1(x_2(t) - x_1(t)) + 0.1(x_2(t) - x_3(t)), \\
\left( D^{0.8,0.6}_{0.8} x_3 \right)(t) &= 0.5(x_3(t) - x_2(t)) + 0.3(x_3(t) - x_4(t)), \\
\left( D^{0.8,0.6}_{0.8} x_4 \right)(t) &= 0.1(x_4(t) - x_3(t)), \\
x_i(0) &= x_i^0, \quad i = 1, 2, 3, 4. 
\end{align*}$$  

\text{(5.4)}

The explicit solution to (5.4) may be found by Lemma 2.14. As shown in Fig. 5.6, errors of state-tracking $|x_i(t) - e^{-\frac{2t}{3}}|$, $i = 1, 2, 3, 4$, are not bounded. It follows that even the impulsive interaction of the leader can cause consensus.
Example 5.6. Finally, similarly to Example 5.3, we analyze the model (5.4) with ordinary derivatives. In this case, the state of the leader is $x_0(t) = x_0, \ t \geq 0$. Therefore, for $x_0 = 1$, the model is

$$
\begin{align*}
  x'_1(t) &= 0.9(x_1(t) - x_2(t)), \\
  x'_2(t) &= 0.1(x_2(t) - x_1(t)) + 0.1(x_2(t) - x_3(t)), \\
  x'_3(t) &= 0.5(x_3(t) - x_2(t)) + 0.3(x_3(t) - x_4(t)), \\
  x'_4(t) &= 0.1(x_4(t) - x_3(t)), \quad t \in \{\xi_k, \xi_{k+1}\}, \ k \in \mathbb{N}, \\
  x_1(\xi_k + 0) &= x_1(\xi_k) - 0.7(x_1(\xi_k) - 1), \\
  x_2(\xi_k + 0) &= x_2(\xi_k) - 1.3(x_2(\xi_k) - 1), \\
  x_3(\xi_k + 0) &= x_3(\xi_k) - 0.7(x_3(\xi_k) - 1), \\
  x_4(\xi_k + 0) &= x_4(\xi_k) - 1.3(x_4(\xi_k) - 1), \quad k \in \mathbb{N}, \\
  x_i(0) &= x_0^i, \quad i = 1, 2, 3, 4. \\
\end{align*}
$$

As shown in Fig. 5.7, errors of state-tracking $|x_i(t) - 1|, i = 1, 2, 3, 4$, are bounded. It follows that we may expect the leader-following consensus. However, without the impulsive interaction of the leader, the multi-agent system

$$
\begin{align*}
  x'_1(t) &= 0.9(x_1(t) - x_2(t)), \\
  x'_2(t) &= 0.1(x_2(t) - x_1(t)) + 0.1(x_2(t) - x_3(t)), \\
  x'_3(t) &= 0.5(x_3(t) - x_2(t)) + 0.3(x_3(t) - x_4(t)), \\
  x'_4(t) &= 0.1(x_4(t) - x_3(t)), \quad t > 0, \\
  x_i(t_0) &= x_0^i, \quad i = 1, 2, 3, 4, \\
\end{align*}
$$

similarly to the fractional-order system, does not achieve consensus (see Fig. 5.8).
6. Conclusions

In this paper, we have studied multi-agent systems with the generalized proportional Caputo fractional derivative. The information exchange between agents occurred only at fixed initially given update times, and the lower limit of the fractional derivative was changing according to the update times. We have obtained an explicit form of the solution to the system of linear generalized proportional Caputo fractional differential equations (see Lemma 2.14) as well as for the solutions to the system of linear generalized proportional Caputo fractional differential equations with impulses (see Lemma 4.6). Both results could be useful for various studies of qualitative properties of solutions to the corresponding linear systems as well as nonlinear systems. Two types of multi-agent systems have been considered, namely without and with a leader. For a generalized proportional Caputo fractional model of multi-agent linear dynamic system, sufficient conditions for exponential stability via impulsive control were obtained. In the case of the presence of a leader in the multi-agent system, we derived sufficient conditions for the leader following consensus via impulsive control based on the leader's influence. Simulation results have verified the essential role of the generalized proportional Caputo fractional derivative and impulsive control in realizing the consensus of multi-agent systems.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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