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Minimization Problems for Functionals Depending on Generalized Proportional Fractional Derivatives

Ricardo Almeida

Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal; ricardo.almeida@ua.pt

Abstract: In this work we study variational problems, where ordinary derivatives are replaced by a generalized proportional fractional derivative. This fractional operator depends on a fixed parameter, acting as a weight over the state function and its first-order derivative. We consider the problem with and without boundary conditions, and with additional restrictions like isoperimetric and holonomic. Herglotz's variational problem and when in presence of time delays are also considered.

Keywords: fractional calculus; calculus of variations; generalized proportional fractional

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1. Introduction

Fractional derivatives play an important role in modeling real-world events, mainly when the past process influences the current state, or when there are uncertainties that can affect the dynamics of the system. The dependence of the derivative on a parameter (the fractional order) allows us to adjust the order of the derivative to the real data, and thus creating more realistic models to describe a system and predict its future dynamics. The question that is always asked is the following: Which fractional derivative are we going to consider? There are several definitions, each with its advantages and disadvantages [1,2]. One way to overcome this problem is to introduce more general concepts of fractional operators, such as the Hilfer operator [3,4], derivatives depending on another function [5–7], or involving arbitrary kernels [8,9].

One area where fractional calculus has been shown to be useful is in the calculus of variations. Here, instead of considering integer-order derivatives, fractional derivatives are considered in the system [10,11]. In recent years, numerous studies have appeared for different types of fractional operators and with different formulations of the problem under study. To cite a few, we can refer the ones dealing with the Riemann–Liouville fractional derivative [12,13], the Caputo derivative [14], symmetric fractional derivative [15], or the Riesz derivative [16]. Due to this high number of definitions, to overcome this problem, more general definitions of fractional derivatives are used. For example, in the works [17–19], some results of the calculus of the variations are presented in the generalized form.

One possible way to generalize the concept of the fractional derivative has recently been presented. Starting from the definition of tempered fractional derivative, and through a convex combination of the state function u and its derivative u', we define the generalized proportional derivative. Despite being a very recent idea, we have already found numerous works on this subject. For example, in [20–22] we find some fundamental properties of it, stability of fractional differential equations were addressed in [23–25], in [26] is studied stochastic differential equations, and a more general form of the derivative, with dependence of an arbitrary kernel, was considered in [27–29]. However, with regard to the calculus of variations, no study has yet been carried out and with this work, we intend to contribute to this area. The aim of our paper is to study optimization conditions dealing with this form of fractional derivative. Thus, we intend to generalize some already known results and obtain new ones that can not be deduced from previous works. We will consider the fundamental problem (minimize a functional, where the Lagrange function depends on a fractional derivative) and then some other cases will be considered. We will study the case when there exist constraints on the formulation of the problem, with the presence of a time delay, or the Herglotz variational problem.

The organization of the paper is as follows. In Section 2 we present some needed definitions and prove a result needed for our proofs. In the following Section 3 we formulate the problem of the calculus of variations and prove the respective Euler–Lagrange equation. Some generalizations are also proven in the last section.

2. Preliminaries

We start by reviewing some needed definitions for our work (see [21]).

Definition 1. *Let $\gamma > 0$ be the fractional order and $\rho \in (0,1]$ be a fixed parameter. Given an integrable function $u : [a,b] \to \mathbb{R}$, we define the left and right generalized proportional fractional integrals of u, of order γ, as*

$$\mathbb{I}_{a+}^{\gamma,\rho} u(t) = \frac{1}{\rho^\gamma \Gamma(\gamma)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{\gamma-1} u(s)\, ds$$

and

$$\mathbb{I}_{b-}^{\gamma,\rho} u(t) = \frac{1}{\rho^\gamma \Gamma(\gamma)} \int_t^b e^{\frac{\rho-1}{\rho}(s-t)} (s-t)^{\gamma-1} u(s)\, ds,$$

respectively.

Definition 2. *Let $\gamma \in (0,1)$ be the fractional order and $\rho \in (0,1]$ be a fixed parameter. Given a C^1 function $u : [a,b] \to \mathbb{R}$, the generalized Caputo proportional fractional derivative of u, of order γ, is defined as*

$$ {}^C\mathbb{D}_{a+}^{\gamma,\rho} u(t) = \mathbb{I}_{a+}^{1-\gamma,\rho} D^\rho u(t) = \frac{1}{\rho^{1-\gamma} \Gamma(1-\gamma)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{-\gamma} D^\rho u(s)\, ds,$$

where $D^\rho u = (1-\rho)u + \rho u'$.

When $\rho = 1$, the generalized Caputo the proportional fractional derivative reduces to the usual Caputo fractional derivative. For the following, we also need a concept similar to generalized Riemann–Liouville proportional right fractional derivative of u:

$$\mathbb{D}_{b-}^{\gamma,\rho} u(t) = \frac{1}{\rho^{1-\gamma} \Gamma(1-\gamma)} {}_{\ominus}D^\rho \left[\int_t^b \mathrm{e}^{\frac{\rho-1}{\rho}(s-t)} (s-t)^{-\gamma} u(s)\, ds \right],$$

where ${}_{\ominus}D^\rho f = (1-\rho) f - \rho f'$.

To end this section, a fractional integration by parts formula is proven, fundamental for the continuation.

Theorem 1. *For two given functions u and v, where u is a continuous function and v a continuously differentiable function, the following formula holds:*

$$\int_a^b u(t) \cdot {}^C\mathbb{D}_{a+}^{\gamma,\rho} v(t)\, dt = \rho \left[\mathbb{I}_{b-}^{1-\gamma,\rho} u(t) \cdot v(t) \right]_a^b + \int_a^b \mathbb{D}_{b-}^{\gamma,\rho} u(t) \cdot v(t)\, dt. \tag{1}$$

Proof. If we interchange the order of integration, we obtain

$$\begin{aligned}
\int_a^b u(t)\cdot {}^C\mathbb{D}_{a+}^{\gamma,\rho} v(t)\,dt &= \int_a^b\int_a^t \frac{1-\rho}{\rho^{1-\gamma}\Gamma(1-\gamma)} u(t) e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-\gamma} v(s)\,ds\,dt \\
&\quad + \int_a^b\int_a^t \frac{\rho}{\rho^{1-\gamma}\Gamma(1-\gamma)} u(t) e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-\gamma} v'(s)\,ds\,dt \\
&= \int_a^b\int_t^b \frac{1-\rho}{\rho^{1-\gamma}\Gamma(1-\gamma)} u(s) e^{\frac{\rho-1}{\rho}(s-t)}(s-t)^{-\gamma} v(t)\,ds\,dt \\
&\quad + \int_a^b\int_t^b \frac{\rho}{\rho^{1-\gamma}\Gamma(1-\gamma)} u(s) e^{\frac{\rho-1}{\rho}(s-t)}(s-t)^{-\gamma} v'(t)\,ds\,dt.
\end{aligned} \tag{2}$$

Integrating by parts the second term, Formula (2) becomes

$$\begin{aligned}
&\int_a^b\int_t^b \frac{1-\rho}{\rho^{1-\gamma}\Gamma(1-\gamma)} u(s) e^{\frac{\rho-1}{\rho}(s-t)}(s-t)^{-\gamma} v(t)\,ds\,dt \\
&\qquad + \left[\left(\int_t^b \frac{\rho}{\rho^{1-\gamma}\Gamma(1-\gamma)} u(s) e^{\frac{\rho-1}{\rho}(s-t)}(s-t)^{-\gamma}\,ds\right) v(t)\right]_a^b \\
&\qquad - \int_a^b \frac{d}{dt}\left(\int_t^b \frac{\rho}{\rho^{1-\gamma}\Gamma(1-\gamma)} u(s) e^{\frac{\rho-1}{\rho}(s-t)}(s-t)^{-\gamma}\,ds\right) v(t)\,dt \\
&\qquad\qquad = \rho\left[\mathbb{I}_{b-}^{1-\gamma,\rho} u(t)\cdot v(t)\right]_a^b + \int_a^b \mathbb{D}_{b-}^{\gamma,\rho} u(t)\cdot v(t)\,dt.
\end{aligned}$$

□

3. Problem Formulation and the Euler–Lagrange Equation

The calculus of variations consists in finding maxima and minima of differentiable functions defined over some functional space. Such functions (or functionals) are usually formed by integrals involving time t, an unknown function u, and its derivative u'. In the fractional calculus of variations, such integer order derivative is replaced by a fractional derivative. Due to the existence of different types of fractional derivatives, we encounter different types of problem formulations of the calculus of variations. In this work, our goal is to generalize some of them, by considering the generalized Caputo proportional fractional derivative. The functional spaces considered here are the sets

$$\Omega = C^1[a,b] \quad \text{and} \quad \Omega_B = \{u \in C^1[a,b] : u(a) = \tau_a, u(b) = \tau_b\},$$

where τ_a and τ_b are the fixed values of the state function at the boundaries. The functional we consider is $\mathcal{F} : \Omega \to \mathbb{R}$ (or $\mathcal{F} : \Omega_b \to \mathbb{R}$) defined as

$$\mathcal{F}(u) = \int_a^b \mathcal{L}(t, u(t), {}^C\mathbb{D}_{a+}^{\gamma,\rho} u(t))\,dt. \tag{3}$$

Function $\mathcal{L} : [a,b]\times\mathbb{R}^2 \to \mathbb{R}$ is assumed to be continuously differentiable. We seek what conditions must a function u^* satisfy in order to be a minimizer of functional (3). The next necessary condition is known as the Euler–Lagrange equation for the variational problem. To simplify notation, we use the following one:

$$\partial_2\mathcal{L} = \frac{\partial\mathcal{L}}{\partial u} \quad \text{and} \quad \partial_3\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\, {}^C\mathbb{D}_{a+}^{\gamma,\rho} u}.$$

Similar notations will appear during the paper, with respect to other functions and variables, with obvious meanings.

Theorem 2. *If $u^* \in \Omega_B$ minimizes functional $\mathcal{F}$, then u^* satisfies the fractional differential equation*

$$\partial_2\mathcal{L}(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t)) + \mathbb{D}^{\gamma,\rho}_{b-}\partial_3\mathcal{L}(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t)) = 0, \quad t\in[a,b]. \tag{4}$$

If $u^* \in \Omega$, that is, $u(a)$ and $u(b)$ are free, then besides Equation (4), the following two transversality conditions

$$\mathbb{I}^{1-\gamma,\rho}_{b-}\partial_3\mathcal{L}(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t)) = 0$$

are satisfied, at $t=a$ and $t=b$.

Proof. If u^* is the optimal solution for the variational problem, consider an auxiliary function f, defined in a neighbourhood of zero, given by the rule $f(\delta) = \mathcal{F}(u^*+\delta\xi)$. Here, $\xi \in \Omega$ (if $u^* \in \Omega_B$, then the conditions $\xi(a) = \xi(b) = 0$ will be imposed). Since u^* extremizes the functional, then $f'(0) = 0$. Computing $f'(0)$, we get

$$\int_a^b \partial_2\mathcal{L}(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t))\xi(t) + \partial_3\mathcal{L}(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t)){}^C\mathbb{D}^{\gamma,\rho}_{a+}\xi(t)\,dt = 0.$$

Using Formula (1), we obtain that

$$\begin{aligned}\int_a^b \Big[\partial_2\mathcal{L}(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t)) + \mathbb{D}^{\gamma,\rho}_{b-}\partial_3\mathcal{L}(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t))\Big]\xi(t)\,dt \\ + \rho\Big[\mathbb{I}^{1-\gamma,\rho}_{b-}\partial_3\mathcal{L}(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t))\xi(t)\Big]_a^b = 0.\end{aligned} \tag{5}$$

If $u^* \in \Omega_B$ then $\xi(a) = \xi(b) = 0$ and so

$$\int_a^b \Big[\partial_2\mathcal{L}(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t)) + \mathbb{D}^{\gamma,\rho}_{b-}\partial_3\mathcal{L}(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t))\Big]\xi(t)\,dt = 0.$$

Since ξ may take any value in the open interval (a,b), we prove that, for all $t\in[a,b]$,

$$\partial_2\mathcal{L}(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t)) + \mathbb{D}^{\gamma,\rho}_{b-}\partial_3\mathcal{L}(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t)) = 0. \tag{6}$$

If the state function u can take any value at $t=a$ and $t=b$, then $\xi(a)$ and $\xi(b)$ are also arbitrary, and by replacing (6) into (5), we conclude that

$$\mathbb{I}^{1-\gamma,\rho}_{b-}\partial_3\mathcal{L}(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t)) = 0$$

at $t=a$ and $t=b$. □

Remark 1. *In order to conclude Equation (6), some continuity assumptions are needed, namely the map*

$$t \mapsto \mathbb{D}^{\gamma,\rho}_{b-}\partial_3\mathcal{L}(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t))$$

must assumed to be continuous in $[a,b]$.

Remark 2. *If $u = (u_1,u_2,\ldots,u_n)$ and $\mathcal{F}_n : \Omega^n \to \mathbb{R}$ is defined as*

$$\mathcal{F}_n(u) = \int_a^b \mathcal{L}_n(t,u_1(t),u_2(t),\ldots,u_n(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u_1(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u_2(t),\ldots,{}^C\mathbb{D}^{\gamma,\rho}_{a+}u_n(t))\,dt,$$

the result is similar: if $u^* \in \Omega^n$ minimizes functional $\mathcal{F}_n$, then u^* satisfies the fractional differential equations

$$\frac{\partial\mathcal{L}_n}{\partial u_i}(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t)) + \mathbb{D}^{\gamma,\rho}_{b-}\frac{\partial\mathcal{L}_n}{\partial\,{}^C\mathbb{D}^{\gamma,\rho}_{a+}u_i}(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t)) = 0, \quad t\in[a,b],$$

and also the two transversality conditions

$$\mathbb{I}_{b-}^{1-\gamma,\rho} \frac{\partial \mathcal{L}_n}{\partial^C \mathbb{D}_{a+}^{\gamma,\rho} u_i}(t, u^*(t), {}^C\mathbb{D}_{a+}^{\gamma,\rho} u^*(t)) = 0$$

are satisfied, at $t = a$ and $t = b$, for every $i \in \{1, 2, \dots, n\}$. For example, for $n = 2$, the functional becomes

$$\mathcal{F}_2(u) = \int_a^b \mathcal{L}_2(t, u_1(t), u_2(t), {}^C\mathbb{D}_{a+}^{\gamma,\rho} u_1(t), {}^C\mathbb{D}_{a+}^{\gamma,\rho} u_2(t))\, dt,$$

and we obtain two fractional differential equations

$$\frac{\partial \mathcal{L}_2}{\partial u_1}(t, u^*(t), {}^C\mathbb{D}_{a+}^{\gamma,\rho} u^*(t)) + \mathbb{D}_{b-}^{\gamma,\rho} \frac{\partial \mathcal{L}_2}{\partial^C \mathbb{D}_{a+}^{\gamma,\rho} u_1}(t, u^*(t), {}^C\mathbb{D}_{a+}^{\gamma,\rho} u^*(t)) = 0, \quad t \in [a, b],$$

and

$$\frac{\partial \mathcal{L}_2}{\partial u_2}(t, u^*(t), {}^C\mathbb{D}_{a+}^{\gamma,\rho} u^*(t)) + \mathbb{D}_{b-}^{\gamma,\rho} \frac{\partial \mathcal{L}_2}{\partial^C \mathbb{D}_{a+}^{\gamma,\rho} u_2}(t, u^*(t), {}^C\mathbb{D}_{a+}^{\gamma,\rho} u^*(t)) = 0, \quad t \in [a, b],$$

$u^* = (u_1^*, u_2^*)$, and four transversality conditions

$$\mathbb{I}_{b-}^{1-\gamma,\rho} \frac{\partial \mathcal{L}_2}{\partial^C \mathbb{D}_{a+}^{\gamma,\rho} u_1}(t, u^*(t), {}^C\mathbb{D}_{a+}^{\gamma,\rho} u^*(t)) = 0,$$

and

$$\mathbb{I}_{b-}^{1-\gamma,\rho} \frac{\partial \mathcal{L}_2}{\partial^C \mathbb{D}_{a+}^{\gamma,\rho} u_2}(t, u^*(t), {}^C\mathbb{D}_{a+}^{\gamma,\rho} u^*(t)) = 0,$$

at $t = a$ and $t = b$.

4. Some Generalizations

We proceed the study when restrictions are imposed in the formulation of the problem, namely the isoperimetric and holonomic constraints. For simplicity, we will assume that $u^* \in \Omega_B$, that is, boundary conditions are imposed on the state functions (if not, transversality conditions similar to the ones presented in Theorem 2 are deduced).

Theorem 3 (Isoperimetric problem). *Let $u^* \in \Omega_B$ be a solution of the following isoperimetric problem: minimize functional $\mathcal{F}$, subject to the integral constraint*

$$\mathcal{G}(u) = \int_a^b \mathcal{M}(t, u(t), {}^C\mathbb{D}_{a+}^{\gamma,\rho} u(t))\, dt = C, \quad C \in \mathbb{R},$$

where $\mathcal{M} : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function. Then, there exists a vector $(\lambda, \lambda_0) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ such that, if we define the Hamiltonian function $\mathcal{H}$ as $\mathcal{H} = \lambda_0 \mathcal{L} + \lambda \mathcal{M}$, u^* satisfies the equation

$$\partial_2 \mathcal{H}(t, u^*(t), {}^C\mathbb{D}_{a+}^{\gamma,\rho} u^*(t)) + \mathbb{D}_{b-}^{\gamma,\rho} \partial_3 \mathcal{H}(t, u^*(t), {}^C\mathbb{D}_{a+}^{\gamma,\rho} u^*(t)) = 0, \quad t \in [a, b].$$

Proof. First, suppose that u^* verifies the condition

$$\partial_2 \mathcal{G}(t, u^*(t), {}^C\mathbb{D}_{a+}^{\gamma,\rho} u^*(t)) + \mathbb{D}_{b-}^{\gamma,\rho} \partial_3 \mathcal{G}(t, u^*(t), {}^C\mathbb{D}_{a+}^{\gamma,\rho} u^*(t)) = 0, \quad t \in [a, b]. \tag{7}$$

In this case, the desired result is proven considering $(\lambda, \lambda_0) = (0, 1)$. Otherwise, define the two functions f and g, in a neighbourhood of $(0, 0)$, as

$$f(\delta_1, \delta_2) = \mathcal{F}(u^* + \delta_1 \xi_1 + \delta_2 \xi_2) \quad \text{and} \quad g(\delta_1, \delta_2) = \mathcal{G}(u^* + \delta_1 \xi_1 + \delta_2 \xi_2) - C,$$

where $\xi_1, \xi_2 \in \Omega$ with $\xi_i(a) = \xi_i(b) = 0$, for $i = 1, 2$. Observe that

$$\begin{aligned}\frac{\partial g}{\partial \delta_2}(0,0) &= \int_a^b \partial_2 \mathcal{M}(t, u^*(t), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(t))\xi_2(t) \\ &\quad + \partial_3 \mathcal{M}(t, u^*(t), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(t)) {}^C\mathbb{D}^{\gamma,\rho}_{a+} \xi_2(t)\, dt \\ &= \int_a^b \left[\partial_2 \mathcal{M}(t, u^*(t), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(t)) + \mathbb{D}^{\gamma,\rho}_{b-} \partial_3 \mathcal{M}(t, u^*(t), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(t))\right]\xi_2(t)\, dt.\end{aligned}$$

Since u^* does not verify Equation (7), we conclude that there exists a function ξ_2 such that $\partial g/\partial \delta_2(0,0) \neq 0$. Also, since $g(0,0) = 0$, applying the Implicit Function Theorem, we ensure the existence of a function χ such that $g(\delta_1, \chi(\delta_1)) = 0$, for all δ_1 in a neighbourhood of zero. In other words, we can ensure the existence of an infinite family of variations $u^* + \delta_1\xi_1 + \delta_2\xi_2$ of the optimal solution that verify the isoperimetric constraint. To prove the desired result, we will apply the Lagrange multiplier method to prove the existence of a real λ such that the pair $(1, \lambda)$ verifies the needed condition. Since $(0,0)$ is a solution of the problem:

$$\text{minimize function } f \text{ s.t. } g(\delta_1, \delta_2) = 0,$$

and since $\nabla g(0,0) \neq (0,0)$, there exists a real λ such that $\nabla(f + \lambda g)(0,0) = (0,0)$. Computing $\partial(f + \lambda g)/\partial\delta_1(0,0)$ and setting it equal to zero, we prove the desired result. □

Theorem 4 (Holonomic constraint). *Let $u^* \in \Omega^2_B$ ($u^* = (u_1^*, u_2^*)$) be a solution of the following problem: minimize functional $\mathcal{F}_2 : \Omega^2_B \to \mathbb{R}$, given by*

$$\mathcal{F}_2(u) = \int_a^b \mathcal{L}_2(t, u(t), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u(t))\, dt,$$

subject to the holonomic constraint

$$g(t, u(t)) = 0, \quad t \in [a,b], \tag{8}$$

where $\mathcal{L}_2 : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ and $g : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ are two continuously differentiable functions. Suppose that

$$\partial_3 g(t, u^*(t)) \neq 0, \quad \forall t \in [a,b].$$

Then, there exists a function $\lambda \in C^0[a,b]$ such that, for $i = 2, 3$ and for all $t \in [a,b]$,

$$\partial_i \mathcal{L}_2(t, u^*(t), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(t)) + \mathbb{D}^{\gamma,\rho}_{b-} \partial_{i+2} \mathcal{L}_2(t, u^*(t), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(t)) + \lambda(t)\partial_i g(t, u^*(t)) = 0.$$

Proof. Define $\lambda : [a,b] \to \mathbb{R}$ as

$$\lambda(t) = -\frac{\partial_3 \mathcal{L}_2(t, u^*(t), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(t)) + \mathbb{D}^{\gamma,\rho}_{b-} \partial_5 \mathcal{L}_2(t, u^*(t), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(t))}{\partial_3 g(t, u^*(t))}.$$

Then, the case $i = 3$ is proven. For the case $i = 2$, we prove it using variational arguments. Consider the curve

$$t \mapsto u^* + \delta\xi = (u_1^* + \delta\xi_1, u_2^* + \delta\xi_2),$$

where $\xi \in \Omega^2_B$, with $\xi(a) = \xi(b) = (0,0)$, and $\delta \in \mathbb{R}$. This variation curve must satisfy Equation (8), that is,

$$g(t, u^* + \delta\xi) = 0, \quad t \in [a,b].$$

Differentiating both sides of this equation with respect to δ, and setting $\delta = 0$, we get

$$\partial_2 g(t, u^*)\xi_1 + \partial_3 g(t, u^*)\xi_2 = 0, \quad t \in [a,b]. \tag{9}$$

Also, since u^* minimizes $\mathcal{F}_2$, its first variation must vanish at u^*:

$$\int_a^b \partial_2\mathcal{L}_2(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t))\xi_1(t)+\partial_3\mathcal{L}_2(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t))\xi_2(t)$$
$$+\partial_4\mathcal{L}_2(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t)){}^C\mathbb{D}^{\gamma,\rho}_{a+}\xi_1(t)+\partial_5\mathcal{L}_2(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t)){}^C\mathbb{D}^{\gamma,\rho}_{a+}\xi_2(t)\,dt=0.$$

Integrating by part (see Equation (1)), we obtain

$$\int_a^b \Big[\partial_2\mathcal{L}_2(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t))+\mathbb{D}^{\gamma,\rho}_{b-}\partial_4\mathcal{L}_2(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t))\Big]\xi_1(t)$$
$$+\Big[\partial_3\mathcal{L}_2(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t))+\mathbb{D}^{\gamma,\rho}_{b-}\partial_5\mathcal{L}_2(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t))\Big]\xi_2(t)\,dt=0.$$

Using Equation (9), we prove that

$$\Big[\partial_3\mathcal{L}_2(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t))+\mathbb{D}^{\gamma,\rho}_{b-}\partial_5\mathcal{L}_2(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t))\Big]\xi_2$$
$$=-\lambda(t)\partial_3 g(t,u^*)\xi_2=\lambda(t)\partial_2 g(t,u^*)\xi_1.$$

Thus,

$$\int_a^b \Big[\partial_2\mathcal{L}_2(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t))+\mathbb{D}^{\gamma,\rho}_{b-}\partial_4\mathcal{L}_2(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t)$$
$$+\lambda(t)\partial_2 g(t,u^*))\Big]\xi_1(t)\,dt=0,$$

and since ξ_1 is arbitrary in (a,b), the other formula is proven. □

We now formulate and solve the problem when the Lagrange function depends on a time delay.

Theorem 5 (Time delays). *Let $\tau>0$ be such that $a+\tau<b$. Consider the space of functions*

$$\Omega^\tau_B=\{u\in C^1[a-\tau,b]:\ u(t)=U(t),\ \text{for all } t\in[a-\tau,a],\ u(b)=\tau_b\},$$

where $U\in C^1[a-\tau,a]$ is a fixed function and $\tau_b\in\mathbb{R}$ a fixed number. Let $u^*\in\Omega^\tau_B$ be a solution of

$$\text{minimize}\quad \begin{array}{rccl}\mathcal{F}_\tau: & \Omega^\tau_B & \to & R\\ & u & \mapsto & \displaystyle\int_a^b \mathcal{L}_\tau(t,u(t),u(t-\tau),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u(t))\,dt\end{array},$$

where $\mathcal{L}_\tau:[a,b]\times\mathbb{R}^3\to\mathbb{R}$ is a function of class C^1. Then, for all $t\in[a,b-\tau]$,

$$\partial_2\mathcal{L}_\tau(t,u^*(t),u^*(t-\tau),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t))+\partial_3\mathcal{L}_\tau(t+\tau,u^*(t+\tau),u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t+\tau))$$
$$+\mathbb{D}^{\gamma,\rho}_{(b-\tau)-}\partial_4\mathcal{L}_\tau(t,u^*(t),u^*(t-\tau),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t))$$
$$+_\ominus D^\rho\Bigg[\frac{1}{\rho^{1-\gamma}\Gamma(1-\gamma)}\int_{b-\tau}^b e^{\frac{\rho-1}{\rho}(s-t)}(s-t)^{-\gamma}$$
$$\times\,\partial_4\mathcal{L}_\tau(s,u^*(s),u^*(s-\tau),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(s))\,ds\Bigg]=0,$$

and for all $t \in [b-\tau, b]$,

$$\partial_2 \mathcal{L}_\tau(t, u^*(t), u^*(t-\tau), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(t)) + \mathbb{D}^{\gamma,\rho}_{(b-\tau)-} \partial_4 \mathcal{L}_\tau(t, u^*(t), u^*(t-\tau), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(t))$$
$$+_\ominus D^\rho \left[\frac{1}{\rho^{1-\gamma}\Gamma(1-\gamma)} \int_{b-\tau}^{b} e^{\frac{\rho-1}{\rho}(s-t)} (s-t)^{-\gamma} \right.$$
$$\left. \times \partial_4 \mathcal{L}_\tau(s, u^*(s), u^*(s-\tau), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(s))\, ds \right] = 0.$$

Proof. Let $\xi \in C^1[a-\tau, b]$ with $\xi(t) = 0$, for all $t \in [a-\tau, a] \cup \{b\}$, and let $f(\delta) = \mathcal{F}_\tau(u^* + \delta\xi)$. From $f'(0) = 0$, we get

$$\int_a^b \partial_2 \mathcal{L}_\tau(t, u^*(t), u^*(t-\tau), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(t)) \xi(t) + \partial_3 \mathcal{L}_\tau(t, u^*(t), u^*(t-\tau), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(t))$$
$$\times \xi(t-\tau) + \partial_4 \mathcal{L}_\tau(t, u^*(t), u^*(t-\tau), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(t)) {}^C\mathbb{D}^{\gamma,\rho}_{a+} \xi(t)\, dt = 0.$$

Obviously

$$\int_a^b \partial_3 \mathcal{L}_\tau(t, u^*(t), u^*(t-\tau), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(t)) \xi(t-\tau)\, dt$$
$$= \int_a^{b-\tau} \partial_3 \mathcal{L}_\tau(t+\tau, u^*(t+\tau), u^*(t), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(t+\tau)) \xi(t)\, dt.$$

On the other hand,

$$\int_a^b \partial_4 \mathcal{L}_\tau(t, u^*(t), u^*(t-\tau), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(t)) {}^C\mathbb{D}^{\gamma,\rho}_{a+} \xi(t)\, dt$$
$$= \int_a^b \mathbb{D}^{\gamma,\rho}_{b-} \partial_4 \mathcal{L}_\tau(t, u^*(t), u^*(t-\tau), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(t)) \cdot \xi(t)\, dt$$
$$= \int_a^b \mathbb{D}^{\gamma,\rho}_{(b-\tau)-} \partial_4 \mathcal{L}_\tau(t, u^*(t), u^*(t-\tau), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(t)) \cdot \xi(t)$$
$$+_\ominus D^\rho \left[\frac{1}{\rho^{1-\gamma}\Gamma(1-\gamma)} \int_{b-\tau}^{b} \mathrm{e}^{\frac{\rho-1}{\rho}(s-t)} (s-t)^{-\gamma} \right.$$
$$\left. \times \partial_4 \mathcal{L}_\tau(s, u^*(s), u^*(s-\tau), {}^C\mathbb{D}^{\gamma,\rho}_{a+} u^*(s))\, ds \right] \cdot \xi(t)\, dt.$$

In conclusion,

$$
\begin{aligned}
&\int_a^{b-\tau}\Bigg[\partial_2\mathcal{L}_\tau(t,u^*(t),u^*(t-\tau),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t)) \\
&\qquad + \partial_3\mathcal{L}_\tau(t+\tau,u^*(t+\tau),u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t+\tau)) \\
&\qquad + \mathbb{D}^{\gamma,\rho}_{(b-\tau)-}\partial_4\mathcal{L}_\tau(t,u^*(t),u^*(t-\tau),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t)) \\
&\qquad +_\ominus D^\rho\Bigg[\frac{1}{\rho^{1-\gamma}\Gamma(1-\gamma)}\int_{b-\tau}^b \mathrm{e}^{\frac{\rho-1}{\rho}(s-t)}(s-t)^{-\gamma} \\
&\qquad \times \partial_4\mathcal{L}_\tau(s,u^*(s),u^*(s-\tau),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(s))\,ds\Bigg]\Bigg]\xi(t)\,dt \\
&\qquad + \int_{b-\tau}^b\Bigg[\partial_2\mathcal{L}_\tau(t,u^*(t),u^*(t-\tau),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t)) \\
&\qquad + \mathbb{D}^{\gamma,\rho}_{(b-\tau)-}\partial_4\mathcal{L}_\tau(t,u^*(t),u^*(t-\tau),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t)) \\
&\qquad +_\ominus D^\rho\Bigg[\frac{1}{\rho^{1-\gamma}\Gamma(1-\gamma)}\int_{b-\tau}^b \mathrm{e}^{\frac{\rho-1}{\rho}(s-t)}(s-t)^{-\gamma} \\
&\qquad\qquad \times \partial_4\mathcal{L}_\tau(s,u^*(s),u^*(s-\tau),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(s))\,ds\Bigg]\Bigg]\xi(t)\,dt = 0.
\end{aligned}
$$

The theorem is proven by the arbitrariness of function ξ. □

In our next problem, we solve the Herglotz problem with dependence on this new fractional derivative. This problem can be regarded as a generalization of the usual calculus of variation problem. For more studies on this topic, we suggest [30].

Theorem 6 (Herglotz problem). *Let $u^* \in \Omega_B$ and $z^* \in \Omega$ be a solution of the problem:*

$$
\begin{cases}
\text{minimize } z(b) \\
z'(t) = \mathcal{L}_H(t,u(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u(t),z(t)),\ t\in[a,b] \\
z(a) = \tau_z,\ \tau_z\in\mathbb{R},
\end{cases}
$$

where $\mathcal{L}_H : [a,b]\times\mathbb{R}^3 \to \mathbb{R}$ is a continuously differentiable function. Then, (u^*,z^*) satisfies

$$
\lambda(t)\partial_2\mathcal{L}_H(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t),z^*(t)) + \mathbb{D}^{\gamma,\rho}_{b-}(\lambda(t)\partial_3\mathcal{L}_H(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t),z^*(t))) = 0,
$$

for all $t\in[a,b]$, where $\lambda : [a,b]\to\mathbb{R}$ is the function defined as

$$
\lambda(t) = \exp\left(-\int_a^t \partial_4\mathcal{L}_H(s,u^*(s),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(s),z^*(s))\,ds\right).
$$

Remark 3. *Observe that function z not only depends on time t, but also on the state function u. So, we emphasize this dependence by writing $z = z(u,t)$ when needed.*

Proof. A variation of the curve u^* is the curve $t \mapsto u^* + \delta\xi$, with $\xi(a) = \xi(b) = 0$. The variation of z^* is defined as

$$
Z(t) = \frac{d}{d\delta}z^*(u^*+\delta\xi,t)\Big|_{\delta=0}.
$$

Computing its derivative with respect to time t, we get

$$\begin{aligned}
Z'(t) = \frac{d}{d\delta}\frac{d}{dt}z^*(u^*+\delta\xi,t)\Big|_{\delta=0} \\
= \frac{d}{d\delta}\mathcal{L}_H(t,u^*+\delta\xi,{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*+\delta{}^C\mathbb{D}^{\gamma,\rho}_{a+}\xi,z^*(u^*+\delta\xi,t))\Big|_{\delta=0} \\
= \partial_2\mathcal{L}_H(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t),z^*(t))\xi(t)+\partial_3\mathcal{L}_H(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t),z^*(t)))^C\mathbb{D}^{\gamma,\rho}_{a+}\xi(t) \\
+\partial_4\mathcal{L}_H(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t),z^*(t))Z(t).
\end{aligned}$$

The solution of the differential equation

$$\begin{aligned}
&Z'(t)-\partial_4\mathcal{L}_H(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t),z^*(t))Z(t) \\
&=\partial_2\mathcal{L}_H(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t),z^*(t))\xi(t)+\partial_3\mathcal{L}_H(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t),z^*(t)))^C\mathbb{D}^{\gamma,\rho}_{a+}\xi(t)
\end{aligned}$$

is the function

$$\begin{aligned}
\lambda(t)Z(t)=\lambda(a)Z(a)+\int_a^t \lambda(s)\partial_2\mathcal{L}_H(s,u^*(s),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(s),z^*(s))\xi(s) \\
+\lambda(s)\partial_3\mathcal{L}_H(s,u^*(s),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(s),z^*(s))^C\mathbb{D}^{\gamma,\rho}_{a+}\xi(s)\,ds.
\end{aligned}$$

Since $z(a)$ is fixed, $Z(a)=0$, and since z attains an extremum at $t=b$, $Z(b)=0$. Thus,

$$\begin{aligned}
\int_a^b \lambda(t)\partial_2\mathcal{L}_H(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t),z^*(t))\xi(t) \\
+\lambda(t)\partial_3\mathcal{L}_H(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t),z^*(t))^C\mathbb{D}^{\gamma,\rho}_{a+}\xi(t)\,dt=0
\end{aligned}$$

and integrating by parts,

$$\begin{aligned}
\int_a^b \Big[\lambda(t)\partial_2\mathcal{L}_H(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t),z^*(t)) \\
+\mathbb{D}^{\gamma,\rho}_{b-}(\lambda(t)\partial_3\mathcal{L}_H(t,u^*(t),{}^C\mathbb{D}^{\gamma,\rho}_{a+}u^*(t),z^*(t)))\Big]\xi(t)\,dt=0,
\end{aligned}$$

proving the result. □

This problem can be extended for functions depending on several independent variables. We denote those variables by $t\in[a,b]$ and $s=(s_1,\dots,s_n)\in S=\prod_{i=1}^n[a_i,b_i]$, for $-\infty<a_i<b_i<\infty$, for each $i\in\{1,\dots,n\}$. Given a function $u\in C^1([a,b]\times S)$, we define its fractional derivative as

$${}^C\mathbb{D}^{\gamma,\rho}_{+}u(t,s)=({}^C\mathbb{D}^{\gamma,\rho}_{a+}u(t,s),{}^C\mathbb{D}^{\gamma,\rho}_{a_1+}u(t,s),\dots,{}^C\mathbb{D}^{\gamma,\rho}_{a_n+})u(t,s)\in\mathbb{R}^{n+1},$$

where each of these fractional derivatives are regarded as partial fractional derivatives with respect to the variables $t,s_1,\dots,s_n$. The space of the state functions is given by

$$\Omega^H_B=\{u\in C^1([a,b]\times S): u(t,s) \text{ is fixed if } t\in\{a,b\} \text{ or } s_i\in\{a_i,b_i\},\ i=1,\dots,n\}.$$

Theorem 7 (Multi-dimensional Herglotz problem)**.** *Let $u^*\in\Omega^H_B$ and $z^*\in\Omega$ be a solution of the problem:*

$$\begin{cases} \text{minimize } z(b) \\ z'(t)=\int_S \mathcal{L}_{H2}(t,s,u(t,s),{}^C\mathbb{D}^{\gamma,\rho}_{+}u(t,s),z(t))\,ds,\ t\in[a,b] \\ z(a)=\tau_z,\ \tau_z\in\mathbb{R}, \end{cases}$$

where $\mathcal{L}_{H2} : [a,b] \times \mathbb{R}^{2n+4} \to \mathbb{R}$ is a continuously differentiable function. Define the function $\lambda : [a,b] \to \mathbb{R}$ as

$$\lambda(t) = \exp\left(- \int_a^t \int_S \partial_{2n+4} \mathcal{L}_{H2}(\tau, s, u(\tau,s), {}^C\mathbb{D}_+^{\gamma,\rho} u(\tau,s), z(\tau))\, ds\, d\tau \right).$$

Then, (u^*, z^*) satisfies

$$\begin{aligned}\int_S \lambda(t)\partial_{n+2}\mathcal{L}_{H2}(t,s,u(t,s),{}^C\mathbb{D}_+^{\gamma,\rho}u(t,s),z(t)) \\ + \mathbb{D}_{b-}^{\gamma,\rho}(\lambda(t)\partial_{n+3}\mathcal{L}_{H2}(t,s,u(t,s),{}^C\mathbb{D}_+^{\gamma,\rho}u(t,s),z(t))) \\ + \sum_{i=1}^n \mathbb{D}_{b_i-}^{\gamma,\rho}(\lambda(t)\partial_{n+i+3}\mathcal{L}_{H2}(t,s,u(t,s),{}^C\mathbb{D}_+^{\gamma,\rho}u(t,s),z(t)))\, ds = 0, \quad t \in [a,b].\end{aligned}$$

Proof. Let $\xi = \xi(t,s) \in C^1([a,b] \times S)$ with $\xi(t,s) = 0$ if $t \in \{a,b\}$ or $s_i \in \{a_i, b_i\}$, for $i = 1, \ldots, n$. Define

$$Z(t) = \frac{d}{d\delta} z^*(u^* + \delta\xi, t)\Big|_{\delta=0}.$$

Again,

$$\begin{aligned}Z'(t) = \int_S \partial_{n+2}\mathcal{L}_{H2}(t,s,u(t,s),{}^C\mathbb{D}_+^{\gamma,\rho}u(t,s),z(t))\xi(t,s) \\ + \partial_{n+3}\mathcal{L}_{H2}(t,s,u(t,s),{}^C\mathbb{D}_+^{\gamma,\rho}u(t,s),z(t)){}^C\mathbb{D}_{a+}^{\gamma,\rho}\xi(t,s) \\ + \sum_{i=1}^n \partial_{n+i+3}\mathcal{L}_{H2}(t,s,u(t,s),{}^C\mathbb{D}_+^{\gamma,\rho}u(t,s),z(t)){}^C\mathbb{D}_{a_i+}^{\gamma,\rho}\xi(t,s) \\ + \partial_{2n+4}\mathcal{L}_{H2}(t,s,u(t,s),{}^C\mathbb{D}_+^{\gamma,\rho}u(t,s),z(t))Z(t)\, ds.\end{aligned}$$

Then, for λ given above, the solution of this equation verifies

$$\begin{aligned}\int_a^b \int_S \lambda(t)\Big(\partial_{n+2}\mathcal{L}_{H2}(t,s,u(t,s),{}^C\mathbb{D}_+^{\gamma,\rho}u(t,s),z(t))\xi(t,s) \\ + \partial_{n+3}\mathcal{L}_{H2}(t,s,u(t,s),{}^C\mathbb{D}_+^{\gamma,\rho}u(t,s),z(t)){}^C\mathbb{D}_{a+}^{\gamma,\rho}\xi(t,s) \\ + \sum_{i=1}^n \partial_{n+i+3}\mathcal{L}_{H2}(t,s,u(t,s),{}^C\mathbb{D}_+^{\gamma,\rho}u(t,s),z(t)){}^C\mathbb{D}_{a_i+}^{\gamma,\rho}\xi(t,s)\, ds\, dt = 0,\end{aligned}$$

and integrating by parts, we prove the desired result. □

5. Conclusions and Future Work

In our present work we proved several conditions that allow us to find the optimal solution for several variational problems, in the fractional calculus context. These are necessary conditions that every extremizer of the function must satisfy, and by solving them we determine the candidates for the problem. Unfortunately, in most situations, there is no way to solve them directly and numerical methods must be applied. To our best knowledge, there is no numerical procedure already available to deal with such fractional operators and so an important question is to develop a proper numerical tool for such derivatives. Another line of investigation, depending on this derivative, is the optimal control with the generalization of the Pontryagin maximum principle. Here, the state equation is of the form ${}^C\mathbb{D}_{a+}^{\gamma,\rho}x(t) = f(t, x(t), u(t))$ and the Lagrange function depends only on time t, the state function x, and the control u. So, the previous results are just a special case. However, this problem is much more complex and prior studies are needed.

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