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Research paper

A generalized proportional Caputo fractional model of multi-agent linear dynamic systems via impulsive control protocol

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ABSTRACT

This paper deals with multi-agent systems that, due to using the generalized proportional Caputo fractional derivative, possess memories. The information exchange between agents does not occur continuously but only at fixed given update times, and the lower limit of the fractional derivative changes according to the update times. Two types of multi-agent systems are studied, namely systems without a leader and systems with a leader. For a generalized proportional Caputo fractional model of multi-agent linear dynamic systems, sufficient conditions for exponential stability via impulsive control are obtained. In the case of the presence of a leader in the multi-agent system, we derive sufficient conditions for the leader-following consensus via impulsive control based on the leader's influence. Simulation results are provided to verify the essential role of the generalized proportional Caputo fractional derivative and impulsive control in realizing the consensus of multi-agent systems.

1. Introduction

Due to the rapid development of embedded systems and communication technology, multi-agent systems have drawn much attention from researchers in science and engineering applications. A multi-agent system is a group of usually autonomous agents that can cooperate with each other in order to accomplish given tasks that a single agent cannot. The problem of synchronization of multi-agent systems has attracted considerable attention in the last decades (see, e.g., [1–6] and survey papers [7,8]). Broadly speaking, synchronization of networked multi-agents mean that by using communication networks and local controllers, the agents should be steered towards a common trajectory. It is obvious that this problem has meaningful applications in multiple areas, such as formation control of mobile robots, target tracking, spacecraft formation flying, and so on [9–11].

It is well known that many natural phenomena in complex environments cannot be accurately explained by using the framework of integer-order dynamics. One of the possible ways to deal with this problem is by expanding

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the existing integer-order models to fractional-order models, which allow for more realistic modeling having much higher freedom to fit possible experimental data, as well as allowing the description of memories and hereditary effects of various materials and processes. In the last thirty years, research concerning applications of fractional-order dynamics has made profound and significant progress. Examples include the time-fractional damage model for hyperelastic body (e.g., to mimic the abdominal aortic aneurysm phenomena [12]), fractional-order models of long memory processes (e.g., the steelmaking process [13]), the fractional-order model of learning [14], fractional-order models in viscoelasticity [15], and economics [16,17]. In particular, fractional calculus was also introduced into the modeling of multi-agent systems. To the best of our knowledge, the first paper devoted to this subject was [18]. Until now, plenty of theoretical results have been obtained. Examples include: studies of coordination algorithms when the fixed interaction graph is directed [19], studies of distributed formation control problems under a dynamic interaction and with absolute/relative damping [20], studies of consensus problems [21,22] and cluster consensus problems [23], studies of leader-following consensus problems [24–27], studies of distributed formation control laws with relative damping and communication delay [28], to name a few. In all aforementioned works, problems were analyzed based on continuous control. It is obvious that unnecessary communication will lead to a waste of energy. On the other hand, continuous communication links among agents are hard to achieve and they may also cause communication resource competition among agents. Therefore, it is reasonable to analyze problems under the assumption that the information exchange among agents occurs only from time to time, at update times. Notably, an adequate tool for modeling such problems is so-called impulsive control. This method was applied, e.g., in [29,30] for integer-order systems, [31,32] for fractional-order systems, and in [33] for randomly occurring update times.

Motivated by this research, we consider multi-agent linear dynamic systems with the generalized proportional Caputo fractional derivative and an impulsive control protocol. The generalized proportional Caputo fractional derivative was introduced in [34] and subsequently studied in [35–38] as an undeviating generalization of the existing Caputo fractional derivative. Namely, in this derivative, we have two parameters: $\alpha \geq 0$ which is the order of the derivative, and $\rho \in (0, 1]$ which could be called the proportionality parameter. Setting the latter parameter equal to 1, we obtain the Caputo fractional derivative. In the former case, we are interested in exponential stability. However, in some practical applications, it is desirable that all the agents track a given trajectory. Therefore, in the latter case, we focus on leader-following consensus. In this paper, we analyze two types of multi-agent linear dynamic systems with generalized proportional Caputo fractional derivative and an impulsive control protocol, namely systems without a leader and with a leader.

The rest of this paper is organized in the following manner. Section 2 includes definitions of the generalized proportional fractional operators, useful lemmas and propositions needed in the sequel. In Section 3, we introduce two types of multi-agent linear dynamic systems with generalized proportional Caputo fractional derivative and an impulsive control protocol, namely systems without a leader and systems with a leader. Section 4 contains our main results and is divided into two parts. In the first part, for a multi-agent system without a leader, sufficient conditions for exponential stability via impulsive control are obtained. In the second part, we derive an explicit form of a solution to a linear impulsive system with the generalized proportional Caputo fractional derivative. Then, for a multi-agent system with a leader, we prove sufficient conditions for the leader-following consensus via impulsive control based on the leader's influence. Numerical simulation examples are given to validate the theoretical analysis in Section 5, and the concluding statements are drawn in Section 6.

2. Preliminaries

We start by recalling definitions of the generalized proportional fractional operators. Let $a, b \in \mathbb{R}$ with $b \leq \infty$ (if $b = \infty$, then the interval is half open), and $\rho \in (0, 1]$ a fixed parameter.

Definition 2.1 (*See [34]*)**.** Let $u : [a, b] \to \mathbb{R}$ and $\alpha \geq 0$. The generalized proportional fractional integral of a function u is defined by

$$\left({}_a\mathcal{I}^{\alpha,\rho}u\right)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}u(s)\mathrm{d}s, \quad t \in (a, b] \tag{2.1}$$

as long as this integral is well defined.

Definition 2.2 (*See [34]*)**.** Let $u : [a, b] \to \mathbb{R}$ and $\alpha \in (0, 1)$. The generalized proportional Caputo fractional derivative of a function u is defined by

$$\begin{aligned}\left({}_a^{\mathrm{C}}\mathcal{D}^{\alpha,\rho}u\right)(t) &= \left({}_a\mathcal{I}^{1-\alpha,\rho}\left(\mathcal{D}^{1,\rho}u\right)\right)(t)\\ &= \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)}\frac{(\mathcal{D}^{1,\rho}u)(s)}{(t-s)^\alpha}\mathrm{d}s, \quad t \in (a, b],\end{aligned} \tag{2.2}$$

as long as this integral is well defined, where $\mathcal{D}^{1,\rho}u = (1-\rho)u + \rho u'$.

Definitions 2.1 and 2.2 can be generalized componentwisely for $u \in \mathrm{C}([a, b], \mathbb{R}^n)$.

Remark 2.3. If $\rho = 1$, then the generalized proportional Caputo fractional derivative is reduced to the classical Caputo fractional derivative.

Now we cite some important results involving generalized proportional fractional operators, which are useful for our further computations.

Lemma 2.4 (*See [34, Theorem 5.3]*)**.** *For $\rho \in (0, 1]$ and $\alpha \in (0, 1)$, we have*

$$\left({}_a\mathcal{I}^{\alpha,\rho}\left({}_a^{\mathrm{C}}\mathcal{D}^{\alpha,\rho}u\right)\right)(t) = u(t) - u(a)e^{\frac{\rho-1}{\rho}(t-a)}. \tag{2.3}$$

Lemma 2.5 (*See [34, Proposition 3.7]*)**.** *For $\rho \in (0, 1]$, $\alpha \in (0, 1)$, and $\beta > 0$, we have*

$$\left({}_a\mathcal{I}^{\alpha,\rho}u\right)(t) = \frac{\Gamma(\beta)}{\rho^\alpha\Gamma(\beta+\alpha)}(t-a)^\alpha u(t), \quad \textit{where} \quad u(t) = e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-1}. \tag{2.4}$$

Remark 2.6. Note that, if $\rho \in (0, 1)$, then the generalized proportional Caputo fractional derivative of a constant is not zero.

Lemma 2.7 (*See [34, Remark 3.2]*)**.** *For $\rho \in (0, 1]$ and $\alpha \in (0, 1)$, we have*

$$\left({}_a^{\mathrm{C}}\mathcal{D}^{\alpha,\rho}u\right)(t) = 0, \quad \textit{where} \quad u(t) = e^{\frac{\rho-1}{\rho}(t-a)}, \quad t > a. \tag{2.5}$$

We will use the explicit form of the solution to the initial value problem for the scalar linear generalized proportional Caputo fractional differential equation, which is given in [34, Example 5.7] (with necessary slight corrections).

Lemma 2.8. *A solution to the scalar linear generalized proportional Caputo fractional initial value problem*

$$\left({}_a^{\mathrm{C}}\mathcal{D}^{\alpha,\rho}u\right)(t) = \lambda u(t), \quad u(a) = u_0, \quad \alpha \in (0, 1), \quad \rho \in (0, 1],$$

is given by

$$u(t) = u_0 e^{\frac{\rho-1}{\rho}(t-a)} E_\alpha\left(\lambda\left(\frac{t-a}{\rho}\right)^\alpha\right),$$

where E_α is the Mittag-Leffler function of one parameter.

For a vector $x \in \mathbb{R}^n$, we denote by $\|x\|$ its Euclidean norm. Let A^{T} be the transpose of a matrix A. For a matrix $A = \{a_{ij}\}_{i,j=1}^n \in \mathbb{R}^{n\times n}$, we use the spectral norm

$$\|A\|_2 = \sqrt{\max_{1\le i\le n}\lambda_i},$$

where λ_i are the eigenvalues of $A^{\mathrm{T}}A$. Then we have

$$\|A\|_2 \le \sum_{i=1}^n\sum_{j=1}^n a_{ij}^2, \quad \left\|e^A\right\|_2 \le e^{\|A\|_2}, \quad \|Ax\| \le \|A\|_2\,\|x\|.$$

Definition 2.9 (*See [39]*)**.** Let A be an arbitrary square matrix. The Mittag-Leffler matrix function with one parameter α is defined as

$$E_\alpha(Az) = \sum_{k=0}^\infty \frac{A^k z^k}{\Gamma(k\alpha+1)}, \quad \alpha > 0.$$

Let us recall some properties of the Mittag-Leffler function.

Proposition 2.10 (*See [40, Theorem 1.2]*)**.** *For every $\alpha \in (0, 1)$, $t \mapsto \frac{e^t}{\alpha} - E_\alpha(t^\alpha)$ is completely monotonic.*

Corollary 2.11. *Let $\alpha \in (0, 1)$. Then*

(i) $0 < \frac{e^t}{\alpha} - E_\alpha(t^\alpha) \le \frac{1}{\alpha} - 1$, $t \ge 0$;
(ii) $\alpha \le 1 - \alpha e^{-t}(\frac{1}{\alpha} - 1) \le \alpha e^{-t}E_\alpha(t^\alpha)$, $t \ge 0$;
(iii) $\lim_{t\to 0}\left(\frac{e^t}{\alpha} - E_\alpha(t^\alpha)\right) = 0$, $t \ge 0$;
(iv) $E_\alpha(t^\alpha) < \frac{e^t}{\alpha}$, $t \ge 0$.

Proposition 2.12. *For a matrix $A \in \mathbb{R}^{n\times n}$ and $\alpha \in (0,1)$, we have the inequality*

$$\|E_\alpha(A)\|_2 \le E_\alpha\left(\|A\|_2\right).$$

Example 2.13. Let

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}.$$

Using the codes provided by Roberto Garrappa [41], we obtain

$$E_{0.8}(A) = \begin{bmatrix} 68.7603 & -88.6273 \\ -29.5424 & 39.2179 \end{bmatrix}.$$

Since the maximum eigenvalue of $A^{\mathrm{T}}A$ is $0.5(15+\sqrt{221})$, it follows that

$$\|A\|_2 = \sqrt{0.5\left(15+\sqrt{221}\right)}.$$

Thus,

$$\|E_{0.8}(A)\|_2 = \sqrt{14993.2} = 122.447 < 281.754 = E_{0.8}(\|A\|_2) = 281.754.$$

The following lemma is used to derive our main results.

Lemma 2.14. *A solution to the initial value problem for the system of linear generalized proportional Caputo fractional differential equations*

$$\left({}_a^{\mathrm{C}}\mathcal{D}^{\alpha,\rho}U\right)(t) = AU(t), \quad U(a) = U_0, \quad \alpha \in (0,1), \quad \rho \in (0,1], \tag{2.6}$$

where $U_0 \in \mathbb{R}^n$ and A is an $n\times n$ dimensional matrix, is given by

$$U(t) = e^{\frac{\rho-1}{\rho}(t-a)} E_\alpha\left(A\left(\frac{t-a}{\rho}\right)^\alpha\right) U_0,$$

where E_α is the one-parametric Mittag-Leffler function.

Proof. We use the Picard iterative process to derive the series solution to (2.6). Applying the operator ${}_a\mathcal{I}^{\alpha,\rho}$ to both sides of the equation $\left({}_a^{\mathrm{C}}\mathcal{D}^{\alpha,\rho}U\right)(t) = AU(t)$ and using the initial condition, by Lemma 2.4, we obtain

$$U(t) = U_0 e^{\frac{\rho-1}{\rho}(t-a)} + A\left({}_a\mathcal{I}^{\alpha,\rho}U\right)(t).$$

Define $\Phi_0 : [a,\infty) \to \mathbb{R}^n$ by $\Phi_0(t) = U(a)e^{\frac{\rho-1}{\rho}(t-a)}$. For $k \in \mathbb{N}$, by the recurrence formula, we calculate the kth approximate solution $\Phi_k : [a,\infty) \to \mathbb{R}^n$:

$$\Phi_k(t) = U_0 e^{\frac{\rho-1}{\rho}(t-a)} + A\left({}_a\mathcal{I}^{\alpha,\rho}\Phi_{k-1}\right)(t).$$

Then, from the recurrence formula, by using Lemma 2.5 with $\beta = 1, \alpha+1, 2\alpha+1\ldots$, and

$$E_\alpha(At^\alpha) = I + \frac{At^\alpha}{\Gamma(\alpha+1)} + \frac{A^2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{A^3t^{3\alpha}}{\Gamma(k\alpha+1)} + \ldots,$$

we obtain

$$\begin{aligned}
\Phi_1(t) &= U_0 e^{\frac{\rho-1}{\rho}(t-a)} + A\left({}_a\mathcal{I}^{\alpha,\rho}\Phi_0\right)(t) \\
&= U_0 e^{\frac{\rho-1}{\rho}(t-a)} + AU_0 e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^\alpha \frac{1}{\rho^\alpha\Gamma(1+\alpha)} \\
&= e^{\frac{\rho-1}{\rho}(t-a)}\left(I + \frac{A(t-a)^\alpha}{\rho^\alpha\Gamma(1+\alpha)}\right)U_0, \\
\Phi_2(t) &= U_0 e^{\frac{\rho-1}{\rho}(t-a)} + A\left({}_a\mathcal{I}^{\alpha,\rho}\Phi_1\right)(t) \\
&= e^{\frac{\rho-1}{\rho}(t-a)}\left(I + \frac{A(t-a)^\alpha}{\rho^\alpha\Gamma(1+\alpha)} + \frac{A^2(t-a)^{2\alpha}}{\rho^{2\alpha}\Gamma(1+2\alpha)}\right)U_0, \\
\Phi_3(t) &= U_0 e^{\frac{\rho-1}{\rho}(t-a)} + A\left({}_a\mathcal{I}^{\alpha,\rho}\Phi_2\right)(t) \\
&= e^{\frac{\rho-1}{\rho}(t-a)}\left(I + \frac{A(t-a)^\alpha}{\rho^\alpha\Gamma(1+\alpha)} + \frac{A^2(t-a)^{2\alpha}}{\rho^{2\alpha}\Gamma(1+2\alpha)} + \frac{A^3(t-a)^{3\alpha}}{\rho^{3\alpha}\Gamma(1+3\alpha)}\right)U_0.
\end{aligned}$$

Taking the limit as $k \to \infty$ for $\Phi_k(t)$ (componentwise), we derive the series expression for a solution

$$U(t) = e^{\frac{\rho-1}{\rho}(t-a)} \sum_{k=0}^{\infty} \frac{A^k (t-a)^{k\alpha}}{\rho^{k\alpha}\Gamma(1+k\alpha)} U_0 = e^{\frac{\rho-1}{\rho}(t-a)} E_\alpha \left(A \left(\frac{t-a}{\rho} \right)^\alpha \right) U_0,$$

completing the proof. □

Remark 2.15. For the particular case $\rho = 1$, the result of Lemma 2.14 reduces to the one for systems of Caputo fractional differential equations [42].

3. Statement of the problem

Let t_0 be a given fixed initial time (usually $t_0 \geq 0$), and the sequence of points $\{\xi_k\}_{k\in\mathbb{N}_0}$ be such that $t_0 = \xi_0 < \xi_k < \xi_{k+1}$, $k \in \mathbb{N}$, with $\lim_{k\to\infty} \xi_k = \infty$. In the literature, there are several different interpretations of solutions of the system of fractional differential equations with impulses (see, for example, [43,44]). Here, in our model, we consider the changeable lower limit of the generalized proportional Caputo fractional derivative at any updated time ξ_k, $k \in \mathbb{N}$. From now on,

$$x(\xi_k + 0) = \lim_{t\to\xi_k+0} x(t), \quad x(\xi_k - 0) = \lim_{t\to\xi_k-0} x(t), \quad k \in \mathbb{N}.$$

3.1. Multi-agent system without a leader

We consider the multi-agent system that consists of $N \in \mathbb{N} \setminus \{1\}$ agents. Each agent has its own scalar variable x_i, $i = 1, 2, \ldots, N$, and it has its own initial condition $x_i(t_0) = x_i^0$. Naturally, agents exchange information among them. Since continuous communication links among agents are hard to achieve in practice, we analyze the case where the information exchange among agents occurs only at update times, i.e., the controller updates of each agent occur at times ξ_k. The agent i will suddenly update its state variable according to the state variables of itself and its neighbors at the instants ξ_k. Thus, the control input is called an *impulsive control protocol*. For any $i = 1, 2, \ldots, N$ and $k \in \mathbb{N}$, we consider the set

$$\mathcal{N}_i(\xi_k) = \{j = 1, 2, \ldots, N : j \neq i \text{ and the state variable } x_j(t) \text{ is available to agent } \ i \text{ at time } t = \xi_k\}.$$

Remark 3.1. The set $\mathcal{N}_i(\xi_k)$ consists of the numbers of all agents which could influence the agent i at the update time ξ_k.

Thus, the control input of agent i at the time ξ_k, $k \in \mathbb{N}$, based on the information it receives from its neighboring agents, is designed by

$$u_i(\xi_k) = \sum_{j\in\mathcal{N}_i(\xi_k)} a_{i,j,k} \left(x_i(\xi_k) - x_j(\xi_k) \right), \quad k \in \mathbb{N}, \tag{3.1}$$

where the weights $a_{i,j,k} \in \mathbb{R}$ are entries of the weighted connectivity matrix

$$A_k = \begin{bmatrix} 0 & a_{1,2,k} & a_{1,3,k} & \ldots & a_{1,N,k} \\ a_{2,1,k} & 0 & a_{2,3,k} & \ldots & a_{2,N,k} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{N,1,k} & a_{N,2,k} & a_{N,3,k} & \ldots & 0 \end{bmatrix}$$

and $a_{i,j,k} = 0$ iff $j \notin \mathcal{N}_i(\xi_k)$. Between two update times ξ_k and ξ_{k+1}, any agent i has information only about its own state. More precisely, the dynamics of agent i is described by

$$\left({}^{C}_{\xi_k}\mathcal{D}^{\alpha,\rho} x_i \right)(t) = b_i x_i(t), \quad t \in (\xi_k, \xi_{k+1}],$$

where $b_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$.

Remark 3.2. We do not assume the weights $a_{i,j,k}$ and coefficients b_i to be positive.

At each time ξ_k, agent i updates its state variable according to the impulsive control protocol defined by (3.1), i.e.,

$$x_i(\xi_k + 0) = u_i(\xi_k), \quad i = 1, 2, \ldots, N, \quad k \in \mathbb{N}.$$

The model described above can be written as a system of differential equations with impulses at times ξ_k and generalized proportional Caputo fractional derivatives

$$\begin{aligned} &\left({}^{C}_{\xi_k}\mathcal{D}^{\alpha,\rho} x_i \right)(t) = b_i x_i(t), \quad i = 1, 2, \ldots, N, \quad t \in (\xi_k, \xi_{k+1}], \quad k \in \mathbb{N}_0, \\ &x_i(\xi_k + 0) = \sum_{j\in\mathcal{N}_i(\xi_k)} a_{i,j,k}(x_i(\xi_k) - x_j(\xi_k)), \quad k \in \mathbb{N}, \\ &x_i(t_0) = x_i^0, \quad i = 1, 2, \ldots, N. \end{aligned} \tag{3.2}$$

We refer to model (3.2) as a *generalized proportional Caputo fractional model of multi-agent linear dynamic system via impulsive control protocol.*

3.2. Multi-agent system with a leader

We consider the multi-agent system with fixed topology that consists of N agents and a leader with state variables $x_i(t)$, $i = 1, 2, \ldots, N$ and $x_0(t)$ at time t, respectively. The agents and the leader exchange information among themselves and there is an impulsive control protocol. Between two update times ξ_k and ξ_{k+1}, the dynamics of any agent i is based only on interaction between itself and other agents; the leader has no interactions with other agents. More precisely, the dynamics is described by

$$\begin{aligned}
&\left({}^{C}_{\xi_k}\mathcal{D}^{\alpha,\rho}x_i\right)(t) = \sum_{j=1}^{N} \ell_{i,j}(x_i(t) - x_j(t)), \quad t \in (\xi_k, \xi_{k+1}],\\
&\left({}^{C}_{\xi_k}\mathcal{D}^{\alpha,\rho}x_0\right)(t) = 0, \quad t \in (\xi_k, \xi_{k+1}], \quad k \in \mathbb{N}_0,
\end{aligned}$$

where the weights $\ell_{i,j} \geq 0$ are such that $\ell_{i,j} = 0$ iff the agent j does not influence agent i. At each update time ξ_k, the leader interacts with some of the agents instantaneously, i.e., the agent i updates its state variable according to the impulsive control protocol, and the state of the leader is continuous, namely

$$x_i(\xi_k + 0) = x_i(\xi_k) + u_i(\xi_k), \quad i = 1, 2, \ldots, N, \quad k \in \mathbb{N},$$

where the control input of agent i at the time ξ_k, $k \in \mathbb{N}$, based on the interaction between the agent and the leader, is designed by

$$u_i(\xi_k) = \mu_{i,k}(x_i(\xi_k) - x_0(\xi_k)), \quad k \in \mathbb{N}.$$

The model described above can be written as a system of differential equations with impulses at times ξ_k and generalized proportional Caputo fractional derivatives

$$\begin{aligned}
&\left({}^{C}_{\xi_k}\mathcal{D}^{\alpha,\rho}x_i\right)(t) = \sum_{j=1}^{N} \ell_{i,j}(x_i(t) - x_j(t)), \quad t \in (\xi_k, \xi_{k+1}],\\
&\left({}^{C}_{\xi_k}\mathcal{D}^{\alpha,\rho}x_0\right)(t) = 0, \quad t \in (\xi_k, \xi_{k+1}], \quad k \in \mathbb{N}_0,\\
&x_i(\xi_k + 0) = x_i(\xi_k) + \mu_{i,k}(x_i(\xi_k) - x_0(\xi_k)), \quad i = 1, 2, \ldots, N,\\
&x_0(\xi_k + 0) = x_0(\xi_k), \quad k \in \mathbb{N},\\
&x_i(t_0) = x_i^0, \quad i = 1, 2, \ldots, N, \quad x_0(t_0) = x_0^0.
\end{aligned} \tag{3.3}$$

Remark 3.3. From Lemma 2.7 with $a = \xi_k$, it follows that the state of the leader is

$$x_0(t) = x_0(\xi_k + 0)e^{\frac{\rho-1}{\rho}(t-\xi_k)} = x_0(\xi_k)e^{\frac{\rho-1}{\rho}(t-\xi_k)} \quad \text{on} \quad (\xi_k, \xi_{k+1}], \quad k \in \mathbb{N}_0.$$

Inductively, from the impulsive condition $x_0(\xi_k + 0) = x_0(\xi_k)$, we get that the leader has the state $x_0(t) = x_0^0 e^{\frac{\rho-1}{\rho}(t-\xi_0)}$ for $t \geq t_0$.

Denote $z_i(t) = x_i(t) - x_0(t) = x_i(t) - x_0^0 e^{\frac{\rho-1}{\rho}(t-\xi_0)}$, $i = 1, 2, \ldots, N$. Then, problem (3.3) can be written in the form of a generalized proportional Caputo fractional differential equation with impulses

$$\begin{aligned}
&\left({}^{C}_{\xi_k}\mathcal{D}^{\alpha,\rho}z_i\right)(t) = \sum_{j=1}^{N} \ell_{i,j}(z_i(t) - z_j(t)), \quad t \in (\xi_k, \xi_{k+1}], \quad k \in \mathbb{N}_0,\\
&z_i(\xi_k + 0) = (1 + \mu_{i,k})z_i(\xi_k), \quad i = 1, 2, \ldots, N, \quad k \in \mathbb{N},\\
&z_i(t_0) = x_i^0 - x_0^0, \quad i = 1, 2, \ldots, N.
\end{aligned} \tag{3.4}$$

Equivalently, (3.4) can be written in matrix form

$$\begin{aligned}
&\left({}^{C}_{\xi_k}\mathcal{D}^{\alpha,\rho}Z\right)(t) = LZ(t), \quad t \in (\xi_k, \xi_{k+1}], \quad k \in \mathbb{N}_0,\\
&Z(\xi_k + 0) = P_k Z(\xi_k), \quad k \in \mathbb{N},\\
&Z(t_0) = Z_0,
\end{aligned} \tag{3.5}$$

where $P_k = \operatorname{diag}(1 + \mu_{1,k}, 1 + \mu_{2,k}, \ldots, 1 + \mu_{N,k})$,

$$L = \begin{bmatrix}
\sum_{j=2}^{N} \ell_{i,j} & -\ell_{1,2} & -\ell_{1,3} & \ldots & -\ell_{1,N}\\
-\ell_{2,1} & \sum_{j=1, j\neq 2}^{N} \ell_{2,j} & -\ell_{2,3} & \ldots & -\ell_{2,N}\\
-\ell_{3,1} & -\ell_{3,2} & \sum_{j=1, j\neq 3}^{N} \ell_{3,j} & \ldots & -\ell_{3,N}\\
\ldots & \ldots & \ldots & \ldots & \ldots\\
-\ell_{N,1} & -\ell_{N,1} & -\ell_{N,3} & \ldots & \sum_{j=1}^{N-1} \ell_{N,j}
\end{bmatrix},$$

and $Z_0 = (x_1^0 - x_0^0, x_2^0 - x_0^0, \ldots, x_N^0 - x_0^0)$.

4. Main results

This section presents our main results. In particular, we prove sufficient conditions for exponential stability in the case of a multi-agent system without a leader (Section 4.1) and sufficient conditions for the leader following consensus in a multi-agent system with a leader (Section 4.2).

4.1. Multi-agent system without a leader

Definition 4.1. We say that the generalized proportional Caputo fractional model of multi-agent linear dynamic systems via impulsive control protocol (3.2) is exponentially stable if there exist positive numbers λ, M such that, for any $x^0 \in \mathbb{R}^N$, the inequality $\|x(t)\| \le Me^{-\lambda(t-t_0)}\|x^0\|$ holds for all $t \ge t_0$.

Remark 4.2. The exponential stability of the generalized proportional Caputo fractional model of multi-agent linear dynamic systems via impulsive control protocol (3.2) implies asymptotic stability, i.e., $\lim_{t\to\infty}\|x(t)\| = 0$.

Before moving on, we need the following result.

Lemma 4.3. *Let $\alpha \in (0,1)$, $\rho \in (0,1]$, $b \in \mathbb{R}$, and $\eta_2 > \eta_1 \ge 0$. Then,*

$$e^{\frac{\rho-1}{\rho}(\eta_2-\eta_1)}E_\alpha\left(b\left(\frac{\eta_2-\eta_1}{\rho}\right)^\alpha\right) < \frac{e^{\left(\rho+\sqrt[\alpha]{|b|}-1\right)\frac{\eta_2-\eta_1}{\rho}}}{\alpha}.$$

Proof. For $b \le 0$, we have

$$e^{\frac{\rho-1}{\rho}(\eta_2-\eta_1)}E_\alpha\left(b\left(\frac{\eta_2-\eta_1}{\rho}\right)^\alpha\right) \le e^{\frac{\rho-1}{\rho}(\eta_2-\eta_1)} \le e^{\frac{\rho-1+\sqrt[\alpha]{|b|}}{\rho}(\eta_2-\eta_1)} < \frac{e^{\frac{\rho-1+\sqrt[\alpha]{|b|}}{\rho}(\eta_2-\eta_1)}}{\alpha}.$$

For $b > 0$, we have

$$E_\alpha\left(b\left(\frac{\eta_2-\eta_1}{\rho}\right)^\alpha\right) = E_\alpha\left(\left[\sqrt[\alpha]{b}\frac{\eta_2-\eta_1}{\rho}\right]^\alpha\right).$$

Thus, by Corollary 2.11 point (iv), it follows that

$$e^{\frac{\rho-1}{\rho}(\eta_2-\eta_1)}E_\alpha\left(b\left(\frac{\eta_2-\eta_1}{\rho}\right)^\alpha\right) < \frac{e^{\left(\rho+\sqrt[\alpha]{b}-1\right)\frac{\eta_2-\eta_1}{\rho}}}{\alpha}$$

holds, completing the proof. □

Theorem 4.4. *Let $\alpha \in (0,1)$ and $\rho \in (0,1)$. If $B = \max_{i=1,2,\dots,N}|b_i| < (1-\rho)^\alpha$ and there exists a positive number $K \le \alpha$ such that*

$$\left|\sum_{j=1}^N a_{i,j,k}\right| + \sum_{j=1}^N |a_{i,j,k}| < K,$$

then the generalized proportional Caputo fractional model of multi-agent linear dynamic systems via impulsive control protocol (3.2) is exponentially stable.

Proof. Let $t \in (\xi_0, \xi_1]$. According to Lemma 2.8 with $a = \xi_0$, $u_0 = x_i^0$, and $\lambda = b_i$, the solution to (3.2) is given by

$$x_i(t) = x_i^0 e^{\frac{\rho-1}{\rho}(t-\xi_0)}E_\alpha\left(b_i\left(\frac{t-\xi_0}{\rho}\right)^\alpha\right), \quad t \in (\xi_0, \xi_1].$$

Hence

$$|x_i(t)| \le |x_i^0| e^{\frac{\rho-1}{\rho}(t-\xi_0)}E_\alpha\left(b_i\left(\frac{t-\xi_0}{\rho}\right)^\alpha\right), \quad t \in (\xi_0, \xi_1].$$

Applying Lemma 4.3 with $\eta_2 = \xi_1$, $\eta_1 = \xi_0$ and $b = b_i$, we obtain

$$\begin{aligned} |x_i(\xi_1)| \le & |x_i^0| e^{\frac{\rho-1}{\rho}(\xi_1-\xi_0)}E_\alpha\left(b_i\left(\frac{\xi_1-\xi_0}{\rho}\right)^\alpha\right) \\ \le & |x_i^0| \frac{e^{\left(\rho+\sqrt[\alpha]{|b_i|}-1\right)\frac{\xi_1-\xi_0}{\rho}}}{\alpha}, \quad i = 1,2,\dots,N. \end{aligned} \tag{4.1}$$

Let $t \in (\xi_1, \xi_2]$. Again, according to Lemma 2.8 with $a = \xi_1$, $u_0 = x_i(\xi_1 + 0)$, and $\lambda = b_i$, the solution to (3.2) is given by

$$x_i(t) = x_i(\xi_1 + 0)e^{\frac{\rho-1}{\rho}(t-\xi_1)}E_\alpha\left(b_i\left(\frac{t-\xi_1}{\rho}\right)^\alpha\right), \quad t \in (\xi_1, \xi_2], \quad i = 1, 2, \dots, N.$$

An application of Lemma 4.3 enables us to write

$$\begin{aligned}|x_i(t)| =&|x_i(\xi_1 + 0)|e^{\frac{\rho-1}{\rho}(t-\xi_1)}E_\alpha\left(b_i\left(\frac{t-\xi_1}{\rho}\right)^\alpha\right)\\ \leq&|x_i(\xi_1 + 0)|\frac{e^{\left(\rho+\sqrt[\alpha]{|b_i|}-1\right)\frac{t-\xi_1}{\rho}}}{\alpha}, \quad t \in (\xi_1, \xi_2].\end{aligned} \tag{4.2}$$

On account of (4.1), at the update time ξ_1, we have

$$\begin{aligned}|x_i(\xi_1 + 0)| &= |u_i(\xi_1)| = \left|x_i(\xi_1)\sum_{j=1}^N a_{i,j,1} - \sum_{j=1,j\neq i}^N a_{i,j,1}x_j(\xi_1)\right|\\ &\leq |x_i^0|\frac{e^{\left(\rho+\sqrt[\alpha]{|b_i|}-1\right)\frac{\xi_1-\xi_0}{\rho}}}{\alpha}\left|\sum_{j=1}^N a_{i,j,1}\right| + \sum_{j=1}^N |a_{i,j,1}|\,|x_j^0|\frac{e^{\left(\rho+\sqrt[\alpha]{|b_j|}-1\right)\frac{\xi_1-\xi_0}{\rho}}}{\alpha}\\ &\leq \|x^0\|\frac{e^{(\rho+\sqrt[\alpha]{B}-1)\frac{\xi_1-\xi_0}{\rho}}}{\alpha}\left(\left|\sum_{j=1}^N a_{i,j,1}\right| + \sum_{j=1}^N |a_{i,j,1}|\right) \leq \|x^0\|\,\frac{K}{\alpha}e^{\left(\rho+\sqrt[\alpha]{B}-1\right)\frac{\xi_1-\xi_0}{\rho}}.\end{aligned} \tag{4.3}$$

Using (4.3) in (4.2) implies

$$|x_i(t)| \leq \left\|x^0\right\|\,\frac{K}{\alpha^2}e^{\left(\rho+\sqrt[\alpha]{B}-1\right)\frac{t-\xi_0}{\rho}}$$

for any $t \in (\xi_1, \xi_2]$ and all $i = 1, 2, \dots, N$. By induction with respect to intervals, we obtain

$$|x_i(t)| \leq \left\|x^0\right\|\,\frac{K^k}{\alpha^k}e^{\frac{\rho-1+\sqrt[\alpha]{B}}{\rho}(t-\xi_{k-1})}\frac{e^{\frac{\rho-1+\sqrt[\alpha]{|b_i|}}{\rho}(t-\xi_k)}}{\alpha} \leq \left\|x^0\right\|\,\frac{K^k}{\alpha^{k+1}}e^{\frac{\rho-1+\sqrt[\alpha]{B}}{\rho}(t-\xi_0)}$$

for any $t \in (\xi_k, \xi_{k+1}]$, $k \in \mathbb{N}_0$, and all $i = 1, 2 \dots, N$. Therefore, the solution to system (3.2) is exponentially stable with $M = \frac{1}{\alpha}$ and $\lambda = \frac{1-\rho-\sqrt[\alpha]{B}}{\rho} > 0$. $\square$

Remark 4.5. It is worth pointing out that the above approach does not allow obtaining of sufficient conditions for exponential stability of (3.2) in the case of Caputo derivative.

4.2. Multi-agent system with a leader

We begin with deriving an explicit form of a solution to linear impulsive system (3.5).

Lemma 4.6. *The exact solution to system (3.5) is*

$$Y(t) =e^{\frac{\rho-1}{\rho}(t-t_0)}E_\alpha\left(L\left(\frac{t-\xi_k}{\rho}\right)^\alpha\right)\left(\prod_{i=0}^{k-1}\left(P_{k-i}E_\alpha\left(L\left(\frac{\xi_{k-i}-\xi_{k-i-1}}{\rho}\right)^\alpha\right)\right)\right)Z_0, \quad t \in (\xi_k, \xi_{k+1}], \quad k \in \mathbb{N}_0. \tag{4.4}$$

Proof. The proof follows by induction. Let $t \in [t_0, \xi_1]$. By Lemma 2.14 with $a = t_0$, $A = L$, $U_0 = Z_0$, we obtain

$$Z(t) = e^{\frac{\rho-1}{\rho}(t-t_0)}E_\alpha\left(L\left(\frac{t-t_0}{\rho}\right)^\alpha\right)Z_0.$$

Therefore,

$$Z(\xi_1 - 0) = e^{\frac{\rho-1}{\rho}(\xi_1-t_0)}E_\alpha\left(L\left(\frac{\xi_1-t_0}{\rho}\right)^\alpha\right)Z_0.$$

Let $t \in (\xi_1, \xi_2]$. Applying Lemma 2.14 with $a = \xi_1$, $A = L$, $U_0 = Z(\xi_1 + 0)$ gives

$$\begin{aligned} Z(t) =& e^{\frac{\rho-1}{\rho}(t-\xi_1)} E_\alpha \left(L \left(\frac{t-\xi_1}{\rho} \right)^\alpha \right) Z(\xi_1 + 0) \\ =& e^{\frac{\rho-1}{\rho}(t-t_0)} E_\alpha \left(L \left(\frac{t-\xi_1}{\rho} \right)^\alpha \right) P_1 E_\alpha \left(L \left(\frac{\xi_1 - t_0}{\rho} \right)^\alpha \right) Z_0. \end{aligned}$$

For $t \in (\xi_2, \xi_3]$, we use Lemma 2.14 with $a = \xi_2,\ A = L,\ U_0 = Z(\xi_2 + 0)$ and get

$$\begin{aligned} & Z(t) \\ =& e^{\frac{\rho-1}{\rho}(t-t_0)} E_\alpha \left(L \left(\frac{t-\xi_2}{\rho} \right)^\alpha \right) P_2 E_\alpha \left(L \left(\frac{\xi_2-\xi_1}{\rho} \right)^\alpha \right) P_1 E_\alpha \left(L \left(\frac{\xi_1 - t_0}{\rho} \right)^\alpha \right) Z_0. \end{aligned}$$

Repeated application of Lemma 2.14 yields (4.4). □

Definition 4.7. We say that for the generalized proportional Caputo fractional model of multi-agent linear dynamic systems via impulsive control protocol, (3.3) achieves the leader following consensus if $\lim_{t\to\infty} |x_i(t) - x_0(t)| = 0$ for all $i = 1, 2, \dots, N$.

Remark 4.8. The important point to note here is that the leader following consensus in the generalized proportional Caputo fractional model of multi-agent linear dynamic systems via impulsive control protocol (3.3) is equivalent to the asymptotic stability of the system of generalized proportional Caputo type fractional differential equations with impulses (3.4).

Theorem 4.9. *Let $\alpha \in (0, 1)$ and $\rho \in (0, 1)$. If there exist numbers $q, \beta > 0$ such that the inequalities*

$$\max_{1\le i\le n} |1 + \mu_{i,k}| E_\alpha \left(\|L\|_2 \left(\frac{\beta}{\rho} \right)^\alpha \right) \le q < 1 \quad \text{and} \quad 0 < \xi_{k+1} - \xi_k \le \beta < \infty, \quad k \in \mathbb{N}_0,$$

hold, then the generalized proportional Caputo fractional model of multi-agent linear dynamic systems via impulsive control protocol (3.3) achieves the leader following consensus.

Proof. Let us denote by Z the solution to (3.4) (or equivalently, to (3.5)). Since

$$P_k^{\mathrm{T}} P_k = \operatorname{diag}((1+\mu_{1,k})^2, (1+\mu_{2,k})^2, \dots, (1+\mu_{N,k})^2),$$

we have $\lambda_i = (1 + \mu_{i,k})^2$ and

$$\|P_k\|_2 = \sqrt{\max_{1\le i\le n} (1+\mu_{i,k})^2} = \max_{1\le i\le n} |1 + \mu_{i,k}|.$$

Hence,

$$\left\| P_k E_\alpha \left(L \left(\frac{\xi_k - \xi_{k-1}}{\rho} \right)^\alpha \right) \right\|_2 \le \max_i |1 + \mu_{i,k}| E_\alpha \left(\|L\|_2 \left(\frac{\beta}{\rho} \right)^\alpha \right) \le q < 1.$$

Applying Lemma 4.6 and Proposition 2.12, we obtain

$$\begin{aligned} \|Z(t)\| \le& e^{\frac{\rho-1}{\rho}(t-t_0)} \left\| E_\alpha \left(L \left(\frac{t-\xi_k}{\rho} \right)^\alpha \right) \right\|_2 \left(\prod_{i=0}^{k-1} \left(\left\| P_{k-i} E_\alpha \left(L \left(\frac{\xi_{k-i} - \xi_{k-i-1}}{\rho} \right)^\alpha \right) \right\|_2 \right) \right) \|Z_0\| \\ \le& e^{\frac{\rho-1}{\rho}(t-t_0)} E_\alpha \left(\|L\|_2 \left(\frac{t-\xi_k}{\rho} \right)^\alpha \right) \|Z_0\| \\ \le& e^{\frac{\rho-1}{\rho}(t-t_0)} E_\alpha \left(\|L\|_2 \left(\frac{\beta}{\rho} \right)^\alpha \right) \|Z_0\|, \end{aligned}$$

for $t \in (\xi_k, \xi_{k+1}]$ and all $k \in \mathbb{N}_0$. Therefore, the generalized proportional Caputo type fractional differential equation with impulses (3.4) is exponentially stable with

$$M = E_\alpha \left(\|L\|_2 \left(\frac{\beta}{\rho} \right)^\alpha \right) \quad \text{and} \quad \lambda = \frac{1-\rho}{\rho} > 0,$$

and this, by Remark 4.8, is the desired conclusion. □

5. Applications

In this section, numerical examples are presented to verify the effectiveness of the proposed impulsive control protocol for generalized proportional Caputo fractional multi-agent linear dynamical systems.

Example 5.1. Let us consider a generalized proportional Caputo fractional model of multi-agent linear dynamic system via impulsive control (see (3.2)) with

$$N = 5, \quad t_0 = \xi_0 = 0, \quad \xi_{k+1} = \xi_k + 1, \quad k \in \mathbb{N}_0, \quad \alpha = 0.65,$$

i.e.,

$$\begin{aligned}
&\left({}_{\xi_k}^{C}\mathcal{D}^{0.65,\rho}x_1\right)(t) = 0.86x_1(t), \quad \left({}_{\xi_k}^{C}\mathcal{D}^{0.65,\rho}x_2\right)(t) = 0.78x_2(t),\\
&\left({}_{\xi_k}^{C}\mathcal{D}^{0.65,\rho}x_3\right)(t) = -0.1x_3(t), \quad \left({}_{\xi_k}^{C}\mathcal{D}^{0.65,\rho}x_4\right)(t) = -0.65x_4(t),\\
&\left({}_{\xi_k}^{C}\mathcal{D}^{0.65,\rho}x_5\right)(t) = 0.85x_5(t), \quad t \in (\xi_k, \xi_{k+1}], \quad k \in \mathbb{N}_0,\\
&x_1(\xi_k + 0) = -\,0.01\,(x_1(\xi_k) - x_2(\xi_k)) + 0.04\,(x_1(\xi_k) - x_3(\xi_k))\\
&\qquad\qquad -\ 0.08\,(x_1(\xi_k) - x_4(\xi_k)) + 0.25\,(x_1(\xi_k) - x_5(\xi_k))\,,\\
&x_2(\xi_k + 0) = -\,0.01\,(x_1(\xi_k) - x_2(\xi_k)) + 0.2\,(x_2(\xi_k) - x_3(\xi_k))\\
&\qquad\qquad +\ 0.09\,(x_2(\xi_k) - x_4(\xi_k)) - 0.08\,(x_2(\xi_k) - x_5(\xi_k))\,,\\
&x_3(\xi_k + 0) = 0.09\,(x_3(\xi_k) - x_1(\xi_k)) - 0.08\,(x_3(\xi_k) - x_2(\xi_k))\\
&\qquad\qquad -\ 0.23\,(x_3(\xi_k) - x_4(\xi_k)) - 0.01\,(x_3(\xi_k) - x_5(\xi_k))\,,\\
&x_4(\xi_k + 0) = -\,0.01\,(x_4(\xi_k) - x_1(\xi_k)) + 0.09\,(x_4(\xi_k) - x_2(\xi_k))\\
&\qquad\qquad -\ 0.08 - (x_4(\xi_k) - x_3(\xi_k)) + 0.23\,(x_4(\xi_k) - x_5(\xi_k))\,,\\
&x_5(\xi_k + 0) = 0.2\,(x_5(\xi_k) - x_1(\xi_k)) - 0.04\,(x_5(\xi_k) - x_2(\xi_k))\\
&\qquad\qquad +\ 0.08\,(x_5(\xi_k) - x_3(\xi_k)) - 0.09\,(x_5(\xi_k) - x_4(\xi_k))\,, \quad k \in \mathbb{N},\\
&x_i(0) = 1, \quad i = 1, 2, 3, 4, 5.
\end{aligned} \tag{5.1}$$

Observe that the weighted connectivity matrix is the same at any updated times, and it is given by

$$A = \begin{bmatrix} 0 & 0.01 & -0.04 & 0.08 & -0.25 \\ 0.01 & 0 & -0.2 & -0.09 & 0.08 \\ -0.09 & 0.08 & 0 & -0.23 & 0.01 \\ 0.01 & -0.09 & 0.08 & 0 & -0.23 \\ -0.2 & 0.04 & -0.08 & 0.09 & 0 \end{bmatrix}.$$

First, we analyze the case when $\rho = 0.2$. Then,

$$B = 0.86 < (1 - 0.2)^{0.65} \approx 0.864985$$

and

$$\begin{aligned}
&\left|\sum_{j=1}^{5} a_{1,j,k}\right| + \sum_{j=1}^{5} |a_{1,j,k}| \;=\; 0.2 + 0.38 = 0.58 \le 0.65,\\
&\left|\sum_{j=1}^{5} a_{2,j,k}\right| + \sum_{j=1}^{5} |a_{2,j,k}| \;=\; 0.2 + 0.38 = 0.58 < 0.65,\\
&\left|\sum_{j=1}^{5} a_{3,j,k}\right| + \sum_{j=1}^{5} |a_{3,j,k}| \;=\; 0.23 + 0.41 = 0.64 < 0.65,\\
&\left|\sum_{j=1}^{5} a_{4,j,k}\right| + \sum_{j=1}^{5} |a_{4,j,k}| \;=\; 0.23 + 0.41 = 0.64 < 0.65,\\
&\left|\sum_{j=1}^{5} a_{5,j,k}\right| + \sum_{j=1}^{5} |a_{5,j,k}| \;=\; 0.15 + 0.41 = 0.56 < 0.65.
\end{aligned}$$

Therefore, the conditions of Theorem 4.4 are satisfied, and the considered generalized proportional Caputo fractional model of multi-agent linear dynamic systems via impulsive control protocol is exponentially stable with

$$\lambda = \frac{1 - 0.2 - \sqrt[0.65]{0.85}}{0.2} \approx 0.106109.$$

The graphs of $|x_i(t)|$ are drawn in Fig. 5.1. It can be seen they are bounded above by $\frac{\sqrt{5}}{0.65}e^{-0.106109t}$. In the second case, we put $\rho = 0.9$. Then the condition

$$B = 0.85 < (1 - 0.9)^{0.65} \approx 0.223872$$

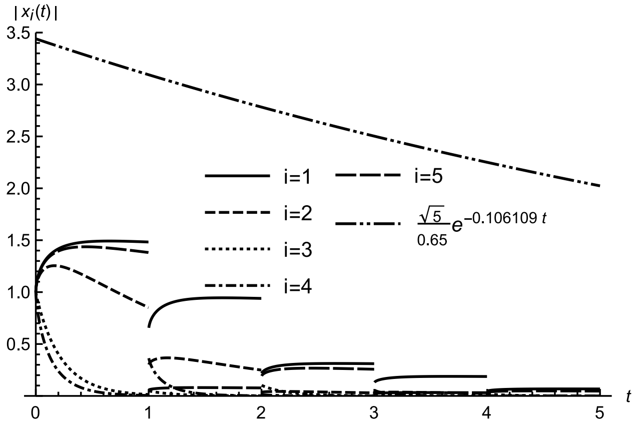

Fig. 5.1. State variables $|x_i(t)|$, $i = 1, 2, 3, 4, 5$, of multi-agent system (5.1) with $\rho = 0.2$.

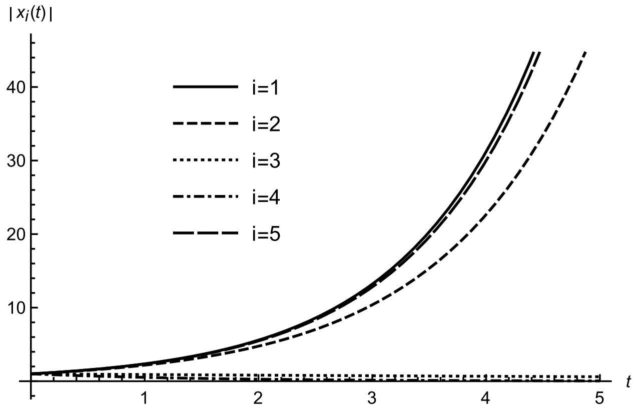

Fig. 5.2. State variables $|x_i(t)|$, $i = 1, 2, 3, 4, 5$, of multi-agent system (5.1) with $\rho = 0.9$.

is not satisfied. From Fig. 5.2, we can see that the system is not exponentially stable.

Example 5.2. Let us consider a generalized proportional Caputo fractional model of multi-agent linear dynamic system without impulsive control protocol with

$$N = 5, \quad t_0 = 0, \quad \alpha = 0.65, \quad \rho = 0.9,$$

i.e.,

$$\begin{aligned}
&\left({}_{\xi_k}^{\mathrm{C}}\mathcal{D}^{0.65,0.2}x_1\right)(t) = 0.86x_1(t), \quad \left({}_{\xi_k}^{\mathrm{C}}\mathcal{D}^{0.65,0.2}x_2\right)(t) = 0.78x_2(t),\\
&\left({}_{\xi_k}^{\mathrm{C}}\mathcal{D}^{0.65,0.2}x_3\right)(t) = -0.1x_3(t), \quad \left({}_{\xi_k}^{\mathrm{C}}\mathcal{D}^{0.65,0.2}x_4\right)(t) = -0.65x_4(t),\\
&\left({}_{\xi_k}^{\mathrm{C}}\mathcal{D}^{0.65,0.2}x_5\right)(t) = 0.85x_5(t), \quad x_i(0) = 1, \quad i = 1, 2, 3, 4, 5.
\end{aligned} \tag{5.2}$$

By Lemma 2.8, the solution of (5.2) is given by

$$x_i(t) = x_i^0 e^{\frac{0.2-1}{0.2}t} E_{0.3}\left(b_i\left(\frac{t}{0.2}\right)^{0.3}\right) = x_i^0 e^{-4t} E_{0.3}(b_i(5t)^{0.3}), \quad i = 1, 2, 3, 4, 5,$$

where $b_1 = 0.86$, $b_2 = 0.78$, $b_3 = -0.1$, $b_4 = -0.65$, $b_5 = 0.85$. The graphs of $|x_i(t)|$, $i = 1, 2, 3, 4, 5$, are shown in Fig. 5.3. It can be seen that all components of the solution approach zero.

Example 5.3. In order to show the importance of the type of derivative appearing in the model, we consider system (5.1), in which the generalized proportional derivatives are replaced by the integer-order ones, i.e., the first-order ordinary derivative is applied in the model instead of a fractional one. The graphs of all components $|x_i(t)|$, $i = 1, 2, 3, 4, 5$, of the corresponding solution are shown in Fig. 5.4.

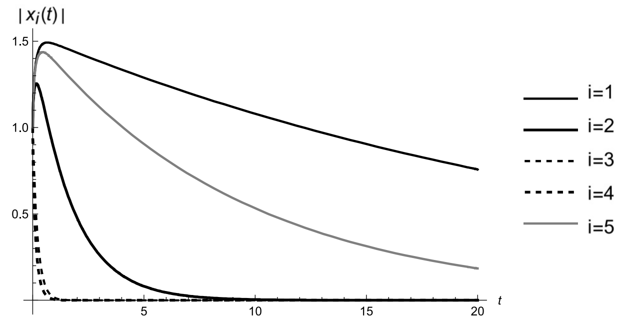

Fig. 5.3. State variables $|x_i(t)|$, $i = 1, 2, 3, 4, 5$, of multi-agent system (5.2) without impulse control protocol.

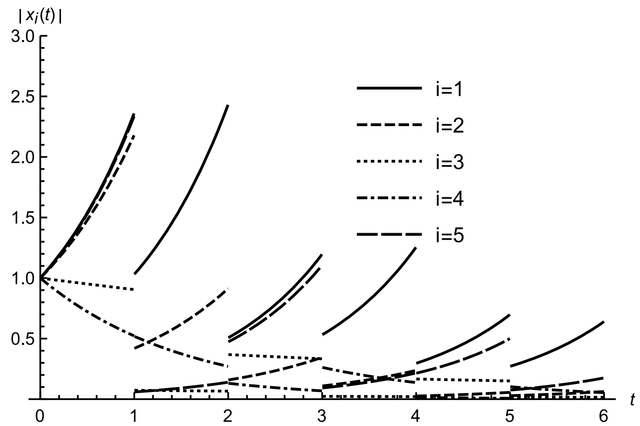

Fig. 5.4. State variables $|x_i(t)|$, $i = 1, 2, 3, 4, 5$, of multi-agent system with ordinary derivatives without impulsive control protocol.

Remark 5.4. From Figs. 5.1-5.4, we may conclude that the type of the applied derivative definitely has an influence on the behavior of the studied multi-agent system. What follows is the necessity of the application of various types of derivatives in models.

Example 5.5. Let us consider the following multi-agent system with a leader

$$
\begin{aligned}
&\left({}^{C}_{\xi_k}\mathcal{D}^{0.8,0.6}x_1\right)(t) = 0.9(x_1(t) - x_2(t)),\\
&\left({}^{C}_{\xi_k}\mathcal{D}^{0.8,0.6}x_2\right)(t) = 0.1(x_2(t) - x_1(t)) + 0.1(x_2(t) - x_3(t)),\\
&\left({}^{C}_{\xi_k}\mathcal{D}^{0.8,0.6}x_3\right)(t) = 0.5(x_3(t) - x_2(t)) + 0.3(x_3(t) - x_4(t)),\\
&\left({}^{C}_{\xi_k}\mathcal{D}^{0.8,0.6}x_4\right)(t) = 0.1(x_4(t) - x_3(t)),\\
&\left({}^{C}_{\xi_k}\mathcal{D}^{0.8,0.6}x_0\right)(t) = 0, \quad t \in (\xi_k, \xi_{k+1}], \quad k \in \mathbb{N},\\
&x_1(\xi_k + 0) = x_1(\xi_k) - 0.7(x_1(\xi_k) - x_0(\xi_k)),\\
&x_2(\xi_k + 0) = x_2(\xi_k) - 1.3(x_2(\xi_k) - x_0(\xi_k)),\\
&x_3(\xi_k + 0) = x_3(\xi_k) - 0.7(x_3(\xi_k) - x_0(\xi_k)),\\
&x_4(\xi_k + 0) = x_4(\xi_k) - 1.3(x_4(\xi_k) - x_0(\xi_k)), \quad k \in \mathbb{N},\\
&x_i(0) = x_i^0, \quad i = 0, 1, 2, 3, 4.
\end{aligned} \tag{5.3}
$$

System (5.3) is of type (3.3) with $N = 4$ agents, $t_0 = \xi_0 = 0$, $\xi_{k+1} = \xi_k + 0.5$, $k \in \mathbb{N}$, $\rho = 0.6$, and $\alpha = 0.8$. In this case, we have $\beta = 0.5$, $P_k = \mathrm{diag}(0.3, -0.3, 0.3, -0.3)$, and

$$
L = \begin{bmatrix} 0.9 & -0.9 & 0 & 0 \\ -0.1 & 0.2 & -0.1 & 0 \\ 0 & -0.5 & 0.8 & -0.3 \\ 0 & 0 & -0.1 & 0.1 \end{bmatrix}.
$$

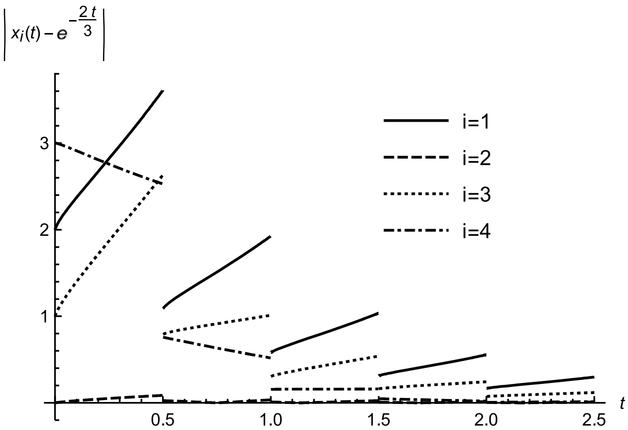

Fig. 5.5. Errors of state-tracking $|x_i(t) - x_0(t)|$, $i = 1, 2, 3, 4$, for the solution to (5.3).

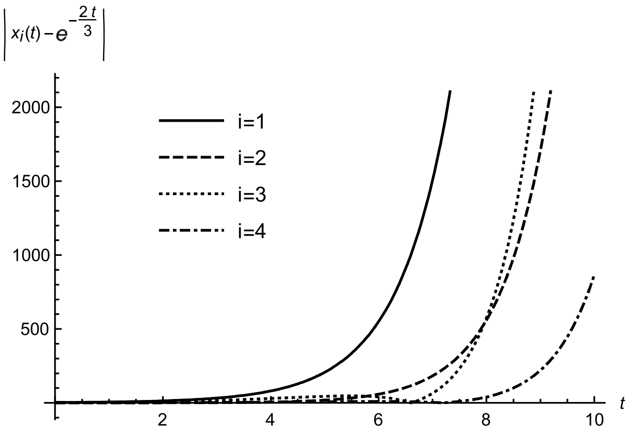

Fig. 5.6. Errors of state-tracking $\left|x_i(t) - e^{-\frac{2}{3}t}\right|$, $i = 1, 2, 3, 4$, for the solution to (5.4).

Hence $x_0(t) = e^{\frac{0.6-1}{0.6}t}$, $\|L\|_2 = 1.0466$, and

$$0.3E_{0.8}\left(1.0466\left(\frac{0.5}{0.6}\right)^{0.8}\right) = 0.873455 = q < 1.$$

According to Theorem 4.9 for the generalized proportional Caputo fractional model of multi-agent linear dynamic systems via impulsive control protocol, (5.3) achieves a leader following consensus, i.e.,

$$\lim_{t\to\infty} |x_i(t) - x_0(t)| = \lim_{t\to\infty} |x_i(t) - e^{-\frac{2}{3}t}| = 0,$$

for $i = 1, 2, 3, 4$ (see Fig. 5.5). Now, let us consider the case without the impulsive interaction of the leader, i.e., the system without impulsive control protocol

$$\begin{aligned}
&\left({}^{C}_{\xi_k}\mathcal{D}^{0.8,0.6}x_1\right)(t) = 0.9(x_1(t) - x_2(t)),\\
&\left({}^{C}_{\xi_k}\mathcal{D}^{0.8,0.6}x_2\right)(t) = 0.1(x_2(t) - x_1(t)) + 0.1(x_2(t) - x_3(t)),\\
&\left({}^{C}_{\xi_k}\mathcal{D}^{0.8,0.6}x_3\right)(t) = 0.5(x_3(t) - x_2(t)) + 0.3(x_3(t) - x_4(t)),\\
&\left({}^{C}_{\xi_k}\mathcal{D}^{0.8,0.6}x_4\right)(t) = 0.1(x_4(t) - x_3(t)),\\
&x_i(0) = x_i^0, \quad i = 1, 2, 3, 4.
\end{aligned} \tag{5.4}$$

The explicit solution to (5.4) may be found by Lemma 2.14. As shown in Fig. 5.6, errors of state-tracking $\left|x_i(t) - e^{-\frac{2}{3}t}\right|$, $i = 1, 2, 3, 4$, are not bounded. It follows that even the impulsive interaction of the leader can cause consensus.

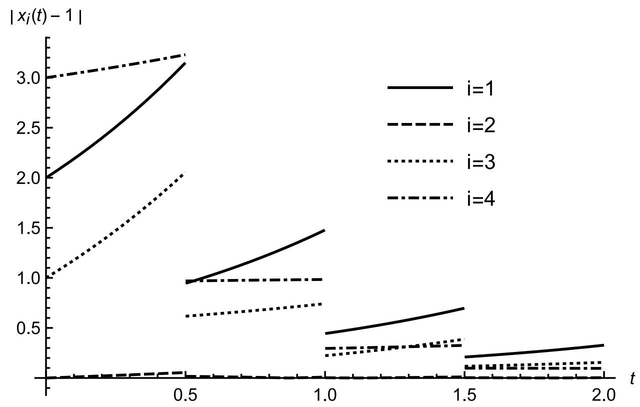

Fig. 5.7. Errors of state-tracking $|x_i(t) - 1|$, $i = 1, 2, 3, 4$, for the solution to (5.5).

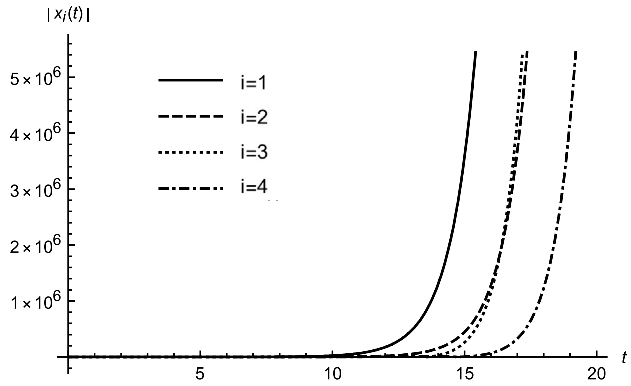

Fig. 5.8. Graph of $|x_i(t)|$, $i = 1, 2, 3, 4$, for the solution to (5.6).

Example 5.6. Finally, similarly to Example 5.3, we analyze the model (5.4) with ordinary derivatives. In this case, the state of the leader is $x_0(t) = x_0$, $t \geq 0$. Therefore, for $x_0 = 1$, the model is

$$
\begin{aligned}
&x_1'(t) = 0.9(x_1(t) - x_2(t)),\\
&x_2'(t) = 0.1(x_2(t) - x_1(t)) + 0.1(x_2(t) - x_3(t)),\\
&x_3'(t) = 0.5(x_3(t) - x_2(t)) + 0.3(x_3(t) - x_4(t)),\\
&x_4'(t) = 0.1(x_4(t) - x_3(t)), \quad t \in (\xi_k, \xi_{k+1}], \quad k \in \mathbb{N},\\
&x_1(\xi_k + 0) = x_1(\xi_k) - 0.7(x_1(\xi_k) - 1),\\
&x_2(\xi_k + 0) = x_2(\xi_k) - 1.3(x_2(\xi_k) - 1),\\
&x_3(\xi_k + 0) = x_3(\xi_k) - 0.7(x_3(\xi_k) - 1),\\
&x_4(\xi_k + 0) = x_4(\xi_k) - 1.3(x_4(\xi_k) - 1), \quad k \in \mathbb{N},\\
&x_i(0) = x_i^0, \quad i = 1, 2, 3, 4.
\end{aligned}
\tag{5.5}
$$

As shown in Fig. 5.7, errors of state-tracking $|x_i(t) - 1|$, $i = 1, 2, 3, 4$, are bounded. It follows that we may expect the leader-following consensus. However, without the impulsive interaction of the leader, the multi-agent system

$$
\begin{aligned}
&x_1'(t) = 0.9(x_1(t) - x_2(t)),\\
&x_2'(t) = 0.1(x_2(t) - x_1(t)) + 0.1(x_2(t) - x_3(t)),\\
&x_3'(t) = 0.5(x_3(t) - x_2(t)) + 0.3(x_3(t) - x_4(t)),\\
&x_4'(t) = 0.1(x_4(t) - x_3(t)), \quad t > 0,\\
&x_i(t_0) = x_i^0, \quad i = 1, 2, 3, 4,
\end{aligned}
\tag{5.6}
$$

similarly to the fractional-order system, does not achieve consensus (see Fig. 5.8).

6. Conclusions

In this paper, we have studied multi-agent systems with the generalized proportional Caputo fractional derivative. The information exchange between agents occurred only at fixed initially given update times, and the lower limit of the fractional derivative was changing according to the update times. We have obtained an explicit form of the solution to the system of linear generalized proportional Caputo fractional differential equations (see Lemma 2.14) as well as for the solutions to the system of linear generalized proportional Caputo fractional differential equations with impulses (see Lemma 4.6). Both results could be useful for various studies of qualitative properties of solutions to the corresponding linear systems as well as nonlinear systems. Two types of multi-agent systems have been considered, namely without and with a leader. For a generalized proportional Caputo fractional model of multi-agent linear dynamic system, sufficient conditions for exponential stability via impulsive control were obtained. In the case of the presence of a leader in the multi-agent system, we derived sufficient conditions for the leader following consensus via impulsive control based on the leader's influence. Simulation results have verified the essential role of the generalized proportional Caputo fractional derivative and impulsive control in realizing the consensus of multi-agent systems.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] Jadbabaie Ali, Lin Jie, Stephen Morse A. Coordination of groups of mobile autonomous agents using nearest neighbor rules. IEEE Trans Automat Control 2003;48(6):988–1001.
[2] Li Chuandong, Feng Gang. Delay-interval-dependent stability of recurrent neural networks with time-varying delay. Neurocomput 2009;72(4):1179–83.
[3] Olfati-Saber Reza, Alex Fax J, Murray Richard M. Consensus and cooperation in networked multi-agent systems. Proc IEEE 2007;95(1):215–33.
[4] Olfati-Saber Reza, Murray Richard M. Consensus problems in networks of agents with switching topology and time-delays. IEEE Trans Automat Control 2004;49(9):1520–33.
[5] Song Qiang, Cao Jinde, Yu Wenwu. Second-order leader-following consensus of nonlinear multi-agent systems via pinning control. Systems Control Lett 2010;59(9):553–62.
[6] Yan Huaicheng, Shen Yanchao, Zhang Hao, Shi Hongbo. Decentralized event-triggered consensus control for second-order multi-agent systems. Neurocomput 2014;133:18–24.
[7] Oh Kwang-Kyo, Park Myoung-Chul, Ahn Hyo-Sung. A survey of multi-agent formation control. Automatica J IFAC 2015;53:424–40.
[8] Qin Jiahu, Ma Qichao, Shi Yang, Wang Long. Recent advances in consensus of multi-agent systems: A brief survey. IEEE Trans Ind Electron 2017;64(6):4972–83.
[9] Gazi Veysel, Fidan Bariş. Coordination and control of multi-agent dynamic systems: models and approaches. In: Proceedings of the 2nd international conference on swarm robotics. SAB'06, Berlin, Heidelberg: Springer-Verlag; 2006, p. 71–102.
[10] Hong Yiguang, Hu Jiangping, Gao Linxin. Tracking control for multi-agent consensus with an active leader and variable topology. Automatica J IFAC 2006;42(7):1177–82.
[11] Liu Tengfei, Jiang Zhong-Ping. Distributed formation control of nonholonomic mobile robots without global position measurements. Automatica J IFAC 2013;49(2):592–600.
[12] Sumelka Wojciech, Łuczak Bartosz, Gajewski Tomasz, Voyiadjis George Z. Modelling of aaa in the framework of time-fractional damage hyperelasticity. Int J Solids Struct 2020;206:30–42.
[13] Petráš Ivo, Terpák Ján. Fractional calculus as a simple tool for modeling and analysis of long memory process in industry. Mathematics 2019;7(6):511.
[14] Dassios George, Fragoyiannis George, Satrazemi Konstantia. A fractional rate model of learning. Fract Differ Calc 2016;6(2):281–92.
[15] Koeller Ralph C. Applications of fractional calculus to the theory of viscoelasticity. Trans ASME J Appl Mech 1984;51(2):299–307.
[16] Tarasova Valentina V, Tarasov Vasily E. Concept of dynamic memory in economics. Commun Nonlinear Sci Numer Simul 2018;55:127–45.
[17] Tarasova Valentina V, Tarasov Vasily E. Logistic map with memory from economic model. Chaos Solitons Fractals 2017;95:84–91.
[18] Cao Yongcan, Li Yan, Ren Wei, Chen YangQuan. Distributed coordination algorithms for multiple fractional-order systems. In: Proceedings of the 47th IEEE conference on decision and control, Cancun, Mexico, December 9–11, 2008. 2008, pp. 2920–5.
[19] Cao Yongcan, Li Yan, Ren Wei, Chen YangQuan. Distributed coordination of networked fractional-order systems. IEEE Trans Syst Man Cybern B 2010;40(2):362–70.

[20] Cao Yongcan, Ren Wei. Distributed formation control for fractional-order systems: dynamic interaction and absolute/relative damping. Systems Control Lett 2010;59(3–4):233–40.
[21] Song Chao, Cao Jinde. Consensus of fractional-order linear systems. In: 2013 9th Asian control conference (ASCC). 2013, p. 1–4.
[22] Wang Fei, Yang Yongqing. On leaderless consensus of fractional-order nonlinear multi-agent systems via event-triggered control. Nonlinear Anal Model Control 2019;24(3):353–67.
[23] Yaghoubi Zahra, Talebi Heidar Ali. Cluster consensus of general fractional-order nonlinear multi agent systems via adaptive sliding mode controller. Arch Control Sci 2019;29(65):643–65.
[24] Almeida Ricardo, Girejko Ewa, Hristova Snezhana, Malinowska Agnieszka B. Leader-following consensus for fractional multi-agent systems. Adv Difference Equ 2019;15, Paper No. 301.
[25] Ren Guojian, Yu Yongguang, Zhang Shuo. Leader-following consensus of fractional nonlinear multiagent systems. Math Probl Eng 2015;919757, 8 pages.
[26] Schmeidel Ewa. The existence of consensus of a leader-following problem with Caputo fractional derivative. Opuscula Math 2019;39(1):77–89.
[27] Yu Zhiyong, Jiang Haijun, Hu Cheng. Leader-following consensus of fractional-order multi-agent systems under fixed topology. Neurocomput 2015;149:613–20.
[28] Bai Jing, Wen Guoguang, Song Yu, Rahmani Ahmed, Yu Yongguang. Distributed formation control of fractional-order multi-agent systems with relative damping and communication delay. Int J Control Autom Syst 2017;15(1):85–94.
[29] Almeida João, Silvestre Carlos, Pascoal António M. Continuous-time consensus with discrete-time communications. Systems Control Lett 2012;61(7):788–96.
[30] Jiang Haibo, Yu Jianjiang, Zhou Caigen. Consensus of multi-agent linear dynamic systems via impulsive control protocols. Internat J Systems Sci 2011;42(6):967–76.
[31] Wang Fei, Yang Yongqing. Leader-following exponential consensus of fractional order nonlinear multi-agents system with hybrid time-varying delay: a heterogeneous impulsive method. Phys A 2017;482:158–72.
[32] Yu Zhiyong, Jiang Haijun, Hu Cheng, Yu Juan. Necessary and sufficient conditions for consensus of fractional-order multiagent systems via sampled-data control. IEEE Trans Cybern 2017;47(8):1892–901.
[33] Almeida Ricardo, Girejko Ewa, Hristova Snezhana, Malinowska Agnieszka B. On leader-following consensus in multi-agent systems with discrete updates at random times. Entropy 2020;22(6):24, Paper No. 650.
[34] Fahd Jarad, Abdeljawad Thabet, Alzabut Jehad. Generalized fractional derivatives generated by a class of local proportional derivatives. Eur Phys J Spec Top 2017;226(16–18):3457–71.
[35] Almeida Ricardo, Agarwal Ravi P, Hristova Snezhana, O'Regan Donal. Stability of gene regulatory networks modeled by generalized proportional Caputo fractional differential equations. Entropy 2022;24(3):372.
[36] Abbas Mohamed I, Hristova Snezhana. On the initial value problems for Caputo-type generalized proportional vector-order fractional differential equations. Mathematics 2021;9(21):2720.
[37] Abbas Mohamed I, Hristova Snezhana. Existence results of nonlinear generalized proportional fractional differential inclusions via the diagonalization technique. AIMS Math 2021;6(11):12832–44.
[38] Bohner Martin, Hristova Snezhana. Stability for generalized Caputo proportional fractional delay integro-differential equations. Bound Value Probl 2022;(14),15 p..
[39] Miller Kenneth S, Ross Bertram. An introduction to the fractional calculus and fractional differential equations. New York: A Wiley-Interscience Publication. John Wiley & Sons, Inc.; 1993.
[40] Simon Thomas. Mittag-Leffler functions and complete monotonicity. Integral Transforms Spec Funct 2015;26(1):36–50.
[41] Garrappa Roberto. MATLAB Code. https://www.mathworks.com/matlabcentral/fileexchange/66272-mittag-leffler-function-with-matrix-arguments.
[42] Duan Junsheng, Chen Lian. Solution of fractional differential equation systems and computation of matrix Mittag–Leffler functions. Symmetry 2018;10(10):503.
[43] Matychyn Ivan, Onyshchenko Viktoriia. Impulsive differential equations with fractional derivatives. Int J Differ Equ 2014;9(1):101–9.
[44] Wang JinRong, Fečkan Michel, Zhou Yong. A survey on impulsive fractional differential equations. Fract Calc Appl Anal 2016;19(4):806–31.