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Minimum Energy Problem in the Sense of Caputo for Fractional Neutral Evolution Systems in Banach Spaces

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Abstract: We investigate a class of fractional neutral evolution equations on Banach spaces involving Caputo derivatives. Main results establish conditions for the controllability of the fractional-order system and conditions for existence of a solution to an optimal control problem of minimum energy. The results are proved with the help of fixed-point and semigroup theories.

Keywords: fractional control systems; neutral evolution systems; controllability; optimal control; minimum energy; Banach fixed point theorem

MSC: 26A33; 34K40; 49J30; 93B05

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1. Introduction

A neutral system is a system where time-delays play an important role. Precisely, such delays appear in both state variables and their derivatives. A delay in the derivative is called "neutral", which makes the system more complex than a classical one where the delays only occur in the state. Neutral delays do not only occur in physical systems, but they also appear in control systems, where they are sometimes added to improve the performance. For instance, a wide range of neutral-type control systems are expressed by

$$\frac{d}{dt}[y(t) - Ky_t] = Ly_t + Bu(t), \quad t \geq 0, \quad y_0(\cdot) = f_0(\cdot), \tag{1}$$

where $y_t : [-1,0] \to \mathbb{C}^n$ is defined by $y_t(s) = y(t+s)$; for $f \in H^1([-1,0],\mathbb{C}^n)$, the difference operator K is given by $Kf = A_{-1}f(-1)$ with A_{-1} a constant $n \times n$ matrix. The delay operator L is defined by

$$Lf = \int_{-1}^{0} [A_2(\theta)f'(\theta) + A_3(\theta)f(\theta)]d\theta$$

with A_2 and A_3 $n \times n$ matrices whose elements belong to $L_2(-1,0)$; B is a constant $n \times r$ matrix; and the control u is an L_2-function [1].

Nowadays, many researchers have investigated neutral differential equations in Banach spaces [2–4]. This interest is explained by the fact that neutral-argument differential equations have interesting applications in real-life problems: they appear, e.g., while modeling networks containing lossless transmission lines or in super-computers. Moreover, second-order neutral equations play an important role in automatic control and in aeromechanical systems, where inertia plays a central role [5–7].

Controllability plays an inherent crucial role in finite and infinite-dimensional systems, being one of the primary concepts in control theory, along with observability and stability. This concept has also attracted many authors; see, for instance, [8–10].

In the last two decades, several researchers have been interested in exploring the concept of controllability for fractional systems [11–13]. This is natural because fractional differential equations are considered a valuable tool in modeling various real-world dynamic systems, including physics, biology, socio-economy, chemistry and engineering [14–16].

It turns out that system (1) can also be studied in the fractional sense, e.g., being expressed by

$$\begin{cases} {}^C D_t^q [y(t) - K y_t] = L y(t) + B u(t), & t \in [0, T], \\ y_0(\cdot) = f_0(\cdot), \end{cases}$$

where ${}^C D_t^q$ denotes the Caputo fractional derivative of order q. The existence of solutions to fractional differential equations for neutral systems involving Caputo or other fractional operators, like Riemann–Liouville fractional derivatives, has been paid much attention [17–19]. Recently, some achievements regarding the existence and uniqueness of mild solutions to fractional stochastic neutral differential systems in a finite dimensional space have been made [20]. Other works are consecrated to demonstrate existence of a mild solution for neutral fractional inclusions of the Sobolev type [21].

In [22], Sakthivel et al. examined the exact controllability of fractional differential neutral systems by establishing sufficient conditions via a fixed-point analysis approach. Later on, Sakthivel et al. investigated the weak controllability of fractional dynamical systems of order $1 < q < 2$ using sectorial operators and Krasnoselskii's fixed-point theorem [23]. Using the same techniques as the previous authors, Qin et al. have studied the controllability and optimal control of fractional dynamical systems of order $1 < q < 2$ in Banach spaces [24]. Yan and Jia used stochastic analysis theory and fixed-point theorems with the strongly continuous α-order cosine family to study an optimal control problem for a class of stochastic fractional equations of order $\alpha \in (1, 2]$ in Hilbert spaces [25]. In 2021, Zhou and He obtained, via the contraction principle and Shauder's fixed-point theorem, a set of sufficient conditions for the exact controllability of a class of fractional systems [26]. More recently, Xi et al. studied the approximate controllability of fractional neutral hyperbolic systems using Sadovskii's fixed point theorem while constructing a Cauchy sequence and a control function [27]. Dineshkumar et al. addressed the problem of approximate controllability for neutral stochastic fractional systems in the sense of Hilfer, treating the problem using Schauder's fixed-point theorem and extending the obtained results to the case of nonlocal conditions [28]. In [29], Ma et al. analyzed the weak controllability of a fractional neutral differential inclusion of the Hilfer type in Hilbert spaces using Bohnenblust–Karlin's fixed point theorem. The concept of complete controllability is studied in [30] by Wen and Xi, where they establish sufficient conditions to assure this type of controllability.

Here, we let $(X, |\cdot|)$ be a Banach space, and we denote the Banach space of continuous functions by $\mathcal{C}(0, T; X)$ with the norm $|x| = \sup_{t \in J} |x(t)|$. Our main goal is to explore the concepts of controllability and optimal control for the following general evolution fractional system:

$$\begin{cases} {}^C D_t^\nu [x(t) - h(t, x_t)] = \mathcal{A} x(t) + \mathcal{B} u(t), & t \in (0, T], \\ x(0) = x_0 \in D(\mathcal{A}), \end{cases} \tag{2}$$

where ${}^C D_t^\nu$ denotes the fractional derivative of order $\nu \in (0, 1)$ in the sense of Caputo, $h : [0, T] \times \mathcal{C}(0, T; X) \to X$ is a given continuous function, and the dynamic of the system $\mathcal{A} : D(\mathcal{A}) \subseteq X \to X$ is a linear, closed operator with dense domain $D(\mathcal{A})$ generating a compact and uniformly bounded C_0 semigroup $\{\mathcal{T}(t)\}_{t \geq 0}$ on X. The control function

$u(\cdot)$ is given in $L^2(0,T;U)$, with U a reflexive Banach space, and the control operator $\mathcal{B} \in \mathcal{L}(U,X)$ is a linear continuous bounded operator, i.e., there exists a constant $M_1 > 0$ such that

$$|\mathcal{B}| \leq M_1. \tag{3}$$

Our main aim is to be able to obtain a set of sufficient conditions assuring the controllability of system (2) and, afterwards, to consider an associated optimal control problem and prove existence of a solution.

The rest of this paper is organized as follows. In Section 2, the definitions of Caputo fractional derivative and mild solutions for system (2) are recalled. Our main result on the controllability of (2) is proved in Section 3. In Section 4, we prove the existence of a control giving minimum energy on a closed convex set of admissible controls. Section 5 is consecrated to the analysis of a concrete example, illustrating the applicability of our main results. We end with Section 6, which contains conclusions and points out some possible future directions of research.

2. Background

In this section, basic definitions, notations, and lemmas are introduced to be used throughout the paper. In particular, we recall the main properties of fractional calculus [31,32] and useful properties of semigroup theory [33].

Throughout the paper, let $\mathcal{A}$ be the infinitesimal generator of a compact and uniformly bounded C_0 semi-group $\{\mathcal{T}(t)\}_{t\geq 0}$. Let $0 \in \varrho(\mathcal{A})$, where $\varrho(\mathcal{A})$ denotes the resolvent of $\mathcal{A}$. Then, for $0 \leq \mu \leq 1$, the fractional power $\mathcal{A}^\mu$ is defined as a closed linear operator on its domain $D(\mathcal{A}^\mu)$. For a compact semi-group $\{\mathcal{T}(t)\}_{t\geq 0}$, the following properties are useful in this paper:

(i) There exists $M_T \geq 1$ such that

$$M_T = \sup_{t\geq 0} |\mathcal{T}(t)|; \tag{4}$$

(ii) For any $\mu \in (0,1]$, there exists $\mathbb{L}_\mu > 0$ such that

$$|\mathcal{A}^\mu \mathcal{T}(t)| \leq \frac{\mathbb{L}_\mu}{t^\mu}, \quad 0 \leq t \leq T. \tag{5}$$

Now we recall the notion of a Caputo fractional derivative.

Definition 1 (See [32]). *The left-sided Caputo fractional derivative of order $\nu > 0$ of a function $z \in L^1([0,T])$ is*

$$ {}_0^C D_t^\nu z(t) = \frac{1}{\Gamma(\kappa-\nu)} \int_0^t (t-s)^{\kappa-\nu-1} \frac{d^\kappa}{ds^\kappa} z(s) ds, \tag{6}$$

where $t \geq 0$, $\kappa - 1 < \nu < \kappa$, $\kappa \in \mathbb{N}$, and $\Gamma(\cdot)$ is the gamma function.

Using the probability density function and its Laplace transform [34] (see also [35,36]), we recall the definition of a mild solution for system (2).

Definition 2 (See [34]). *Let $u \in U$ for $t \in]0,T]$. A function $x \in \mathcal{C}(0,T;X)$ is said to be a mild solution of system (2) if*

$$\begin{aligned} x(t,u) = {} & S_\nu(t)[x_0 - h(0,x_0)] + h(t,x_t) + \int_0^t (t-s)^{\nu-1} \mathcal{A} K_\nu(t-s) h(s,x_s) \mathrm{d}s \\ & + \int_0^t (t-s)^{\nu-1} K_\nu(t-s) \mathcal{B} u(s) \mathrm{d}s, \end{aligned} \tag{7}$$

where $S_\nu(\cdot)$ and $K_\nu(\cdot)$ are the characteristic solution operators defined by

$$S_\nu(t) = \int_0^\infty \phi_\nu(\Theta)\mathcal{T}(t^\nu\Theta)\,\mathrm{d}\Theta \quad \textit{and} \quad K_\nu(t) = \nu\int_0^\infty \Theta\phi_\nu(\Theta)\mathcal{T}(t^\nu\Theta)\,\mathrm{d}\Theta$$

with

$$\phi_\nu(\Theta) = \frac{1}{\nu}\Theta^{-1-\frac{1}{\nu}}\psi_\nu\left(\Theta^{-\frac{1}{\nu}}\right)$$

and

$$\psi_\nu(\Theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\Theta^{-\nu n-1}\frac{\Gamma(n\nu+1)}{n!}\sin(n\pi\nu), \quad \Theta \in (0,\infty),$$

the probability density. In addition, we have

$$\int_0^\infty \psi_\nu(\Theta)d\Theta = 1 \textit{ and } \int_0^\infty \Theta^\Lambda\phi_\nu(\Theta)d\Theta = \frac{\Gamma(1+\Lambda)}{\Gamma(1+\nu\Lambda)}, \quad \Lambda \in [0,1].$$

Remark 1. *The solution $x(t,u)$ of (2) is considered in the weak sense, and, when there are no ambiguities, it is denoted by $x_u(t)$. We denote by $x_u(T)$ the mild solution of system (2) at the final time T.*

The following properties of $S_\nu(\cdot)$ and $K_\nu(\cdot)$ will be used throughout the paper.

Lemma 1 (See [34])**.**

1. *For any $t \geq 0$, the operators $S_\nu(t)$ and $K_\nu(t)$ are linear and bounded, i.e.,*

$$|S_\nu(t)y| \leq M_T|y| \quad \textit{and} \quad |K_\nu(t)y| \leq \frac{\nu M_T}{\Gamma(1+\nu)}|y|$$

for any $y \in X$ where $M_T = \sup_{t\geq 0}|\mathcal{T}(t)|$.

2. *For $t > 0$, if $\mathcal{T}(t)$ is compact, then $S_\nu(t)$ and $K_\nu(t)$ are both compact operators.*

Lemma 2 (See [34])**.** *For any $x \in X$, $\varsigma \in (0,1)$ and $\mu \in (0,1]$ we have*

(i) $\mathcal{A}K_\nu(t)x = \mathcal{A}^{1-\varsigma}K_\nu(t)\mathcal{A}^\varsigma x$, $0 \leq t \leq a$;

(ii) $|\mathcal{A}^\mu K_\nu(t)| \leq \frac{\nu\mathbb{L}_\mu}{t^{\nu\mu}}\frac{\Gamma(2-\mu)}{\Gamma(1+\nu(1-\mu))}$, $0 < t \leq a$.

3. Controllability

Following [37], let us define the meaning of controllability for our system (2).

Definition 3. *System (2) is said to be controllable in X on $[0,T]$ if for any given initial state $x_0 \in X$ and any desired final state $x_d \in X$, there exists a control $u(\cdot) \in L^2(0,T;U)$ such that the mild solution $x \in \mathcal{C}(0,T;X)$ of system (2) satisfies $x_u(T) = x_d$.*

To prove controllability, we make use of the following assumptions (A_1) and (A_2):

(A_1) $\mathcal{T}(t)$ is compact for every $t > 0$;

(A_2) The function $h : [0,T] \times \mathcal{C}(0,T;X) \to X$ is continuous, and there exists a constant $\varsigma \in]0,T[$ and $H, H_1 > 0$ such that $h \in D(\mathcal{A}^\varsigma)$, and for any $z, y \in \mathcal{C}(0,T;X)$, $t \in [0,T]$, the function $\mathcal{A}^\varsigma h(\cdot,z)$ is strongly measurable and $\mathcal{A}^\varsigma h(t,\cdot)$ satisfies the Lipschitz condition

$$|\mathcal{A}^\varsigma h(t,z) - \mathcal{A}^\varsigma h(t,y)| \leq H\|z-y\| \tag{8}$$

and

$$|\mathcal{A}^\varsigma h(t,z)| \leq H_1(\|z\|+1). \tag{9}$$

Let $H_\nu : L^2(0,T;U) \to X$ be the linear operator defined by

$$H_\nu u = \int_0^T (T-s)^{\nu-1} K_\nu(T-s)\mathcal{B}u(s)\mathrm{d}s.$$

By construction, this operator is invertible. Indeed, because H_ν takes values in the cokernel $L^2(0,T;U)/kerH_\nu$, then it is injective. It is also surjective because $L^2(0,T;U)/kerH_\nu \simeq ImH_\nu$ (see [38,39]). The inverse operator H_ν^{-1} takes values in $L^2(0,T;U)/\ker H_\nu$. Thus, there exists a positive constant $M_2 \geq 0$ such that

$$\left|H_\nu^{-1}\right|_{\mathcal{L}\left(X, L^2(0,T;U)/\ker H_\nu\right)} \leq M_2. \tag{10}$$

Let $r \geq 0$. Note that $B_r = \{x \in \mathcal{C}(0,T;X) : \|x\| \leq r\}$ is a bounded closed and convex subset in $\mathcal{C}(0,T;X)$.

Theorem 1. *If* (A_1) *and* (A_2) *are fulfilled, then the evolution system* (2) *is controllable in* $[0,T]$ *provided*

$$\left[|\mathcal{A}^{-\varsigma}| + \frac{\mathbb{L}_{1-\varsigma}\Gamma(1+\varsigma)}{\varsigma\Gamma(1+\nu\varsigma)}T^{\nu\varsigma} + \frac{MM_TM_1}{\Gamma(1+\nu)}T^\nu\left(|\mathcal{A}^{-\varsigma}| + \frac{\mathbb{L}_{1-\varsigma}\Gamma(1+\varsigma)}{\varsigma\Gamma(1+\nu\varsigma)}T^{\nu\varsigma}\right)\right]H < 1. \tag{11}$$

Proof. For any function x, we define the control

$$\begin{aligned} u_x(t) = H_\nu^{-1}\Big[&x_d - S_\nu(t)[x_0 - h(0,x_0)] - h(T,x_T) \\ &- \int_0^T (T-s)^{\nu-1}\mathcal{A}K_\nu(T-s)h(s,x_s)\mathrm{d}s\Big](t). \end{aligned} \tag{12}$$

We shall prove that $G : \mathcal{C}(0,T;X) \to \mathcal{C}(0,T;X)$, defined by

$$\begin{aligned} (Gx(t)) = {}& S_\nu(t)[x_0 - h(0,x_0)] + h(t,x_t) + \int_0^t (t-s)^{\nu-1}\mathcal{A}K_\nu(t-s)h(s,x_s)\mathrm{d}s \\ &+ \int_0^t (t-s)^{\nu-1}K_\nu(t-s)\mathcal{B}u_x(s)\mathrm{d}s, \quad t \in [0,T], \end{aligned} \tag{13}$$

has a fixed point x for the control u_x steering system (2) from x_0 to x_d in time T. From (3), (10), Lemma 1 and (i) of Lemma 2, we have

$$\begin{aligned} |\mathcal{B}u_x(t)| \leq MM_1\Big(&|x_d| + M_T\Big[|x_0| + |h(0,x_0)|\Big] + |h(T,x_T)| \\ &+ \int_0^T (T-s)^{\nu-1}\Big|\mathcal{A}^{1-\varsigma}K_\nu(T-s)\mathcal{A}^{\varsigma}h(s,x_s)\Big|ds\Big). \end{aligned}$$

In view of (9) and (ii) of Lemma 2, it follows that

$$\begin{aligned} |\mathcal{B}u_x(t)| &\leq MM_1\Big(|x_d| + M_T\Big[|x| + \big(r+1\big)H_1|\mathcal{A}^{-\varsigma}|\Big] + \big(r+1\big)H_1|\mathcal{A}^{-\varsigma}| \\ &\qquad + \frac{\nu\mathbb{L}_{1-\varsigma}\Gamma(1+\varsigma)}{\Gamma(1+\nu\varsigma)}H_1\big(r+1\big)\int_0^T (T-s)^{\nu\varsigma-1}ds\Big) \\ &\leq MM_1\Big(|x_d| + M_T\Big[|x| + \big(r+1\big)H_1|\mathcal{A}^{-\varsigma}|\Big] + \big(r+1\big)H_1|\mathcal{A}^{-\varsigma}| \\ &\qquad + \frac{\mathbb{L}_{1-\varsigma}}{\Gamma(1+\varsigma)}H_1\big(r+1\big)T^{\nu\varsigma}\Big). \end{aligned}$$

Let

$$\mathcal{Y} = MM_1\bigg(|x_d| + M_T\Big[|x| + \Big(r+1\Big)H_1|\mathcal{A}^{-\varsigma}|\Big] + \Big(r+1\Big)H_1|\mathcal{A}^{-\varsigma}| + \frac{\mathbb{L}_{1-\varsigma}}{\Gamma(1+\varsigma)}H_1\Big(r+1\Big)T^{\nu\varsigma}\bigg).$$

It follows that

$$|\mathcal{B}u_x(t)| \leq \mathcal{Y}. \tag{14}$$

In order to show that G has a unique fixed point on B_r, we will proceed in two steps.
Step I: $Gx \in B_r$ whenever $x \in B_r$. For any fixed $x \in B_r$ and $0 \leq t \leq T$, we have

$$\begin{aligned} |(Gx(t))| \leq & |S_\nu(t)[x_0 - h(0,x_0)]| + |h(t,x_t)| + \int_0^t \Big|(t-s)^{\nu-1}\mathcal{A}K_\nu(t-s)h(s,x_s)\Big|\,ds \\ & + \int_0^t (t-s)^{\nu-1}|K_\nu(t-s)\mathcal{B}u_x(s)|ds. \end{aligned}$$

From Lemma 1, (9), and (i) of Lemma 2, it results that

$$\begin{aligned} |(Gx(t))| \leq & M_T\Big[r + \Big(r+1\Big)H_1|\mathcal{A}^{-\varsigma}|\Big] + \Big(r+1\Big)H_1|\mathcal{A}^{-\varsigma}| \\ & + \int_0^t (t-s)^{\nu-1}\Big|\mathcal{A}^{1-\varsigma}K_\nu(t-s)\mathcal{A}^{\varsigma}h(s,x_s)\Big|\,ds \\ & + \frac{\nu M_T}{\Gamma(1+\nu)}\int_0^t (t-s)^{\nu-1}|\mathcal{B}u_x|ds. \end{aligned}$$

Now, by using (ii) of Lemma 2, we get

$$\begin{aligned} |(Gx(t))| \leq & M_T[r + H|\mathcal{A}^{-\varsigma}|\Big(r+1\Big)] + H|\mathcal{A}^{-\varsigma}|\Big(r+1\Big)\Big| \\ & + \frac{\nu\mathbb{L}_{1-\varsigma}\Gamma(1+\varsigma)}{\Gamma(1+\nu\varsigma)}H\Big(r+1\Big)\int_0^t (t-s)^{\nu\varsigma-1}ds \\ & + \frac{\nu M_T}{\Gamma(1+\nu)}\int_0^t (t-s)^{\nu-1}\Big|\mathcal{B}u_x(s)\Big|ds. \end{aligned}$$

According to (14), one has

$$\begin{aligned} |(Gx(t))| \leq & M_T\Bigg[r + H|\mathcal{A}^{-\varsigma}|\Big(r+1\Big)\Bigg] + H|\mathcal{A}^{-\varsigma}|\Big(r+1\Big)| \\ & + \frac{\nu\mathbb{L}_{1-\varsigma}\Gamma(1+\varsigma)}{\varsigma\Gamma(1+\nu\varsigma)}H\Big(r+1\Big)T^{\nu\varsigma} + \frac{M_T}{\Gamma(1+\nu)}\mathcal{Y}T^{\nu}. \end{aligned}$$

By choosing

$$\begin{aligned} r = & M_T\Bigg[r + \Big(r+1\Big)H_1|\mathcal{A}^{-\varsigma}|\Bigg] + \Big(r+1\Big)H_1|\mathcal{A}^{-\varsigma}| \\ & + \frac{\nu\mathbb{L}_{1-\varsigma}\Gamma(1+\varsigma)}{\varsigma\Gamma(1+\nu\varsigma)}H_1\Big(r+1\Big)T^{\nu\varsigma} + \frac{\nu M_T}{\Gamma(1+\nu)}\mathcal{Y}T^{\nu}, \end{aligned}$$

we get that $Gx \in B_r$ whenever $x \in B_r$.
Step II: G is a contraction on B_r. For any $v, w \in B_r$ and $0 \leq t \leq T$, in accordance with (12), we obtain

$$
\begin{aligned}
|(Gv)(t)-(Gw)(t)| \leq & \Big|h(t,v_t)-h(t,w_t)\Big| \\
& + \int_0^t (t-s)^{\nu-1}\Big|\mathcal{A}r_\nu(t-s)\Big(h\big(s,v(s)\big)-h\big(s,w(s)\big)\Big)\Big|\,ds \\
& + \int_0^t (t-s)^{\nu-1}\Big|r_\nu(t-s)\mathcal{B}H_\nu^{-1}\Big[h(T,v_T)-h(T,w_T)+\int_0^T (T-\tau)^{\nu-1} \\
& \times \mathcal{A}K_\nu(T-\tau)\Big(h\big(\tau,v(\tau)\big)-h\big(\tau,w(\tau)\big)\Big)d\tau\Big](s)\Big|ds.
\end{aligned}
$$

Considering Lemma 2 and (A_2), we get

$$
\begin{aligned}
|(Gv)(t)-(Gw)(t)| \leq & H|\mathcal{A}^{-\varsigma}|v-w| + \frac{\nu\mathbb{L}_{1-\varsigma}\Gamma(1+\varsigma)}{\Gamma(1+\nu\varsigma)}H|v-w|\int_0^t (t-s)^{\nu\varsigma-1}ds \\
& + \frac{\nu M M_T M_1}{\Gamma(1+\nu)}\int_0^t (t-s)^{\nu-1}\Big[\Big|h(T,v_T)-h(T,w_T)\Big| \\
& + \int_0^t (T-\tau)^{\nu-1}\Big|\mathcal{A}^{1-\varsigma}K_\nu(t-\tau)\mathcal{A}^{\varsigma}\Big[h(\tau,v(\tau))-h(\tau,w(\tau))\Big]\Big|d\tau\Big]ds.
\end{aligned}
$$

From (8), we obtain that

$$
\begin{aligned}
|(Gv)(t)-(Gw)(t)| \leq & H|\mathcal{A}^{-\varsigma}|v-w| + \frac{\mathbb{L}_{1-\varsigma}\Gamma(1+\varsigma)}{\varsigma\Gamma(1+\nu\varsigma)}H|v-w|T^{\nu\varsigma} \\
& + \frac{\nu M M_T M_1}{\Gamma(1+\nu)}\int_0^T (t-s)^{\nu-1}\Big[H|\mathcal{A}^{-\varsigma}|v-w| \\
& + \frac{\mathbb{L}_{1-\varsigma}\Gamma(1+\varsigma)}{\varsigma\Gamma(1+\nu\varsigma)}H|v-w|T^{\nu\varsigma}\Big]ds \\
\leq & H|\mathcal{A}^{-\varsigma}|v-w| + \frac{\mathbb{L}_{1-\varsigma}\Gamma(1+\varsigma)}{\varsigma\Gamma(1+\nu\varsigma)}H|v-w|T^{\nu\varsigma} \\
& + \frac{M M_T M_1}{\Gamma(1+\nu)}T^{\nu}\Big[|\mathcal{A}^{-\varsigma}| + \frac{\mathbb{L}_{1-\varsigma}\Gamma(1+\varsigma)}{\varsigma\Gamma(1+\nu\varsigma)}T^{\nu\varsigma}\Big]H|v-w| \\
= & \Big[|\mathcal{A}^{-\varsigma}| + \frac{\mathbb{L}_{1-\varsigma}\Gamma(1+\varsigma)}{\varsigma\Gamma(1+\nu\varsigma)}T^{\nu\varsigma} + \frac{M M_T M_1}{\Gamma(1+\nu)}T^{\nu} \\
& \Big(|\mathcal{A}^{-\varsigma}| + \frac{\mathbb{L}_{1-\varsigma}\Gamma(1+\varsigma)}{\varsigma\Gamma(1+\nu\varsigma)}T^{\nu\varsigma}\Big)\Big]H|v-w|.
\end{aligned}
$$

From Theorem 1, we have

$$
\Big[|\mathcal{A}^{-\varsigma}| + \frac{\mathbb{L}_{1-\varsigma}\Gamma(1+\varsigma)}{\varsigma\Gamma(1+\nu\varsigma)}T^{\nu\varsigma} + \frac{M M_T M_1}{\Gamma(1+\nu)}T^{\nu}\Big(|\mathcal{A}^{-\varsigma}| + \frac{\mathbb{L}_{1-\varsigma}\Gamma(1+\varsigma)}{\varsigma\Gamma(1+\nu\varsigma)}T^{\nu\varsigma}\Big)\Big]H < 1;
$$

it follows that

$$
|(Gv)(t)-(Gw)(t)| < |v-w|,
$$

that is, G is a contraction on B_r. We conclude from the Banach fixed-point theorem that G has a unique fixed point x in $\mathcal{C}(0,T;X)$. Then, by injecting u_x in (7), we have

$$\begin{aligned} x_{u_x}(T) &= S_\nu(T)[x_0 - h(0,x_0)] + h(T,x_T) + \int_0^T (T-s)^{\nu-1}\mathcal{A}K_\nu(T-s)h(s,x_s)\mathrm{d}s \\ &\quad + \int_0^T (T-s)^{\nu-1}K_\nu(T-s)\mathcal{B}u_x(s)\mathrm{d}s, \\ &= S_\nu(T)[x_0 - h(0,x_0)] + h(T,x_T) + \int_0^T (T-s)^{\nu-1}\mathcal{A}K_\nu(T-s)h(s,x_s)\mathrm{d}s \\ &\quad + H_\nu H_\nu^{-1}\Big[x_d - S_\nu(T)[x_0 - h(0,x_0)] - h(T,x_T) \\ &\quad - \int_0^T (T-s)^{\nu-1}\mathcal{A}K_\nu(T-s)h(s,x_s)\mathrm{d}s\Big] \\ &= x_d \end{aligned}$$

and system (2) is exactly controllable, which completes the proof. □

We have shown, under assumptions (A_1) and (A_2), and with the help of Schauder's fixed-point theorem, that the neutral system (2) is controllable when condition (11) holds. It would be interesting to clarify if the obtained control is unique in the sense that any control that allows reaching the state x_d is such that the associated state x is a fixed point of the operator G. This uniqueness question is relevant but remains open.

4. Optimal Control

Now, we consider the problem of steering system (2) from the state x_0 to a target state x_d in time T with minimum energy. We prove the existence of solution to such an optimal control problem when the set of admissible controls is closed and convex.

Let $\mathcal{U}_{ad}$ be the nonempty set of admissible controls defined by

$$\mathcal{U}_{ad} = \Big\{u \in L^2(0,T;U) : x_u(T) = x_d\Big\}.$$

We shall prove that $\mathcal{U}_{ad}$ is closed. For that, let us consider a sequence u_n in $\mathcal{U}_{ad}$ such that $u_n \to u$ strongly in $L^2(0,T;U)$, so

$$\begin{aligned} x_{u_n}(T) &= S_\nu(T)[x_0 - h(0,x_0)] + h(T,x_T) + \int_0^T (T-s)^{\nu-1}\mathcal{A}K_\nu(T-s)h(s,x_s)\mathrm{d}s \\ &\quad + \int_0^T (T-s)^{\nu-1}K_\nu(T-s)\mathcal{B}u_n(s)\mathrm{d}s. \end{aligned}$$

Put

$$\mathcal{Q}u = \int_0^T (T-s)^{\nu-1}\mathcal{A}K_\nu(T-s)h(s,x_s)\mathrm{d}s + \int_0^T (T-s)^{\nu-1}K_\nu(T-s)\mathcal{B}u_n(s)\mathrm{d}s.$$

Since $\mathcal{Q}u$ is continuous, then $\mathcal{Q}u_n \to \mathcal{Q}u$ strongly in X. We also have that $h : [0,T] \times \mathcal{C}(0,T;X) \to X$ is continuous; then $x_{u_n}(T) \to x_u(T)$ in X, but $x_{u_n}(T) \in \{x_d\}$, which is closed. Therefore, $x_u(T) \in \{x_d\}$, which means that $u \in \mathcal{U}_{ad}$. Hence, $\mathcal{U}_{ad}$ is closed.

For a desired state x_d, our optimal control problem consists of finding within $\mathcal{U}_{ad}$ a control minimizing the functional

$$J(u) = \frac{\varsigma}{2}\int_0^T |x_u(t) - x_d|_X^2 dt + \frac{\varepsilon}{2}\int_0^T |u(t)|_U^2 dt,$$

where $x_u(\cdot)$ is the mild solution of system (2) associated with u. The parameters ε and ς are non-negative constants. Precisely, our optimal control problem is:

$$\left\{ \begin{array}{l} \inf\limits_{u\in\mathcal{U}_{ad}} J(u), \\ \text{s.t. } (2). \end{array} \right. \tag{15}$$

The following result gives a necessary condition for the existence of an optimal control to our minimum energy problem.

Theorem 2. *Let $\mathcal{U}_{ad}$ be closed and convex. If $1 - H|\mathcal{A}^{-\varsigma}| > 0$, then there exists a $u^\star \in \mathcal{U}_{ad}$ solution to the optimal control problem* (15).

Proof. Let $|u_p|^2 \leq \frac{2}{\varepsilon} J(u_p)$ with $(u_p)_{p\in\mathbb{N}}$ bounded. Then there exists a subsequence, still denoted $(u_p)_{p\in\mathbb{N}}$, that converges weakly to a limit $u^\star$. If $\mathcal{U}_{ad}$ is closed and convex, then $\mathcal{U}_{ad}$ is closed for the weak topology, which implies that $u^\star \in \mathcal{U}_{ad}$. Let x_p be the unique solution of system (2) associated with u_p, and let $x^\star$ be the unique solution of system (2) associated with $u^\star$. Then,

$$
\begin{aligned}
|x_p(t) - x^*(t)| \leq\ & |h(t, x_p(t)) - h(t, x^\star(t))| \\
& + \left| \int_0^t (t-s)^{\nu-1} \mathcal{A} K_\nu(t-s) [h(s, x_p(s)) - h(s, x^\star(s))] \mathrm{d}s \right| \\
& + \left| \int_0^t (t-s)^{\nu-1} K_\nu(t-s) \mathcal{B}[u_p(s) - u^\star(s)] \mathrm{d}s \right| \\
\leq\ & H|\mathcal{A}^{-\varsigma}||x_p(t) - x^\star(t)| \\
& + \int_0^t (t-s)^{\nu-1} \left| \mathcal{A}^{1-\varsigma} K_\nu(t-s) [\mathcal{A}^\varsigma h(s, x_p(s)) - \mathcal{A}^\varsigma h(s, x^\star(s))] \right| \mathrm{d}s \\
& + \left| \int_0^t (t-s)^{\nu-1} K_\nu(t-s) \mathcal{B}[u_p(s) - u^\star(s)] \mathrm{d}s \right|, \quad t \in [0, T].
\end{aligned}
\tag{16}
$$

This leads us to

$$
\begin{aligned}
(1 - H|\mathcal{A}^{-\varsigma}|)|x_p(t) - x^*(t)| \leq\ & \frac{\nu \Gamma(1+\varsigma)}{\Gamma(1+\nu\varsigma)} \mathbb{L}_{1-\varsigma} \int_0^t (t-s)^{\nu\varsigma-1} H |x_p(t) - x^\star(t)| \mathrm{d}s \\
& + \left| \int_0^t (t-s)^{\nu-1} K_\nu(t-s) \mathcal{B}[u_p(s) - u^\star(s)] \mathrm{d}s \right|,
\end{aligned}
\tag{17}
$$

$t \in [0, T]$. Set $\mathcal{K}' = \dfrac{1}{1 - H|\mathcal{A}^{-\varsigma}|}$. Then,

$$
\begin{aligned}
|x_p(t) - x^*(t)| \leq\ & \mathcal{K}' \frac{\nu \Gamma(1+\varsigma)}{\Gamma(1+\nu\varsigma)} \mathbb{L}_{1-\varsigma} \int_0^t (t-s)^{\nu\varsigma-1} H |x_p(t) - x^\star(t)| \mathrm{d}s \\
& + \mathcal{K}' \left| \int_0^t (t-s)^{\nu-1} K_\nu(t-s) \mathcal{B}[u_p(s) - u^\star(s)] \mathrm{d}s \right|, \quad t \in [0, T].
\end{aligned}
\tag{18}
$$

Using the Gronwall lemma, we obtain that

$$
\begin{aligned}
|x_p(t) - x^*(t)| \leq\ & \mathcal{K}' \left| \int_0^t (t-s)^{\nu-1} K_\nu(t-s) \mathcal{B}[u_p(s) - u^\star(s)] \mathrm{d}s \right| \\
& \exp\left(\mathcal{K}' \frac{\nu \Gamma(1+\varsigma)}{\Gamma(1+\nu\varsigma)} \mathbb{L}_{1-\varsigma} H \int_0^t (t-s)^{\nu\varsigma-1} \mathrm{d}s \right) \\
\leq\ & \mathcal{K}' \left| \int_0^t (t-s)^{\nu-1} K_\nu(t-s) \mathcal{B}[u_p(s) - u^\star(s)] \mathrm{d}s \right| \\
& \exp\left(\mathcal{K}' \frac{\Gamma(1+\varsigma)}{\varsigma \Gamma(1+\nu\varsigma)} \mathbb{L}_{1-\varsigma} H T^{\nu\varsigma} \right).
\end{aligned}
\tag{19}
$$

Now, by the weak convergence, $u_p \rightharpoonup u^*$ in $L^2(0,T,U)$, and from Lemma 1, we obtain that

$$\left|\int_0^t (t-s)^{\nu-1}K_\nu(t-s)\mathcal{B}[u_p(s)-u^\star(s)]\mathrm{d}s\right| \\ \leq \frac{\nu M_T M_1}{\Gamma(1+\nu)}\int_0^t (t-s)^{\nu-1}|u_p(s)-u^\star(s)|_{L^2(0,T,U)}\mathrm{d}s, \quad (20)$$

from which $x_p \to x^\star$ strongly in $L^2(0,T;X)$. Hence,

$$\lim_{n\to\infty}\int_0^T |x_p(t)-x_d|_X^2 dt = \int_0^T |x(t)-x_d|_X^2 dt.$$

Using the lower semi-continuity of norms, the weak convergence of $(u_p)_n$ gives

$$|u^\star| \leq \lim_{n\to\infty}\inf|u_p|.$$

Therefore, $J(u^\star) \leq \lim_{n\to\infty}\inf J(u_p)$, leading to $J(u^\star) = \inf_{u\in\mathcal{U}_{ad}} J(u_p)$, which establishes the optimality of $u^\star$. □

We have just proved the existence of an optimal control for a closed convex set of admissible controls. In Section 5, our main results are illustrated with the help of an example.

5. An Application

In this section we illustrate the results given by our Theorems 1 and 2.

Let $X = L^2((0,1);\mathbb{R})$ and consider the fractional differential system

$$\begin{cases} {}^C D_t^{1/2}\Big(y(t,z)-h(t,y_t)\Big) = \Delta y(t,z)+\mathcal{B}u(t,z), & t\in[0,1], \\ y(t,0)=y(t,1)=0, & t\in[0,1], \end{cases} \quad (21)$$

where the order ν of the fractional derivative is equal to $\frac{1}{2}$, and the function $h : [0,1]\times\mathcal{C}\to X$ is given by

$$h(t,y_t)(x) = \int_0^1 \mathcal{F}(x,z)u_t(v,z)dz, \quad (22)$$

where $\mathcal{F}$ is assumed to satisfy the following conditions:

(a) The function $\mathcal{F}(x,z)$, $x,z\in[0,1]$, is measurable and

$$\int_0^1\int_0^1 \mathcal{F}^2(x,z)dz < \infty;$$

(b) The function $\partial x\mathcal{F}(x,z)$ is measurable, $\mathcal{F}(0,z)=\mathcal{F}(1,z)=0$, and

$$\left(\int_0^1\int_0^1 \big(\partial x\mathcal{F}(x,z)\big)^2 dzdx\right)^{1/2} < \infty.$$

Let $\mathcal{A} : D(\mathcal{A}) \subseteq X \to X$ be defined by $\mathcal{A}x = -x''$ with the domain

$$D(\mathcal{A}) = \big\{x(\cdot)\in X : x, x' \text{ absolutely continuous }, x''\in X, x(0)=x(1)=0\big\}.$$

We begin by proving that the assumption (A_1) holds. Indeed, operator $\mathcal{A}$ is self-adjoint, with a compact resolvent, and generating an analytic compact semi-group $\mathcal{T}(t)$. Furthermore, the eigenvalues of $\mathcal{A}$ are $\Lambda_p = p^2\pi^2$, $p\in\mathbb{N}$, with corresponding normalized eigenvectors $e_p(z) = \sqrt{\frac{2}{\pi}}\sin(p\pi z)$, $\{e_i\}_{i=1}^\infty$ forming an orthonormal basis of X. Then,

$$\mathcal{A}x = -\sum_{p=1}^{p=\infty} \Lambda_p (x, e_p) e_p, \quad x \in D(\mathcal{A}),$$

and

$$\mathcal{T}(t)x(s) = \sum_{i=1}^{i=\infty} \exp(\Lambda_i t)(x, e_i)e_i(s), \quad x \in X.$$

Note that $\mathcal{T}(\cdot)$ is a uniformly stable semi-group and $\|\mathcal{T}(t)\|_{L^2[0,1]} \leq \exp(-t)$. The following properties hold:

(i) $\mathcal{A}^{-\frac{1}{2}}x = \sum_{p=1}^{\infty} \frac{1}{p}(x, e_p)e_p$;

(ii) The operator $\mathcal{A}^{\frac{1}{2}}$ is given by

$$\mathcal{A}^{\frac{1}{2}}x = \sum_{p=1}^{\infty} p(x, e_p)e_p$$

and $D(\mathcal{A}^{\frac{1}{2}}) = \left\{ x(\cdot) \in X, \sum_{p=1}^{\infty} p(x, e_p)e_p \in X \right\}$.

Clearly, (4), (5), and (A_1) are satisfied.

Under our assumptions (a) and (b) on $\mathcal{F}$, (8) and (9) are also satisfied, and assumption (A_2) also holds.

Let U be a reflexive Banach space. We consider the control operator $\mathcal{B} : U \to X$ defined by

$$\mathcal{B}u = \sum_{p=1}^{p=\infty} \Lambda_p (\bar{u}, e_p) e_p,$$

where

$$\bar{u} = \begin{cases} u_p, & p = 1, 2, \ldots N, \\ 0, & p = N+1, N+2, \ldots \end{cases}$$

We see that $\mathcal{B}$ is a bounded continuous operator with $M_1 = N\Lambda_N$. For $N \in \mathbb{N}$ and $H_{1/2} : L^2([0,1], U) \to X$ given by

$$H_{1/2}u = \int_0^1 (1-s)^{1/2} P_{1/2}(1-s)\mathcal{B}u(s)ds,$$

we have

$$\begin{aligned}
H_{1/2}u &= \int_0^1 (1-s)^{1/2} \frac{1}{2} \int_0^{\infty} \Theta\phi_{1/2}(\Theta) T((1-s)^{1/2}\Theta)\mathcal{B}u(s) d\Theta\, ds \\
&= \int_0^1 (1-s)^{1/2} \frac{1}{2} \int_0^{\infty} \Theta\phi_{1/2}(\Theta) \sum_{i=1}^{i=\infty} \exp(\Lambda_i (1-s)^{1/2}\Theta)(\mathcal{B}u, e_i)e_i(s) d\Theta\, ds \\
&= \int_0^1 (1-s)^{1/2} \sum_{i=1}^{\infty} \int_0^{\infty} \frac{1}{2}\Theta\phi_{1/2}(\Theta) \sum_{j=0}^{\infty} \frac{\Lambda_i(1-s)^{1/2}\Theta)^j}{j!}(u, e_i)e_i(s) d\Theta\, ds \\
&= \int_0^1 (1-s)^{1/2} \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \frac{(\Lambda_i(1-s)^{1/2})^j}{\Gamma(1/2 + \frac{1}{2}j)}(u, e_i)e_i(s) ds \\
&= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \int_0^1 \frac{\Lambda_i^j}{\Gamma(\frac{1}{2} + \frac{1}{2}j)}(1-s)^{\frac{1+j}{2}}(u, e_i)e_i(s) \\
&= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \frac{2\Lambda_i^j}{\Gamma(\frac{1}{2} + \frac{1}{2}j)(3+j)}(u, e_i)e_i(s).
\end{aligned}$$

Applying Theorem 1, we deduce that the fractional differential system (21) is controllable. Moreover, for function h defined as in (22) with the Lipshitz constant $H < \frac{1}{\left|\mathcal{A}^{-\frac{1}{2}}\right|}$, we conclude from Theorem 2 that there exists a control steering the system, in one unit of time, from a given initial state to a given terminal state with minimum energy.

6. Conclusions

Using the Banach fixed-point theorem, we have obtained a set of sufficient conditions for the controllability of a class of fractional neutral evolution equations involving the Caputo fractional derivative of order $\alpha \in]0,1[$ (cf. Theorem 1). The result is proved in two major steps: (i) in the first step, we proved that the operator G defined by (13) is an element of the bounded closed and convex subset B_r, (ii) while in the second, we proved that G is a contraction on the same subset B_r. Moreover, we formulated a minimum energy optimal control problem and proved conditions assuring the existence of a solution for the optimal control problem $\inf_{u \in \mathcal{U}_{ad}} J(u)$ subject to (2) (cf. Theorem 2). An example was given illustrating the two main results.

Our work can be extended in several directions: (i) to a case of enlarged controllability using different fractional derivatives; (ii) by developing methods to determine the control predicted by our existence theorem, e.g., by using RHUM and penalization approaches [10,40,41]; (iii) or by giving applications of neutral systems to epidemiological problems [42,43]. Many other questions remain open, as is the case of regional controllability and regional discrete controllability for problems of the type considered here. A strong motivation behind the investigation of neutral evolution systems, such as (2) considered here, comes from physics, since they describe well various physical phenomena as fractional diffusion equations. However, neutral systems are difficult to study, since such control systems contain time-delays not only in the state but also in the velocity variables, which make them intrinsically more complicated. The limitations of the method we proposed here is that we are not able to provide conditions under which the optimal control is unique. Additionally, we do not have an explicit form for it.

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