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Sensitivity analysis of water supply systems hydraulic models

Análise de sensibilidade de modelos hidráulicos de redes de abastecimento de água

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Dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Mestre em Engenharia Mecânica, realizada sob orientação científica de António Gil D'Orey de Andrade Campos, Professor Auxiliar com Agregação, do Departamento de Engenharia Mecânica da Universidade de Aveiro.

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Keywords

Water supply system, Operations, Energy efficiency, Optimization, Sensitivity analysis, Hydraulic modelling

Abstract

Nowadays, water is one of the most important assets of human society. Although the Earth's surface is covered by 71% of water, just a small percentage of 0.0075% can be consumed. To reach the population, the drinking water must be transported from the source by the Water Supply Systems (WSS). This transportation process requires a large amount of energy, particularly with hydraulic pumps. Therefore, this process is not fully efficient, as it generally does not consider the demand for water by consumers in the different times of day and seasons of the year, nor the prices of electricity, among others.

For the operational efficiency problem mentioned above, the WSS can be improved with the use of hydraulic models and optimization algorithms. This approach supports the operation and management with important details, which allows for obtaining the optimal solution. Nevertheless, the combination of hydraulic models and optimization algorithms requires an effective sensitivity analysis.

This work aims to perform an expedited methodology to solve the energy-efficient operation of WSS. Thus, it is focused on the reproduction of hydraulic systems in models, to be analysed and become more efficient with a gradient-based algorithm (analytical sensitivity analysis). The analytical sensitivity analysis was compared to finite difference, and it was validated with two numerical case studies, a simple network and the AnyTown Modified benchmark.

As expected, the results show that the analytical sensitivities have more accurate results than the finite difference method, as well as need fewer computational resources. This supports the advantage of using this analytical methodology to replace what is in use by the hydraulic simulators, namely the finite difference method.

Palavras-chave Sistema de abastecimento de água, Operações, Eficiência energética, Optimização, Análise de sensibilidade, Modelação hidráulica

Resumo Hoje em dia, a água é um dos bens mais importantes da sociedade humana. Embora a superfície terrestre seja coberta por 71% de água, apenas uma pequena percentagem de 0,0075% pode ser consumida. Para chegar à população, a água potável deve ser transportada a partir da fonte pelos Sistemas de Abastecimento de Água (SAA). Este processo de transporte requer uma grande quantidade de energia, particularmente com bombas hidráulicas. Portanto, este processo não é totalmente eficiente, uma vez que geralmente não considera a procura de água pelos consumidores ao longo do dia e nas diferentes estações do ano, nem os preços da electricidade, entre outros.

Para o problema de eficiência operacional acima mencionado, o SAA pode ser melhorado com a utilização de modelos hidráulicos e algoritmos de optimização. Esta abordagem apoia a operação e gestão com detalhes importantes, o que permite obter a solução óptima. No entanto, a combinação de modelos hidráulicos e algoritmos de optimização requer uma análise de sensibilidade eficaz.

Este trabalho visa realizar uma metodologia expedita para resolver a operação energeticamente eficiente de SAA. Assim, concentra-se na reprodução de sistemas hidráulicos em modelos, para ser analisado e tornar-se mais eficiente com um algoritmo baseado em gradientes (análise de sensibilidade analítica). A análise de sensibilidade analítica foi comparada a diferenças finitas, e foi validada com dois estudos de caso numéricos, uma rede simples e o benchmark AnyTown Modificado.

Como esperado, os resultados mostram que as sensibilidades analíticas têm resultados mais precisos do que o método da diferença finita, bem como necessitam de menos recursos computacionais. Isto suporta a vantagem de utilizar esta metodologia analítica para substituir o que está em uso pelos simuladores hidráulicos, nomeadamente o método da diferença finita.

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Nomenclature

$\$_{i,p}$	- tariff (€/kWh)
Δt	- time increment
$\Delta t^{\mathrm{h}}_{i,p}$	- time horizon i in the total time horizon T
$\Delta t^{\mathrm{h,on}}_{i,p}$	- pump operating time
η	- pump's efficiency
$\mathbf{A}_{10}$	- matrix of fixed head nodes
$\mathbf{A}_{11}$	- head loss matrix
$\mathbf{A}_{12}$	- connectivity matrix
$\mathbf{x}$	- optimization decision variable
EPS	- extended period simulation
WSS	- water supply system
ρ	- water's density ($\rho = 1000$ kg/m)
a	- shutoff head (pump)
A_{R}	- tank area
$C(x)$	- operation cost
C	- Hazen-Williams roughness coefficient (unitless)
d	- pipe diameter
$F(Q)$	- head loss function
f	- friction factor
g	- gravity acceleration ($g = 9.81$ m^2/s)
H	- nodal head
H_0	- fixed nodal heads
$H_{\mathrm{in/out}}$	- nodal head in/out

h_{loss}	- head loss pipes
h_{pump}	- head loss pumps
h_{R}	- water tank level
i	- index of time horizon ($i = 1, \ldots, n_{\text{h}}$)
L	- pipe length
m	- minor loss coefficient
n	- flow exponent
n_{h}	- number of time horizons
n_{nodes}	- number of nodes
n_{pipes}	- number of pipes
n_{pump}	- number of pumps
p	- index of number of pumps ($p = 1, \ldots, n_{\text{pump}}$)
Q	- flow rate
q_{external}	- external flow demand
$Q_{\text{in/out}}$	- input/output flow rate
$q_{\text{r}}, q_{\text{vc}}, Q_{\text{D}}$	- external flow demands used Section 4.1
R	- roughness coefficient
R_{pump}	- pump's curve roughness coefficient
T	- total time horizon
t^{h}	- time horizon
t_{α}	- generic time in $[t_0, T]$
w	- pump relative speed setting
W_p	- pump power

Chapter 1

Introduction

1.1 Background and motivation

Water is an essential commodity for the survival of human beings. Although the Earth's surface is covered by 71% of water, only 2.5% of the total amount is freshwater. This freshwater is distributed over glaciers and ice caps, groundwater and surface water (such as rivers, lakes, atmosphere, etc). The only part that can be consumed is the liquid surface water, equivalent to 0.0075% of the initial amount [USGS 2019]; [Water Science School 2018]; [Wikipedia Foundation b].

Drinking water is the water that can actually be consumed for drinking and food preparation. The amount required by each person depends on his physical activity, health, age and environmental conditions, however, an average daily amount of 2.0L - 2.5L per person can be stated. According to UNICEF, 2.2 billion people do not have access to safe drinking water [UNICEF 2020]. The safe drinking water goes from the source to the customer due to a Water Supply System (WSS), which guarantees the quality of the water.

The WSS can be divided into the transmission and the distribution parts [Wikipedia Foundation a]. Therefore, water transmission and distribution are important systems to consider since those systems transport drinking water to consumers. Additionally, with the population growing, supply systems need to expand, which will increase water losses during the water transportation. Some literature [Coelho and Andrade-Campos 2014] states that in the world, the estimated water losses are around 30%. These losses can be explained by low efficiency regarding system complexity, inefficient pumping stations, old pipes, poor installations and maintenance, among others [Coelho and Andrade-Campos 2014].

The WSS cycle (Figure 1.1) encompasses many processes such as abstraction, treatment, transport, storage and distribution, and all of those processes require large amount of energy. The energy spent on water production corresponds to 35% of the total expenses and, for example, in Portugal with 10 M inhabitants the pumping energy consumption is more than 644 GWh per year [Coelho 2016]. A large part of this energy consumption is due to poorly planned strategies in pumping stations. The most typical used rule is the minimum tank water level, which manage whether pumping is necessary or not. This does not take into consideration electricity prices, variable-speed pumps and water demand by the consumers.

In addition, WSS is indirectly fuel dependent, mainly on fossil fuel which is not

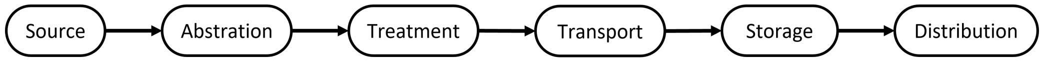


Figure 1.1: Water supply systems cycle

environmentally sustainable due to greenhouse gas (GHG) emissions. To counter this situation, some renewable energy sources such as solar, wind and hydro-power generation are becoming more common. However, despite a large number of advantages, the main obstacle of the cost of implementation remains [Coelho and Andrade-Campos 2014].

1.2 Aim

One of the solutions to the energy efficiency problems, mentioned above, is the hydraulic simulation and optimization of the operation of WSS. The hydraulic simulators can support the operation and management by giving important details, such as optimal pump operation according to water demand and energy prices, and solving hydraulic equations. Almost all types of models use hydraulic equations, for example, regression models, mass-balance models, simplified hydraulic models and full hydraulic simulation models. A Geographic Information System (GIS) combined with hydraulic simulators can also be used to model data and support decision making [Coelho and Andrade-Campos 2014].

Optimization algorithms evaluates and compares values of engineering systems to obtain the optimal solution. In this particular case, it takes into consideration pump systems optimization, water demand prediction, storage and production reservoir systems optimization. With this method, the systems can respond adequately to the growing demand for water by the population and industry, and saving resources such as energy and money. There are a few different types of optimization methodologies, depending on the objective function (single or multi-objective) and the analysis applied. For the case of operational optimization, the objective function is the operation cost. However, demand, pressure and tank level constraints must be taken into account [Wikipedia Foundation c].

The continues use of simulation and optimization is not simple. Therefore, the majority of the solutions to increase operational efficiency uses trial and error approaches, without taking into account sensitivities (derivatives). The calculation and implementation of numerical sensitivities is difficult. Although approximation methods, such as the finite difference method, can be used, their computational effort and approximation error can lead to non-efficient optimization procedures. Automatic differentiation methods cannot be used when commercial hydraulic simulators, such as EPANET, are used and their code is black-boxed. Therefore, analytical sensitivity analysis seem to be the best approach, which is the main goal of this work.

The proposed work consists of applying an expedited methodology to solve the energy efficiency problem of WSS operation using a gradient-based optimization method. To make them computationally efficient, the derivatives (also known as sensitivities) must be calculated.

1.3 Assessing the needs and importance of the water sector

To measure the importance of this topic, an online questionnaire was performed to determine the influence of product features on customer satisfaction [Scubic 2021]. The product consisted of a decision support tool. The target population was related to water supply and wastewater sanitation. There were 37 features that were part of the online questionnaire, the most illustrative for this particular case being the following:

- Pump operating setpoints are optimized to reduce energy consumption and costs;
- All energy consumed was from renewable production (to achieve the energy neutrality through self-consumption);
- Operation performance indicators (including those required by the regulator) automatically available;
- Quickly identify water losses;
- Automatic calibration tool for hydraulic models;
- Automatic hydraulic models were always updated;
- Possibility of real-time operation simulators;
- Possibility of operation simulators;
- Something to help the operator to manage the system intelligently.

These features are designated by one-dimensional or attractive requirements, which means that these features are something expected or appealing to choose this product over another, according to the surveyed persons.

The feature mainly related to this project in the last one mentioned above, which consists of something to help the operator to manage the system intelligently. It demonstrates the importance of it, following a proportional relationship between customer satisfaction and the level of fulfilment/functionality.

This demonstrates the importance of this work in the water sector and for water supply system companies.

1.4 Reading guide

The present document is divided into five parts: Introduction and background, State of the art review, Methodology, Validation and Conclusions. Chapter 1, Introduction and background is dedicated to explaining the importance of the Water Supply System (WSS), and its energy efficiency through optimization processes.

Chapter 2, State of the art review, aims at reviewing the current knowledge over water supply systems, at both scientific, industrial, and commercial levels. In the literature review, some articles about this optimization topic are analysed. The commercial review is carried out and some hydraulic simulators are presented, in particular EPANET, which is one of the most used. Then, the industrial review is done and shows some patents, which are not exactly similar to the proposed work but do the optimization using flow controllers.

In Chapter 3, Methodology, the hydraulic formulation is shown as well as the optimization formulation, describing the decision variable, objective function and constraints. Still in this chapter, the sensitivities equations of the optimization problem are deducted. Additionally, in the Validation chapter, a generic hydraulic model is introduced, as well as two numerical case studies are validated the analytical sensitivities approach (deducted in Chapter 3) compared to the finite difference method. Chapter 5 sums up the major conclusions of the work and provides some insight into future challenges.

Chapter 2

State of the art review

2.1 Literature review

Hydraulic optimization problem is a nonlinear problem because hydraulic equations are nonlinear convex equations. There are some authors, such as Alvisi and Franchini, and Price and Ostfeld, who studied before this type of system and its optimization.

[Alvisi and Franchini 2014] studies the replacement of the standard hydraulic simulator by a simpler simulator based on linearized energy balance equations. The optimization problem is based on the efficient global gradient method and quantifies performance indicators for various solutions. The hydraulic simulation model, with nonlinear equations, is characterized with nodal pressures and pipe discharges and velocities, which has different results regarding open/close pipes different combinations. The linearized hydraulic simulator gives an approximate hydraulic characterisation with less processing time, but it does not provide accurate verification.

There are two cases studies explained: a) optimal placement of flow meters and b) optimal design problem, where the linearized method is applied to arrive at the five best solutions, before the verification with the hydraulic simulator. Thus, the main conclusions are that the linearized simulator gives a reasonable result when compared with the hydraulic simulator solution, special when supported by a high Spearman ranking correlation coefficient, and because it allows a quick identification of the near-optimal solutions. Although, it needs an *a posteriori* verification with the hydraulic simulator.

In the paper of [Price and Ostfeld 2013a], the optimal operating time for each pump, maximising water tank utilization, knowing the hourly flows between supply nodes and the hourly demand constraints are studied. It also describes what type of linearization is used, which is the iteration method. This method corresponds to a first iteration to know the starting conditions between the origin and the maximum value, while the rest of iterations are between the flow rate resulting from the previous iteration (Q_r) and a fixed optimized flow rate point (Q_{fix}). The iteration solution converges for two similar solutions, always checking that the operative cost values are between the iteration – penalty constraint to minimize the flow rate. Thus, the objective function is to minimize the annual operation cost and flow change with constraint such as node water balance, pump node balance, hourly and annual water tank balance, head change at water tank, demand water head, pump total head, dynamic head-loss in pipes and penalty for flow rate change from the previous one. This optimization process uses GAMS/CLP. Application examples are presented, including some basic examples with only water balance and the combination with head-loss and minimal head constraints or a more complex example. The main conclusions of this paper are that the optimal solution is the one that combines minimum annual electric operating costs and hydraulic constraints, this algorithm is capable of iterative linearization of convex nonlinear equations and incorporation in LP optimization models.

[Price and Ostfeld 2013b] uses the same method as the previous one (iteration method using GAMS/CLP), studying the effect of head-loss, leakage, total head and source cost on the operation of the minimum cost optimal system. Head-loss is a non-linear convex relationship between flow and head, given by Hazen-William or Darcy-Weibach formulas. The water leakage depends on the size and age and increases the cost of operation because it pumps out excess water. Therefore, it may be a non-linearly relationship as it relates the average water pressure along the pipe. The pump's energy consumption is dependent of the total pump head and there is a nonlinear relation between it and the flow rate. To simplify the calculations, a fixed average linear energy consumption is assumed, which depends on the hourly flow rate through the pump. Finally, the source cost is a fixed price (per unit of water passing through a pipe) and the source cost "penalty" represents the overhead cost of conveying the water. It can be applied to pumping stations that supply water via a tank or directly with a VFD pump.

This article methodology combines the three convex nonlinear equation of head-loss, water leakage and pump energy consumption in order to minimize the annual water system's operating costs and water flow change penalty. The decision variables are flow rate in pipe, flow change losses, leakage at destination pipe node, water tank node water volume at the tank node, water head at origin pipe node and artificial gain or loss head along the pipe. In addition, it is necessary to have some constraints, for example: water balance at node with leakage, demand node water balance, pump node balance, water leakage equation, flow change penalty, water tank hourly and annual water balance, dynamic head loss in pipes, tank volume to water head constraint (cross section constant).

The optimization algorithm is demonstrated on examples of applications with hypothetical WSS, such as 1) water balance and head-loss, 2a) water balance and minimum water head constraints, 2b) water balance, minimum and maximum water head constraints, 3) water balance, minimum water head constraint and single pipe leakage, 4a) water balance, minimum water head leakage and positive source cost, 4b) water balance, minimum water head leakage and negative source cost. The main result of this work is that the algorithm successfully finds the optimal operating solutions, so it is possible to have a minimum cost optimal operation with an iterative LP model that includes nonlinear Hazen-Williams's head-loss, water leakage, variable pump energy consumption and linear source costs. It also concludes that minimum/maximum water head constraints cause a change in pump operations to maintain a minimum/maximum water level and that reducing hourly flow r to lower dynamic head loss will increases water heads at constrained nodes.

Although the above-mentioned papers are able to solve the problem of WSS optimization, they use a heuristic process that is not as efficient as using gradient-based methods, such as what is proposed in this work.

2.2 Hydraulic simulators

Currently, there are several different simulators and optimization solvers to improve the WSS optimization problem. Some examples of this software programs are listed below:

- OpenFlows FlowMaster is a hydraulic calculation software from Bentley/Virtuosity that performs hydraulic calculations for different types of elements, with useful design and cost savings [Systems 2022a].
- OpenFlows WaterGEMS (WaterCAD) is a water distribution analysis and design software that helps to react with demands increase, operational strategies, etc [Systems 2022b].
- Fluidit Water is an optimized water supply system that helps to minimize energy consumption with a GIS interface. It is usually implemented in smart cities networks to analyse flow characteristics, network problems, design and simulate WSS control parameters [Fluidit 2022].
- Info360 from Innovyze, is an operational analytics software for water distribution to reduce water loss and maximize operational efficiencies and resources [Innovyze 2022b].
- H2ONET / H2OMAP from Innovyze is a modelling, analysis and design software integrated with AutoCAD. It performs hydraulic modelling, energy management and real-time simulation aided with SCADA and GIS interface [Innovyze 2022a].
- Aquis is a water network simulation platform to improve hydraulic performance and perform operational analysis such as minimizing operation costs [Schneider Electric 2022].
- OptiDesigner is an optimal design of WSS software that uses genetic algorithms (GA) to find the minimum cost. It is based on EPANET to design and analyse the system [Salomons 2022].
- STANET is a stationary and dynamic calculator that analyses data from another system (GIS) in order to correct invalid data and modelling [Fischer-Uhrig Engineering 2021].
- WADISO is an application for analyse and optimise WSS design using EPANET and GIS interface [GLS Software 2022].
- Pipe2020 is a Kypipe software that models, calibrates and optimises systems, such as pumping operation [KYPipe LLC 2022].

The most used software is the EPANET. It is a public domain (open source) software application for modelling water supply systems. It is used to design and size new water infrastructure, optimise tank and pump operations, reduce energy consumption, retrofit existing infrastructure, investigate water quality problems, among others.

This simulator also performs extended-period simulation of the hydraulic behaviour in pressurized pipe networks (pipes, nodes/junctions, pumps, valves, storage tanks and reservoirs). It takes into account, during a simulation period, the water flow in each pipe, the pressure in each node, the water height in each tank, the chemical concentration, etc. The main applications are helping and improving WSS with the design sampling program, planning and improving the hydraulic performance of a system, energy minimization and vulnerability studies.

EPANET has extensions/models to enhance performance, such as visual network editor, hydraulic modelling, water quality modelling, water security and resilience modelling and programmer's toolkit. The visual network editor simplifies the process and editing of properties and data. Hydraulic modelling has an accurate hydraulic analysis engine with the ability of different friction head loss methods, different types of equipment (valves, shape of tanks, constant or variable speed pumps), computation pumping energy and costs, and no size limit analysed. On the other hand, water quality modelling allows to know the percentage of flow between two nodes, reactions on the pipe wall, mass transfer limitations, among others. The modelling of water security and resilience has an extension to simulate the interactions between chemical and biological agents with the pipe walls and the bulk water. Finally, the programmer's toolkit is a dynamic link library (DLL) to customize EPANET, where C/C++, visual basic, phyton, and other can be used. This is especially used for optimization or automated calibration models. Some features like CAD, GIS and database packages can be added [EPA 2021].

2.3 Patents

It is important to research already for patented methods to check the market. For this specific case, none has been found that fulfils 100% of the project's objective. However, patents WO2018031911 [KLEIN *et al.* 2018b] and US20180042189 [Klein *et al.* 2018a], that are already in the same patent family, present an "optimized flow control for water infrastructure". These patents claim an optimization method that adjusts water use with flow controllers. The optimization process includes flow controller receiving water consumption data, adjusting the water consumption schedule by a processor and finally activating water outlets by flow controllers.

Chapter 3

Methodology

As already mentioned, the main objective of this work is the calculation of sensitivities of a water supply network optimization problem. Aiming to use these analytical equations to replace the approximate methods usually used, such as finite difference method.

Thus, this chapter begins by describing the general operation of a hydraulic system and its main equations, relating them to the Newton-Raphson optimisation method, which is a numerical method. Also, the operation under unsteady flow is described, since main approach is a quasi-static analysis. This system can be also solved with the help of hydraulic simulators. The non-linear solution uses the method of Todini and Pilati, which is based on the calculation of the derivative by the gradient algorithm to arrive at the solution iteratively.

For the formulation of the operational optimization problem it is important to study the decision variable, i.e. the duration of the status of n_{pump} pumps, as well as the objective function and the constraints.

Therefore, the sensitivities of both the constraints and the objective function are calculated to achieve the optimization solution in a more efficient way. In this case, the time and the space derivatives are deduced, taking into account the different time horizons and the number of tanks. Concerning the derivatives of the cost function, it takes account of the tariff for the different time horizons and the number of pumps in operation.

3.1 Hydraulic formulation

3.1.1 Hydraulic problem

The challenge of the hydraulic system problem is to determine the flow rate Q in each pipe (link) and the nodal head H in each node (junction). The method used in almost every scenario is the Todini and Pilati (1988) [Todini and Pilati 1988], which is based on the Newton-Raphson approach and solves simultaneously the equations of unknown flows and nodal heads. This system has a dimension equal to the number of unknowns ($n_{\text{pipes}} \times n_{\text{nodes}}$) [Boulos *et al.* 2006], where n_{pipes} and n_{nodes} are the number of pipes and nodes, respectively.

In order to solve the system of equations, it is necessary to define the conservation relations of mass and energy that is shown in Figure 3.1. The conservation of mass at each node takes into account the input/output flow rate $Q_{\text{in/out}}$ and external flows

q_{external}, given as

$$q_{\text{in}} = q_{\text{out}} \Leftrightarrow \sum Q_{\text{in}} - \sum Q_{\text{out}} = \sum q_{\text{external}}. \tag{3.1}$$

The conservation of energy in each pipe involves the nodal head in/out $H_{\text{in/out}}$ and head losses in pipes h_{loss} and/or pumps h_{pump},

$$H_{\text{out}} - H_{\text{in}} = -h_{\text{loss}} + h_{\text{pump}}. \tag{3.2}$$

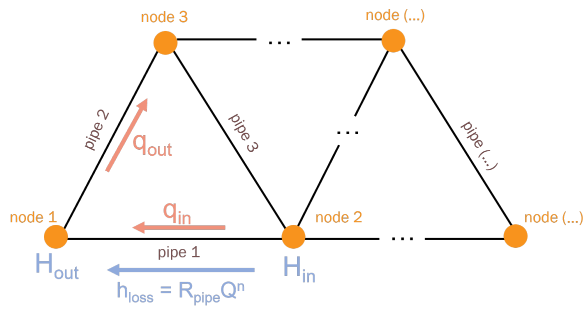


Figure 3.1: Illustrative representation of the mass and energy conservation balance for each node and pipe.

The head losses are a function of flow ($h_{\text{loss}} = F(Q)$),i.e.

$$h_{\text{loss},i} = R_i \times Q_i^n, \quad \text{with} \quad i = 1, \ldots, n_{\text{pipes}} \tag{3.3}$$

where R_i is the roughness coefficient at each pipe i and for the pump's case it is given by the pump curve $h_{\text{pump}} = a - bQ - R_{\text{pump}}Q^2$, which is more frequently represented by Equation 3.4 and Figure 3.2,

$$h_{\text{pump}} = a - R_{\text{pump}} \times Q^2, \tag{3.4}$$

where a is the pump's shutoff head and R_{pump} is the pump's curve roughness coefficient.

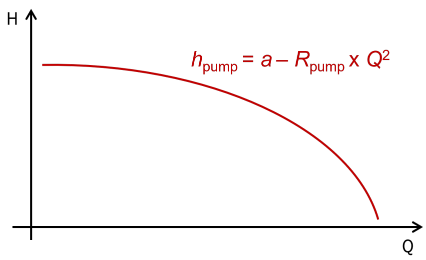


Figure 3.2: Pump curve representation.

A generic hydraulic system is given by

$$\begin{cases} \mathbf{A}_{12} \times \mathbf{H} + \mathbf{F}(Q) = -\mathbf{A}_{10} \times \mathbf{H}_0 \\ \mathbf{A}_{21} \times \mathbf{Q} = \mathbf{q} \end{cases} \Leftrightarrow \begin{cases} \mathbf{A}_{12} \times \mathbf{H} + \mathbf{A}_{11} \times \mathbf{Q} = -\mathbf{A}_{10} \times \mathbf{H}_0 \\ \mathbf{A}_{21} \times \mathbf{Q} = \mathbf{q} \end{cases}, \tag{3.5}$$

or written in an algebraic system of equation:

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & 0 \end{bmatrix} \times \begin{bmatrix} \mathbf{Q} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_{10}\mathbf{H}_0 \\ \mathbf{q} \end{bmatrix}, \tag{3.6}$$

where n_{pipes} is number of pipes, n_{nodes} is number of nodes, $\mathbf{A}_{12} = \mathbf{A}_{21}^{\text{T}}$ (dimension $n_{\text{pipes}} \times n_{\text{nodes}}$) is the connectivity matrix, $\mathbf{Q} = [Q_1, ..., Q_{n_{\text{pipes}}}]^{\text{T}}$ (dimension $1 \times n_{\text{pipes}}$) is the unknown flow rate in each pipe, $\mathbf{H} = [H_1, ..., H_{n_{\text{nodes}}}]^{\text{T}}$ (dimension $1 \times n_{\text{nodes}}$) is the unknown nodal head in each node. $\mathbf{A}_{10}$ (dimension $n_{\text{pipes}} \times n_0$) is matrix of fixed head nodes, $\mathbf{H}_0 = [H_{0_1}, ..., H_{0_{n_{\text{nodes}}}}]^{\text{T}}$ (dimension $1 \times n_0$) are the fixed nodal heads, which also includes the constant value for the pump equation, $\mathbf{q} = [q_1, ..., q_{n_{\text{nodes}}}]^{\text{T}}$ (dimension $1 \times n_{\text{nodes}}$) are the external nodal demands. And $\mathbf{F}(Q) = [f_1, ..., f_{n_{\text{pipes}}}]^{\text{T}}$ (dimension $1 \times n_{\text{pipes}}$) is head loss matrix in pipes and pumps with

$$f_i = R_i|Q_i|^{(n-1)} \times Q_i \Rightarrow \mathbf{F}(Q) = \mathbf{A}_{11} \times \mathbf{Q} \quad \text{and}$$

$$\mathbf{A}_{11} = \begin{bmatrix} R_1|Q_1|^{(n-1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R_{n_{\text{pipes}}}|Q_{n_{\text{pipes}}}|^{(n-1)} \end{bmatrix}. \tag{3.7}$$

Equation 3.6 characterizes the space equilibrium of the system and must be achieved during the all the time.

The hydraulic problem is represented by a system of non-linear equations that needs to be solved by iterative methods, such as for example, the Newton-Raphson method. This method finds the roots of a non-linear function by using the value of the previous iteration (x_n), the function ($F(x)$) and the derivative (Jacobian matrix) of the function ($F'(x) = J_F$). The iteration increment is $\Delta x = x_{n+1} - x_n$ [Andrade-Campos *et al.* 2015], and it is given for a system of equations as

$$x_{n+1} = x_n - J_F^{-1}(x_n)F(x_n), \text{ where } n \in \mathbb{R}^n. \tag{3.8}$$

In order to apply Newton's method it is necessary to derive the previous system (energy conservation - F_{pipes} and mass conservation - F_{nodes}), i.e.

$$\begin{cases} F_{\text{pipes}}(Q,H) = \mathbf{A}_{11} \times \mathbf{Q} + \mathbf{A}_{12} \times \mathbf{H} \\ F_{\text{nodes}}(Q,H) = \mathbf{A}_{21} \times \mathbf{Q} \end{cases}. \tag{3.9}$$

The derived equations of the energy and mass conservation are given by:

$$\begin{cases} \frac{\mathrm{d}F_{\text{pipes}}(Q,H)}{\mathrm{d}\mathbf{Q}} = (\mathbf{A}_{11} \times \mathbf{Q})' + (\mathbf{A}_{12} \times \mathbf{H})' \\ \frac{\mathrm{d}F_{\text{pipes}}(Q,H)}{\mathrm{d}\mathbf{H}} = (\mathbf{A}_{11} \times \mathbf{Q})' + (\mathbf{A}_{12} \times \mathbf{H})' \end{cases} \Rightarrow \mathrm{d}F_{\text{pipes}}(Q,H) = n\mathbf{A}_{11}\mathrm{d}\mathbf{Q} + \mathbf{A}_{12}\mathrm{d}\mathbf{H}; \tag{3.10}$$

and

$$\begin{cases} \frac{\mathrm{d}F_{\text{nodes}}(Q,H)}{\mathrm{d}\mathbf{Q}} = (\mathbf{A}_{21} \times \mathbf{Q})' - (\mathbf{q})' \\ \frac{\mathrm{d}F_{\text{nodes}}(Q,H)}{\mathrm{d}\mathbf{H}} = (\mathbf{A}_{21} \times \mathbf{Q})' - (\mathbf{q})' \end{cases} \Rightarrow \mathrm{d}F_{\text{nodes}}(Q,H) = \mathbf{A}_{21}\mathrm{d}\mathbf{Q}. \tag{3.11}$$

In this way, it can be also represented in a matrix form:

$$\begin{bmatrix} \mathbf{n}\mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & 0 \end{bmatrix} \times \begin{bmatrix} \mathrm{d}\mathbf{Q} \\ \mathrm{d}\mathbf{H} \end{bmatrix} = \begin{bmatrix} \mathrm{d}\mathbf{E} \\ \mathrm{d}\mathbf{q} \end{bmatrix}, \tag{3.12}$$

where $\mathbf{n}$ is a diagonal matrix of dimension $n_{\text{pipes}} \times n_{\text{pipes}}$, d$\mathbf{Q}$ and d$\mathbf{H}$ are the unknown increment, and d$\mathbf{E}$ and d$\mathbf{q}$ are defined with the residual equations.

In each iteration k, the values of the flow rate and nodal head need to be updated according to the following equation:

$$\begin{cases} H^{k+1} = H^k + \Delta H^k \\ Q^{k+1} = Q^k + \Delta Q^k \end{cases}, \tag{3.13}$$

and the residual equations:

$$\begin{cases} \mathrm{d}\mathbf{E} = \mathbf{A}_{11}\mathbf{Q}^k + \mathbf{A}_{12}\mathbf{H}^k + \mathbf{A}_{10}\mathbf{H}_0 \\ \mathrm{d}\mathbf{q} = \mathbf{A}_{21}\mathbf{Q}^k - \mathbf{q} \end{cases}. \tag{3.14}$$

3.1.2 Unsteady flow

The majority of these systems are used in unsteady flow. Unsteady flow is characterized by time variations in flow and pressure. There are different approaches according to the variations, such as the Extended Period Simulation - EPS (quasi-static state analysis), Dynamic Simulation (gradually varying flow) and Transient Simulation Modelling (abrupt changes) [Boulos *et al.* 2006].

The most applied approach in WSS is the EPS, which represents a steady state simulation with a time increment between each iteration. This approach is used with tanks where the tank level (h_R) is not unknown, since it is time dependent and is updated at each time increment.

In order to calculate the mass balance to update the tank data is necessary to know the initial tank level, the demand distribution and the duration between iterations, which is defined by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{CV} \mathrm{d}V = \frac{\mathrm{d}(A_\text{R} h_\text{R})}{\mathrm{d}t} = A_\text{R} \times \frac{\mathrm{d}h_\text{R}}{\mathrm{d}t} = Q_{\text{R,in}} - Q_{\text{R,out}}, \tag{3.15}$$

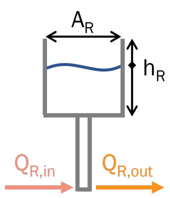


Figure 3.3: Schematic representation on the functioning of a tank and the relation between the water level (h_R) and inflow/outflow ($Q_{\text{in/out}}$).

where $Q_{\text{R,in}}$ and $Q_{\text{R,out}}$ are the tank inflow and outflow, respectively, A_R is the area of the tank and h_R is the water tank level. Equation 3.15 governs the time in the engineering hydraulic system, and for small time increment Δt, the linerization of Equation 3.15 becomes:

$$A_{\mathrm{R}} \times \frac{h_{\mathrm{R},t+\Delta t} - h_{\mathrm{R},t}}{\Delta t} = Q_{\mathrm{R,in}} - Q_{\mathrm{R,out}} \Leftrightarrow h_{\mathrm{R},t+\Delta t} = h_{\mathrm{R},t} + (Q_{\mathrm{R,in}} - Q_{\mathrm{R,out}}) \times \frac{\Delta t}{A_{\mathrm{R}}}, \quad (3.16)$$

where $h_{\mathrm{R},t}$ and $h_{\mathrm{R},t+\Delta t}$ are the tank level before and after the time increment Δt.

The demand patterns used for analysis typically cover a 24 hour cycle to analyse changes in tank level and temporal variations in demand.

3.1.3 Hydraulic simulators approach

The most used hydraulic simulator EPANET uses Todini and Pilati method to solve hydraulic problems [Rossman 2000]. It use some equations of flow and pressure relationships for pipes and pumps to know the head losses, between node i and node j ($i = 1, \ldots, n_{\mathrm{nodes}}$ and $j = 1, \ldots, n_{\mathrm{nodes}}$). For pipes:

$$H_i - H_j = h_{ij} = RQ_{ij}|Q_{ij}|^{n-1} + m|Q_{ij}||Q_{ij}|, \quad (3.17)$$

where R is the resistance coefficient, n is the flow exponent and m is the minor loss coefficient. For pumps:

$$h_{ij} = -w^2 \left[a - R_{\mathrm{pump}} \left(\frac{Q_{ij}}{w} \right)^n \right], \quad (3.18)$$

with w as the relative speed setting, a as the shutoff head and n and R_{pump} as the pump curve coefficients.

EPANET already uses the derived equations and the procedure for solving the hydraulic system starts with an initial estimation of the flow, then finding out the nodal head with the matrix $\mathbf{A} \times \mathbf{H} = \mathbf{F}$ and finally calculating the new flow used in the next iteration. For each iteration, the matrix $\mathbf{A} \times \mathbf{H} = \mathbf{F}$ needs to be solved, where $\mathbf{H}$ is the unknown nodal head, $\mathbf{A}$ is the Jacobian matrix with $A_{ii} = \sum p_{ij}$ and $A_{ij} = -p_{ij}$, where p_{ij} represents the head loss inverse derivative of the between node i and node j, and assumes a different equation for pipes or pumps, respectively given as

$$p_{ij} = \frac{1}{nr|Q_{ij}|^{n-1} + 2m|Q_{ij}|}, \text{ and} \quad (3.19)$$

$$p_{ij} = \frac{1}{nw^2 R_{\mathrm{pump}}(Q_{ij}/w)^{n-1}}. \quad (3.20)$$

The vector $\mathbf{F}$ represents the flow imbalance at the node and a correction factor, which is

$$F_i = \left(\sum Q_{ij} - D_i \right) + \sum y_{ij} + \sum p_{ij} \times H_f, \quad (3.21)$$

where D_i is the flow demand at node i and y_{ij} is the correction flow factor that assumes different expressions for pipes or pumps, respectively:

$$y_{ij} = p_{ij} \times \left(r|Q_{ij}|^n + m|Q_{ij}|^2 \right) \times sgn(Q_{ij}), \text{ and} \quad (3.22)$$

$$y_{ij} = p_{ij} \times w^2 \left[h_0 - R_{\mathrm{pump}} \left(\frac{Q_{ij}}{w} \right)^n \right], \quad (3.23)$$

with $sng(x) = 1$ if $x > 0$ and $sng(x) = -1$ if $x < 0$, to define the sign of the flow rate Q_{ij}.

Finally, the updated flow rate for iteration $k+1$ is calculated by

$$Q_{ij}^{k+1} = Q_{ij}^{k} - \left[y_{ij} - p_{ij} \times (H_i - H_j)\right]. \tag{3.24}$$

3.2 Formulation of the operational optimization problem

In order to improve the optimisation process with the calculation of analytical derivatives it is necessary to formulate the WSS operational problem. Thus, the main objective is to find the status of n_{pump} pumps to minimise the pump costs. In this work, it was defined a decision variable as a continuous variable $\mathbf{x} \in [0,1]$ that represents the duration of the status (on/off) of the n_{pump} pumps in the time horizon $t^{\text{h}} \in [0,T]$. This variable is defined with:

$$\mathbf{x} = \begin{bmatrix} x_{1,1} & \dots & x_{1,n_{\text{pump}}} \\ \vdots & \ddots & \vdots \\ x_{n_h,1} & \dots & x_{n_h,n_{\text{pump}}} \end{bmatrix}, \text{ (dimension } n_h \times n_{\text{pump}}), \tag{3.25}$$

which is represented at Figure 3.4 and given by

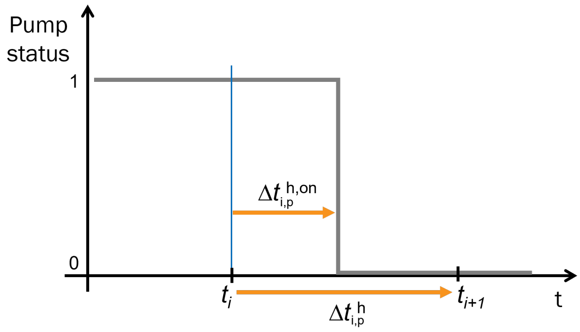


Figure 3.4: Graphic representation of the optimization decision variable $\mathbf{x}$.

$$x_{i,p} = \frac{\Delta t_{i,p}^{\text{h,on}}}{\Delta t_{i,p}^{\text{h}}} \Leftrightarrow \Delta t_{i,p}^{\text{h,on}} = x_{i,p} \Delta t_{i,p}^{\text{h}}, \quad \text{with } i = 1, \dots, n_h \text{ and } p = 1, \dots, n_{\text{pump}}, \tag{3.26}$$

where $\Delta t_{i,p}^{\text{h}}$ is the time horizon i in the total time horizon T, $\Delta t_{i,p}^{\text{h,on}}$ is the pump operating time and n_h is the number of time horizons.

The operation cost $C(x)$ is dependent of the tariff $\$_{i,p}$ and the pump power $\dot{W}_p(t)$ can be written as

$$\begin{aligned} C(x) &= \int_0^T \$_{i,p} \times \dot{W}_p(t) dt \\ &= \sum_{i=1}^{n_h} \sum_{p=1}^{n_{\text{pump}}} \left[\int_0^{\Delta t_{i,p}^{\text{h,on}}} \$_{i,p} \times \dot{W}_p(t) dt + \int_{\Delta t_{i,p}^{\text{h,on}}}^{\Delta t_{i,p}} \$_{i,p} \times \dot{W}_p(t) dt \right] \\ &= \sum_{i=1}^{n_h} \sum_{p=1}^{n_{\text{pump}}} \left[\int_0^{x_{i,p} \Delta t_{i,p}^{\text{h}}} \$_{i,p} \times \dot{W}_p(t) dt + \int_{x_{i,p} \Delta t_{i,p}^{\text{h}}}^{\Delta t_{i,p}} \$_{i,p} \times \dot{W}_p(t) dt \right] \end{aligned} \tag{3.27}$$

and $\dot{W}_p(t)$ is given by

$$\dot{W}_p(t) = \frac{\rho g}{\eta(Q_p)} \times Q_p(t) \times H_p(t) = \frac{\rho g}{\eta(Q_p)} \times Q_p(t) \times [a - R_{\text{pump}} Q_p(t)^2], \tag{3.28}$$

where ρ is the water's density, g is gravity acceleration, η is the pump's efficiency, Q_p is the pump flow rate, H_p is the pump's head, a is the shutoff head and R_{pump} is the pump head loss coefficient.

The problem constraints can be summarized as:

- Water demand. The water demand are satisfied using the hydraulic equilibrium system
$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & 0 \end{bmatrix} \times \begin{bmatrix} \mathbf{Q} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_{10}\mathbf{H}_0 \\ \mathbf{q} \end{bmatrix}; \tag{3.29}$$

- Water tank levels, which can be mathematically written as
$$\mathbf{h}_{\text{R,min}} < \mathbf{h}_{\text{R}} < \mathbf{h}_{\text{R,max}} \Leftrightarrow \begin{bmatrix} g_{1_{i,p}} \\ g_{2_{i,p}} \end{bmatrix} = \begin{bmatrix} h_{\text{R,min}} - h_{\text{R},i,p} \\ h_{\text{R},i,p} - h_{\text{R,max}} \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \tag{3.30}$$

- Continuity, given as
$$\mathbf{h}_{\text{R}}(\text{T}) \geq \mathbf{h}_{\text{R}}(t_0); \tag{3.31}$$

- Node pressure. These constraints can be settled as
$$\mathbf{H}_{\text{no,min}} < \mathbf{H}_{\text{no}} < \mathbf{H}_{\text{no,max}} \Leftrightarrow \begin{bmatrix} g_{3_{i,p}} \\ g_{4_{i,p}} \end{bmatrix} = \begin{bmatrix} H_{\text{no,min}} - H_{\text{no},i,p} \\ H_{\text{no},i,p} - H_{\text{no,max}} \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.32}$$

Thus the optimization problem can be formulated as:

Search for $\mathbf{x}$ of the n_{pump} pumps in order to

$$\begin{aligned}
&\text{minimise} \;\; C(x) = \sum_{i=1}^{n_h} \sum_{p=1}^{n_{\text{pump}}} \left[\int_0^{x_{i,p}\Delta t^{\text{h}}_{i,p}} \$_{i,p} \times \dot{W}_p(t)dt + \int_{x_{i,p}\Delta t^{\text{h}}_{i,p}}^{\Delta t_{i,p}} \$_{i,p} \times \dot{W}_p(t)dt \right] \\
&\text{subject to} \;\; \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & 0 \end{bmatrix} \times \begin{bmatrix} \mathbf{Q} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_{10}\mathbf{H}_0 \\ \mathbf{q} \end{bmatrix} \\
&\qquad \begin{bmatrix} g_{1_{i,p}} \\ g_{2_{i,p}} \end{bmatrix} = \begin{bmatrix} h_{\text{R,min}} - h_{\text{R},i,p} \\ h_{\text{R},i,p} - h_{\text{R,max}} \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&\qquad \mathbf{h}_{\text{R}}(\text{T}) \geq \mathbf{h}_{\text{R}}(t_0) \\
&\qquad \begin{bmatrix} g_{3_{i,p}} \\ g_{4_{i,p}} \end{bmatrix} = \begin{bmatrix} H_{\text{no,min}} - H_{\text{no},i,p} \\ H_{\text{no},i,p} - H_{\text{no,max}} \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ with } i = 1, \ldots, n_h \text{ and} \\
&\qquad\qquad p = 1, \ldots, n_{\text{pump}}.
\end{aligned} \tag{3.33}$$

3.2.1 Sensitivity analysis

A sensitivity analysis aim to determine the effect of a variation in the total value, providing information on the behaviour of the function at a certain point. This process can be done with an approximation method, such as the finite difference method, or an analytic method through the derivatives, which is the main goal of this work.

The finite difference method (FDM) is one of the numerical techniques used to solve differential equations with the derivative approximation by finite difference [Andrade-Campos *et al.* 2015], given by:

$$f' \approx \frac{f(x+b) - f(x+a)}{b-a}. \tag{3.34}$$

The FDM considers three different approaches for finite difference: forward ($\Delta_h[f](x)$), backward ($\nabla_h[f](x)$) and central ($\delta_h[f](x)$), where h is the value of the perturbation made:

$$\begin{aligned} f'(x) &\approx \frac{f(x+h) - f(x)}{h} = \frac{\Delta_h[f](x)}{h}, \\ f'(x) &\approx \frac{f(x) - f(x-h)}{h} = \frac{\nabla_h[f](x)}{h}, \text{ and} \\ f'(x) &\approx \frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{h} = \frac{\delta_h[f](x)}{h}. \end{aligned} \tag{3.35}$$

Being an approximation method, the results are not as accurate as those resulting from analytical calculations and depends on the value of h. Another disadvantage of this process is that it has to be evaluated for all the perturbations performed, and therefore requires large computational effort.

The analytical method, including the calculation of derivatives, has better results since sensitivities only have to be calculated once for the system, which leads to faster and accurate results. For that purpose, the sensitivities are calculated for each constraint presented before and for the objective function presented in Equation 3.33.

Water tank level sensitivity - Time derivatives

In relation to the constraints, the water tank level is represented in Figure 3.5, and it can be seen that the water level is a continuous function while the flow rate is discontinuous with the change of the pump status (point $t_i + x_{i,p}\Delta t^{\mathrm{h}}_{i,p}$).

The tank level updating equation is defined for a single tank R and a single pump as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{CV} \mathrm{d}V = A_{\mathrm{R}} \frac{\mathrm{d}h_{\mathrm{R},t_i}}{\mathrm{d}t} = Q_{\mathrm{R,in}} - Q_{\mathrm{R,out}} \Leftrightarrow \frac{\mathrm{d}h_{\mathrm{R},t_i}}{\mathrm{d}t} = \frac{1}{A_{\mathrm{R}}} \times [Q_{\mathrm{R,in}}(t) - Q_{\mathrm{R,out}}(t)] \tag{3.36}$$

for time increment t_i. Then, integrating Equation 3.36, we obtain

$$h_{\mathrm{R},t_i+\Delta t_i}(x_i) = h_{\mathrm{R},t_i}(x_0, \ldots, x_{i-1}) + \frac{1}{A_{\mathrm{R}}}\left[\int_{t_i}^{t_i+\Delta t_i} Q_{\mathrm{R,in}}(t) - Q_{\mathrm{R,out}}(t)\mathrm{d}t\right], \tag{3.37}$$

where h_{R} represents the water tank level, which is influenced by both the inflow and outflow and by the previous level of the tank, and it is therefore a function of the time increment.

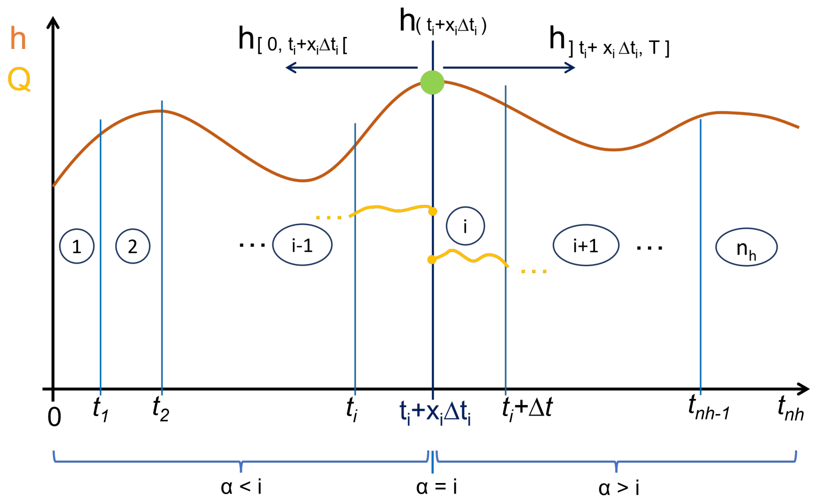


Figure 3.5: Water tank level and flow at the time horizon i.

The sensitivities of the water tank level constraints can be generically summarized by the following matrix for the entire global time horizon T, looking more carefully at the instant of pump state change $t_i + x_{i,p}\Delta t^{\mathrm{h}}_{i,p}$,

$$\frac{\mathrm{d}h_{\mathrm{R}}}{\mathrm{d}\mathbf{x}} = \begin{bmatrix} & & \vdots & & \\ \cdots & \frac{\mathrm{d}h_{\mathrm{R},[0,t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p}[}}{\mathrm{d}x_{i-1,p}} & \frac{\mathrm{d}h_{\mathrm{R},(t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})}}{\mathrm{d}x_{i-1,p}} & \frac{\mathrm{d}h_{\mathrm{R},]t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p},T]}}{\mathrm{d}x_{i-1,p}} & \cdots \\ \cdots & \frac{\mathrm{d}h_{\mathrm{R},[0,t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p}[}}{\mathrm{d}x_{i,p}} & \frac{\mathrm{d}h_{\mathrm{R},(t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})}}{\mathrm{d}x_{i,p}} & \frac{\mathrm{d}h_{\mathrm{R},]t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p},T]}}{\mathrm{d}x_{i,p}} & \cdots \\ \cdots & \frac{\mathrm{d}h_{\mathrm{R},[0,t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p}[}}{\mathrm{d}x_{i+1,p}} & \frac{\mathrm{d}h_{\mathrm{R},(t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})}}{\mathrm{d}x_{i+1,p}} & \frac{\mathrm{d}h_{\mathrm{R},]t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p},T]}}{\mathrm{d}x_{i+1,p}} & \cdots \\ & & \vdots & & \end{bmatrix} \tag{3.38}$$

Thus, the updating equation of the tank level (for tank R) can be settled for a generic $t_\alpha \in [t_0, T]$ as:

$$\begin{aligned} h_{\mathrm{R}}(t_\alpha) &= h_{\mathrm{R}}(t_0) + \frac{1}{A_{\mathrm{R}}}\int_{t_0}^{T} \left[Q_{\mathrm{in}}(t) - Q_{\mathrm{out}}(t)\right]\mathrm{d}t = h_{\mathrm{R}}(t_0) + \frac{1}{A_{\mathrm{R}}}\int_{t_0}^{T} Q_{\mathrm{in/out}}(t)\mathrm{d}t \\ &= h_{\mathrm{R}}(t_0) + \frac{1}{A_{\mathrm{R}}}\left[\int_{t_0}^{t_i} Q_{\mathrm{in/out}}(t)\mathrm{d}t + \int_{t_i}^{t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p}} Q_{\mathrm{in/out}}(t)\mathrm{d}t\right. \\ &\quad \left. + \int_{t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p}}^{t_{i+1}} Q_{\mathrm{in/out}}(t)\mathrm{d}t + \int_{t_{i+1}}^{T} Q_{\mathrm{in/out}}(t)\mathrm{d}t\right], \end{aligned} \tag{3.39}$$

and the derivative with respect to $x_{i,p}$ is given by

$$\frac{\mathrm{d}h_\mathrm{R}}{\mathrm{d}x_{i,p}}(t_\alpha) = \overbrace{\cancel{\frac{\mathrm{d}}{\mathrm{d}x_{i,p}}(h_\mathrm{R}(t_0))}}^{0} + \frac{1}{A_\mathrm{R}}\frac{\mathrm{d}}{\mathrm{d}x_{i,p}}\left[\int_{t_0}^{t_i} Q_{\mathrm{in/out}}(t)\mathrm{d}t + \int_{t_i}^{t_i+x_{i,p}\Delta t^\mathrm{h}_{i,p}} Q_{\mathrm{in/out}}(t)\mathrm{d}t \right.$$
$$\left. + \int_{t_i+x_{i,p}\Delta t^\mathrm{h}_{i,p}}^{t_{i+1}} Q_{\mathrm{in/out}}(t)\mathrm{d}t + \int_{t_{i+1}}^{T} Q_{\mathrm{in/out}}(t)\mathrm{d}t\right]. \tag{3.40}$$

The previous terms depend on t_α and the discontinuity of $Q_{\mathrm{in/out}}$ is highlighted using multiple integration terms. Therefore the derivative can be given by:

- For $t_\alpha < t_i + x_{i,p}\Delta t^\mathrm{h}_{i,p}$:

$$\frac{\mathrm{d}h_\mathrm{R}(t_\alpha)}{\mathrm{d}x_{i,p}} = \frac{1}{A_\mathrm{R}} \times \frac{\mathrm{d}}{\mathrm{d}x_{i,p}}\left(\int_{t_0}^{t_i} Q_{\mathrm{in/out}}(t)\mathrm{d}t + \int_{t_i}^{t_\alpha} Q_{\mathrm{in/out}}(t)\mathrm{d}t\right) = 0, \tag{3.41}$$

since it concerns to the time horizon before any influence of the pump status change $x_{i,p}$,

- For $t_\alpha = t_i + x_{i,p}\Delta t^\mathrm{h}_{i,p}$:

$$\begin{aligned}\frac{\mathrm{d}h_\mathrm{R}(t_\alpha)}{\mathrm{d}x_{i,p}} &= \frac{1}{A_\mathrm{R}}\frac{\mathrm{d}}{\mathrm{d}x_{i,p}}\int_{t_i}^{t_i+x_{i,p}\Delta t^\mathrm{h}_{i,p}} Q_{\mathrm{in/out}}(t)\mathrm{d}t \\ &= \frac{1}{A_\mathrm{R}}\frac{\mathrm{d}}{\mathrm{d}x_{i,p}}\int_{t_i}^{(t_i+x_{i,p}\Delta t^\mathrm{h}_{i,p})^-} Q_{\mathrm{in/out}}(t)\mathrm{d}t \\ &= \frac{1}{A_\mathrm{R}}\frac{\mathrm{d}}{\mathrm{d}x_{i,p}}\left[\lim_{a\to(t_i+x_{i,p}\Delta t^\mathrm{h}_{i,p})^-}\int_{t_i}^{a} Q_{\mathrm{in/out}}(t)\mathrm{d}t\right] \\ &= \frac{1}{A_\mathrm{R}}\left[Q_{\mathrm{in/out}}((t_i+x_{i,p}\Delta t^\mathrm{h}_{i,p})^-) \times \frac{\mathrm{d}}{\mathrm{d}x_{i,p}}(t_i+x_{i,p}\Delta t^\mathrm{h}_{i,p})^-\right] \\ &= \frac{1}{A_\mathrm{R}}\left[Q_{\mathrm{in/out}}((t_i+x_{i,p}\Delta t^\mathrm{h}_{i,p})^-) \times \Delta t^\mathrm{h}_{i,p}\right].\end{aligned} \tag{3.42}$$

This term is solved with the negative limit of the integral, since the function $Q_{\mathrm{in/out}}$ is a discontinuous point in $t_i + x_{i,p}\Delta t^\mathrm{h}_{i,p}$. The previous equation considers $Q_{\mathrm{in/out}}$ constant for $[t_i,\ t_i + x_{i,p}\Delta t^\mathrm{h}_{i,p}[$

- For $t_\alpha > t_i + x_{i,p}\Delta t^{\mathrm{h}}_{i,p}$:

$$
\begin{aligned}
&\frac{\mathrm{d}h_{\mathrm{R}}(t_\alpha)}{\mathrm{d}x_{i,p}} = \\
&= \frac{1}{A_{\mathrm{R}}}\frac{\mathrm{d}}{\mathrm{d}x_{i,p}}\left[\int_{t_i}^{t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p}} Q_{\mathrm{in/out}}(t)\mathrm{d}t + \int_{t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p}}^{t_{i+1}} Q_{\mathrm{in/out}}(t)\mathrm{d}t + \int_{t_{i+1}}^{T} Q_{\mathrm{in/out}}(t)\mathrm{d}t\right] \\
&= \frac{1}{A_{\mathrm{R}}}\frac{\mathrm{d}}{\mathrm{d}x_{i,p}}\left[\int_{t_i}^{(t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})^-} Q_{\mathrm{in/out}}(t)\mathrm{d}t - \int_{t_{i+1}}^{(t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})^+} Q_{\mathrm{in/out}}(t)\mathrm{d}t \right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. + \int_{t_{i+1}}^{T} Q_{\mathrm{in/out}}(t)\mathrm{d}t\right] \\
&= \frac{1}{A_{\mathrm{R}}}\frac{\mathrm{d}}{\mathrm{d}x_{i,p}}\left[\lim_{a\to(t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})^-}\int_{t_i}^{a} Q_{\mathrm{in/out}}(t)\mathrm{d}t - \lim_{a\to(t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})^+}\int_{t_{i+1}}^{a} Q_{\mathrm{in/out}}(t)\mathrm{d}t \right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. + \int_{t_{i+1}}^{T} Q_{\mathrm{in/out}}(t)\mathrm{d}t\right],
\end{aligned}
\tag{3.43}
$$

where the first two integrals are solved with the fundamental theorem of calculus and can be approximated by a Riemann sum, using the negative and positive limit of the integral[1]. Therefore,

$$
\begin{aligned}
&\frac{1}{A_{\mathrm{R}}}\frac{\mathrm{d}}{\mathrm{d}x_{i,p}}\left[\lim_{a\to(t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})^-}\int_{t_i}^{a} Q_{\mathrm{in/out}}(t)\mathrm{d}t - \lim_{a\to(t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})^+}\int_{t_{i+1}}^{a} Q_{\mathrm{in/out}}(t)\mathrm{d}t\right] \\
&=\frac{1}{A_{\mathrm{R}}}\left[Q_{\mathrm{in/out}}((t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})^-)\times\frac{\mathrm{d}}{\mathrm{d}x_{i,p}}(t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})^- \right.\\
&\qquad\qquad \left. -Q_{\mathrm{in/out}}((t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})^+)\times\frac{\mathrm{d}}{\mathrm{d}x_{i,p}}(t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})^+\right] \\
&=\frac{1}{A_{\mathrm{R}}}\left[Q_{\mathrm{in/out}}((t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})^-)\times\Delta t^{\mathrm{h}}_{i,p} - Q_{\mathrm{in/out}}((t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})^+)\times\Delta t^{\mathrm{h}}_{i,p}\right],
\end{aligned}
\tag{3.44}
$$

and the last term of Equation 3.43 represents the implicit influence of $x_{i,p}$ after

[1] According to the fundamental theorem of calculus the derivative of the integral when the variable is in the top limit can be applied to continuous functions with the chain rule (substituting it at the expression and multiply by the expression derivative, e.g. $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{xb} f(t)\mathrm{d}t = f(xb)\times\frac{\mathrm{d}}{\mathrm{d}x}(xb) = f(xb)\times b$). When it is a discontinuous function, this theorem can be used at the discontinuous point if it can be approximated by Riemann sum, using the negative and positive limit of the integral [Strang and Herman 2022]

the pump status change that affect the flow rate $Q_{\text{in/out}}$, and is given by

$$
\begin{aligned}
&\frac{1}{A_{\mathrm{R}}}\frac{\mathrm{d}}{\mathrm{d}x_{i,p}}\left[\int_{t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p}}^{t_{i+1}} Q_{\text{in/out}}(h_{\mathrm{R}})\mathrm{d}t+\int_{t_{i+1}}^{T} Q_{\text{in/out}}(h_{\mathrm{R}})\mathrm{d}t\right]\\
=&\frac{1}{A_{\mathrm{R}}}\frac{\mathrm{d}}{\mathrm{d}x_{i,p}}\left[\int_{t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p}}^{t_{i+1}} Q_{\text{in/out}}(h_{\mathrm{R}})\mathrm{d}t+\int_{t_{i+1}}^{t_{i+2}} Q_{\text{in/out}}(h_{\mathrm{R}})\mathrm{d}t+\ldots+\int_{t_{n_h-1}}^{T} Q_{\text{in/out}}(h_{\mathrm{R}})\mathrm{d}t\right]\\
=&\frac{1}{A_{\mathrm{R}}}\frac{\mathrm{d}}{\mathrm{d}x_{i,p}}\left[\mathrm{P}\left[Q_{\text{in/out}}(h_{\mathrm{R}})\right]_{t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p}}^{t_{i+1}}+\mathrm{P}\left[Q_{\text{in/out}}(h_{\mathrm{R}})\right]_{t_{i+1}}^{t_{i+2}}+\ldots+\mathrm{P}\left[Q_{\text{in/out}}(h_{\mathrm{R}})\right]_{t_{n_h-1}}^{T}\right]\\
=&\frac{1}{A_{\mathrm{R}}}\frac{\mathrm{d}}{\mathrm{d}x_{i,p}}\left[Q_{\text{in/out}}(h_{\mathrm{R}})\times(t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p}-t_{i+1})+Q_{\text{in/out}}(h_{\mathrm{R}})\times(t_{i+2}-t_{i+1})+\ldots+\right.\\
&\left.\qquad\qquad\qquad\qquad\qquad\qquad+Q_{\text{in/out}}(h_{\mathrm{R}})\times(t_\alpha-t_{n_h-1})\right]\\
=&\frac{\Delta t_{t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p}}}{A_{\mathrm{R}}}\times\frac{\mathrm{d}Q_{\text{in/out}}(t_i)}{\mathrm{d}h_{\mathrm{R}}}\frac{\mathrm{d}h_{\mathrm{R}}}{\mathrm{d}x_{i,p}}+\frac{\Delta t_{i+1}}{A_{\mathrm{R}}}\times\frac{\mathrm{d}Q_{\text{in/out}}(t_{i+1})}{\mathrm{d}h_{\mathrm{R}}}\frac{\mathrm{d}h_{\mathrm{R}}}{\mathrm{d}x_{i,p}}+\ldots+\\
&\qquad\qquad\qquad\qquad+\frac{\Delta t_{n_h-1}}{A_{\mathrm{R}}}\times\frac{\mathrm{d}Q_{\text{in/out}}(t_{n_h-1})}{\mathrm{d}h_{\mathrm{R}}}\frac{\mathrm{d}h_{\mathrm{R}}}{\mathrm{d}x_{i,p}}\\
=&\sum_{k=1}^{n_{\mathrm{h}}-1}\left[\frac{\Delta t_k}{A_{\mathrm{R}}}\times\frac{\mathrm{d}Q_{\text{in/out}}(t_k)}{\mathrm{d}h_{\mathrm{R}}}\frac{\mathrm{d}h_{\mathrm{R}}}{\mathrm{d}x_{i,p}}\right].
\end{aligned}
\tag{3.45}
$$

Consequently the global equation for the water tank level sensitivity $\left(\frac{\mathrm{d}h_{\mathrm{R}}}{\mathrm{d}x_{i,p}}\right)$ represents the time sensitivity. However, Equation 3.45 show that it depends on it value itself. Using an explicit approach (forward differences at time), i.e., using the value of the previous instant of tank level, one obtains:

$$
\begin{aligned}
\frac{\mathrm{d}h_{\mathrm{R}}(t_\alpha)}{\mathrm{d}x_{i,p}}=0+\frac{1}{A_{\mathrm{R}}}&\left[Q_{\text{in/out}}((t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})^-)\ \times\Delta t^{\mathrm{h}}_{i,p}-Q_{\text{in/out}}((t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})^+)\times\Delta t^{\mathrm{h}}_{i,p}\right]\\
&+\sum_{k=1}^{t_\alpha}\left[\frac{\Delta t}{A_{\mathrm{R}}}\times\frac{\mathrm{d}Q_{\text{in/out}}(t_k)}{\mathrm{d}h_{\mathrm{R}}}\frac{\mathrm{d}h_{\mathrm{R}}(t_{k-1})}{\mathrm{d}x_{i,p}}\right].
\end{aligned}
\tag{3.46}
$$

From an implicit point of view (backward differences at time), i.e., using the tank level value for the same instant, the water tank level sensitivity equation is given by:

$$
\begin{aligned}
&\frac{\mathrm{d}h_{\mathrm{R}}(t_\alpha)}{\mathrm{d}x_{i,p}}=0+\frac{1}{A_{\mathrm{R}}}\left[Q_{\text{in/out}}((t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})^-)\times\Delta t^{\mathrm{h}}_{i,p}-Q_{\text{in/out}}((t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})^+)\times\Delta t^{\mathrm{h}}_{i,p}\right]\\
&\qquad\qquad+\sum_{k=1}^{t_\alpha}\left[\frac{\Delta t_k}{A_{\mathrm{R}}}\times\frac{\mathrm{d}Q_{\text{in/out}}(t_k)}{\mathrm{d}h_{\mathrm{R}}}\frac{\mathrm{d}h_{\mathrm{R}}(t_k)}{\mathrm{d}x_{i,p}}\right]\\
&\Leftrightarrow\left(1-\frac{\Delta t_\alpha}{A_{\mathrm{R}}}\times\frac{\mathrm{d}Q_{\text{in/out}}(t_\alpha)}{\mathrm{d}h_{\mathrm{R}}}\right)\frac{\mathrm{d}h_{\mathrm{R}}(t_\alpha)}{\mathrm{d}x_{i,p}}=\frac{1}{A_{\mathrm{R}}}\left[Q_{\text{in/out}}((t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})^-)\times\Delta t^{\mathrm{h}}_{i,p}\right.\\
&\qquad\left.-Q_{\text{in/out}}((t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p})^+)\times\Delta t^{\mathrm{h}}_{i,p}\right]+\sum_{k=1}^{t_{\alpha-1}}\left[\frac{\Delta t_k}{A_{\mathrm{R}}}\times\frac{\mathrm{d}Q_{\text{in/out}}(t_k)}{\mathrm{d}h_{\mathrm{R}}}\frac{\mathrm{d}h_{\mathrm{R}}(t_k)}{\mathrm{d}x_{i,p}}\right]\\
&\Leftrightarrow\frac{\mathrm{d}h_{\mathrm{R}}(t_\alpha)}{\mathrm{d}x_{i,p}}=\frac{\frac{\mathrm{d}h_{\mathrm{R}}(t_{\alpha-1})}{\mathrm{d}x_{i,p}}}{\left(1-\frac{\Delta t_\alpha}{A_{\mathrm{R}}}\times\frac{\mathrm{d}Q_{\text{in/out}}(t_\alpha)}{\mathrm{d}h_{\mathrm{R}}}\right)}.
\end{aligned}
\tag{3.47}
$$

For the case of a hydraulic network with more than one tank and using an implicit approach, the tank level sensitivity can be represented by the matrix:

$$\begin{bmatrix} 1-\frac{\Delta t}{A_\mathrm{R}}\frac{\mathrm{d}Q_1}{\mathrm{d}h_{\mathrm{R}1}} & -\frac{\Delta t}{A_\mathrm{R}}\frac{\mathrm{d}Q_1}{\mathrm{d}h_{\mathrm{R}2}} & \cdots & -\frac{\Delta t}{A_\mathrm{R}}\frac{\mathrm{d}Q_1}{\mathrm{d}h_{\mathrm{nr}}} \\ -\frac{\Delta t}{A_\mathrm{R}}\frac{\mathrm{d}Q_2}{\mathrm{d}h_{\mathrm{R}1}} & 1-\frac{\Delta t}{A_\mathrm{R}}\frac{\mathrm{d}Q_2}{\mathrm{d}h_{\mathrm{R}2}} & \cdots & -\frac{\Delta t}{A_\mathrm{R}}\frac{\mathrm{d}Q_2}{\mathrm{d}h_{\mathrm{nr}}} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\Delta t}{A_\mathrm{R}}\frac{\mathrm{d}Q_{n_\mathrm{pipes}}}{\mathrm{d}h_{\mathrm{R}1}} & -\frac{\Delta t}{A_\mathrm{R}}\frac{\mathrm{d}Q_{n_\mathrm{pipes}}}{\mathrm{d}h_{\mathrm{R}2}} & \cdots & 1-\frac{\Delta t}{A_\mathrm{R}}\frac{\mathrm{d}Q_{n_\mathrm{pipes}}}{\mathrm{d}h_{\mathrm{nr}}} \end{bmatrix} \times \begin{bmatrix} \frac{\mathrm{d}h_{\mathrm{R}1}(t_\alpha)}{\mathrm{d}x_{i,p}} \\ \frac{\mathrm{d}h_{\mathrm{R}2}(t_\alpha)}{\mathrm{d}x_{i,p}} \\ \vdots \\ \frac{\mathrm{d}h_{\mathrm{nr}}(t_\alpha)}{\mathrm{d}x_{i,p}} \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}h_{\mathrm{R}1}(t_{\alpha-1})}{\mathrm{d}x_{i,p}} \\ \frac{\mathrm{d}h_{\mathrm{R}2}(t_{\alpha-1})}{\mathrm{d}x_{i,p}} \\ \vdots \\ \frac{\mathrm{d}h_{\mathrm{nr}}(t_{\alpha-1})}{\mathrm{d}x_{i,p}} \end{bmatrix}, \tag{3.48}$$

where nr is the number of tanks.

The water tank level sensibility can be summarized by Figure 3.6, which represents the optimization diagram process for each time increment, where the sensitivity in calculated with the results from EPANET simulator.

Continuity sensitivity

Concerning the continuity constraint of Equation 3.31, it is a particular case of the water tank level since it compares the initial tank level $h_\mathrm{R}(t_0)$ with the one of the end of the time horizon $h_\mathrm{R}(T)$. This is considered in the situation in which $t_\alpha > t_i + x_{i,p}\Delta t^\mathrm{h}_{i,p}$, so the derivative also has an influence of the pump status change,

$$\frac{\mathrm{d}h_\mathrm{R}(T)}{\mathrm{d}x_{i,p}} = \frac{1}{A_\mathrm{R}}\left[Q_{\mathrm{in/out}}((t_i + x_{i,p}\Delta t^\mathrm{h}_{i,p})^-) \times\Delta t^\mathrm{h}_{i,p} - Q_{\mathrm{in/out}}((t_i + x_{i,p}\Delta t^\mathrm{h}_{i,p})^+) \times \Delta t^\mathrm{h}_{i,p}\right] + \sum_{k=1}^{T}\left[\frac{\Delta t_k}{A_\mathrm{R}} \times \frac{\mathrm{d}Q_{\mathrm{in/out}}(t_k)}{\mathrm{d}h_\mathrm{R}}\frac{\mathrm{d}h_\mathrm{R}(t_{k-1})}{\mathrm{d}x_{i,p}}\right] \tag{3.49}$$

Node pressure sensitivity - Space derivatives

The pressure node constraint can be defined using the chain rule:

$$\frac{\mathrm{d}H}{\mathrm{d}x_{i,p}} = \frac{\mathrm{d}H}{\mathrm{d}h_\mathrm{R}}\frac{\mathrm{d}h_\mathrm{R}}{\mathrm{d}x_{i,p}} \tag{3.50}$$

where $\frac{\mathrm{d}h_\mathrm{R}}{\mathrm{d}x_{i,p}}$ represents sensitivity of the water tank level already deducted before.

The $\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}h_\mathrm{R}}$ is represented by:

$$\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}h_\mathrm{R}} = \begin{bmatrix} \frac{\mathrm{d}H_1}{\mathrm{d}h_{\mathrm{R}1}} & \cdots & \frac{\mathrm{d}H_1}{\mathrm{d}h_{nr}} \\ \vdots & \ddots & \vdots \\ \frac{\mathrm{d}H_{n_\mathrm{nodes}}}{\mathrm{d}h_{\mathrm{R}1}} & \cdots & \frac{\mathrm{d}H_{n_\mathrm{nodes}}}{\mathrm{d}h_{nr}} \end{bmatrix}, \tag{3.51}$$

where nr represents the number of tanks.

In order to know $\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}h_\mathrm{R}}$ and $\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}h_\mathrm{R}}$, it is needed to solve the complete derivative system (based on system 3.6) and $\mathbf{A}_{10}\frac{\mathrm{d}\mathbf{H}_0}{\mathrm{d}h_\mathrm{R}}$ is a vector with the value of 1 at the nodes where exist a tank,

$$\begin{bmatrix} \mathbf{nA}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & 0 \end{bmatrix} \times \begin{bmatrix} \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}h_\mathrm{R}} \\ \frac{\mathrm{d}\mathbf{H}}{\mathrm{d}h_\mathrm{R}} \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_{10}\frac{\mathrm{d}\mathbf{H}_0}{\mathrm{d}h_\mathrm{R}} \\ 0 \end{bmatrix}. \tag{3.52}$$

It is relevant to note that the data for each pumps when they are switched off needs to be omitted as the $\frac{\mathrm{d}Q}{\mathrm{d}h_{\mathrm{R}}}$ is zero. Thus, it is as if the pump's pipe has disconnected.

The previous calculation is the space derivative of Q and H in respect to the level of the tanks h_{R} and must be calculated for each tank R and for each time instant. Thus $\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}h_{\mathrm{R}}}$ is given by

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}h_{\mathrm{R}}} = \begin{bmatrix} \frac{\mathrm{d}Q_1}{\mathrm{d}h_{\mathrm{R}1}} & \cdots & \frac{\mathrm{d}Q_1}{\mathrm{d}h_{nr}} \\ \vdots & \ddots & \vdots \\ \frac{\mathrm{d}Q_{n_{\mathrm{pipes}}}}{\mathrm{d}h_{\mathrm{R}1}} & \cdots & \frac{\mathrm{d}Q_{n_{\mathrm{pipes}}}}{\mathrm{d}h_{nr}} \end{bmatrix}. \tag{3.53}$$

Objective function

Considering the pump power as a time dependent function

$$\dot{W}_p(t) = \frac{\rho g}{\eta(Q_p)} \times Q_p(t) \times [a - bQ_p(t) - R_{\mathrm{pump}}Q_p(t)^2], \tag{3.54}$$

the derivative of the cost function (Equation 3.27) in relation to the decision variable $x_{i,p}$ is given by:

$$\begin{aligned}
\frac{\mathrm{d}C(x)}{\mathrm{d}x_{i,p}} &= \sum_{i=1}^{n_h}\sum_{p=1}^{n_{\mathrm{pump}}} \$_{i,p} \int_0^T \dot{W}_p(t)\mathrm{d}t \\
&= \sum_{i=1}^{n_h}\sum_{p=1}^{n_{\mathrm{pump}}} \$_{i,p} \left[\int_0^{t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p}} \dot{W}_p(t)\mathrm{d}t + \int_{t_i+x_{i,p}\Delta t^{\mathrm{h}}_{i,p}}^{t_i+1} \dot{W}_p(t)\mathrm{d}t + \int_{t_i+1}^{T} \dot{W}_p(t)\mathrm{d}t\right] \\
&= \sum_{i=1}^{n_h}\sum_{p=1}^{n_{\mathrm{pump}}} \$_{i,p} \left[\dot{W}_p(t_i + x_{i,p}\Delta t^{\mathrm{h}}_{i,p})\frac{\mathrm{d}(t_i + x_{i,p}\Delta t^{\mathrm{h}}_{i,p})}{\mathrm{d}x_{i,p}}\right. \\
&\qquad\qquad \left. + \frac{\mathrm{d}\left(\dot{W}_p(t_i + x_{i,p}\Delta t^{\mathrm{h}}_{i,p})\right)}{\mathrm{d}x_{i,p}} \times (t_i + x_{i,p}\Delta t^{\mathrm{h}}_{i,p})\right] \\
&= \sum_{i=1}^{n_h}\sum_{p=1}^{n_{\mathrm{pump}}} \$_{i,p}\dot{W}_p(t_i + x_{i,p}\Delta t^{\mathrm{h}}_{i,p}) \times \Delta t^{\mathrm{h}}_{i,p} + \sum_{i=1}^{n_h}\sum_{p=1}^{n_{\mathrm{pump}}} \$_{i,p}\frac{\mathrm{d}\dot{W}_p(t)}{\mathrm{d}x_{i,p}} \times \Delta t,
\end{aligned} \tag{3.55}$$

where $\$_{i,p} \times \dot{W}_p(t_i + x_{i,p}\Delta t^{\mathrm{h}}_{i,p}) \times \Delta t^{\mathrm{h}}_{i,p}$ represents the term when the pump p change the status and $\sum_{i=1}^{n_h}\sum_{p=1}^{n_{\mathrm{pump}}} \$_{i,p} \times \frac{\mathrm{d}}{\mathrm{d}x_{i,p}}\dot{W}_p(t) \times \Delta t$ is the sum of the terms when the pump is afterwards running and can be established by the chain rule of the derivative of the composite function:

$$\frac{\mathrm{d}}{\mathrm{d}x_{i,p}}\dot{W}_p(t) = \frac{\mathrm{d}W}{\mathrm{d}Q} \times \frac{\mathrm{d}Q}{\mathrm{d}h_{\mathrm{R}}} \times \frac{\mathrm{d}h_{\mathrm{R}}}{\mathrm{d}x_{i,p}}, \tag{3.56}$$

where $\frac{\mathrm{d}Q}{\mathrm{d}h_{\mathrm{R}}}$ it is defined with flow sensitivity (Equation 3.52) and $\frac{\mathrm{d}h_{\mathrm{R}}}{\mathrm{d}x_{i,p}}$ is the water tank level sensitivity (see Equation 3.40). The remainder part is defined by the pump power $\left(\frac{\mathrm{d}W}{\mathrm{d}Q}\right)$.

For each pump and knowing that the pump hydraulic curve is given as $H(Q_p) = a - bQ_p - R_{\text{pump}}Q_p^2$, the sensitivity of the pump power can be summarize:

$$
\begin{aligned}
\frac{\mathrm{d}W_p}{\mathrm{d}Q} &= \frac{\mathrm{d}}{\mathrm{d}Q}\left(\frac{\rho g}{\eta(Q_p)} \times Q_p \times H(Q_p)\right) = \frac{\mathrm{d}}{\mathrm{d}Q}\left(\frac{\rho g}{\eta(Q_p)} \times Q_p \left[a - bQ_p - R_{\text{pump}}Q_p^2\right]\right) \\
&= \cancelto{0}{\frac{\mathrm{d}}{\mathrm{d}Q}(\rho g)} \times \frac{1}{\eta(Q_p)}\left[aQ_p - bQ_p^2 - R_{\text{pump}}Q_p^3\right] + \rho g \times \frac{\mathrm{d}}{\mathrm{d}Q}\left(\frac{1}{\eta(Q_p)}\right) \times \left[aQ_p - bQ_p^2 - R_{\text{pump}}Q_p^3\right] \\
&\qquad + \frac{\rho g}{\eta(Q_p)} \times \frac{\mathrm{d}}{\mathrm{d}Q}\left(aQ_p - bQ_p^2 - R_{\text{pump}}Q_p^3\right) \\
&= -\rho g\left[\eta(Q_p)\right]^{-2} \times \frac{\mathrm{d}\eta(Q_p)}{\mathrm{d}Q} \times \left[aQ_p - bQ_p^2 - R_{\text{pump}}Q_p^3\right] + \frac{\rho g}{\eta(Q_p)}\left[\left(a - bQ_p - 3R_{\text{pump}}Q_p^2\right)\frac{\mathrm{d}Q_p}{\mathrm{d}Q}\right] \\
&= \frac{\rho g}{\eta(Q_p)}\left[\frac{aQ_p - bQ_p^2 - R_{\text{pump}}Q_p^3}{\eta(Q_p)} \times \frac{-\mathrm{d}\eta(Q_p)}{\mathrm{d}Q} + \left(a - 2bQ_p - 3R_{\text{pump}}Q_p^2\right)\frac{\mathrm{d}Q_p}{\mathrm{d}Q}\right].
\end{aligned}
\tag{3.57}
$$

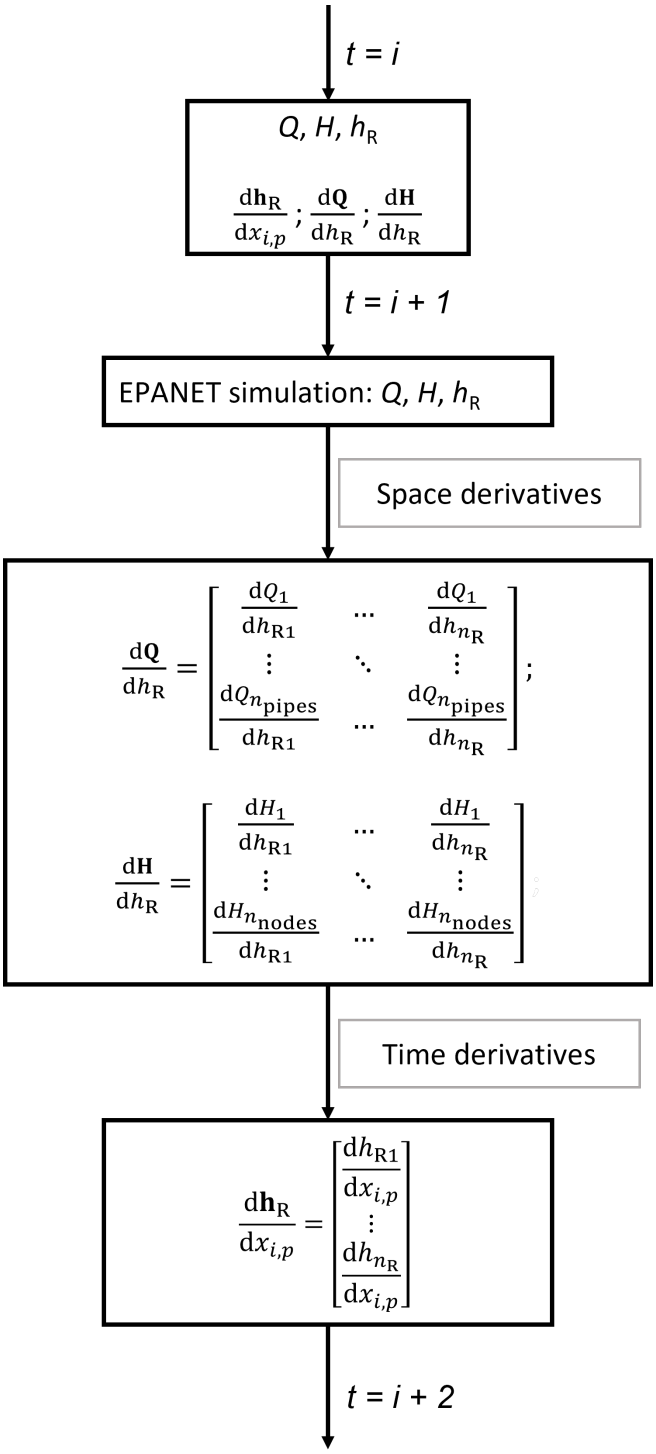


Figure 3.6: Optimization diagram to calculate $\frac{\mathrm{d}\mathbf{h}_\mathrm{R}}{\mathrm{d}x_{i,p}}$ fo each time increment using an explicit approach (forward difference at time).

Chapter 4

Validation

This chapter aims to validate the formulation previously deduced with the help of numerical models. The results of the analytical derivatives are compared with the values obtained through the finite difference method to prove the numerous advantages of the analytical method.

Two examples are presented, the first one to show the generic model of a hydraulic system, while in the second case the main objective is to validate the calculation of the analytical sensitivities. Furthermore, a third case study is shown, considering the Any-Town Modified benchmark in order to validate the same equations for a more complex system.

4.1 Generic hydraulic system model

The example shown in Figure 4.1 is a water distribution system with 5 nodes, 5 pipes, 1 pump at pipe 2 and 1 tank at node 4. There are also some external flow demands represented as q_{r}, q_{vc} and Q_{D}.

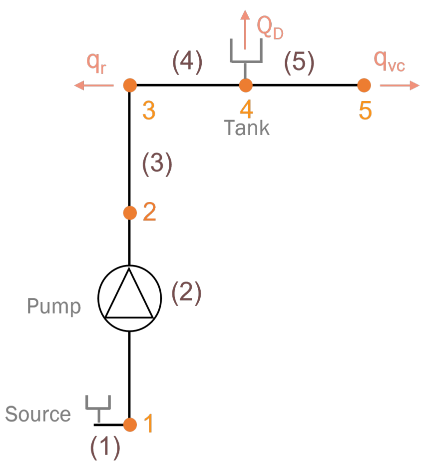


Figure 4.1: Generic water distribution network with a source, a tank and a pump, based on the benchmark 2018, ONLE [Andrade-Campos and Dias-de Oliveira 2019].

To solve the hydraulic system, it is necessary to perform a mass balance at each node and an energy balance at each pipe. The mass balance equations are:

$$Q_{\text{in}} - Q_{\text{out}} = q_{\text{external}} \Leftrightarrow \begin{cases} N1: Q_1 - Q_2 = 0 \\ N2: Q_2 - Q_3 = 0 \\ N3: Q_3 - Q_4 - q_{\text{r}} = 0 \\ N4: Q_4 - Q_5 - Q_{\text{D}} = 0 \\ N5: Q_5 - q_{\text{vc}} = 0 \end{cases}, \tag{4.1}$$

which can be written in a matrix form by

$$\begin{aligned} &\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ q_{\text{r}} \\ Q_{\text{D}} \\ q_{\text{vc}} \end{bmatrix} \\ &\Leftrightarrow \mathbf{A}_{21} \times \mathbf{Q} = \mathbf{q}, \end{aligned} \tag{4.2}$$

The energy balance equations are written as

$$H_{\text{out}} - H_{\text{in}} + F(Q) = 0 \Leftrightarrow \begin{cases} P1: H_1 - H_0 + h_{\text{loss}}(Q_1) = 0 \\ P2: H_2 - H_1 - h_{\text{pump}}(Q_2) = 0 \\ P3: H_3 - H_2 + h_{\text{loss}}(Q_3) = 0 \\ P4: H_4 - H_3 + h_{\text{loss}}(Q_4) = 0 \\ P5: H_5 - H_4 + h_{\text{loss}}(Q_5) = 0 \end{cases}, \tag{4.3}$$

that can be seen in the following matrix form

$$\begin{aligned} &\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{bmatrix} + \begin{bmatrix} h_{\text{loss}}(Q_1) \\ -h_{\text{pump}}(Q_2) \\ h_{\text{loss}}(Q_3) \\ h_{\text{loss}}(Q_4) \\ h_{\text{loss}}(Q_5) \end{bmatrix} = \begin{bmatrix} H_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &\Leftrightarrow \mathbf{A}_{12} \times \mathbf{H} + \mathbf{F}(Q) = \mathbf{A}_{10}\mathbf{H}_0. \end{aligned} \tag{4.4}$$

After a careful evaluation, an EPS analysis should be performed since node 4 is a tank where the level depends on the time. For this, the balance equations need to be updated for every small increments of time Δt according to the following equation:

$$H_4^{t+\Delta t} = H_4^t + Q_{\text{D}} \times \frac{\Delta t}{A_{\text{R}}} \Leftrightarrow H_4^{t+\Delta t} = H_4^t + (Q_4 - Q_5) \times \frac{\Delta t}{A_{\text{R}}} \tag{4.5}$$

The algebraic data with node 4 is removed, such as row 4 of $\mathbf{A}_{21}$, column 4 of $\mathbf{A}_{12}$ and nodal head H_4 which becomes a known value from the previous iteration. This results in the following mass balance:

$$
\begin{aligned}
\mathbf{A}_{21} \times \mathbf{Q} = \mathbf{q} \Leftrightarrow & \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{-1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ q_{\mathrm{r}} \\ \cancel{Q_{\mathrm{D}}} \\ q_{\mathrm{vc}} \end{bmatrix} \\
\Leftrightarrow & \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ q_{\mathrm{r}} \\ q_{\mathrm{vc}} \end{bmatrix}.
\end{aligned}
\tag{4.6}
$$

The corresponding energy balance is given by

$$
\begin{aligned}
& \mathbf{A}_{12} \times \mathbf{H} + \mathbf{F}(Q) = \mathbf{A}_{10}\mathbf{H}_0 \\
\Leftrightarrow & \begin{bmatrix} 1 & 0 & 0 & \cancel{0} & 0 \\ -1 & 1 & 0 & \cancel{0} & 0 \\ 0 & -1 & 1 & \cancel{0} & 0 \\ 0 & 0 & -1 & \cancel{1} & 0 \\ 0 & 0 & 0 & \cancel{-1} & 1 \end{bmatrix} \times \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ \cancel{H_4} \\ H_5 \end{bmatrix} + \begin{bmatrix} h_{\mathrm{loss}}(Q_1) \\ -h_{\mathrm{pump}}(Q_2) \\ h_{\mathrm{loss}}(Q_3) \\ h_{\mathrm{loss}}(Q_4) \\ h_{\mathrm{loss}}(Q_5) \end{bmatrix} = \begin{bmatrix} H_0 \\ 0 \\ 0 \\ \cancelto{-H_4}{0} \\ \cancelto{H_4}{0} \end{bmatrix} \\
\Leftrightarrow & \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ H_5 \end{bmatrix} + \begin{bmatrix} h_{\mathrm{loss}}(Q_1) \\ -h_{\mathrm{pump}}(Q_2) \\ h_{\mathrm{loss}}(Q_3) \\ h_{\mathrm{loss}}(Q_4) \\ h_{\mathrm{loss}}(Q_5) \end{bmatrix} = \begin{bmatrix} H_0 \\ 0 \\ 0 \\ -H_4 \\ H_4 \end{bmatrix}.
\end{aligned}
\tag{4.7}
$$

A global matrix, which governs all the engineering systems, and including both mass and energy balance is obtained for a static system:

$$
\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & 0 \end{bmatrix} \times \begin{bmatrix} \mathbf{Q} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_{10}\mathbf{H}_0 \\ \mathbf{q} \end{bmatrix}; \text{ i. e.} \tag{4.8}
$$

$$\left[\begin{array}{ccccc|ccccc}
R_1|Q_1|^{n-1} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & R_{\mathrm{pump}}|Q_2|^{n-1} & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & R_3|Q_3|^{n-1} & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & R_4|Q_4|^{n-1} & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & R_5|Q_5|^{n-1} & 0 & 0 & 0 & -1 & 1 \\
\hline
1 & -1 & 0 & 0 & 0 & & & & & \\
0 & 1 & -1 & 0 & 0 & & & & & \\
0 & 0 & 1 & -1 & 0 & & & 0 & & \\
0 & 0 & 0 & 1 & -1 & & & & & \\
0 & 0 & 0 & 0 & 1 & & & & &
\end{array}\right] \times \left[\begin{array}{c} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ \hline H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{array}\right] = \left[\begin{array}{c} H_0 \\ a \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ q_{\mathrm{r}} \\ Q_{\mathrm{D}} \\ q_{\mathrm{vc}} \end{array}\right]. \tag{4.9}$$

This non-linear system of equations can be solved using the Newton method. For this purpose, the derivative matrix is given by

$$\left[\begin{array}{cc} \mathbf{nA}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & 0 \end{array}\right] \times \left[\begin{array}{c} \mathrm{d}\mathbf{Q} \\ \mathrm{d}\mathbf{H} \end{array}\right] = \left[\begin{array}{c} \mathrm{d}\mathbf{E} \\ \mathrm{d}\mathbf{q} \end{array}\right]; \text{ i. e.} \tag{4.10}$$

$$\left[\begin{array}{ccccc|ccccc}
nR_1|Q_1|^{n-1} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & nR_{\mathrm{pump}}|Q_2|^{n-1} & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & nR_3|Q_3|^{n-1} & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & nR_4|Q_4|^{n-1} & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & nR_5|Q_5|^{n-1} & 0 & 0 & 0 & -1 & 1 \\
\hline
1 & -1 & 0 & 0 & 0 & & & & & \\
0 & 1 & -1 & 0 & 0 & & & & & \\
0 & 0 & 1 & -1 & 0 & & & 0 & & \\
0 & 0 & 0 & 1 & -1 & & & & & \\
0 & 0 & 0 & 0 & 1 & & & & &
\end{array}\right] \times \left[\begin{array}{c} \mathrm{d}\mathbf{Q} \\ \mathrm{d}\mathbf{H} \end{array}\right] = \left[\begin{array}{c} \mathbf{A}_{11}\mathbf{Q}^k + \mathbf{A}_{12}\mathbf{H}^k + \mathbf{A}_{10}\mathbf{H}_0 \\ \mathbf{A}_{21}\mathbf{Q}^k - \mathbf{q} \end{array}\right]. \tag{4.11}$$

For an EPS analysis, the system is represented as

$$\left[\begin{array}{ccccc|cccc} R_1|Q_1|^{n-1} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & R_{\mathrm{pump}}|Q_2|^{n-1} & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & R_3|Q_3|^{n-1} & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & R_4|Q_4|^{n-1} & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & R_5|Q_5|^{n-1} & 0 & 0 & 0 & 1 \\ \hline 1 & -1 & 0 & 0 & 0 & & & & \\ 0 & 1 & -1 & 0 & 0 & & & 0 & \\ 0 & 0 & 1 & -1 & 0 & & & & \\ 0 & 0 & 0 & 0 & 1 & & & & \end{array}\right] \times \left[\begin{array}{c} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ \hline H_1 \\ H_2 \\ H_4 \\ H_5 \end{array}\right] = \left[\begin{array}{c} H_0 \\ a \\ 0 \\ -H_4 \\ H_4 \\ \hline 0 \\ 0 \\ q_{\mathrm{r}} \\ q_{\mathrm{vc}} \end{array}\right], \tag{4.12}$$

and the derivative matrix of the variables Q and H

$$\left[\begin{array}{ccccc|cccc} nR_1|Q_1|^{n-1} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & nR_{\mathrm{pump}}|Q_2|^{n-1} & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & nR_3|Q_3|^{n-1} & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & nR_4|Q_4|^{n-1} & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & nR_5|Q_5|^{n-1} & 0 & 0 & 0 & 1 \\ \hline 1 & -1 & 0 & 0 & 0 & & & & \\ 0 & 1 & -1 & 0 & 0 & & & 0 & \\ 0 & 0 & 1 & -1 & 0 & & & & \\ 0 & 0 & 0 & 0 & 1 & & & & \end{array}\right] \times \left[\begin{array}{c} \mathrm{d}\mathbf{Q} \\ \mathrm{d}\mathbf{H} \end{array}\right] = \left[\begin{array}{c} \mathbf{A}_{11}\mathbf{Q}^k + \mathbf{A}_{12}\mathbf{H}^k + \mathbf{A}_{10}\mathbf{H}_0 \\ \mathbf{A}_{21}\mathbf{Q}^k - \mathbf{q} \end{array}\right]. \tag{4.13}$$

In this section, the general equations of a hydraulic system were developed, and presented in a global matrix form. Also, it was described the transformation to an EPS system, as well as its derivative, in order to provide an understanding of this system resolution, which is often used by hydraulic simulators software.

4.2 Sensitivity analysis for the simple pump-tank network - Validation

4.2.1 Description

The analysed system of this section consists of a simple network composed of a water source, one pump and a storage tank (reservoir) that supplies a node with a variable consumption (q_{vc}). The system is schematically represented in the Figure 4.2.

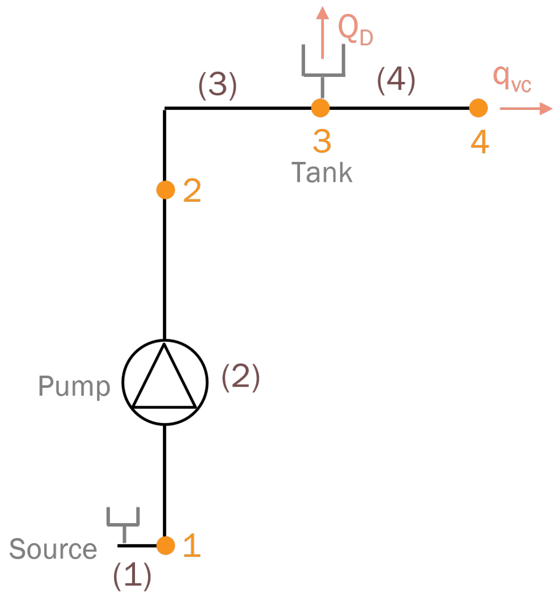


Figure 4.2: Pump-tank network. Simple network used for validation.

The tank has an area of $A = 155$ m^2 and an elevation of $H_0 = 100$ m relatively to the pump, with an initial water level of 4 m. The pump is characterised by a curve type $h_p[\text{m}] = a - R_{\text{pump}} Q_p^2 = 280 - 0.0027Q^2$ (with Q in m^3/h) and has a constant efficiency of $\eta = 75\%$ (see Figure 4.3). With regard to the pipes connecting the several components, these have a diameter $d = 0.3$ m, differing only in length, which are $L_1 = 100$ m, $L_3 = 3500$ m and $L_4 = 3500$ m respectively for pipes 1, 3 and 4.

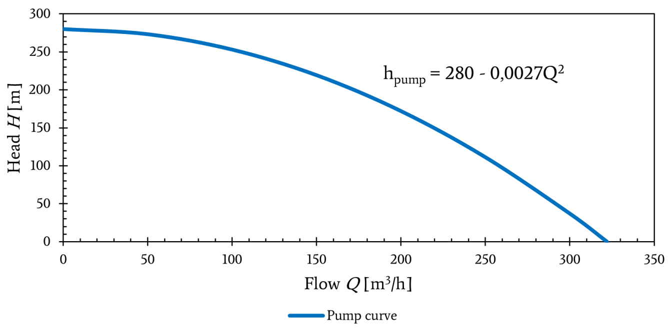


Figure 4.3: Pump curve for the simple system.

The head loss is calculated as $h_{\text{loss}} = RQ^2$, with R representing the loss given by Darcy-Weisbach formulation:

$$R = \frac{32f}{g\pi^2} \times \frac{L}{d^5}, \tag{4.14}$$

where f represents the friction factor, which is considered as $f = 0.02$.

The water consumption at node 4 is time dependent and it is defined by:

$$\begin{aligned} q_{\text{vc}}[\text{m}^3/\text{h}] = -5.728 \times 10^-5t^6 + 3.9382 \times 10^-3t^5 - 9.8402 \times 10^-2t^4 + 1.0477t^3 \\ -3.862t^2 - 1.1695t + 75.393, \end{aligned} \tag{4.15}$$

as illustrated in Figure 4.4.

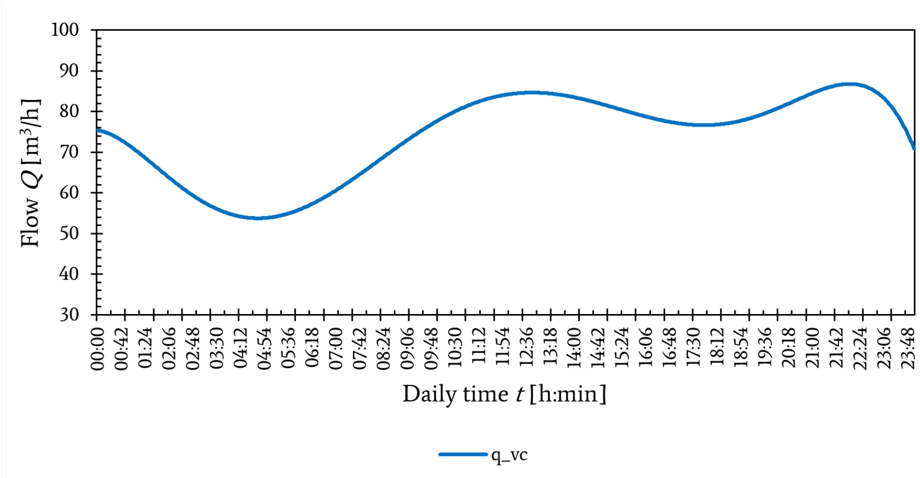


Figure 4.4: Water consumption q_{vc} at node 4.

A daily energy tariff pattern, which has six time periods with four different prices, is also defined in Table 4.1.

Table 4.1: Daily energy tariff for the simple network.

Time period [h]	Cost [€/kWh]
[0, 2[	0.07370
[2, 6[	0.06618
[6, 7[	0.07370
[7, 9[	0.10094
[9, 12[	0.18581
[12, 24[	0.10094

Furthermore, eight time horizons were defined taking into consideration the energy tariff, which are $[0, 2[$, $[2, 6[$, $[6, 7[$, $[7, 9[$, $[9, 12[$, $[12, 17[$, $[17, 21[$, $[21, 24[$. For the optimization process, a decision variable $x_{2,p} = 0.833$, which corresponding to the time of $5:20$, was chosen.

The system pump-tank previously described was simulated for one day (24 hours) with time steps of 1 minute.

4.2.2 Case study resolution

The system described in Figure 4.2 can be defined in matrix form and simplified to calculate the analytical derivatives by the equations demonstrated in the previous chapters. It can be summarised in a generic matrix by:

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & 0 \end{bmatrix} \times \begin{bmatrix} \mathbf{Q} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_{10}\mathbf{H}_0 \\ \mathbf{q} \end{bmatrix}. \tag{4.16}$$

Replacing the variables for this specific case, the system is:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ Q_\mathrm{D} \\ q_\mathrm{vc} \end{bmatrix}, \tag{4.17}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{bmatrix} + \begin{bmatrix} R_1Q_1^n & 0 & 0 & 0 \\ 0 & R_\mathrm{pump}Q_2^n - a & 0 & 0 \\ 0 & 0 & R_3Q_3^n & 0 \\ 0 & 0 & 0 & R_4Q_4^n \end{bmatrix} = \begin{bmatrix} H_0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{4.18}$$

and for an EPS status:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ q_\mathrm{vc} \end{bmatrix}, \tag{4.19}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} H_1 \\ H_2 \\ H_4 \end{bmatrix} + \begin{bmatrix} R_1Q_1^n & 0 & 0 & 0 \\ 0 & R_\mathrm{pump}Q_2^n & 0 & 0 \\ 0 & 0 & R_3Q_3^n & 0 \\ 0 & 0 & 0 & R_4Q_4^n \end{bmatrix} = \begin{bmatrix} H_0 \\ a \\ -h_\mathrm{R} \\ h_\mathrm{R} \end{bmatrix}. \tag{4.20}$$

Some simplifications can be made for this particular system obtaining

$$\begin{aligned} &\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} H_2 \\ H_4 \end{bmatrix} + \begin{bmatrix} R_1Q_1^n + R_\mathrm{pump}Q_2^n & 0 & 0 \\ 0 & R_3Q_3^n & 0 \\ 0 & 0 & R_4Q_4^n \end{bmatrix} = \begin{bmatrix} H_0 + a \\ -h_\mathrm{R} \\ h_\mathrm{R} \end{bmatrix} \\ \Leftrightarrow &\begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} H_2 \\ H_4 \end{bmatrix} + \begin{bmatrix} R_1Q_1^n + R_\mathrm{pump}Q_2^n + R_3Q_3^n & 0 & 0 \\ 0 & R_3Q_3^n & 0 \\ 0 & 0 & R_4Q_4^n \end{bmatrix} = \begin{bmatrix} H_0 + a - h_\mathrm{R} \\ -h_\mathrm{R} \\ h_\mathrm{R} \end{bmatrix}, \end{aligned} \tag{4.21}$$

and with the equation of mass conservation which gives $Q_1 = Q_2 = Q_3$, $Q_4 = q_\mathrm{vc}$ and $Q_3 = Q_\mathrm{D} + q_\mathrm{vc}$, the energy balance can be summarized as

$$\begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} H_2 \\ H_4 \end{bmatrix} + \begin{bmatrix} (R_1 + R_\mathrm{pump} + R_3)Q_3^n & 0 & 0 \\ 0 & R_3Q_3^n & 0 \\ 0 & 0 & R_4q_\mathrm{vc}^n \end{bmatrix} = \begin{bmatrix} H_0 + a - h_\mathrm{R} \\ -h_\mathrm{R} \\ h_\mathrm{R} \end{bmatrix}, \tag{4.22}$$

$$\Leftrightarrow Q_3 = \left(\frac{H_0 + a - h_\mathrm{R}}{R_1 + R_\mathrm{pump} + R_3} \right)^{\frac{1}{n}}. \tag{4.23}$$

4.2.3 Analysis

The system analysis used a calculation tool in order to validate the previous equations when compared with the finite difference methodology for a pump's operation change at $5:20$.

The method used to calculate the finite difference was the forward approach, with one and five minutes perturbation. This analysis used the hydraulic software EPANET to have the different variable values, such as the flow in each pipe and the head in each node. Simulations were run for the original system and the finite difference perturbations.

For the analytical approach, it was used the implicit point of view, already described at the 3.2.1 (Equation 3.47), to calculate the water tank level sensitivity (time derivative), that is summarized as:

$$\frac{\mathrm{d}h_{\mathrm{R}}(t_\alpha)}{\mathrm{d}x_{i,p}} = \frac{\frac{\mathrm{d}h_{\mathrm{R}}(t_{\alpha-1})}{\mathrm{d}x_{i,p}}}{\left(1 - \frac{\Delta t_\alpha}{A_{\mathrm{R}}} \times \frac{\mathrm{d}Q_{\mathrm{in/out}}(t_\alpha)}{\mathrm{d}h_{\mathrm{R}}}\right)}. \tag{4.24}$$

Figure 4.5 shows the sensitivity of the water tank level to the change of the pump operation for both methods. In Figure 4.5, the tank level for a change in pump status at $5:20$, $5:21$, $5:25$ is illustrated, which data is required for this finite difference approach. It can be concluded that the time of pump operation is proportional to the water tank level, since if it pumps more time the level increases as expected. Also it can be seen that the results achieved by finite difference method and the analytical one perfectly matched, what validates the equations deducted previously for this sensitivity.

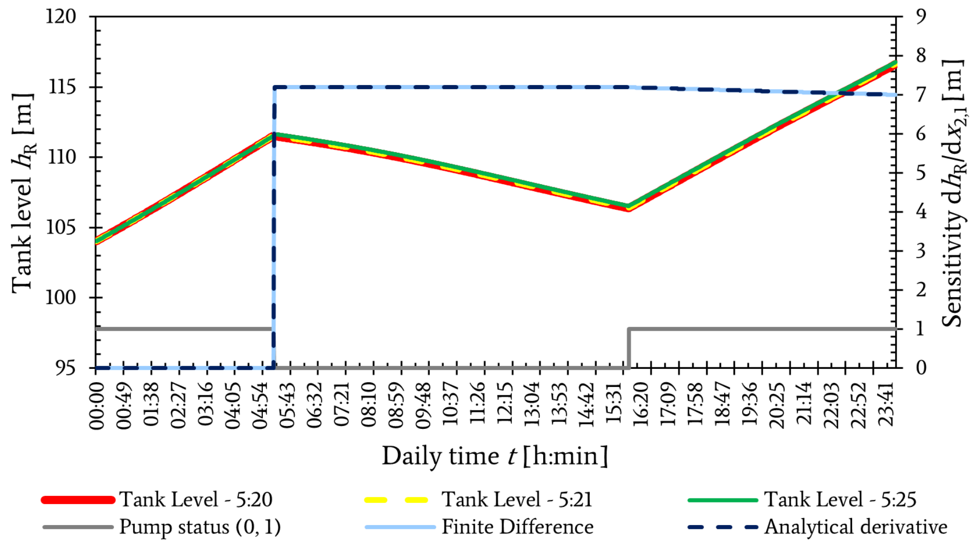


Figure 4.5: Sensitivity of water tank level to the pump operation $x_{2,1} = 0.83$ represented by $\frac{\mathrm{d}h_{\mathrm{R}}}{\mathrm{d}x_i}$.

In order to calculate the node pressure sensitivity, it was used the system described at Equation 3.52, that is sum up for this particular case as:

$$\left[\begin{array}{cccc|ccc} nR_1Q_1^{n-1} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & nR_{\text{pump}}Q_2^{n-1} & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & nR_3Q_3^{n-1} & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & nR_4Q_4^{n-1} & 0 & 0 & 1 \\ \hline 1 & -1 & 0 & 0 & & & \\ 0 & 1 & -1 & 0 & & 0 & \\ 0 & 0 & 0 & 1 & & & \end{array}\right] \times \begin{bmatrix} \frac{\mathrm{d}Q_1}{\mathrm{d}h_\mathrm{R}} \\ \frac{\mathrm{d}Q_2}{\mathrm{d}h_\mathrm{R}} \\ \frac{\mathrm{d}Q_3}{\mathrm{d}h_\mathrm{R}} \\ \frac{\mathrm{d}Q_4}{\mathrm{d}h_\mathrm{R}} \\ \frac{\mathrm{d}H_1}{\mathrm{d}h_\mathrm{R}} \\ \frac{\mathrm{d}H_2}{\mathrm{d}h_\mathrm{R}} \\ \frac{\mathrm{d}H_4}{\mathrm{d}h_\mathrm{R}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{4.25}$$

from this system the value of $\frac{\mathrm{d}Q_2}{\mathrm{d}h_\mathrm{R}}$ is used to calculate the pump's power and finally the operation cost.

The objective function, i.e. the operation cost $C(x)$, is calculated with Equation 3.55, where the term related to the pump change status can be found (Equation 4.26) and the afterwards influence is represented with the sum (Equation 4.27). For this case, the cost at the changing point is given by:

$$C(x) = \$_{i,p}\dot{W}_p(t_i + x_{i,p}\Delta t^{\mathrm{h}}_{i,p}) \times \Delta t^{\mathrm{h}}_{i,p} = 0.006618 \times 71.10 \times 4 = 18.82 \text{ euros}; \tag{4.26}$$

and the propagation term can be written as

$$\sum_{i=1}^{n_h} \sum_{p=1}^{n_{\text{pump}}} \$_{i,p} \frac{\mathrm{d}\dot{W}_p(t)}{\mathrm{d}x_{i,p}} \times \Delta t. \tag{4.27}$$

In Figure 4.6, the costs calculated by the previous analytical expression and the finite difference method are compared. As can be observed, it can be concluded that the results are identical no matter the used method, which validates the equations deducted before. Figure 4.6 also shows that the major impact in the cost function is the time of status changing pump operation when compared with the residual values (approximately 0.008 €) given by the sum operation.

Finally, to prove that the matrix form and the equations of the simplified system are equivalent and give the same results, the intermediate derivatives were calculated by both approaches. As expected, the results are the same, which can be confirmed by the overlapping lines in Figure 4.7.

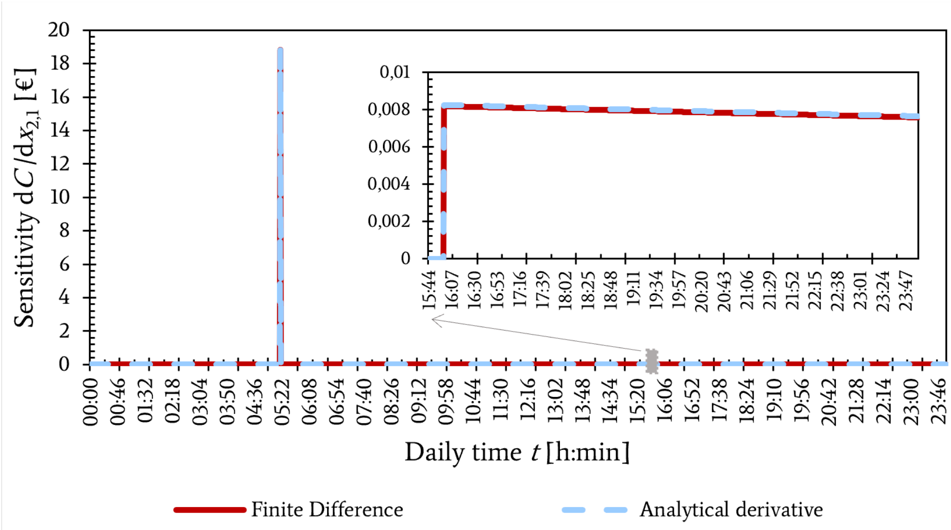


Figure 4.6: Sensitivity of the objective function $C(x)$ to the pump operation $x_{2,1} = 0.83$ alterations, represented by $\frac{\mathrm{d}C}{\mathrm{d}x_{2,1}}$.

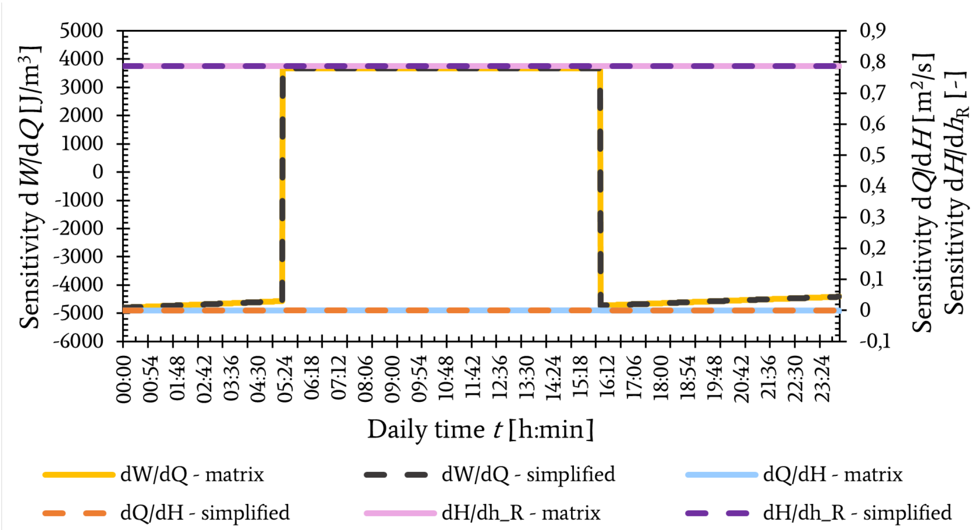


Figure 4.7: Comparison of the sensitivity calculated by both approaches: matrix form and reduced system (simplified approach).

4.3 AnyTown modified model

4.3.1 Description

This case study is the well-known benchmark named AnyTown Modified - ATM [University of Exeter 2022]. This system is a complex network with 44 pipes and 25 nodes that have a water source (at node 10), three fixed-rate pumps (b1, b2 and b3) and three storage tanks (node 65, 165 and 265). The system is represented schematically in Figure 4.8.

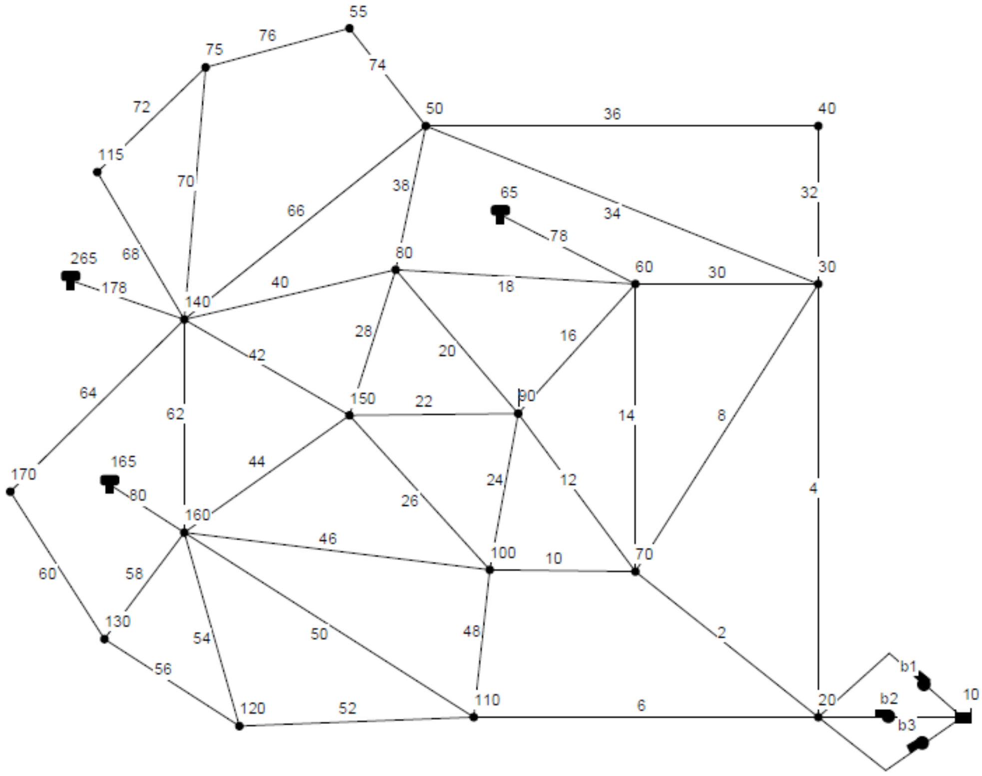


Figure 4.8: Schematics of the AnyTown modified network.

Concerning the tanks, it has predefined a maximum, a minimum and an initial level, given as 71.53 m, 66.53 m and 66.93 m, respectively. Each tank has an area of $A = 365$ m^2, without an initial elevation ($H_0 = 0$ m).

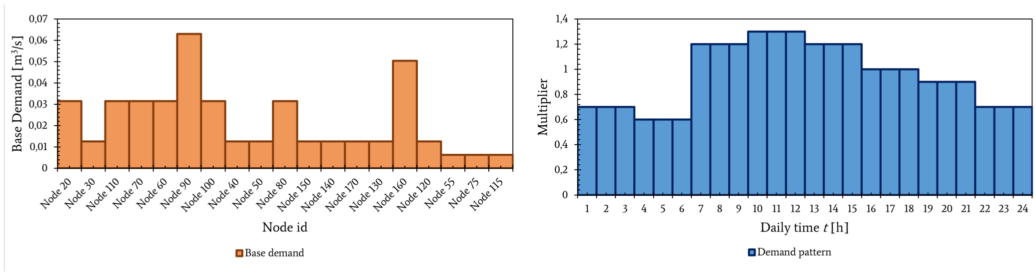
With respect to the nodes, all of them have an external demand with different base demand and follows the multiplier pattern shown at Figure 4.9.

Concerning the head loss in each pipe, it is calculated as $h_{\text{loss}} = RQ^n$, where R is the resistance coefficient giving by Hazen-Williams formulation in imperial units:

$$R = 4.727C^{-1.852}d^{-4.871}L, \tag{4.28}$$

where $n = 1.852$ is the flow exponent, C is the unitless roughness coefficient, d is the pipe diameter in foot (ft) and L is the pipe length also in foot (ft). For SI units, i.e. meters (m), the same equation is given by:

$$R = 10.665C^{-1.852}d^{-4.871}L. \tag{4.29}$$



(a) External base demand. (b) Demand multiplier pattern.

Figure 4.9: External base demand and demand multiplier for each node.

The pump characteristics, such as pump's curve and efficiency, are defined piece wise in a linear way, and are shown in Figure 4.10.

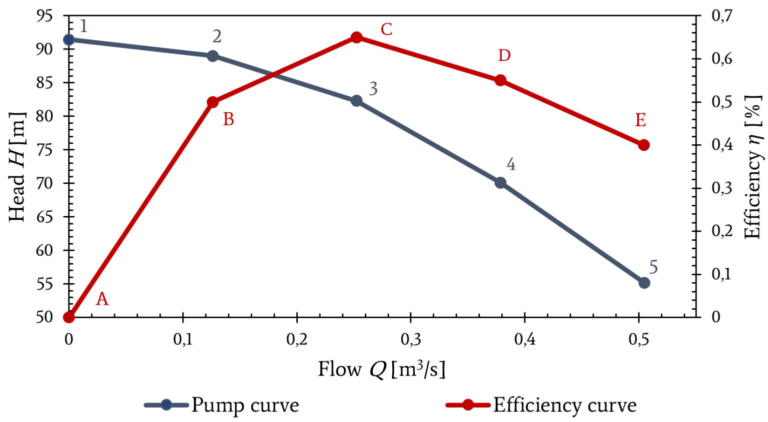


Figure 4.10: Characteristic curves of the pumps b1, b2 and b3. $\mathrm{H}_{12} = -19.325Q + 91.44$, $\mathrm{H}_{23} = -53.143Q + 95.707$, $\mathrm{H}_{34} = -96.624Q + 106.68$, $\mathrm{H}_{45} = -118.36Q + 114.91$, $\eta_{\mathrm{AB}} = 3.962Q$, $\eta_{\mathrm{BC}} = 1.189Q + 0.35$, $\eta_{\mathrm{CD}} = -0.793Q + 0.85$, $\eta_{\mathrm{DE}} = -1.189Q + 1$.

Furthermore, this system follows a daily energy tariff patterns as mentioned in Table 4.2, which has four time periods with four different prices It is defined six time horizons taking into consideration the energy tariff which are $[0,4[$, $[4,7[$, $[7,12[$, $[12,17[$, $[17,21[$, $[21,24[$. The chosen decision variable $x_{i,p}$, as in Section 4.2.1, appears at the second time horizon $[4,7[$ with the value of $5:20$, corresponding to $x_{2,1} = 0.44$.

Table 4.2: Daily energy tariff for the AnyTown Modified network.

Time period [h]	Cost [€/kWh]
[0, 7[	0.1814
[7, 17[	0.3528
[17, 21[	0.8097
[21, 24[	0.1814

4.3.2 Case study resolution

The hydraulic system described previously can be defined by the equation system:

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & 0 \end{bmatrix} \times \begin{bmatrix} \mathbf{Q} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_{10}\mathbf{H}_0 \\ \mathbf{q} \end{bmatrix}, \tag{4.30}$$

where A_{12} is a matrix of 44 pipes $\times$ 23 nodes and A_{11} has the size of 44 pipes $\times$ 44 pipes.

In order to calculate the sensitivities, the equations described at the Chapter 3 were compared with the results obtained by finite difference method. This method chosen is the forward approach as described in the previous example (4.2.3) and follows the same perturbation (one minute), however it was observed little influence of the perturbation value.

For the water tank level sensitivity, the implicit approach given by Equation 3.48 was followed and can be summarized for this particular case, i.e. for tank R65, R165 and R265 which are connected to the rest of the network by the pipes 78, 80 and 178, by:

$$\begin{bmatrix} 1-\frac{\Delta t}{A_\mathrm{R}}\frac{\mathrm{d}Q_{78}}{\mathrm{d}h_{\mathrm{R65}}} & -\frac{\Delta t}{A_\mathrm{R}}\frac{\mathrm{d}Q_{78}}{\mathrm{d}h_{\mathrm{R165}}} & -\frac{\Delta t}{A_\mathrm{R}}\frac{\mathrm{d}Q_{78}}{\mathrm{d}h_{\mathrm{R265}}} \\ -\frac{\Delta t}{A_\mathrm{R}}\frac{\mathrm{d}Q_{80}}{\mathrm{d}h_{\mathrm{R65}}} & 1-\frac{\Delta t}{A_\mathrm{R}}\frac{\mathrm{d}Q_{80}}{\mathrm{d}h_{\mathrm{R165}}} & -\frac{\Delta t}{A_\mathrm{R}}\frac{\mathrm{d}Q_{80}}{\mathrm{d}h_{\mathrm{R265}}} \\ -\frac{\Delta t}{A_\mathrm{R}}\frac{\mathrm{d}Q_{178}}{\mathrm{d}h_{\mathrm{R65}}} & -\frac{\Delta t}{A_\mathrm{R}}\frac{\mathrm{d}Q_{178}}{\mathrm{d}h_{\mathrm{R165}}} & 1-\frac{\Delta t}{A_\mathrm{R}}\frac{\mathrm{d}Q_{178}}{\mathrm{d}h_{\mathrm{R265}}} \end{bmatrix} \times \begin{bmatrix} \frac{\mathrm{d}h_{\mathrm{R65}}(t_\alpha)}{\mathrm{d}x_{i,p}} \\ \frac{\mathrm{d}h_{\mathrm{R165}}(t_\alpha)}{\mathrm{d}x_{i,p}} \\ \frac{\mathrm{d}h_{\mathrm{R265}}(t_\alpha)}{\mathrm{d}x_{i,p}} \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}h_{\mathrm{R65}}(t_{\alpha-1})}{\mathrm{d}x_{i,p}} \\ \frac{\mathrm{d}h_{\mathrm{R165}}(t_{\alpha-1})}{\mathrm{d}x_{i,p}} \\ \frac{\mathrm{d}h_{\mathrm{R265}}(t_{\alpha-1})}{\mathrm{d}x_{i,p}} \end{bmatrix}, \tag{4.31}$$

Concerning the node pressure sensitivity, it is used Equation 3.52 in order to know $\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}h_\mathrm{R}}$, specifically the $\frac{\mathrm{d}Q_{78}}{\mathrm{d}h_{\mathrm{R65}}}$, $\frac{\mathrm{d}Q_{80}}{\mathrm{d}h_{\mathrm{R165}}}$ and $\frac{\mathrm{d}Q_{178}}{\mathrm{d}h_{\mathrm{R265}}}$. It should be noted that this system has to be rewritten for each time increment, removing the data relating to switched off pumps, when existing.

$$\begin{bmatrix} \mathbf{nA}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & 0 \end{bmatrix} \times \begin{bmatrix} \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}h_\mathrm{R}} \\ \frac{\mathrm{d}\mathbf{H}}{\mathrm{d}h_\mathrm{R}} \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_{10}\frac{\mathrm{d}\mathbf{H}_0}{\mathrm{d}h_\mathrm{R}} \\ 0 \end{bmatrix}, \text{ (dimension } 63 \times 63). \tag{4.32}$$

The objective function sensitivity is calculated by Equation 3.55 summarized as:

$$\frac{\mathrm{d}C(x)}{\mathrm{d}x_{i,p}} = \sum_{i=1}^{n_h}\sum_{p=1}^{n_{\mathrm{pump}}} \$_{i,p}\dot{W}_p(t_i + x_{i,p}\Delta t^{\mathrm{h}}_{i,p}) \times \Delta t^{\mathrm{h}}_{i,p} + \sum_{i=1}^{n_h}\sum_{p=1}^{n_{\mathrm{pump}}} \$_{i,p}\frac{\mathrm{d}\dot{W}_p(t)}{\mathrm{d}x_{i,p}} \times \Delta t, \tag{4.33}$$

where $\frac{\mathrm{d}W_p(t)}{\mathrm{d}x_{i,p}}$ for each pump p is given by

$$\frac{\mathrm{d}W_p(t)}{\mathrm{d}x_{i,p}} = \frac{\mathrm{d}W}{\mathrm{d}Q_p} \times \left[\frac{\mathrm{d}Q_p}{\mathrm{d}h_{\mathrm{R65}}} \times \frac{\mathrm{d}h_{\mathrm{R65}}}{\mathrm{d}x_{i,p}} + \frac{\mathrm{d}Q_p}{\mathrm{d}h_{\mathrm{R165}}} \times \frac{\mathrm{d}h_{\mathrm{R165}}}{\mathrm{d}x_{i,p}} + \frac{\mathrm{d}Q_p}{\mathrm{d}h_{\mathrm{R265}}} \times \frac{\mathrm{d}h_{\mathrm{R265}}}{\mathrm{d}x_{i,p}}\right]. \tag{4.34}$$

4.3.3 Analysis

For this system analysis, were compared the finite difference method and the analytical sensitivities, calculated in Chapter 3. The EPANET hydraulic simulator was used to obtain the flow and head values for each time increment (both for the original system and for each perturbation) and a Python program was built in this project to find, for each time increment, the space derivatives, i.e. the sensitivities $\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}h_\mathrm{R}}$ and $\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}h_\mathrm{R}}$.

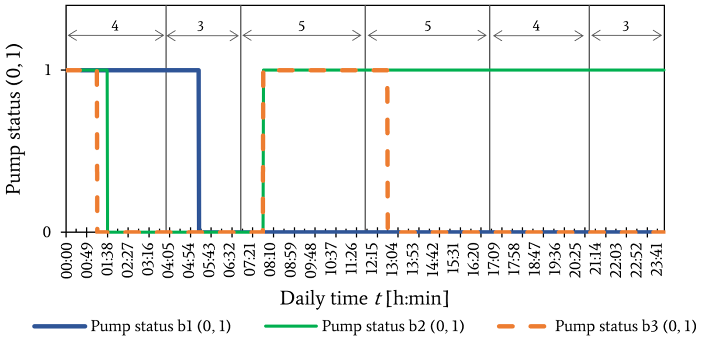
The water tank sensitivity, for each tank, is shown in Figure 4.11. As it can be seen, the analytical derivatives have the same results as the finite difference method. Furthermore, it can be shown in the Figure the effect of the pump operation on the tank levels. When the three pumps are switched off at 5 : 20, the tank level decreases to compensate the external demand at the nodes. Once it reaches the point where the pumps switch back on, the level at all tank increases as expected, since the pumped water is enough to power the node demands. In addition, the tanks 165 and 265 have a similar behaviour throughout the whole process.

Therefore, it is also relevant to analyse the different derivatives of the flow in the pipe connecting the system to the tank in relation to level of each tank. The connecting pipes are: pipe 78 to tank 65, pipe 80 to tank 165 and pipe 178 to tank 265 (Figure 4.8). The results are shown in Figure 4.12. As expected, the sensitivity which relates the connecting pipe to the tank is negative, since the outflow has a direct influence to the connected tank, while the others sensitivities are positive. Unsurprisingly, this sensitivity change with the change of the pump operation, but the multipliers of the external demand also have their influence. This can be observed by the peaks in the sensitivity analysis.

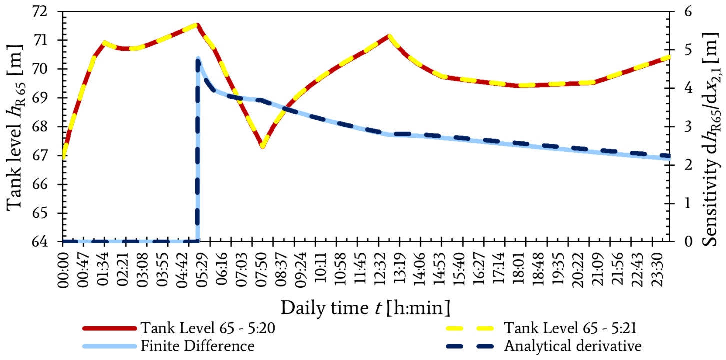
Hence, the same derivatives but relating the same pipe flow to each tank are shown at Figure 4.13. It can be seen that the derivatives of the pipe 78 which connects tank 65 have a lower values when compared with the ones for tanks 165 and 265. Also, tanks 165 and 265 have a symmetrical sensitivity between each other.

Concerning the objective function sensitivity, the results are shown at Figure 4.14. As expected, the cost of the pumping operation is proportional to the working time and has an impact on the pump status changing, for an extra minute of operation, with the value of 252.58€. However, the propagation values, which are given by the sum operation, have a greater impact on the results as the sum is a higher value. This alteration at the pump status results in a negative total cost sensitivity of -7.32€, which represents that this alteration reduces the total cost at the end of the total time. This result is justified due to the change in tank level, which will reduce the pumped flow rate in the following increments.

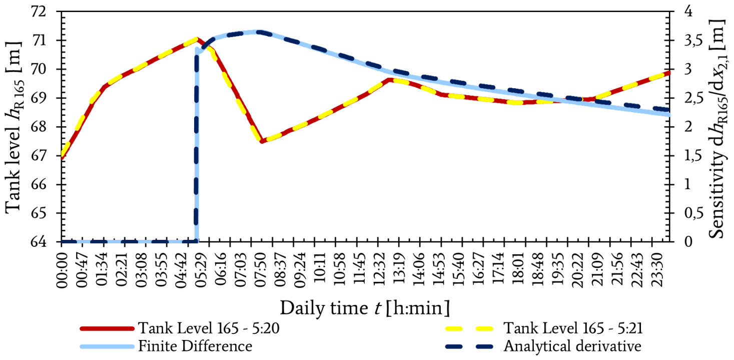
$$C(x) = \$_{i,p}\dot{W}_p(t_i + x_{i,p}\Delta t^{\mathrm{h}}_{i,p}) \times \Delta t^{\mathrm{h}}_{i,p} = 0.1814 \times 464.14 \times 3 = 252.58 \text{ euros.} \tag{4.35}$$



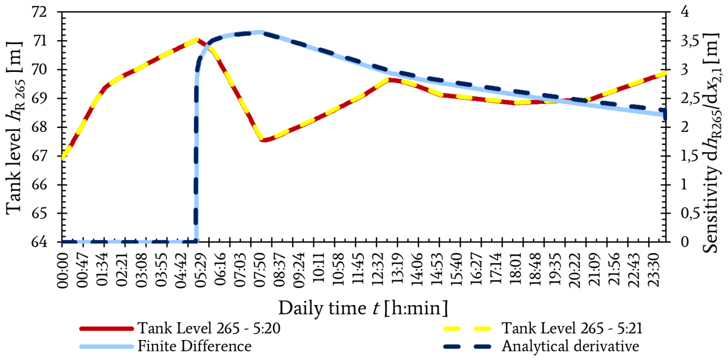
(a) Pump status and time horizon



(b) Tank 65

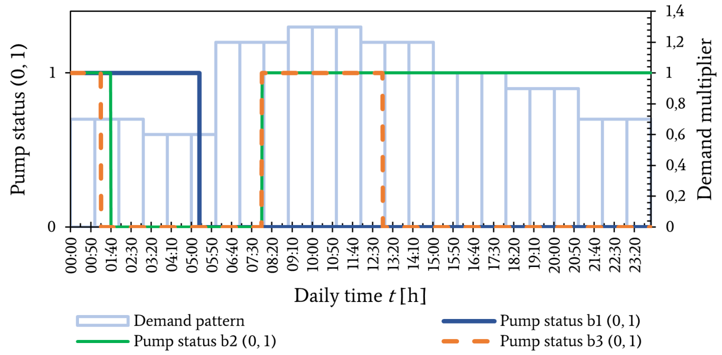


(c) Tank 165

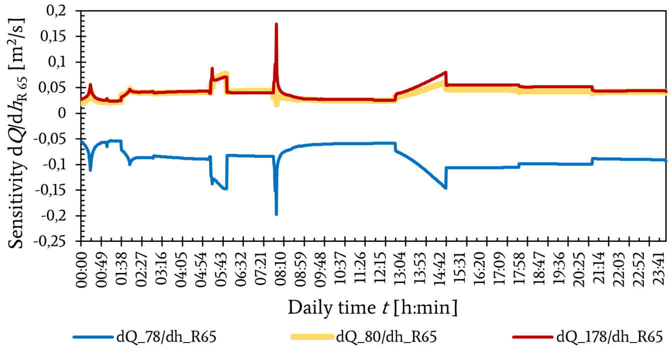


(d) Tank 265

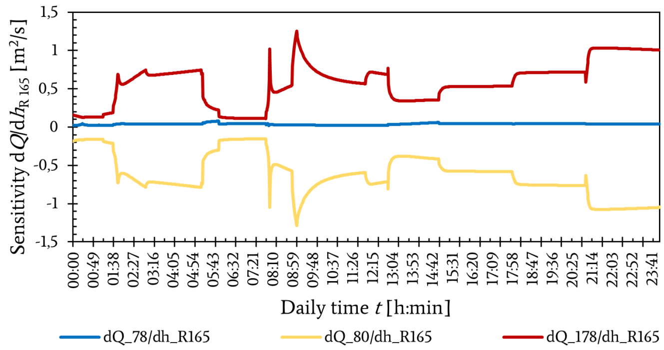
Figure 4.11: Tank level sensitivity to the change of the pump operation $x_{2,1}$ at $5:20$.



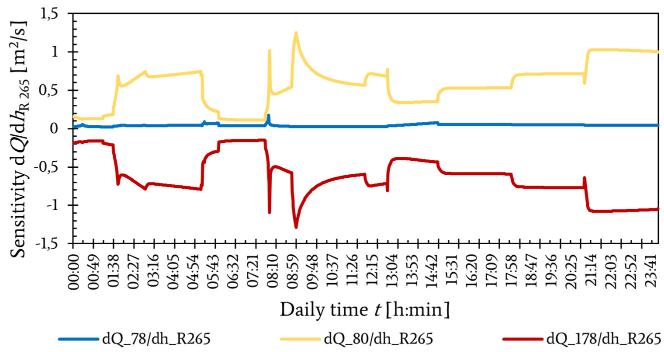
(a) Pump status and demand multipliers



(b) Tank 65

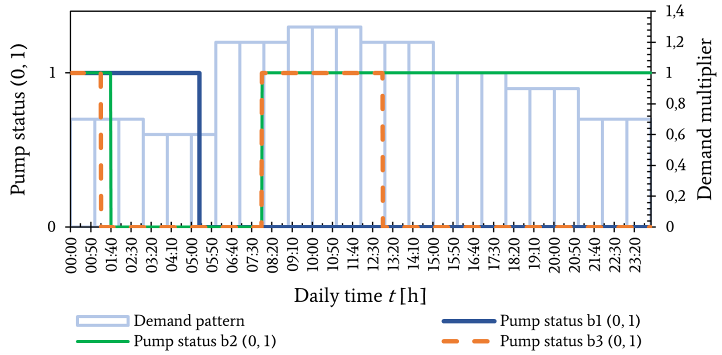


(c) Tank 165

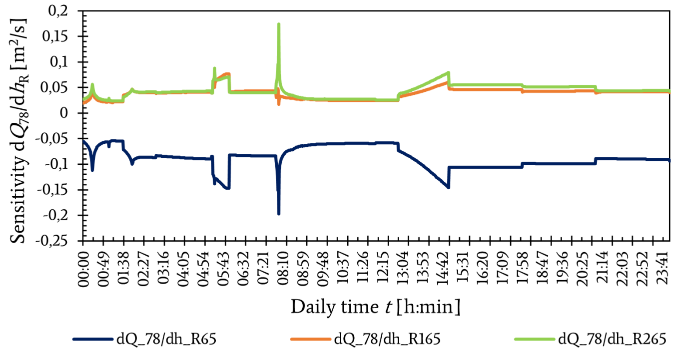


(d) Tank 265

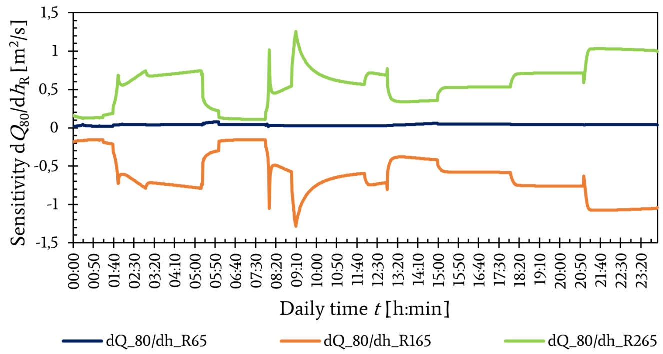
Figure 4.12: Sensitivity of the flow at the pipe that connects the system with each tank.



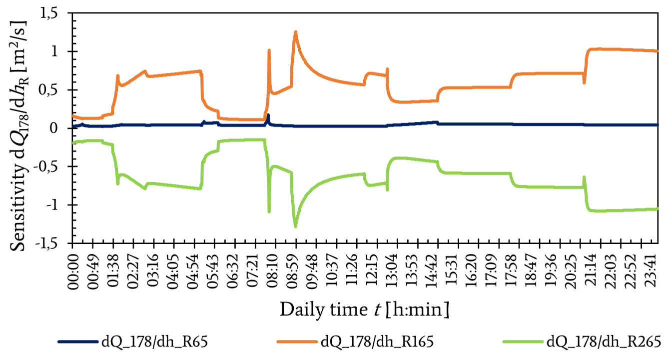
(a) Pump status and demand multipliers



(b) Pipe 78

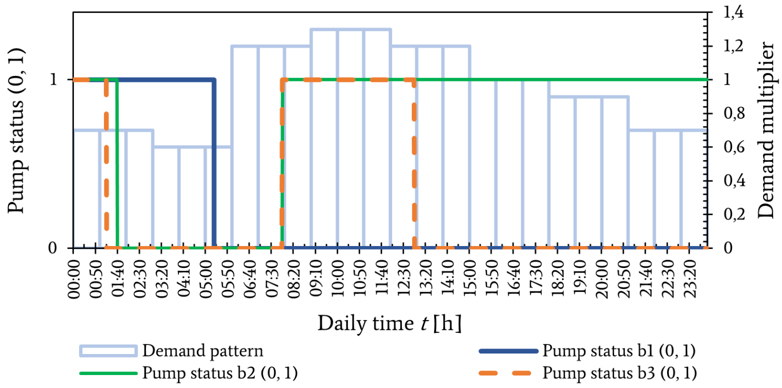


(c) Pipe 80

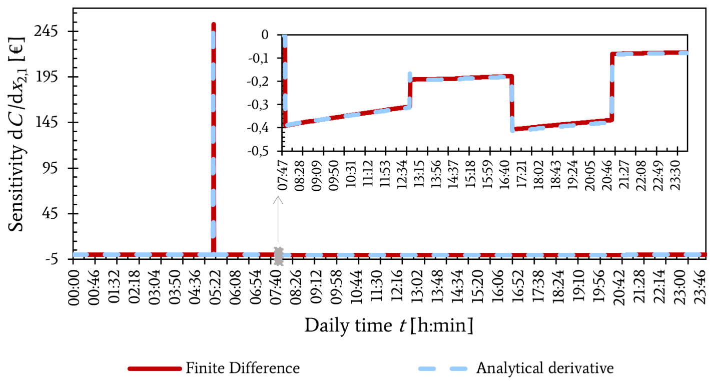


(d) Pipe 178

Figure 4.13: Sensitivity of the flow at a specific pipe relative with every tank.



(a) Pump status and demand multipliers



(b) Total cost.

Figure 4.14: Sensitivity of the objective function $C(x)$ to the pump operation $x_{2,1} = 0.44$ alterations, represented by $\frac{\mathrm{d}C}{\mathrm{d}x_{2,1}}$

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Chapter 5

Conclusion

At this moment, the planet's resources, such as energy and water, are precious commodities and need to be managed more efficiently. For this, it is necessary to optimize the management of the water supply systems.

Taking this issue into account, the main objective of carrying out an expedient methodology was to find a water supply operation solution that ensures energy efficiency. Therefore, the work consisted of the deduction of the analytical sensitivities for the general case of hydraulic systems. Initially, the hydraulic systems were studied to know the operating system and the optimization process. The present work aimed to develop an analytical method that allows replacing the current existing method of controlling the operation of water supply.

Currently, this supply operation control process is performed by approximate methods, such as the finite difference method. This method needs constant evaluations and adaptations depending on each new perturbation, in this work the goal was to create an autonomous and adaptive method to new realities. In addition, the analytical sensitivity methodology implemented is based on a gradient-based method, which have the ability to present more accurate results, without having to resort to high computational resources, as it only needs to evaluate the initial system.

The optimization problem of the WSS was identified, as well as the analytical sensitivity analysis was implemented for the problem constraints and objective function. In particular, the optimization problem consists in finding the status of n_{pump} pumps to minimise pump costs, subject to water demand, water tank level, continuity, and node pressure constraints. These constraints were divided into two categories, time derivatives and space derivatives, depending on whether these govern the temporal relationship with the decision variable $x_{i,p}$, i.e. the water tank level, or at the spatial level.

The validation of the analytical methodology was applied in three case studies. The first consisted of showing the generic model of the hydraulic system and the EPS transformation, while the second and third examples consisted of comparing the results obtained through the analytical sensitivity calculation and the finite difference method (with a perturbation of one and five minutes). In both of these analyses, the networks studies were resolved as explained earlier in Chapters 3 and 4.

It can be concluded that the pump operation directly affects the water tank level, i.e. the time perturbation in pump operation is proportional to the increase/decrease in water level, which can be seen in Figures 4.5 and 4.11, for both case studies. Also, the sensitivity of the objective function shows that the major impact is on the change in pump operation for an extra minute of operation, Figures 4.6 and 4.14, however for the AnyTown Modified benchmark, the propagation values have a greater impact on the total value when compared to the simple network, where those values don't change significantly the total cost value sensitivity.

Furthermore, the results obtained by the analytical method and the finite difference method are similar, which leads to the conclusion that the possible solution to this problem consists in replacing the approximate method with the analytical approach developed in this work.

Thus, the method developed in this work will allow the management of water supply networks to be more efficient, as optimisation can be carried out more expeditiously and leading to more efficient management of the planet's resources such as water and energy.

In future work, the analytical sensitivities studied earlier can be applied to a test program in Python and real scenarios to effectively validate and implement the work developed. Also, it can be important to implement this system in real cases and evaluate its behaviour as a support tool for digital twin network.

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Appendix A

AnyTown modified benchmark characteristics given by EPANET

Table A.1: Node characteristics

Node id	Node elevation [m]	External demand [m^3/s]	Demand pattern
20	6.096	0.03155	DEM
30	15.24	0.01262	DEM
110	15.24	0.03155	DEM
70	15.24	0.03155	DEM
60	15.24	0.03155	DEM
90	15.24	0.06309	DEM
100	15.24	0.03155	DEM
40	15.24	0.01262	DEM
50	15.24	0.01262	DEM
80	15.24	0.03155	DEM
150	36.576	0.01262	DEM
140	24.384	0.01262	DEM
170	36.576	0.01262	DEM
130	36.576	0.01262	DEM
160	36.576	0.05047	DEM
120	36.576	0.01262	DEM
55	24.384	0.00631	DEM
75	24.384	0.00631	DEM
115	24.384	0.00631	DEM

Table A.2: Demand pattern multiplier DEM

[0,1[	[1,2[	[2,3[	[3,4[	[4,5[	[5,6[	[6,7[	[7,8[	[8,9[	[9,10[	[10,11[	[11,12[
0.7	0.7	0.7	0.6	0.6	0.6	1.2	1.2	1.2	1.3	1.3	1.3

[12,13[	[13,14[	[14,15[	[15,16[	[16,17[	[17,18[	[18,19[	[19,20[	[20,21[	[21,22[	[22,23[	[23,24[
1.2	1.2	1.2	1	1	1	0.9	0.9	0.9	0.7	0.7	0.7

Table A.3: Reservoir characteristics

Reservoir id	Head H_0 [m]
10	3.048

Table A.4: Tank characteristics

Tank id	Elevation [m]	Initial level [m]	Mininum level [m]	Maximum level [m]	Diameter [m]
65	0	66.93	66.53	71.53	21.55
165	0	66.93	66.53	71.53	21.55
265	0	66.93	66.53	71.53	21.55

Table A.5: Pump characteristics

Pump id	Node inflow	Node outflow
b1	10	20
b2	10	20
b3	10	20

Table A.6: Pipes characteristics

Pipe id	Node inflow	Node outflow	Length [m]	Diameter [m]	Roughness C [unitless]	Hazen-Williams coefficient R
4	20	30	3657.6	609.6	120	61.31
30	30	60	1828.8	508.0	120	74.51
16	60	90	1828.8	152.4	120	26252.07
14	70	60	1828.8	406.4	120	220.94
12	70	90	1828.8	254.0	120	2180.41
2	20	70	3657.6	609,6	120	61.31
6	20	110	3657.6	457.2	120	248.96
48	110	100	1828.8	304.8	120	897.11
24	100	90	1828.8	508.0	120	74.51
10	70	100	1828.8	609.6	120	30.65
32	30	40	1828.8	203.2	120	6465.27
36	40	50	1828.8	304.8	120	897.11
38	50	80	1828.8	203.2	120	6465.27
18	80	60	1828.8	457.2	120	124.48
20	80	90	1828.8	406.4	120	220.94
66	50	140	3657.6	203.2	120	12930.55
40	140	80	1828.8	304.8	120	897.11
28	80	150	1828.8	457.2	120	124.48
22	150	90	1828.8	152.4	120	26252.07
26	150	100	1828.8	609.6	120	30.66
42	150	140	1828.8	508.0	120	74.51
64	140	170	3657.6	304.8	120	1794.22
60	170	130	1828.8	304.8	120	897.11
58	130	160	1828.8	406.4	120	220.94
44	160	150	1828.8	355.6	120	423.40
50	160	110	1828.8	304.8	120	897.11
52	110	120	1828.8	355.6	120	423.40
56	120	130	1828.8	304.8	120	897.11
62	140	160	1828.8	508.0	120	74.51
46	100	160	1828.8	457.2	120	124.48
34	50	30	2743.2	355.6	120	635.10
78	60	65	30.48	508.0	120	1.24
80	160	165	30.48	406.4	120	3.68
8	70	30	2743.2	355.6	120	635.10
74	50	55	1828.8	304.8	120	897.11
76	55	75	1828.8	304.8	120	897.11
72	75	115	1828.8	304.8	120	897.11
68	115	140	1828.8	304.8	120	897.11
70	75	140	1828.8	304.8	120	897.11
178	140	265	30.48	406.4	120	3.68
54	160	120	2743.2	304.8	120	1345.66

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Appendix B

ATM system matrix

$$\begin{bmatrix} n \times \mathbf{KQ}^{n-1} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^{\mathrm{T}} & 0 \end{bmatrix} \tag{B.1}$$

$$
\left[\begin{array}{c|c}
1.852 \times \begin{bmatrix}
61.31 \times Q_{1}^{1.852-1} \\
74.51 \times Q_{2}^{1.852-1} \\
26252.07 \times Q_{3}^{1.852-1} \\
220.94 \times Q_{4}^{1.852-1} \\
2180.41 \times Q_{5}^{1.852-1} \\
61.31 \times Q_{6}^{1.852-1} \\
248.96 \times Q_{7}^{1.852-1} \\
897.11 \times Q_{8}^{1.852-1} \\
74.51 \times Q_{9}^{1.852-1} \\
30.65 \times Q_{10}^{1.852-1} \\
6465.27 \times Q_{11}^{1.852-1} \\
897.11 \times Q_{12}^{1.852-1} \\
6465.27 \times Q_{13}^{1.852-1} \\
124.48 \times Q_{14}^{1.852-1} \\
220.94 \times Q_{15}^{1.852-1} \\
12930.55 \times Q_{16}^{1.852-1} \\
7.11 \times Q_{17}^{1.852-1} \\
124.48 \times Q_{18}^{1.852-1} \\
26252.07 \times Q_{19}^{1.852-1} \\
30.66 \times Q_{20}^{1.852-1} \\
74.51 \times Q_{21}^{1.852-1} \\
1794.22 \times Q_{22}^{1.852-1} \\
897.11 \times Q_{23}^{1.852-1} \\
220.94 \times Q_{24}^{1.852-1} \\
423.40 \times Q_{25}^{1.852-1} \\
897.11 \times Q_{26}^{1.852-1} \\
423.40 \times Q_{27}^{1.852-1} \\
897.11 \times Q_{28}^{1.852-1} \\
74.51 \times Q_{29}^{1.852-1} \\
124.48 \times Q_{30}^{1.852-1} \\
635.10 \times Q_{31}^{1.852-1} \\
1.24 \times Q_{32}^{1.852-1} \\
3.68 \times Q_{33}^{1.852-1} \\
635.10 \times Q_{34}^{1.852-1} \\
897.11 \times Q_{35}^{1.852-1} \\
897.11 \times Q_{36}^{1.852-1} \\
897.11 \times Q_{37}^{1.852-1} \\
897.11 \times Q_{38}^{1.852-1} \\
897.11 \times Q_{39}^{1.852-1} \\
3.68 \times Q_{40}^{1.852-1} \\
1345.66 \times Q_{41}^{1.852-1} \\
a/1.852 \\
a/1.852 \\
a/1.852
\end{bmatrix} \times I
&
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \\
\hline
A_{12}^{\mathrm{T}} & 0
\end{array}\right] \tag{B.2}
$$