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Generalizações Fracionais Estocásticas no Controlo Ótimo

Stochastic Fractional Generalizations in Optimal Control

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Stochastic Fractional Generalizations in Optimal Control

Tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática Aplicada, realizada sob a orientação científica do Doutor Delfim Fernando Marado Torres, Professor Catedrático do Departamento de Matemática da Universidade de Aveiro

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palavras-chave

Fórmula de integração fracionária estocástica por partes, equação fracionária estocástica de Euler-Lagrange, modelo estocástico da COVID-19 com tempo de atraso, controlo ótimo, reação-difusão, condições de otimalidade necessárias e suficientes, generalizações do teorema de Taylor.

Resumo

Duas tendências matemáticas são visadas nesta tese de doutoramento. A primeira está relacionada ao estabelecimento de certos resultados teóricos dentro da teoria do cálculo das variações, por um lado uma fórmula de integração fracionária estocástica por partes e uma equação fracionária estocástica de Euler-Lagrange são obtidas, por outro lado, são estudados uma generalização do teorema de Taylor fracionário ponderado e a equação de Euler-Lagrange com um núcleo não singular. A segunda vertente visa o enriquecimento da literatura com várias aplicações matemáticas em diferentes campos, com a ideia de fornecer soluções adequadas a problemas sociais complexos, nomeadamente os originados com o surto da pandemia da COVID-19, para além da descrição de algumas soluções adequadas tanto em biomatemática como em bioeconomia.

keywords

Stochastic fractional integration by parts formula, Stochastic fractional Euler-Lagrange equation, stochastic time delayed COVID-19 model, optimal control, reaction-diffusion, necessary and sufficient optimality conditions, extended Taylor's theorem.

abstract

Two mathematical tendencies are targeted in this doctoral thesis, the first is related to the establishment of certain theoretical results within the theory of the calculus of variations, on one hand, a stochastic fractional integration by parts formula and a stochastic fractional Euler-Lagrange equation are obtained, on the other hand, weighted generalized fractional Taylor's theorem and Euler Lagrange equation with non-singular kernel sense are studied. The second trend was aimed at enriching the literature with several mathematical applications in different fields, with the idea of providing various appropriate solutions to complex social problems, namely those constructed with the outbreak of the COVID-19 pandemic, in addition, the description of some adequate solutions in both bio-mathematics and bio-economics.

Contents

Introduction

The subject of my research areas varies between different concerns, starting by an accommodate contribution within the calculus of variations theory related to the generalization of what is called Euler-Lagrange equation to the Stochastic Fractional counterpart, for which the second chapter is devoted.

We shall at first give a simple overview in language of the theory of calculus of variations as invented by Euler and Lagrange, as well as an amount of the history of its invention, which is most interesting to solve a wide range of the optimization problems and how it was useful in the mathematics, physics and related areas up to the present day.

As it is known, Euler and Lagrange are the founders of the calculus of variations. A simple and magnificent idea that revolutionized the manner of solving numerous problems of optimization and having an enormous action on how partial derivative equations are handled touching all sorts of domains of application. The calculus of variations reflects the basis of the mechanics known as Lagrangian, without which modern physics could not exist.

Now, we are looking at how Lagrange was led to his result in these problems, discussing the simplest components and principles of his discovery, and lastly, mentioning the repercussions they have had up to the present time.

The history of the calculus of variations and Lagrange's contribution to it is well documented. A good point of departure is the work of Catherine Goldstein [113]. The reader can also consult [58] for a more advanced epistemological analysis, as well as [31] regarding biographical elements.

In 1754, at the earlier age of eighteen, Lagrange read the article "Une méthode pour trouver des lignes courbes jouissant de propriétés de maximum ou de minimum"[54] by the great Euler. Inspired by this study, he investigated his first original mathematical result, and dared to communicate it via letter to Euler, already at the time a leading figure in science. His letter remained unanswered. Lagrange, however, was so ambitious, and continued to think about Euler's article. In 1755 he wrote a second letter to Euler in which he explained the novel method that he had developed, that is, his proper manner for dealing with the problem studied by Euler. The latter method would be named by Euler himself in one of his letters:"calculus of variations ". This time, Euler responded to Lagrange, in terms of praise: (Your solution to the isoperimetric problem leaves nothing to be desired, and I rejoice that this subject, of which I was almost the only one who dealt with it since the first attempts, has been taken by you to the highest degree of perfection. The importance of the matter has led me to outline, with the aid of your light, an analytical solution to which I will give no publicity until you yourself have published the whole of your research, so that I do not take away any part of the glory that is due to you.)

This letter was enough of a recommendation to secure Lagrange a position as a teacher at the Royal School of Artillery in Turin. The beginning of the young Lagrange was a period of great activity. In 1758 he co-founded what would become the Academy of Sciences in Turin. He published at that time numerous articles in the Miscellanea Taurinensia, the first one of which, in 1762, was entitled "Essai dúne nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies"[121]. It is this article that interests us the most because it already contains the foundations of the calculus of variations and the methods of multipliers called "Lagrange multipliers", ideas that would both be developed over the course of his career. However, Lagrange had a broad mathematical range and also wrote other articles at that time on different topics, including the vibrating string and differential equations.
In general, the history of science traces the genesis of the calculus of variations to the problem that Newton posed in 1685: find the shape of a solid of revolution offering the least resistance (in the direction of its axis) to a fluid. Newton himself proposed a purely geometric solution of it; for some very recent developments on this problem the reader can consult [37]. The second problem that genuinely enthralled the mathematicians, and which was the true birth of the calculus of variations, is that of the brachistochrone (Greek: brachis = short, brachiston = the shortest, chrone = time). This was a challenge (with the promise of a prize!) launched in 1699 by Johann Bernoulli1: find among all of the curves connecting two points A and B the one along which a particle falling from A and gliding under the effect of gravity arrives at B in the shortest time. It is thus asking us to determine, among all possible shapes of the ski slopes (for example) connecting points A and B, which one will permit the fastest run (ideally, without friction). The greatest mathematicians of the day, that is Johann Bernoulli, his brother Jacob, Newton, then Leibniz, Euler and finally Lagrange, attacked the problem and gave solutions to it.

In [54], Euler is the first to propose a systematic treatment of this kind of problem: instead of concerning himself only with the problem of the brachistochrone, he seeks a method to find a curve that minimises or maximises any quantity expressed by an integral, and to derive the equation that must be satisfied by the minima. That equation, which has become known as the Euler–Lagrange equation, takes for example the form (in the notation of physics):

$$\frac{d}{dt}\left(\frac{dL}{\dot{q}}\right) - \frac{dL}{dq} = 0$$

The Euler-Lagrange equation started with the latter deterministic form, passed by the stochastic one, extended to the fractional counterpart, and generalized to our stochastic fractional Euler–Lagrange equation contribution (Zine et al.) see [165].
A theory of stochastic calculus of variations is presented which generalizes the ordinary calculus of variations to stochastic processes. Generalizations of the Euler equation and Noether's theorem are obtained and several conservation laws are discussed. An application to Nelson's probabilistic framework of quantum mechanics is also given. [152]

An Euler–Lagrange equation for this problem has been derived first in (Riewe, 1996, 1997), see also (Agrawal, 2002). A generalization of the problem to include fractional integrals, the transversality conditions and many other aspects can be found in the literature of recent years. See (Almeida and Torres, 2010; Atanacković, Konjik and Pilipović, 2008; Malinowska and Torres, 2012) and references therein. Indirect methods for fractional variational problems have a vast background in the literature and can be considered a well-studied subject: see (Agrawal, 2002; Almeida, Pooseh and Torres, 2012; Atanacković, Konjik and Pilipović, 2008; Frederico and Torres, 2010; Jelicic and Petrovacki, 2009; Klimek, 2001; Odzijewicz, Malinowska and

Torres, 2012b; Riewe, 1997) and references therein that study different variants of the problem and discuss a range of possibilities in the presence of fractional terms, Euler–Lagrange equations and boundary conditions. With respect to results on fractional variational calculus via Caputo operators, we refer the reader to (Agrawal, 2007b; Almeida, Malinowska and Torres, 2012; Almeida and Torres, 2011; Frederico and Torres, 2010; Malinowska and Torres, 2010e; Mozyrska and Torres, 2010; Odzijewicz, Malinowska and Torres, 2012a) and references therein. At a recent time, Zine et al [165] have introduced a stochastic fractional calculus. As an application, we have presented a stochastic fractional calculus of variations, in which a stochastic fractional Euler–Lagrange equation is obtained, extending those available in the literature for the classical, fractional, and stochastic calculus of variations. To illustrate our main theoretical result, we have discussed two examples: one derived from quantum mechanics, the second validated by an adequate numerical simulation.
We move now, in the next chapters, towards the presentation of some detailed separately deterministic, stochastic and fractional works related to the dynamical systems studies switching between the biomathematics field and the economic framework, in addition, some fractional theoretical results are performed.

Some special works [88, 162] are provided with the appearance of COVID-19 pandemic so as to outline the mathematical point of view towards this phenomenon, intending to help sanitary authorities to adopt the right decisions reaching the requested desires.

The third chapter contains firstly, our first deterministic version of the studied pandemic COVID-19 model with delays in Morocco. As known, the arrival of the COVID-19 pandemic at 2019 which posed a great threat to public health and economy worldwide pushed us to think about publishing some collaboratively works to fight against its spread. Unfortunately, there is yet no effective drug for this disease at this moment. For this, several countries have adopted multiple preventive interventions to avoid the spread of COVID-19. Here, we propose, firstly, a delayed mathematical model to predict the epidemiological trend of COVID-19 in Morocco. Parameter estimation and sensitivity analysis of the proposed model are rigorously studied. Moreover, numerical simulations are presented in order to test the effectiveness of the preventive measures and strategies that were imposed by the Moroccan authorities and also help policy makers and public health administration to develop such strategies.
And secondly, an adequate contribution linked to the extension of the SICA model to the mathematical spatiotemporal epidemic SICA model with optimal control strategy, in which we have modeled the spatial behavior by adding a diffusion term with the Laplace operator for which we have devoted one section to justify and interpret, mathematically and physically, its use in this context. The aim objective is to prove Existence and Uniqueness of the global positive spatiotemporal solution of the system, using semi-group theory and ordinary differential equations. An appropriate constructive sequential method is proposed to find the optimal control pair that minimizes the number of infected individuals and the corresponding cost. We construct further, an explicit necessary optimality condition. This work is concluded by some numerical simulations supporting our main results.

Extending our last deterministic pandemic COVID-19 model with several delays in Morocco to the stochastic ones with and without confinement as summarized in the sequel and presented in the fourth chapter.
The first case of COVID-19 in Morocco was reported on 2 March 2020, and the number of

reported cases has increased day by day. We construct secondly, another work in which we extend the well-known SIR compartmental model to deterministic and stochastic time-delayed models in order to predict the epidemiological trend of COVID-19 in Morocco and to assess the potential role of multiple preventive measures and strategies imposed by Moroccan authorities. The main features of the work include the well-posedness of the models and conditions under which the COVID-19 may become extinct or persist in the population. Parameter values have been estimated from real data and numerical simulations are presented for forecasting the COVID-19 spreading as well as verification of theoretical results.

The COVID-19 pandemic evolves in many countries to a second stage, characterized by the need for the liberation of the economy and relaxation of the human psychological effects. To this end, numerous countries decided to implement adequate deconfinement strategies. After the first extension of the established confinement, Morocco moves to the deconfinement stage on May 20, 2020. The relevant question concerns the impact on the COVID-19 spreading by considering an additional degree of realism related to stochastic noises due to the effectiveness level of the adapted measures. In this section, we propose thirdly, a delayed stochastic mathematical model to predict the epidemiological trend of COVID-19 in Morocco after the deconfinement. To ensure the well-posedness of the model, we prove the existence and uniqueness of a positive solution. Based on the large number theorem for martingales, we discuss the extinction of the disease under an appropriate threshold parameter. Moreover, numerical simulations are performed in order to test the efficiency of the deconfinement strategies chosen by the Moroccan authorities to help the policy makers and public health administration to make suitable decisions in the near future.

It is not convenient to adopt the assumption of the precise parameters since models are usually exposed to the natural fluctuations, which leads us to the consideration of the near-optimal control with imprecise parameters
We present some theoretical studies based on the stochastic SICA model (Silva and Torres), on which we have applied an adequate optimal control for minimizing or quasi-minimizing the cost regarded functional, and for which we have established different inequalities related to the solutions in question and the boundedness of the appropriate adjoint functions in order to prove necessary and sufficient conditions for near- optimal control with imprecise parameters. Moreover we also build a numerical simulation to validate the foregoing result.

The stochastic epidemic system governed by the Brownian motion process as a unique source of perturbation is not forever relevant, therefore, it is with great interest to examine models driven by both brownian motion and jump Lévy noise in order to take into account continuous and discontinuous components in the model.
To deal this, we have proposed and studied, in the same chapter, a shifted SICA epidemic model, extending that due to Silva and Torres (2017) to the stochastic setting driven by both Brownian motion processes and jump Lévy noise. Lévy noise perturbations are usually ignored by existing works of mathematical modelling in epidemiology, but its incorporation into the SICA epidemic model is worth to be considered because of the presence of strong fluctuations in HIV/AIDS dynamics, often leading to the emergence of a number of discontinuities in the processes under investigation. Our work was organised as follows: (i) we began by presenting our model, by clearly justifying its used form, namely the component related to the Lévy noise; (ii) we proved existence and uniqueness of a global positive solution by constructing

a suitable stopping time; (iii) under some assumptions, we showed extinction of HIV/AIDS; (iv) we obtained sufficient conditions assuring persistence of HIV/AIDS; (v) we illustrated our mathematical results through numerical simulations.
Because of the most impotant role played by the economic side worldwide, we have thought about translating different working mathematical tools from the biomathematics field to the economic framework.
We have also proposed and studied a stochastic capital-labour model with logistic growth function. First, we have shown that the model has a unique positive global solution. Then, using the Lyapunov analysis method, we have obtained conditions for the extinction of the total labour force. Furthermore, we have also proven sufficient conditions for their persistence in the mean. Finally, we have illustrated our theoretical results through numerical simulations.

The fact that the attraction and the mathematical interest of doing research on the elaboration of a new theoretical works within the fractional derivative with non singular kernel leads me devoting the remainder time before the completion of my thesis report to move towards establishing certain papers presenting in the fifth chapter.

For this, an appropriate study, in the fifth chapter, presents new estimates for fractional derivatives without singular kernels defined by some specific functions. Based on some obtained inequalities, we give a useful method to establish the global stability of steady states for fractional-order systems and generalize some works existing in the literature. Finally, we apply our results to prove the global stability of a fractional-order SEIR model with a general incidence rate.

To enrich the fractional calculus theory which is nowadays widely addressed in different scientific areas in order to describe accurately real world problems with effect memory, one of our works is focused on the discovering of some classical extended formula, namely, the generalized Taylor's theorem related to the generalized weighted fractional derivative with non singular kernel is investigated using some demonstrated lemmas.
As an application, some corollaries are obtained as well as the generalized mean value theorem is carried out.

Integration by parts formulas plays a crucial role in mathematical analysis, e.g., during the proof of necessary optimality conditions in the calculus of variations and optimal control. Motivated by this fact, we construct a new right-weighted generalized fractional derivative in Riemann–Liouville sense with its associated integral. We rewrite these operators equivalently in effective series, obtaining some interesting properties relating the left and the right fractional operators. These achievements permit us to prove different versions of an adequate integration by parts formula. With the new general formula, we obtain an appropriate weighted Euler–Lagrange equation for dynamic optimization. We end with an application in the quantum mechanics framework.

I would like to mention that the last declared works are separately governed by the deterministic, stochastic, fractional, and stochastic fractional approaches, as detailed below, accompanied by some needed mathematical prerequisites.

Chapter 1

Stochastic fractional approach: Mathematical prerequisites and original results

The original work of this chapter is published at [165].

1.1 Mathematical prerequisites

1.1.1 Integration by parts formula

Integration by parts formulas play a fundamental role in the calculus of variations and optimal control.

Theorem 1.1 (Fractional formulas of integration by parts). *Let $\alpha > 0$, $p, q \geq 1$, and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case where $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$).*

(i) If $X_t \in L_p(a, b)$ and $Y_t \in L_q(a, b)$ for every $t \in [a, b]$, then

$$\int_a^b X_t({}_aI_t^\alpha)Y_t dt = \int_a^b Y_t({}_tI_b^\alpha X_t)dt.$$

(ii) If $Y_t \in {}_tI_b^\alpha(L_p)$ and $X_t \in {}_aI_t^\alpha(L_q)$ for every $t \in [a, b]$, then

$$\int_a^b X_t({}_aD_t^\alpha Y_t)dt = \int_a^b Y_t({}_tD_b^\alpha X_t)dt.$$

(iii) For the Caputo fractional derivatives, one has

$$\int_a^b X_t({}_a^CD_t^\alpha Y_t)dt = \int_a^b Y_t({}_tD_b^\alpha X_t)dt + [{}_tI_b^{1-\alpha}X_t) \cdot Y_t)]_a^b$$

and

$$\int_a^b (X_t)({}_t^CD_b^\alpha Y_t)dt = \int_a^b (Y_t)({}_aD_t^\alpha X_t)dt - [({}_aI_t^{1-\alpha}X_t) \cdot Y_t]_a^b$$

for $\alpha \in (0, 1)$.

1.1.2 Fractional Euler-Lagrange equation

Many generalizations of the classical calculus of variations and optimal control have been made, in order to extend the theory to the field of fractional variational and fractional optimal control. A simple fractional variational problem consists in finding a function x that minimizes the functional

$$J[x] = \int_a^b L(t, x(t),_a D_t^\alpha x(t))dt$$

where ${}_aD_t^\alpha$ is the left $Riemann - Liouville$ fractional derivative. Typically, some boundary conditions are prescribed as $x(a) = x_a$ *and/or* $x(b) = x_b$.
Classical techniques have been adopted to solve such problems. The Euler–Lagrange equation for a Lagrangian of the form

$$J[x] = \int_a^b L(t, x(t),_a D_t^\alpha x(t),_t D_b^\alpha x(t))dt$$

has been derived in [4, 2002]. Many variants of necessary conditions of optimality have been studied. A generalization of the problem to include fractional integrals, i.e.,

$$J[x] = \int_a^b L(t, x(t),_a I_t^{1-\alpha} x(t),_a D_t^\alpha x(t))dt,$$

the transversality conditions of fractional variational problems and many other aspects can be found in the literature of recent years. See (Almeida and Torres, 2009a, 2010; Atanackovic, Konjik and Pilipovic, 2008; Riewe, 1996, 1997) and references therein. Furthermore, it has been shown that a variational problem with fractional derivatives can be reduced to a classical problem using an approximation of the Riemann–Liouville fractional derivatives in terms of a finite sum, where only derivatives of integer order are present (Atanackovic, Konjik and Pilipovic, 2008).

Theorem 1.2 ([4]). *Let J be a functional of the form*

$$J[x] = \int_a^b L(t, x(t),_a D_t^\alpha x(t))dt$$

defined on the set of functions x which have continuous left and right Riemann–Liouville derivatives of order $\alpha \in [a, b]$, and satisfy the boundary conditions $x(a) = x_a$ and $x(b) = x_b$. A necessary condition for J to have an extremum for a function x is that x satisfy the following Euler–Lagrange equation:

$$\frac{\partial L}{\partial x} +_t D_b^\alpha \left[\frac{\partial L}{\partial_a D_t^\alpha x}\right] = 0.$$

1.2 A Stochastic Fractional Calculus with Applications to Variational Principles

1.2.1 Introduction

A stochastic calculus of variations, which generalizes the ordinary calculus of variations to stochastic processes, was introduced in 1981 by Yasue, generalizing the Euler–Lagrange equation and giving interesting applications to quantum mechanics [95]. Recently, stochastic variational differential equations have been analyzed for modeling infectious diseases [59, 49], and stochastic processes have shown to be increasingly important in optimization [110].

In 1996, fifteen years after Yasue's pioneer work [95], the theory of the calculus of variations evolved in order to include fractional operators and better describe non-conservative systems in mechanics [6]. The subject is currently under strong development [13]. We refer the interested reader to the introductory book [6] and to [15, 20, 21] for numerical aspects on solving fractional Euler–Lagrange equations. For applications of fractional-order models and variational principles in epidemics, biology, and medicine, see [9, 21, 117, 155] and references therein. Given the importance of both stochastic and fractional calculi of variations, it seems natural to join the two subjects. That is the main goal of our current work, i.e., to introduce a stochastic-fractional calculus of variations. For that, we start our work by introducing new definitions: left and right stochastic fractional derivatives and integrals of Riemann–Liouville and Caputo types for stochastic processes of second order, as a deterministic function resulting from the intuitive action of the expectation, on which we can compute its fractional derivative several times to obtain additional results that generalize analogous classical relations. Our definitions differ from those already available in the literature by the fact that they are applied on second order stochastic processes, whereas known definitions, for example, those in [19, 53, 52, 63], are defined only for mean square continuous second order stochastic process, which is a short family of operators. Moreover, available results in the literature have not used the expectation, which we claim to be more natural, easier to handle and estimate, when applied to fractional derivatives by different methods of approximation, like those developed and cited in [15]. More than different, our definitions are well posed and lead to numerous results generalizing those in the literature, like integration by parts and Euler–Lagrange variational equations. The section is organized as follows. we introduce, first, the new stochastic fractional operators. Their fundamental properties are given in the second part. In particular, we prove stochastic fractional formulas of integration by parts. We consider then, the basic problem of the stochastic fractional calculus of variations and obtain the stochastic Riemann–Liouville and Caputo fractional Euler–Lagrange equations theorems. the last step gives two illustrative examples.

1.2.2 The Stochastic Fractional Operators

Let (Ω, F, P) be a probabilistic space, where Ω is a nonempty set, F is a σ-algebra of subsets of Ω, and P is a probability measure defined on Ω. A mapping X from an open time interval I into the Hilbert space $H = L_2(\Omega, P)$ is a stochastic process of second order in $\mathbb{R}$. We introduce the stochastic fractional operators by composing the classical fractional operators with the expectation E. In what follows, the classical fractional operators are denoted using standard notations [118]: ${}_aD_t^\alpha$ and ${}_tD_b^\alpha$ denote the left and right Riemann–Liouville fractional derivatives of order α; ${}_aI_t^\alpha$ and ${}_tI_b^\alpha$ the left and right Riemann–Liouville fractional integrals of order α; while the left and right Caputo fractional derivatives of order α are denoted by ${}_a^CD_t^\alpha$ and ${}_t^CD_b^\alpha$, respectively. The new stochastic operators add to the standard notations an 's' for "stochastic".

Definition 1.3 (Stochastic fractional operators)**.** *Let X be a stochastic process on $[a, b] \subset I$, $\alpha > 0$, $n = [\alpha] + 1$, such that $E(X(t)) \in AC^n([a, b] \to \mathbb{R})$ with AC the class of absolutely continuous functions. Then,* *(D1) the left stochastic Riemann–Liouville fractional derivative of order α is given by*

$$\begin{aligned} {}_a^s D_t^\alpha X(t) &= {}_a D_t^\alpha [E(X_t)] \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-1-\alpha} E(X_\tau) d\tau, \quad t > a; \end{aligned}$$

(D2) the right stochastic Riemann–Liouville fractional derivative of order α by

$$\begin{aligned} {}_t^s D_b^\alpha X(t) &= {}_t D_b^\alpha [E(X_t)] \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{-d}{dt}\right)^n \int_t^b (\tau-t)^{n-1-\alpha} E(X_\tau) d\tau, \quad t < b; \end{aligned}$$

(D3) the left stochastic Riemann–Liouville fractional integral of order α by

$$\begin{aligned} {}_a^s I_t^\alpha X(t) &= {}_a I_t^\alpha [E(X_t)] \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} E(X_\tau) d\tau, \quad t > a; \end{aligned}$$

(D4) the right stochastic Riemann–Liouville fractional integral of order α by

$$\begin{aligned} {}_t^s I_b^\alpha X(t) &= {}_t I_b^\alpha [E(X_t)] \\ &= \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} E(X_\tau) d\tau, \quad t < b; \end{aligned}$$

(D5) the left stochastic Caputo fractional derivative of order α by

$$\begin{aligned} {}_a^{sC} D_t^\alpha X(t) &= {}_a^C D_t^\alpha [E(X_t)] \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-1-\alpha} E(X(\tau))^{(n)} d\tau; \quad t > a. \end{aligned}$$

(D6) and the right stochastic Caputo fractional derivative of order α by

$$\begin{aligned} {}_t^{sC} D_b^\alpha X(t) &= {}_t^C D_b^\alpha [E(X_t)] \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-1-\alpha} E(X(\tau))^{(n)} d\tau, \quad t < b. \end{aligned}$$

Remark 1.4. *The stochastic processes $X(t)$ used along the manuscript can be of any type satisfying the announced conditions of existence of the novel stochastic fractional operators. For example, we can consider Levy processes as a particular case, provided one considers some intervals where $E(X(t))$ is sufficiently smooth [60].*

As we shall prove in the following sections, the new stochastic fractional operators just introduced provide a rich calculus with interesting applications.

1.2.3 Associated Properties

Several properties of the classical fractional operators, like boundedness or linearity, also hold true for their stochastic counterparts.

Proposition 1.5. *If $t \to E(X_t) \in L_1([a,b])$, then ${}_a^sI_t^\alpha(X_t)$ is bounded.*

Proof. The property follows easily from definition (D3):

$$|{}_a^sI_t^\alpha(X_t)| = \left|\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}E(X_\tau)d\tau\right| \leq k\left\|E(X_t)\right\|_1,$$

which shows the intended conclusion. □

Proposition 1.6. *The left and right stochastic Riemann–Liouville and Caputo fractional operators given in Definition 1.3 are linear operators.*

Proof. Let c and d be real numbers and assume that ${}_a^sD_t^\alpha X_t$ and ${}_a^sD_t^\alpha Y_t$ exist. It is easy to see that ${}_a^sD_t^\alpha(c\cdot X_t + d\cdot Y_t)$ also exists. From Definition 1.3 and by linearity of the expectation and the linearity of the classical/deterministic fractional derivative operator, we have

$$\begin{aligned}{}_a^sD_t^\alpha(c\cdot X_t + d\cdot Y_t) &= {}_aD_t^\alpha E(c\cdot X_t + d\cdot Y_t)\\ &= c\cdot {}_aD_t^\alpha E(X_t) + d\cdot {}_aD_t^\alpha E(Y_t)\\ &= c\cdot {}_a^sD_t^\alpha(X_t) + d\cdot {}_a^sD_t^\alpha(Y_t).\end{aligned}$$

The linearity of the other stochastic fractional operators is obtained in a similar manner. □

Our next proposition involves both stochastic and deterministic operators. Let $\mathscr{O} \in \left\{D, I, {}^CD\right\}$. Recall that if ${}_a^s\mathscr{O}_t^\beta$ is a left stochastic fractional operator of order β, then ${}_a\mathscr{O}_t^\beta$ is the corresponding left classical/deterministic fractional operator of order β; similarly for right operators.

Note that the proofs of Propositions 1.7 and 1.8 and Lemma 1.9 are not hard to prove in the sense that they are based on well-known results available for deterministic fractional derivatives (observe that $E(X(t))$ is deterministic).

Proposition 1.7. *Assume that ${}_a^sI_t^\beta X_t$, ${}_t^sI_b^\beta X_t$, ${}_a^sI_t^\alpha X_t$, ${}_aD_t^\alpha\left[{}_a^sI_t^\alpha X_t\right]$, ${}_aI_t^\alpha\left[{}_a^sI_t^\beta X_t\right]$ and ${}_tI_b^\alpha\left[{}_t^sI_b^\beta X_t\right]$ exist. The following relations hold:*

$$\begin{aligned}{}_aI_t^\alpha\left[{}_a^sI_t^\beta X_t\right] &= {}_a^sI_t^{\alpha+\beta}X_t,\\ {}_tI_b^\alpha\left[{}_t^sI_b^\beta X_t\right] &= {}_t^sI_b^{\alpha+\beta}X_t,\\ {}_aD_t^\alpha\left[{}_a^sI_t^\alpha X_t\right] &= E(X_t).\end{aligned}$$

Proof. Using Definition 1.3 and well-known properties of the deterministic Riemann–Liouville fractional operators [14], one has

$$\begin{aligned}{}_aI_t^\alpha\left[{}_a^sI_t^\beta X_t\right] &= {}_aI_t^\alpha\left[{}_aI_t^\beta E(X_t)\right]\\ &= {}_aI_t^{\alpha+\beta}E(X_t)\\ &= {}_a^sI_t^{\alpha+\beta}X_t.\end{aligned}$$

The second and third equalities are easily proved in a similar manner. □

Proposition 1.8. *Let $\alpha > 0$. If $E(X_t) \in L_\infty(a,b)$, then*

$$ {}_a^C D_t^\alpha \left[{}_a^s I_t^\alpha X_t\right] = E(X_t) $$

and

$$ {}_t^C D_b^\alpha \left[{}_t^s I_b^\alpha X_t\right] = E(X_t). $$

Proof. Using Definition 1.3 and well-known properties of the deterministic Caputo fractional operators [14], we have

$$ \begin{aligned} {}_a^C D_t^\alpha \left[{}_a^s I_t^\alpha X_t\right] &= {}_a^C D_t^\alpha \left[{}_a I_t^\alpha E(X_t)\right] \\ &= E(X_t). \end{aligned} $$

The second formula is shown with the same argument. □

1.2.4 Integration by parts formula

Formulas of integration by parts play a fundamental role in the calculus of variations and optimal control [18, 106]. Here we make use of Lemma 1.9 to prove our stochastic fractional Euler–Lagrange necessary optimality condition.

Lemma 1.9 (Stochastic fractional formulas of integration by parts)**.** *Let $\alpha > 0$, $p, q \geq 1$, and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case where $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$).*

(i) If $E(X_t) \in L_p(a,b)$ and $E(Y_t) \in L_q(a,b)$ for every $t \in [a,b]$, then

$$ E\left(\int_a^b (X_t)\,{}_a^s I_t^\alpha Y_t dt\right) = E\left(\int_a^b (Y_t)\,{}_t^s I_b^\alpha X_t dt\right). $$

(ii) If $E(Y_t) \in {}_t I_b^\alpha(L_p)$ and $E(X_t) \in {}_a I_t^\alpha(L_q)$ for every $t \in [a,b]$, then

$$ E\left(\int_a^b (X_t)({}_a^s D_t^\alpha Y_t) dt\right) = E\left(\int_a^b (Y_t)({}_t^s D_b^\alpha X_t) dt\right). $$

(iii) For the stochastic Caputo fractional derivatives, one has

$$ \begin{aligned} E\left[\int_a^b (X_t)({}_a^{sC} D_t^\alpha Y_t) dt\right] &= E\left[\int_a^b (Y_t)({}_t^s D_b^\alpha X_t) dt\right] \\ &+ E\left[({}_t^s I_b^{1-\alpha} X_t) \cdot Y_t\right]_a^b \end{aligned} $$

and

$$ \begin{aligned} E\left[\int_a^b (X_t)({}_t^{sC} D_b^\alpha Y_t) dt\right] &= E\left[\int_a^b (Y_t)({}_a^s D_t^\alpha X_t) dt\right] \\ &- E\left[({}_a^s I_t^{1-\alpha} X_t) \cdot Y_t\right]_a^b \end{aligned} $$

for $\alpha \in (0,1)$.

Proof. (i) We have

$$\begin{aligned}
E\left(\int_a^b (X_t)\,{}_a^sI_t^\alpha Y_t dt\right) &= \int_a^b E\left((X_t)\,{}_a^sI_t^\alpha Y_t\right) dt \quad \text{(by Fubini–Tonelli's theorem)}\\
&= \int_a^b E\left((X_t)\,{}_aI_t^\alpha E(Y_t)\right) dt \quad (by\ (D3))\\
&= \int_a^b E((X_t))\,{}_aI_t^\alpha E(Y_t) dt \quad \text{(the expectation is deterministic)}\\
&= \int_a^b {}_tI_b^\alpha E(X_t)\cdot E(Y_t) dt \quad \text{(by fractional integration by parts)}\\
&= E\left(\int_a^b {}_t^sI_b^\alpha (X_t)(Y_t) dt\right) \quad \text{(by Fubini–Tonelli's theorem).}
\end{aligned}$$

(ii) With similar arguments as in item (i), we have

$$\begin{aligned}
E\left(\int_a^b (X_t)\,{}_a^sD_t^\alpha Y_t dt\right) &= \int_a^b E\left((X_t)\,{}_a^sD_t^\alpha Y_t\right) dt\\
&= \int_a^b E\left((X_t)\,{}_aD_t^\alpha E(Y_t)\right) dt \quad (by\ (D_1))\\
&= \int_a^b E((X_t))\,{}_aD_t^\alpha E(Y_t) dt\\
&= \int_a^b {}_tD_b^\alpha E(X_t)\cdot E(Y_t) dt\\
&= E\left(\int_a^b {}_t^sD_b^\alpha (X_t)(Y_t) dt\right).
\end{aligned}$$

(iii) By using Caputo's fractional integration by parts formula we obtain that

$$\begin{aligned}
E\left[\int_a^b (X_t)({}_a^{sC}D_t^\alpha Y_t) dt\right] &= \int_a^b E\left[(X_t)\right]({}_a^CD_t^\alpha E\left[(Y_t)\right]) dt\\
&= \int_a^b ({}_tD_b^\alpha E\left[(X_t)\right]\cdot E\left[(Y_t)\right]) dt + \left[({}_tI_b^{1-\alpha}E(X_t)\cdot E(Y_t)\right]_a^b\\
&= \int_a^b ({}_t^sD_b^\alpha (X_t)\cdot E\left[(Y_t)\right]) dt + \left[({}_tI_b^{1-\alpha}E(X_t)\cdot E(Y_t)\right]_a^b\\
&= E\left[\int_a^b ({}_t^sD_b^\alpha (X_t)\cdot (Y_t)) dt\right] + E\left[({}_tI_b^{1-\alpha}E(X_t)\cdot (Y_t)\right]_a^b.
\end{aligned}$$

The first equality of (iii) is proved. By using a similar argument and applying the integration by parts formula associated with the right Caputo fractional derivative [14], we easily get the second equality of (iii). □

1.2.5 Stochastic Fractional Euler–Lagrange Equations

Let us denote by $C^1(I \to H)$ the set of second order stochastic processes X such that the left and right stochastic Riemann–Liouville fractional derivatives of X exist, endowed with the norm

$$\|X\| = \sup_{t\in I} (\|X(t)\|_H + \mid {}_a^sD_t^\alpha X(t) \mid + \mid {}_t^sD_b^\alpha X(t) \mid),$$

where $\|\cdot\|_H$ is the norm of H. Let $L \in C^1(I \times H \times \mathbb{R} \times \mathbb{R} \to \mathbb{R})$ and consider the following minimization problem:

$$J[X] = E\left(\int_a^b L\left(t, X(t), {}_a^sD_t^\alpha X(t), {}_t^sD_b^\alpha X(t)\right) dt\right) \longrightarrow \min \tag{1.1}$$

subject to the boundary conditions

$$E(X(a)) = X_a, \quad E(X(b)) = X_b, \tag{1.2}$$

where X verifies the above conditions and L is a smooth function. Taking into account the method used in [15] for the fractional setting, and according to stochastic fractional integration by parts given by our Lemma 1.9, we obtain the following necessary optimality condition for the fundamental problem (1.1)–(1.2) of the stochastic fractional calculus of variations.

Theorem 1.10 (The stochastic Riemann–Liouville fractional Euler–Lagrange equation)**.** *If $J \in C^1(H \times \mathbb{R} \times \mathbb{R} \to \mathbb{R})$ and $X \in C^1(I \to H)$ is an F-adapted stochastic process on $[a, b]$ with $E(X(t)) \in AC([a, b])$ that is a minimizer of* (1.1) *subject to the fixed end points* (1.2)*, then X satisfies the following stochastic fractional Euler–Lagrange equation:*

$$\frac{\partial L}{\partial X} + {}_t^sD_b^\alpha\left[\frac{\partial L}{\partial_a^s D_t^\alpha}\right] + {}_a^sD_t^\alpha\left[\frac{\partial L}{\partial_t^s D_b^\alpha}\right] = 0.$$

Proof. We have

$$J[X] = E\left(\int_a^b L(t, X(t), {}_a^sD_t^\alpha X(t), {}_t^sD_b^\alpha X(t) dt\right).$$

Assume that X^* is the optimal solution of problem (1.1)–(1.2). Set

$$X = X^* + \varepsilon\eta,$$

where η is an F-adapted stochastic process on $[a, b]$ in $C^1(I \to H)$. By linearity of the stochastic fractional derivatives (Proposition 1.6), we get

$${}_a^sD_t^\alpha X = {}_a^sD_t^\alpha X^* + \varepsilon\left({}_a^sD_t^\alpha \eta\right)$$

and

$${}_t^sD_b^\alpha X = {}_t^sD_b^\alpha X^* + \varepsilon\left({}_t^sD_b^\alpha \eta\right).$$

Consider now the following function:

$$j(\varepsilon) = E\left(\int_a^b L\left(t, X^* + \varepsilon\eta, {}_a^sD_t^\alpha X^* + \varepsilon\left({}_a^sD_t^\alpha \eta\right), {}_t^sD_b^\alpha X^* + \varepsilon\left({}_t^sD_b^\alpha \eta\right)\right) dt\right).$$

We deduce, by the chain rule, that

$$\frac{d}{dt} j(\varepsilon) \mid_{\varepsilon=0} = E\left(\int_a^b (\partial_2 L \cdot \eta + \partial_3 L \cdot {}_a^sD_t^\alpha \eta + \partial_4 L \cdot {}_t^sD_b^\alpha \eta) dt\right) = 0,$$

where $\partial_i L$ denotes the partial derivative of the Lagrangian L with respect to its ith argument. Using Lemma 1.9 of stochastic fractional integration by parts, we obtain

$$E\left(\int_a^b \left(\partial_2 L \cdot \eta + {}_t^s D_b^\alpha(\partial_3 L) \cdot \eta + {}_a^s D_t^\alpha(\partial_4 L) \cdot \eta\right) dt\right) = 0.$$

We claim that if Y is a stochastic process with continuous paths of second order such that

$$E\left[\int_a^b Y(t) \cdot \eta(t) dt\right] = 0$$

for any stochastic process with continuous paths η, then

$$Y = 0 \quad \text{almost surely (a.s.)}.$$

Indeed, suppose that $Y(s) > 0$ a.s. for a certain $s \in (a, b)$. By continuity, $Y(t) > c > 0$ a.s. in a neighborhood of s, $a < s - r < s < s + r < b, r > 0$. Consider the process η such that $\eta(t) = 0$ a.s. on $[a, s-r] \cup [s+r, b]$ and $\eta(t) > 0$ a.s. on $(s-r, s+r)$, and $\eta(t) = 1$ a.s. on $\left(s - \frac{r}{2}, s + \frac{r}{2}\right)$. Then, $\int_a^b Y(t) \cdot \eta(t) dt \geq rc > 0$ a.s. Consequently, $E\left[\int_a^b Y(t) \cdot \eta(t) dt\right] > 0$, which completes the proof of our claim. Taking into account this result, and the fact that η is an arbitrary process, we deduce the desired stochastic fractional Euler–Lagrange equation:

$$\partial_2 L + {}_t^s D_b^\alpha \left[\partial_3 L\right] + {}_a^s D_t^\alpha \left[\partial_4 L\right] = 0.$$

The proof is complete. □

By adopting the same method as in the proof of Theorem 1.10 and using our result of integration by parts for stochastic Caputo fractional derivatives, i.e., item (iii) of Lemma 1.9, we obtain the appropriate stochastic Caputo fractional Euler–Lagrange necessary optimality condition.

Theorem 1.11 (The stochastic Caputo fractional Euler–Lagrange equation)**.** *If $J \in C^1(H \times \mathbb{R} \times \mathbb{R} \to \mathbb{R})$ and $X \in C^1(I \to H)$ is an F-adapted stochastic process on $[a, b]$ with $E(X(t)) \in AC([a, b])$ that is a minimizer of*

$$J[X] = E\left(\int_a^b L\left(t, X(t), {}_a^{sC} D_t^\alpha X(t), {}_t^{sC} D_b^\alpha X(t)\right) dt\right)$$

subject to the fixed end points $E(X(a)) = X_a$ and $E(X(b)) = X_b$, then X satisfies the following stochastic fractional Euler–Lagrange equation:

$$\frac{\partial L}{\partial X} + {}_t^{sC} D_b^\alpha \left[\frac{\partial L}{\partial {}_a^{sC} D_t^\alpha}\right] + {}_a^{sC} D_t^\alpha \left[\frac{\partial L}{\partial {}_t^{sC} D_b^\alpha}\right] = 0.$$

Remark 1.12. *Note that the conclusions of Theorems 1.10 and 1.11 are not contradictory: one conclusion is valid for Riemann–Liouville derivative problems, while the other holds true for Caputo-type problems. The conclusions are proved in a similar manner by remarking that the additional quantity with parentheses, in the integration by parts theorem linked to the Caputo approach, vanishes under the condition that X and X^* verify the same initial and final conditions. Note also that the assumptions of Theorems 1.10 and 1.11 are necessary for the existence of left and right stochastic Riemann–Liouville/Caputo fractional derivative operators.*

Our Theorems 1.10 and 1.11 give an extension of the Euler–Lagrange equations of the classical calculus of variations [140], stochastic calculus of variations [152], and fractional calculus of variations [4]. The best way to illustrate a new theory is by choosing simple examples. We give two illustrative examples of the stochastic Riemann–Liouville fractional Euler–Lagrange equation: the first one inspired from quantum mechanics; the second chosen to allow a simple numerical solution to the obtained stochastic Riemann–Liouville fractional Euler–Lagrange equation.

1.2.6 Generalization and application in quantum mechanics

Let us consider the stochastic fractional variational problem (1.1)–(1.2) with

$$L\left(t, X(t), {}_a^sD_t^\alpha X(t), {}_t^sD_b^\alpha X(t)\right) = \frac{1}{2}\left(\frac{1}{2}m \mid {}_a^sD_t^\alpha X(t) \mid^2 + \frac{1}{2}m \mid {}_t^sD_b^\alpha X(t) \mid^2\right) - V(X(t)),$$

where X is a stochastic process of second order with $E(X(t)) \in AC([a,b])$ and V maps $C'(I \to H)$ to $\mathbb{R}$. Note that

$$\frac{1}{2}\left(\frac{1}{2}m \mid {}_a^sD_t^\alpha X(t) \mid^2 + \frac{1}{2}m \mid {}_t^sD_b^\alpha X(t) \mid^2\right)$$

can be viewed as a generalized kinetic energy in the quantum mechanics framework. By applying our Theorem 1.10 to the current variational problem, we get

$$\frac{1}{2}m\left[{}_a^sD_t^\alpha({}_t^sD_b^\alpha X(t)) + {}_t^sD_b^\alpha({}_a^sD_t^\alpha X(t))\right] = gradV(X(t)), \tag{1.3}$$

where $gradV$ is the gradient of V, which in this case means the derivative of the potential energy of the system. We observe that if α tends to zero and X is a deterministic function, then relation (1.3) becomes what is known in the physics literature as Newton's dynamical law: $m\ddot{X}(t) = gradV(X(t))$.

The calculus of variations can assist us both analytically and numerically. Now we give a numerical example, carried out with the help of the MATLAB computing environment [51].

1.2.7 Example with fractional computational method

Let $\alpha := 0.25$, $a := 0.01$, $b := 0.99$, $X_a := 1.00$, and $X_b := 1.00$. Consider the following variational problem (1.1)–(1.2):

$$J[X] = \int_a^b {}_a^sD_t^\alpha X(t) \times {}_t^sD_b^\alpha X(t)\, dt \longrightarrow \min,$$
$$E(X(a)) = X_a, \quad E(X(b)) = X_b,$$

where $X \in C^1(I \to H)$ with $E(X(t)) \in AC$ and ${}_a^sD_\cdot^\alpha X$ and ${}_\cdot^sD_b^\alpha X$ denote, respectively, the left and the right stochastic fractional Riemann–Liouville derivatives of order α. Resorting again to Theorem 1.10, we obtain the following stochastic fractional Euler–Lagrange differential equation:

$${}_a^sD_t^{2\alpha} X(t) + {}_t^sD_b^{2\alpha} X(t) = 0.$$

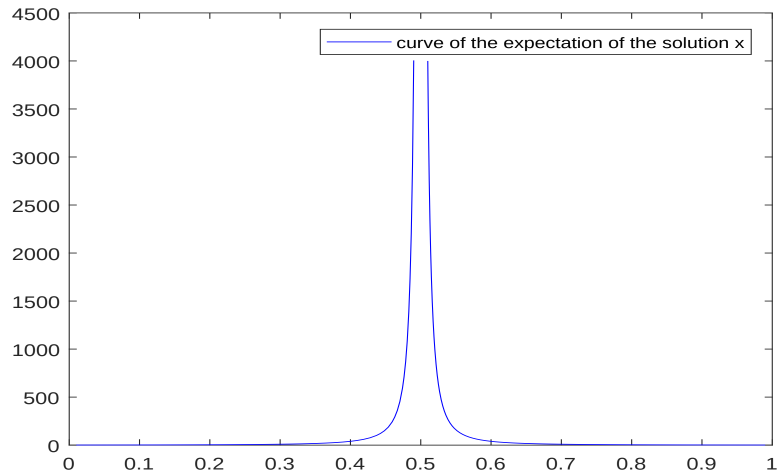

Figure 1.1: Expectation of the extremizer to the stochastic fractional problem of the calculus of variations related to the current example.

Following [15], we observe that ${}_a^sD_t^\alpha X(t)$ and ${}_t^sD_b^\alpha X(t)$ can be approximated as follows:

$$ {}_a^sD_t^\alpha X(t) = {}_aD_t^\alpha E(X(t)) \simeq \sum_{k=0}^{N} \frac{(-1)^{(k-1)}\alpha(E(X(t)))^{(k)}}{k!(k-\alpha)\Gamma(1-\alpha)}(t-a)^{(k-\alpha)} $$

and

$$ {}_t^sD_b^\alpha X(t) = {}_tD_b^\alpha E(X(t)) \simeq \sum_{k=0}^{N} \frac{-\alpha(E(X(t)))^{(k)}}{k!(k-\alpha)\Gamma(1-\alpha)}(b-t)^{(k-\alpha)}. $$

Choosing $N = 1$, we get the curve for $E(X(t))$ as shown in Figure 1.1.

One can increase the value of N under the condition one adds a sufficient number of initial values related to some degrees of derivatives of $E(X(t))$. This particular question is similar to the standard fractional calculus and we refer the interested reader to the book [15].

Chapter 2

Deterministic approach: Mathematical prerequisites and original results

2.1 Mathematical prerequisites

2.1.1 The next generation matrix method

Consider a heterogeneous population whose individuals are distinguishable by age, behaviour, spatial position and/or stage of disease, but can be grouped into n homogeneous compartments. A general epidemic model for such a population is developed in this section. Let $x = (x_1; ...; x_n)'$, with each x_i, be the number of individuals in each compartment. For clarity we sort the compartments so that the first m compartments correspond to infected individuals. The distinction between infected and uninfected compartments must be determined from the epidemiological interpretation of the model and cannot be deduced from the structure of the equations alone, as we shall discuss below. It is plausible that more than one interpretation is possible for some models.
The basic reproduction number can not be determined from the structure of the mathematical model alone, but depends on the definition of infected and uninfected compartments. We define X_s to be the set of all disease free states, that is,

$$X_s = \{x \geq 0 | x_i = 0, i = 1, \ldots, m\}.$$

In order to compute the basic reproduction number R_0, it is important to distinguish new infections from all other changes in population. Let $\mathbb{F}_i(x)$ be the rate of appearance of new infections in compartment i,$\mathbb{V}_i^+(x)$ be the rate of transfer of individuals into compartment i by all other means, and $\mathbb{V}_i^-(x)$ be the rate of transfer of individuals out of compartment i. It is assumed that each function is continuously differentiable at least twice in each variable. The disease transmission model consists of nonnegative initial conditions together with the following system of equations:

$$\dot{x}_i = f_i(x) = \mathbb{F}_i(x) - \mathbb{V}_i(x) \tag{2.1}$$

where $\mathbb{V}_i(x) = \mathbb{V}_i^+(x) - \mathbb{V}_i^-(x)$ and the functions satisfy assumptions (A_1)–(A_5) described below. Since each function represents a directed transfer of individuals, they are all nonnegative. Thus,

(A_1) if $x \geq 0$ then $\mathbb{F}_i(x), \mathbb{V}_i^+(x), \mathbb{V}_i^-(x) \geq 0$ for $i = 1, 2, ..., n$.
(A_2) if $x_i = 0$ then $\mathbb{V}_i^-(x) = 0$, in particular, if $x \in X_s$ then $\mathbb{V}_i^-(x) = 0$, for $i = 1, 2..., m$.
(A_3) $\mathbb{F}_i = 0$ if $i > m$.
(A_4) if $x \in X_s$ then $\mathbb{F}_i = 0$ and $\mathbb{V}_i^+(x) = 0$ for $i = 1, ..., m$.
(A_5) If $\mathbb{F}(x)$ is set to zero, then all eigenvalues of $Df(x_0)$ have negative real parts.
The conditions listed above allow us to partition the matrix $Df(x_0)$ as shown by the following lemma.

Lemma 2.1. *If x_0 is a Disease Free Equilibrium (DFE) of* (2.1) *and $f_i(x)$ satisfies* $(A_1)-(A_5)$, *then the derivatives $D\mathbb{F}(x_0)$ and $D\mathbb{V}(x_0)$ are partitioned as*

$$D\mathbb{F}(x_0) = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$$

$$D\mathbb{V}(x_0) = \begin{pmatrix} V & 0 \\ J_3 & J_4 \end{pmatrix}$$

where F and V are the $m \times m$ matrices defined by

$$F = \left[\frac{\partial \mathbb{F}_i}{\partial x_j}(x_0)\right]; \quad V = \left[\frac{\partial \mathbb{V}_i}{\partial x_j}(x_0)\right]$$

for $i \geq 1$, $j \leq m$. Further, F is non-negative, V is a non-singular M-matrix and all eigenvalues of J_4 have positive real part.

The basic reproduction number, denoted R_0, is "the expected number of secondary cases produced, in a completely susceptible population, by a typical infective individual". If $R_0 < 1$, then on average an infected individual produces less than one new infected individual over the course of its infectious period, and the infection cannot grow. Conversely, if $R_0 > 1$, then each infected individual produces, on average, more than one new infection, and the disease can invade the population. For the case of a single infected compartment, R_0 is simply the product of the infection rate and the mean duration of the infection. However, for more complicated models with several infected compartments this simple heuristic definition of R_0 is insufficient. A more general basic reproduction number can be defined as the number of new infections produced by a typical infective individual in a population at a DFE.
Following Diekmann et al.[48], we call FV^{-1} the next generation matrix for the model and set

$$R_0 = \rho(FV^{-1}) \tag{2.2}$$

where $\rho(A)$ denotes the spectral radius of a matrix A.

Theorem 2.2. *Consider the disease transmission model given by* (2.1) *with $f(x)$ satisfying conditions* $(A1) - (A5)$. *If x_0 is a DFE of the model, then x_0 is locally asymptotically stable if $R_0 < 1$, but unstable if $R_0 > 1$, where R_0 is defined by* (2.2).

2.1.2 Least square procedure to estimate infection rate

Let $y(t), t = 0, 1, \ldots, 45$ be the number of daily reported cases. We perform the following $least - square - based \quad procedure$ with Poisson noise to estimate the infection rate β.
`Description of the procedure.`

(P_1) Fix $\beta > 0$ and calculate the numerical value of $Y(t), t = 0, 1, ..., 45$.
(P_2) Calculate $\tilde{Y}(t) = Y(t) + \sqrt{Y(t)}e(t) = Y(t) + (Poissonnoise), t = 0, 1, ..., 45$, where $e(t), t = 0, 1, ..., 45$ denote random variables from a normal distribution with mean zero and variance 1.
(P_3) Calculate $J(\beta) = \sum_{t=0}^{45} \left[y(t) - \tilde{Y}(t)\right]^2$.
(P_4) Run $(P1) - (P3)$ for $0.2 \leq \beta \leq 0.4$ and find β^* such that $J(\beta^*) = min_{0.2\leq\beta\leq0.4} J(\beta)$.
(P_5) Repeat $(P1) - (P4)$ 10000 times and obtain the distribution of β^*.
(P_6) Approximate the distribution of β^* by a normal distribution and obtain a 95% confidence interval.

2.1.3 Existence and uniqueness's theorem within semi-group theory

Consider the initial value problem:

$$\begin{cases} \dfrac{\partial y}{\partial t} = Ay(t) + g(t, y(t)), & t \in [0, T] \\ & y(0) = y_0 \end{cases} \tag{2.3}$$

where A is a linear operator defined on a Banach space$X$, with the domain $D(A)$ and $g : [0, T] \times X \to X$ is a given function. If X is a Hilbert space endowed with the scalar product $(.,.)$, then the linear operator A is called dissipative if $(Ay, y) \geq 0, \quad (y \in D(A))$.

Theorem 2.3. *X be a real Banach space, $A : D(A) \subseteq X \longrightarrow X$ be the infinitesimal generator of a $C_0 -$ semigroup of linear contractions $S(t), t \geq 0$ on X, and $g : [0, T] \times X \to X$ be a measurable function in t and Lipschitz continuous in $x \in X$, uniformly with respect to $t \in [0, T]$.*
(i) If $y_0 \in X$, then problem (2.3) *admits a unique mild solution, i.e.a function $y \in C([0, T]; X)$ which verifies the equality*

$$y(t) = S(t)y_0 + \int_0^t S(t-s)g(s, y(s))ds, \quad t \in [0, T].$$

(ii) If X is a Hilbert space,A is $self-adjoint$ and dissipative on X and $y_0 \in D(A)$, then the mild solution is in fact a strong solution and

$$y \in W^{1,2}([0, T]; X) \cap L^2(0; T; D(A)).$$

2.2 Modeling the spread of COVID-19 pandemic in Morocco

The original results of this section are published in [162].

2.2.1 Introduction

Coronavirus disease 2019 (COVID-19) is an infectious disease that appeared in China at the end of 2019. It is caused by a new type of virus belonging to the coronaviruses family and recently named *severe acute respiratory syndrome coronavirus 2* (SARS-CoV-2) [109]. On March 11, 2020, COVID-19 was reclassified as a pandemic by the World Health Organization (WHO). The disease spreads rapidly from country to country, causing enormous economic damage and many deaths worldwide. The first case of COVID-19 in Morocco was confirmed on March 2, 2020 in city of Casablanca. It involved a Moroccan expatriate residing in Italy and who came from Italy on February 27, 2020. As of April 17, 2020, the confirmed cases reached 2564 and the number of recoveries reached 281 with a total number of 135 deaths [107].

Moroccan authorities have implemented multiple preventive measures and strategies to control the spread of disease, such as the closing of borders, suspension of schools and universities, closing coffee shops, the shut-down of all mosques in the country, etc. Further, Morocco has declared a state of health emergency during the period from March 20 to April 20, 2020, to avoid the spread of COVID-19. During this period, movement during the day should be limited to work, shopping, medical care, purchasing medicine, medical supplies, and emergency situations only. In addition, and from April 6, 2020, the wearing of a mask became compulsory for all persons authorized to move.

Mathematical modeling of COVID-19 transmission has attracted the attention of many scientists. Tang et al. [135] used a Susceptible–Exposed–Infectious–Recovered (SEIR) compartmental model to estimate the basic reproduction number of COVID-19 transmission based on data obtained for the confirmed cases of the disease in mainland China. Wu et al. [150] provided an estimate of the size of the epidemic in Wuhan on the basis of the number of cases exported from Wuhan to cities outside mainland China by using a SEIR model. In [79], Kuniya applied the SEIR compartmental model for the prediction of the epidemic peak for COVID-19 in Japan by using the real-time data from January 15 to February 29, 2020. Fanelli and Piazza [55] analyzed and forecasted COVID-19 spreading in China, Italy and France by using a simple Susceptible–Infected–Recovered–Death (SIRD) model. The authors of [102] present a mathematical model and study the dynamics of COVID-19 that emerged recently in Wuhan, China. For a fractional (non-integer order) model see [74].

In the models cited above, the transmission of the disease was assumed to be instantaneous and therefore they are formulated by ordinary differential equations (ODEs), without time delays. In this study, we propose a mathematical model governed by delay differential equations (DDEs) to predict the epidemiological trend of COVID-19 in Morocco and taking into account multiple preventive measures and strategies implemented by Moroccan authorities, related to the confinement period between March 2 and June 20, 2020, in order to control the spread of disease. To do this, the formulation of the model is obtained, further, the parameters estimation and sensitivity analysis are handled, in addition, a forecast of COVID-19 spreading in Morocco is presented in the next part of study, We end lastly our work by an adequate discussion of the results.

2.2.2 Formulation of the model

Around the world, all the countries that are attacked by the COVID-19 have imposed several strategies, with different degrees, to fight against it, namely the reduction of some rights by adopting the quarantine method in order to prevent contacts between vulnerable and infected individuals, closing the geographical borders of the countries, and enforcing the capacity of the sanitary system. Similarly, the Kingdom of Morocco quickly followed all of the previous strategies when the pandemic was in its early stages.

Remark 2.4. *The terms "susceptibility" and "vulnerability" are often used interchangeably for populations with disproportionate health burdens [28]. The distinction between vulnerability and susceptibility marks the difference between being intact but fragile–vulnerable and being*

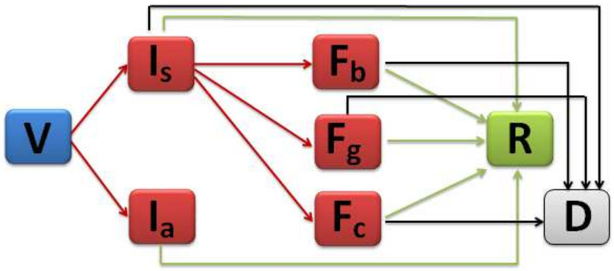

Figure 2.1: Schematic diagram of our extended model.

injured and predisposed to compound additional harm–susceptible [78]. Here, we refer to "the potential to contract the COVID-19" as vulnerability, to emphasize the environmental nature of the disease.

After the first reported positive case in Morocco, March 2, 2020, the closing of schools and universities is done at March 16, 2020; the state of health emergency (containment) is imposed to contain the outbreak from March 20, 2020; and the closure of the borders is performed at March 24, 2020. Additionally, the face mask is obligatory used in the general population at April 6, 2020. Based on these preventive measures and strategies, we model the dynamics of the transmission of COVID-19 in Morocco by extending the classical SIR model. Precisely, the population is divided into eight classes, denoted by V, I_s, I_a, F_b, F_g, F_c, R and D, where V represents the vulnerable sub-population, which is not infected and has not been infected before, but is susceptible to develop the disease if exposed to the virus; I_s is the symptomatic infected sub-population, which has not yet been treated, it transmits the disease, and outside of proper support it can progress to spontaneous recovery or death; I_a is the asymptomatic infected sub-population who is infected but does not transmit the disease, it is not known by the health system and progresses spontaneously to recovery; F_b, F_g and F_c are the patients diagnosed, supported by the Moroccan health system and under quarantine, and subdivided into three categories: benign, severe, and critical forms, respectively. Finally, R and D are the recovered and died classes. The schematic diagram of our extended model is illustrated in Figure 2.1. Therefore, the extended model can be governed by the following system of DDEs:

$$
\begin{cases}
\dfrac{dV(t)}{dt} &= -\beta(1-u)V(t)I_s(t),\\
\dfrac{dI_s(t)}{dt} &= \beta\epsilon(1-u)V(t-\tau_1)I_s(t-\tau_1) - (\mu_s+\eta_s+\alpha(\gamma_b+\gamma_g+\gamma_c))\,I_s(t),\\
\dfrac{dI_a(t)}{dt} &= \beta(1-\epsilon)(1-u)V(t-\tau_1)I_s(t-\tau_1) - \eta_a I_a(t),\\
\dfrac{dF_b(t)}{dt} &= \alpha\gamma_b I_s(t-\tau_2) - \big(\mu_b + r_b\big)F_b(t),\\
\dfrac{dF_g(t)}{dt} &= \alpha\gamma_g I_s(t-\tau_2) - \big(\mu_g + r_g\big)F_g(t),\\
\dfrac{dF_c(t)}{dt} &= \alpha\gamma_c I_s(t-\tau_2) - \big(\mu_c + r_c\big)F_c(t),\\
\dfrac{dR(t)}{dt} &= \eta_a I_a(t) + \eta_s I_s(t) + r_b F_b(t) + r_g F_g(t) + r_c F_c(t),\\
\dfrac{dD(t)}{dt} &= \mu_s I_s(t) + \mu_b F_b(t) + \mu_g F_g(t) + \mu_c F_c(t),
\end{cases} \tag{2.4}
$$

where u represents the level of control strategies on the vulnerable population. We adopt the bilinear incidence rate to describe the infection of the disease and use parameter β to denote the transmission rate. It is reasonable to assume that the infected individuals are subdivided into individuals with symptoms and others without symptoms, for which we employ the parameter ϵ to denote the proportion for the symptomatic individuals and $1-\epsilon$ for the asymptomatic ones. The parameter α measures the efficiency of public health administration for hospitalization. Diagnosed symptomatic infected population moves to the three forms: benign, severe and critical, by the rates γ_b, γ_g and γ_c, respectively. The mean recovery period of these forms are denoted by $1/r_b$, $1/r_g$ and $1/r_c$, respectively. The later forms die also with the rates μ_b, μ_g and μ_c, respectively. Symptomatic infected population, which is not diagnosed, moves to the recovery compartment with a rate η_s or dies with a rate μ_s. On the other hand, asymptomatic infected population moves to the recovery compartment with a rate η_a. The times delay τ_1 and τ_2 denote the incubation period and the period time needed before hospitalization, respectively.

For biological reasons, we assume that the initial conditions of system (2.4) satisfy:

$$
\begin{array}{llll}
V(\theta) & = \phi_1(\theta) \geq 0, & I_s(\theta) = \phi_2(\theta) \geq 0, & I_a(\theta) = \phi_3(\theta) \geq 0,\\
F_b(\theta) & = \phi_4(\theta) \geq 0, & F_g(\theta) = \phi_5(\theta) \geq 0, & F_c(\theta) = \phi_6(\theta) \geq 0,\\
R(\theta) & = \phi_7(\theta) \geq 0, & D(\theta) = \phi_8(\theta) \geq 0, & \theta \in [-\tau, 0],
\end{array}
$$

where $\tau = \max\{\tau_1, \tau_2\}$. Let $C = C([-\tau, 0], \mathbb{R}^8)$ be the Banach space of continuous functions from the interval $[-\tau, 0]$ into $\mathbb{R}^8$, equipped with the uniform topology. It follows from the theory of functional differential equations [64] that system (2.4) with initial conditions $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8) \in C$ has a unique solution.

On the other hand, the basic reproduction number is an important threshold parameter that determines the spread of infection when the disease is introduced into the population [48]. This number is defined as the expected number of secondary cases produced, in a completely susceptible population, by a typical infective individual. By using the next generation matrix approach [141], the basic reproduction number $\mathscr{R}_0$ of system (2.4) is given by

$$
\mathscr{R}_0 = \rho(FV^{-1}) = \frac{\beta\epsilon(1-u)}{\eta_s + \mu_s + \alpha(\gamma_b+\gamma_g+\gamma_c)}, \tag{2.5}
$$

where ρ is the spectral radius of the next generation matrix FV^{-1} with

$$F = \begin{pmatrix} \beta\epsilon(1-u) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} \eta_s + \mu_s + \alpha(\gamma_b + \gamma_g + \gamma_c) & 0 \\ 0 & \eta_a \end{pmatrix}.$$

2.2.3 Parameter estimation and sensitivity analysis

Based on the daily published Moroccan data [96], we estimate the values of some parameters of the model. The proportion of asymptomatic forms can vary from 20.6% of infected population to 39.9% [97]. Then, $\epsilon \in [0.61, 0.794]$. The progression rates γ_b, γ_g and γ_c from symptomatic infected individuals to the three forms are assumed to be 80% of diagnosed cases for benign form, 15% of diagnosed cases for severe form, and 5% of diagnosed cases for critical form, respectively [148]. The true mortality of COVID-19 will take some time to be fully understood. The data we have so far indicate that the crude mortality ratio (the number of reported deaths divided by the reported cases) is between 3 and 4% [148]. As the Moroccan health system is not overloaded at the moment, it is assumed that deaths mainly come from critical cases with a percentage of 40% for an average period of 13.5 days [148]. Since the mortality rate of symptomatic individuals differs from country to country [55], we assume that 1% of symptomatic individuals die for an average period of 21 days, whereas the recovery rate for asymptomatic cases is 100% and is the same for severe and benign forms if a proper medical care is taken with an average period of 21 days. We employ a least-square procedure with Poisson noise as in [79] to estimate the transmission rate. The incubation period is estimated to be 5.5 days [26] while the time needed before hospitalization is estimated to be 7.5 days [73, 144]. The estimation of the above parameters is given in the table 2.2.

Sensitivity analysis is commonly used to determine the robustness of model predictions to some parameter values. It is used to discover parameters that have a high impact on $\mathscr{R}_0$ and should be targeted by intervention strategies. The main objective of this section is to examine the sensitivity of the basic reproduction number $\mathscr{R}_0$ with respect to model parameters by the so-called *sensitivity index*. The normalized forward sensitivity index of a variable ν, that depends differentially on a parameter ρ, is defined as

$$\Upsilon_\rho^\nu := \frac{\partial \nu}{\partial \rho} \times \frac{\rho}{\nu}.$$

According to the above definition, we derive the normalized forward sensitivity index of $\mathscr{R}_0$ with respect to β, ϵ, η_s, μ_s, γ_b, γ_g, γ_c, and α, which is summarized in Table 2.2.

As we observe in Table 2.2, the most sensitive parameters, which have a higher impact on $\mathscr{R}_0$, are β and ϵ, since $\Upsilon_\beta^{\mathscr{R}_0}$ and $\Upsilon_\epsilon^{\mathscr{R}_0}$ are independent of any parameter of system (2.4) with $\Upsilon_\beta^{\mathscr{R}_0} = \Upsilon_\epsilon^{\mathscr{R}_0} = +1$. In addition, the parameter α has a middle negative impact on $\mathscr{R}_0$, while $\mathscr{R}_0$ is slightly impacted by the rest of the parameters.

2.2.4 Prevision of COVID-19 in Morocco

In this section, we present the forecasts of COVID-19 in Morocco relating with different preventive measures and strategies implemented by Moroccan authorities on the confinement

Table 2.1: Parameter values for our model (2.4).

Parameter	Value	Source
β	$0.4517\ (95\%CI, 0.4484-0.455)$	Estimated
u	$0-1$	Varied
ϵ	0.794	[97]
γ_b	0.8	[148]
γ_g	0.15	[148]
γ_c	0.05	[148]
α	0.06	Assumed
η_a	1/21	Calculated
η_s	0.8/21	Calculated
μ_s	0.01/21	Calculated
μ_b	0	Assumed
μ_g	0	Assumed
μ_c	0.4/13.5	Calculated
r_b	1/13.5	Calculated
r_g	1/13.5	Calculated
r_c	0.6/13.5	Calculated
τ_1	5.5	[26]
τ_2	7.5	[73, 144]

period between March 2 and June 20, 2020. Then the parameter u can be defined as follows:

$$u = \begin{cases} u_1, & \text{on (March 2, March 10];} \\ u_2, & \text{on (March 10, March 20];} \\ u_3, & \text{on (March 20, April 6];} \\ u_4, & \text{after April 6,} \end{cases}$$

where $u_i \in (0,1]$, $i = 1,2,3,4$, measures the effectiveness of applying the multiple preventive interventions imposed by Moroccan authorities presented in Table 2.3.

To make a better illustration of the different strategies, we test the four decisions made at the government level in Figure 2.2.

We see in Figure 2.2 the evolution of the number of diagnosed infected positive individuals with different sets of measures: low, middle, high, and strict interventions. Up to April 15, the curves corresponding to the first three sets of measures increase exponentially, while the curve corresponding to the fourth set of measures has lost its initial exponential character and tends to flatten over time. In addition, the last daily reported cases in Morocco from March 2 to April 17, confirm the biological tendency of our model. Thus, our model is efficient to describe the spread of COVID-19 in Morocco. However, we note that some clinical data is a little far from the values of the model due to certain foci that appeared in some large areas or at the level of certain industrial areas.

Next, we give the graphical results related to delays parameters to prove their biological importance.

Table 2.2: The normalized forward sensitivity index of $\mathscr{R}_0$.

Parameters	Sensitivity index of $\mathscr{R}_0$	Value
β	$\Upsilon_{\beta}^{R_0} = +1$	+1
ϵ	$\Upsilon_{\epsilon}^{R_0} = +1$	+1
η_s	$\Upsilon_{\eta_s}^{R_0} = -\dfrac{\eta_s}{\eta_s + \mu_s + (\gamma_b + \gamma_g + \gamma_c)\alpha}$	-0.3864
μ_s	$\Upsilon_{\mu_s}^{R_0} = -\dfrac{\mu_s}{\eta_s + \mu_s + (\gamma_b + \gamma_g + \gamma_c)\alpha}$	-0.0048
γ_b	$\Upsilon_{\gamma_b}^{R_0} = -\dfrac{\alpha\gamma_b}{\eta_s + \mu_s + (\gamma_b + \gamma_g + \gamma_c)\alpha}$	-0.487
γ_g	$\Upsilon_{\gamma_g}^{R_0} = -\dfrac{\alpha\gamma_g}{\eta_s + \mu_s + (\gamma_b + \gamma_g + \gamma_c)\alpha}$	-0.0913
γ_c	$\Upsilon_{\gamma_c}^{R_0} = -\dfrac{\alpha\gamma_c}{\eta_s + \mu_s + (\gamma_b + \gamma_g + \gamma_c)\alpha}$	-0.0304
α	$\Upsilon_{\alpha}^{R_0} = -\dfrac{\alpha(\gamma_b + \gamma_g + \gamma_c)}{\eta_s + \mu_s + (\gamma_b + \gamma_g + \gamma_c)\alpha}$	-0.687

Table 2.3: Summary of non-pharmaceutical interventions considered.

Policies	**Control values**
Without any intervention measures	$u = 0$, after March 2
First set of measures	$u = 0.2$, after March 2
Second set of measures	$u = 0.2$, on (March 2, March 10] $u = 0.3$, after March 10
Third set of measures	$u = 0.2$, on (March 2, March 10], $u = 0.3$, on (March 10, March 16] $u = 0.4$, after March 16
Fourth set of measures	$u = 0.2$, on (March 2, March 10], $u = 0.3$, on (March 10, March 16] $u = 0.4$, on (March 16, April 6] $u = 0.8$, after April 6,

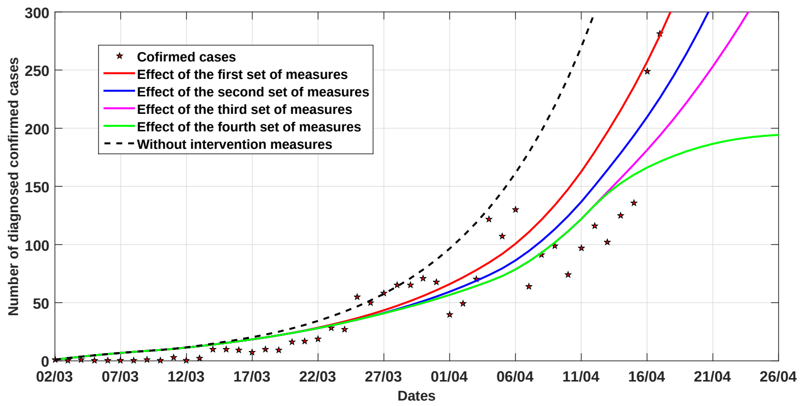

Figure 2.2: Comparison of the non-pharmaceutical interventions considered and the daily reported cases of COVID-19 in Morocco from March 2 to April 17, 2020.

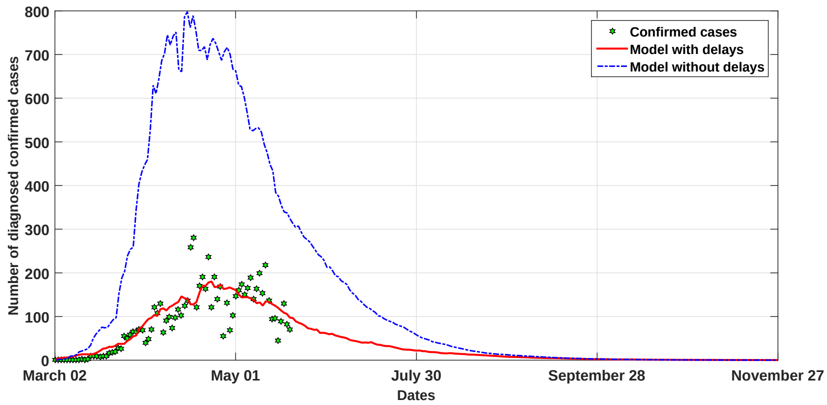

Figure 2.3: Effect of delays on the diagnosed confirmed cases.

We observe in Figure 2.3 a highly impact of delays on the number of diagnosed positive cases, thereby the plot of model (2.4) without delays ($\tau_1 = \tau_2 = 0$) is very far from the clinical data.

2.2.5 Peak prediction

Now, we indicate the predicted relative impact of the model and especially the diagnosed infective individuals with and without interventions applied progressively in Morocco.

Before finding the first positive infected case in Morocco, the authorities have begun with a suspension of international air lines to and from China, and installed health control check-points at the borders but without any interventions into the Moroccan population. For this, we simulate model (2.4) in the case $u = 0$, which is illustrated by Figures 2.4 and 2.5.

We remark from Figure 2.4 that the estimated epidemic peak is $t^* = 142$ ($95\% CI, 141 - 143$), that is, starting from March 2, 2020 ($t = 0$), the estimated epidemic peak is July 21,

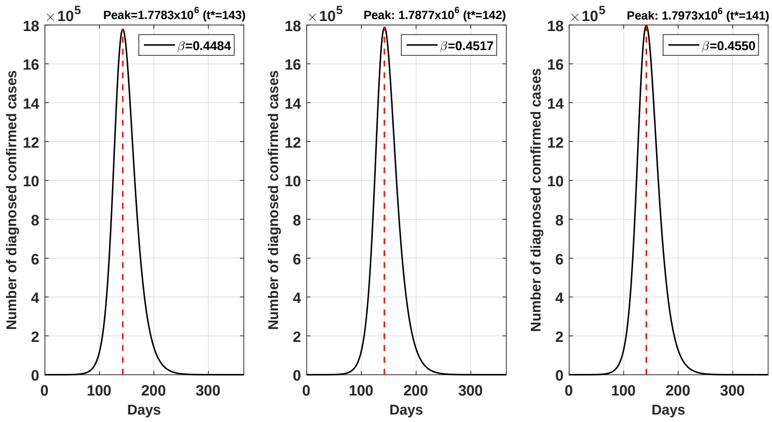

Figure 2.4: Time variation of the diagnosed infective individuals without any intervention on the Moroccan population with different values of β $(95\%CI, 0.4484 - 0.455)$.

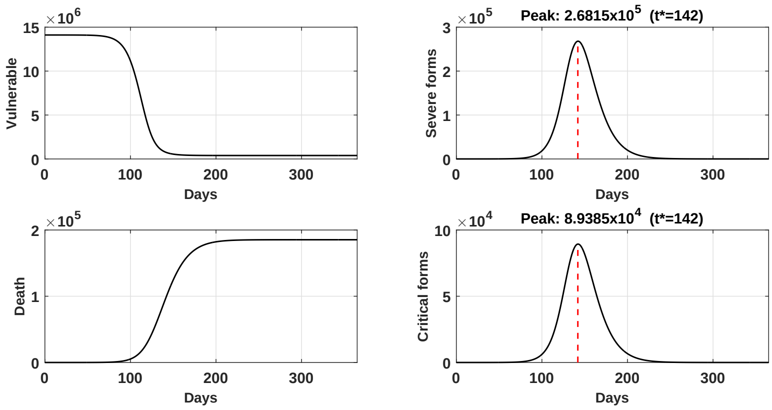

Figure 2.5: Time variation of the model with $\beta = 0.4517$ and $\mathscr{R}_0 = 3.6385$.

2020 ($t = 142$).

In the absence of any government intervention, the disease persists strongly and almost all of the vulnerable population will be reached by the infection (Figure 2.5).

After the first imported positive infected case, Moroccan authorities began to establish some preventive interventions between the 2 and the 10th of March, namely isolation of positive cases, contact tracing, hygiene measures, prevention measures in workplaces, and ban of mass gathering events. For this reason, we have selected in this period $u = 0.2$. From March 10 up to March 20, 2020, additional preventive measures were established: gradual suspension of all international sea, ground, and air lines (including with Spain, Italia, Algeria, France, Germany, Netherlands, Belgium, and Portugal), closure of coffees, restaurants, cinemas, theaters, party rooms, clubs, sport centers, hammams, game rooms and sport fields, closure of mosques, schools and universities, disinfection of public transportation means, reduction of the carrying capacity of taxis, buses and tramways, movement/travel restrictions, and containment measures of the general population. These measures correspond to the choice of the

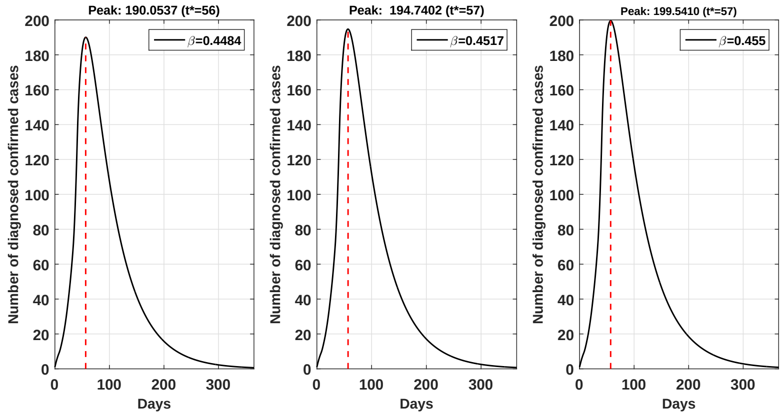

Figure 2.6: Time variation of the diagnosed infective individuals with hight level respect of measures for different values of β $(95\%CI, 0.4484 - 0.455)$.

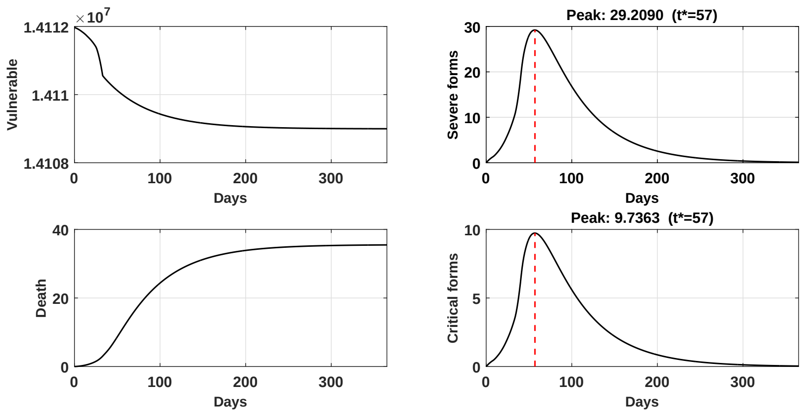

Figure 2.7: Time variation of the model with $\beta = 0.4517$ and $\mathscr{R}_0 = 2.9108$ (March 2–10), $\mathscr{R}_0 = 2.5469$ (March 10–20), $\mathscr{R}_0 = 2.1831$ (March 20–April 6), $\mathscr{R}_0 = 0.7277$ (from April 6, 2020).

control $u = 0.3$. From March 20 up to April 6, the Moroccan authority declared a state of emergency with a complete lockdown, night-time curfew, movement restrictions 24/24, ban of human movements between cities, suspension of railway lines, streets disinfection, and extensive cleaning and disinfection of port and airport facilities. For this, we assume that $u = 0.4$. From April 6, the authority decided compulsory wearing of masks in public spaces, which implies a significant positive influence on the above interventions and an increase of their efficiency level. In this case, we assume that $u = 0.8$. Tacking into account all these policies, we present the following Figures 2.6 and 2.7.

We remark from Figure 2.6 that the estimated epidemic peak is $t^* = 57$ $(95\%CI, 56 - 57)$, that is, starting from March 2 $(t = 0)$, the estimated epidemic peak is April 28, 2020 $(t = 57)$.

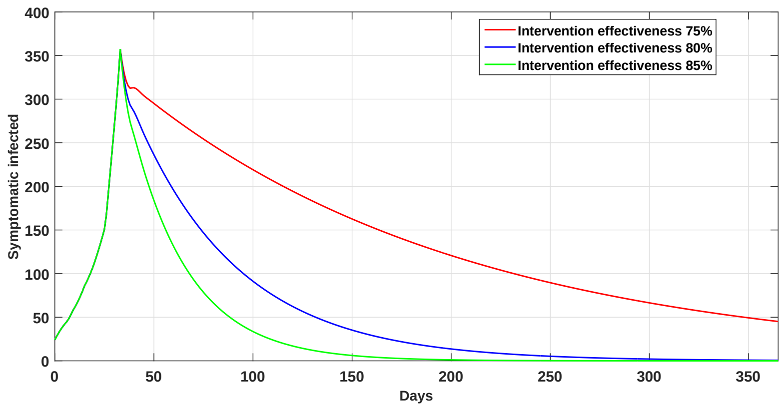

Figure 2.8: Evolution of the symptomatic individuals with different effectiveness degrees.

From Figure 2.7, we see that all the measures taken into this second strategy have a significant impact on the number of new positive diagnosed cases per day. Compared to Figure 2.5, the time required to reach the peak is reduced by 85 days, avoiding globally an interesting number of new infections and new deaths. Furthermore, the computed basic reproduction number $\mathscr{R}_0$ is less than 1, which means the extinction of the disease if the measures cited above are strictly implemented.

2.2.6 Intervention effectiveness

Here, on one hand, we compare the impact of different degrees of effectiveness on the evolution of the number of positive infected diagnosed individuals, symptomatic individuals, and deaths (see Figures 2.8 and 2.9). In addition, we present the cumulative cases in Figure 2.10 and we summarize it in Table 2.4. We remark that the effectiveness of the policies plays an important role to reduce, or not, the human damage and ensure the eradication of the illness. However, mitigation measures must be strictly respected to maintain a good level of control over the spread of the virus.

On the other hand, we are carrying out a statistical study on a national scale and we note that the trend at the beginning was exponential and will undergo a break due to the multiple interventions of the government, which is globally a good sign (see Figure 2.11), whereas it is needful to pay attention at the evolution of the curves in the different regions in Morocco. Since the clinical data of COVID-19 was not available on a daily basis at the start of the spread of the epidemic in Morocco, we proceeded with a choice of unit of three days. We also remark that almost all the regions have a homogeneous tendency with the national one, except Tangier–Tetouan–Al Hoceima (TTA), Oriental, Marrakech–Safi (MS), and Casablanca–Settat (CS), which show a mitigation of the epidemic that does not seem very stable (see Figures 2.12 and 2.13).

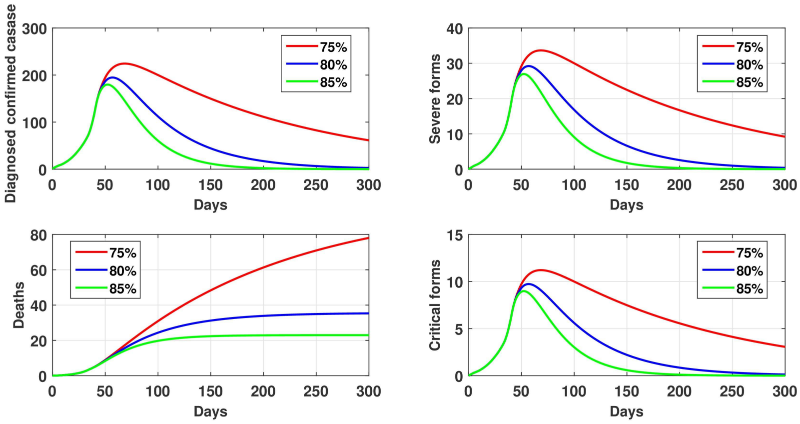

Figure 2.9: Evolution of the positive infected diagnosed individuals and deaths with different effectiveness degrees.

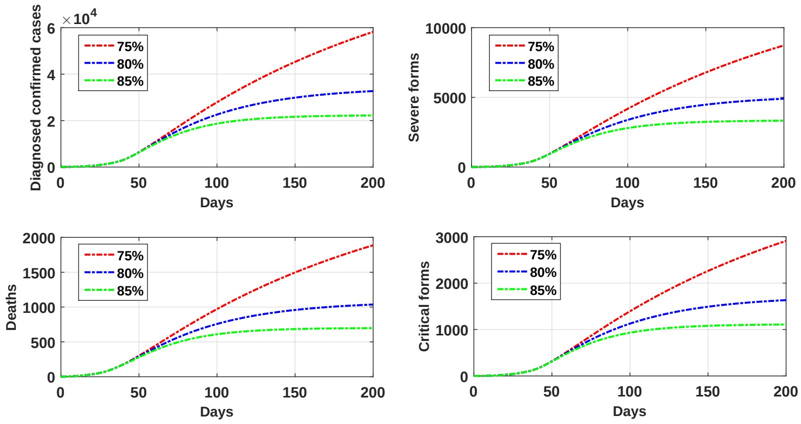

Figure 2.10: Cumulative diagnosed cases, severe forms, critical forms, and deaths, with different effectiveness degrees.

Table 2.4: Cumulative diagnosed cases, severe forms, critical forms, and deaths, after 150 days of the start of the pandemic in Morocco.

Effectiveness	75%	80%	85%
Diagnosed	42834	29116	21432
Severe forms	6419	4361	3209
Critical forms	2139	1453	1069
Deaths	1500	993	661

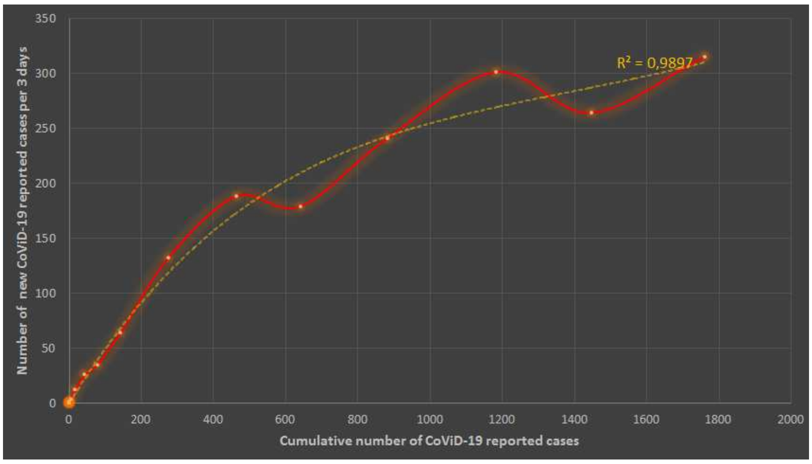

Figure 2.11: Trends in the number of new COVID-19 reported cases per three days in Morocco, compared to the cumulative number of COVID-19 reported cases with correlation coefficient $R^2 = 0.9897$.

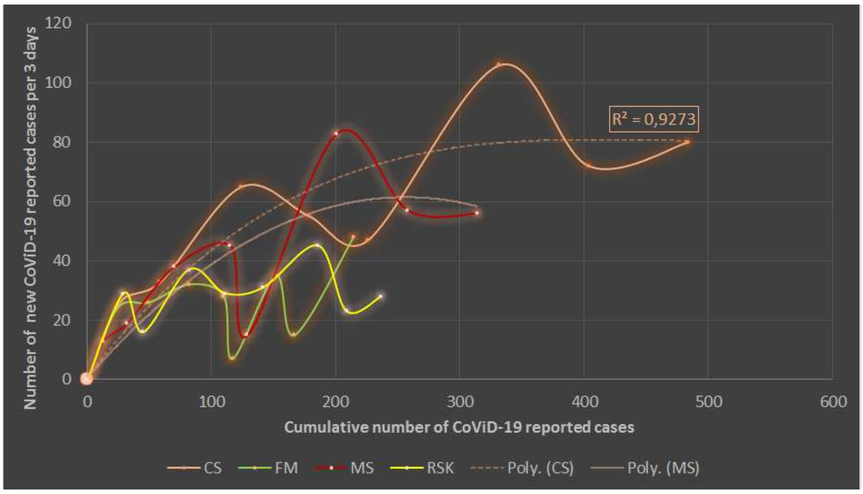

Figure 2.12: Trends in the number of new COVID-19 reported cases per three days in Morocco, by regions, compared to the cumulative number of COVID-19 reported cases (CS: Casablanca–Settat; FM: Fes–Meknes; MS: Marrakech–Safi; RSK: Rabat–Sale–Kenitra).

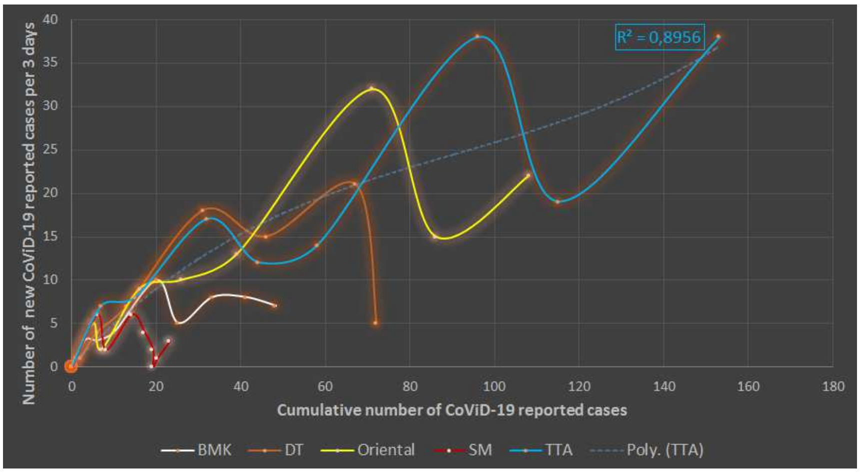

Figure 2.13: Trends in the number of new COVID-19 reported cases per three days in Morocco, by regions, compared to the cumulative number of COVID-19 reported cases (BMK: Beni Mellal–Khenifra; DT: Daraa–Tafilalet; SM: Souss–Massa; TTA: Tetouan–Tangier–Assillah).

2.3 Mathematical Analysis, Forecasting and Optimal Control of HIV-AIDS Spatiotemporal Transmission with a Reaction Diffusion SICA Model

This article is published in [161]

2.3.1 Introduction

The human immunodeficiency virus (HIV) is a serious disease causing death to humans worldwide, it is one of the most infectious and deadly infectious factors in the terrestrial globe. The deterministic SICA model was firstly introduced as a sub-model of a TB-HIV/AIDS co-infection model and published in 2015, (see [124]). After, it was extended to fractional (see [127]) and stochastic systems of differential equations (see [49]) and adjusted to the HIV/AIDS epidemic situation in Morocco.

One of the fundamental purposes of SICA models is to illustrate that an adequate mathematical model can help to specify some of the essential epidemiological factors leading to the spread of the AIDS disease .

The susceptible population is nourished by the recruitment of individuals into the population assumed, at a rate λ. All individuals is exposed to the natural death, at a constant rate μ. Susceptible individuals S receive HIV infection from an effective contact with individual infected carrying HIV, at the rate $\dfrac{\beta}{N(t,x)}\left(I(t,x)+\eta_C C(t,x)+\eta_A A(t,x)\right)$.

It is well known that reaction-diffusion equations are commonly used to model a variety of physical and biological phenomena. These equations describe how the concentration or density distributed in space varies under the influence of two processes: local interactions of species, and diffusion which causes the spread of species in space. Recently, reaction-diffusion equations have been used by many authors in epidemiology as well as virology, (see in instance [145])

We consider an optimal control to curb the mortality and reduce the number of the infected people in order to recuperate the normal course of life and suppress some of the psychological constraints. We proceed by defining and characterizing an optimal control which minimizes both, the number of infected people and the cost of treatment. Existence and uniqueness of positive solution for the system is proved. We also prove existence of the desired optimal control by means of the characterization of the infimum of the objective functional and we give an explicite necessary optimality condition in terms of state function and adjoint function. To illustrate the effectiveness of our theoretical results, we also presented some numerical simulations for several scenarios. The obtained results represent a good framework for interventions and strategies to fight against the transmission of the AIDS epidemic.

2.3.2 Physical interpretation of the Laplacian

The Laplacian in two dimensions is expressed by:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Suppose that at a point O, taken as the origin of the system of axises Oxy, the field f takes the value f_0. Consider an elementary square with side a, whose edges are parallel to the coordinate axises and whose center merges with the origin O. The average value of f in this elementary cube, in other words, *the mean value* of "f "in the neighborhood of the point O, is given by the expression

$$\overline{f} = \frac{1}{a^2} \int_{\mathscr{C}} f(x,y)\, \mathrm{d}x\mathrm{d}y,$$

where the two integrations each relate to the rectangle $C = [-\frac{a}{2}, \frac{a}{2}]^2$. At an arbitrary point $P(x,y)$ in the neighborhood of $O(0,0)$, we develop f in Taylor-Maclaurin series. Thus,

$$\begin{aligned} f(x,y) &= f_0 + \left(\tfrac{\partial f}{\partial x}\right)_0 x + \left(\tfrac{\partial f}{\partial y}\right)_0 y \\ &+\tfrac{1}{2}\left[\left(\tfrac{\partial^2 f}{\partial x^2}\right)_0 x^2 + \left(\tfrac{\partial^2 f}{\partial y^2}\right)_0 y^2\right] + \left(\tfrac{\partial^2 f}{\partial x \partial y}\right)_0 xy + o(x^2+y^2). \end{aligned}$$

On the one hand, the odd functions in this expression provide, by integration from $-\frac{a}{2}$ to $\frac{a}{2}$, a zero contribution to $\overline{f}$. For example,

$$\int_{\mathscr{C}} x\, \mathrm{d}x\mathrm{d}y = \left(\frac{\left(\frac{a}{2}\right)^2}{2} - \frac{\left(\frac{-a}{2}\right)^2}{2} \right)\left(\frac{a}{2} - \frac{-a}{2} \right) = 0.$$

On the other hand, the even functions each provide a contribution of $\frac{a^4}{12}$. For example,

$$\int_{\mathscr{C}} x^2\, \mathrm{d}x\mathrm{d}y = \left(\frac{\left(\frac{a}{2}\right)^3}{3} - \frac{\left(\frac{-a}{2}\right)^3}{3} \right)\left(\frac{a}{2} - \frac{-a}{2} \right) = \frac{a^4}{12}.$$

Using Fubini-Tonnelli's theorem, we get

$$\int_{\mathscr{C}} xy\, \mathrm{d}x\mathrm{d}y = 0.$$

We deduce that

$$\overline{f} \approx f_0 + \frac{a^4}{24}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)_0$$

and

$$\overline{f} \approx f_0 + \frac{a^4}{24}\left(\nabla^2 f\right)_0.$$

As the point O has been chosen arbitrarily, we can assimilate it to the current point P and drop the index 0. We therefore obtain the following expression, the interpretation of which is immediate:

$$\nabla^2 f \approx \frac{24}{a^4}\left(\overline{f} - f\right),$$

that is, the quantity $\nabla^2 f$ is approximately proportional to the difference $\overline{f} - f$. The constant of proportionality is worth $\frac{24}{a^4}$ in Cartesian axises. In other words, the quantity $\nabla^2 f$ is a measure of the difference between the value of f at any point P and the mean value $\overline{f}$ in the neighborhood of point P.

Remark 2.5. *The Laplacian of a function can also be interpreted as the local mean curvature of the function, which is easily visualized for a function f with only one variable. We can easily verify that the reasoning proposed here for the Laplacian applies to a function f and to its second derivative. The second derivative (or curvature) represents the local deviation of the mean from the value at the point considered.*

2.4 The spatiotemporal mathematical SICA model

In [125], Silva and Torres proposed the following epidemic SICA model:

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \beta\left(I(t) + \eta_C \cdot C(t) + \eta_A \cdot A(t)\right) \cdot S(t) - \mu S(t), \\ \dfrac{dI(t)}{dt} = \beta\left(I(t) + \eta_C \cdot C\left(t\right) + \eta_A \cdot A(t)\right) \cdot S(t) - \xi_3 I(t) + \gamma A(t) + \omega C(t), \\ \dfrac{dC(t)}{dt} = \phi I(t) - \xi_2 C\left(t\right), \\ \dfrac{dA(t)}{dt} = \rho I(t) - \xi_1 A\left(t\right), \end{cases}$$

The limitation of the temporal dynamic systems to give a good description of the spread of the virus in the space is obvious. To bridge this gap, we suggest the use of the Laplacian operator, so the previous interpretation is illustrated in the following extended deterministic epidemic SICA model:

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = d_S \Delta S(t,x) + \Lambda - \beta \left(I(t,x) + \eta_C \cdot C(t,x) + \eta_A \cdot A(t,x) \right) \cdot S(t,x) - \mu S(t,x) \\ \qquad\qquad + u(t,x) I(t,x), \\ \dfrac{\partial I(t,x)}{\partial t} = d_I \Delta I(t,x) + \beta \left(I(t,x) + \eta_C \cdot C(t,x) + \eta_A \cdot A(t,x) \right) \cdot S(t,x) \\ \qquad\qquad - \xi_3 I(t,x) + \gamma A(t,x) + \omega C(t,x) - u(t,x) I(t,x), \\ \dfrac{\partial C(t,x)}{\partial t} = d_C \Delta C(t,x) + \phi I(t,x) - \xi_2 C(t,x), \\ \dfrac{\partial A(t,x)}{\partial t} = d_A \Delta A(t,x) + \rho I(t,x) - \xi_1 A(t,x), \end{cases} \tag{2.6}$$

subject to the homogeneous Neumann boundary conditions

$$\frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial C}{\partial n} = \frac{\partial A}{\partial n} = 0$$

and initial conditions $S(0,x) = S_0$, $I(0,x) = I_0$, $C(0,x) = C_0$ and $A(0,x) = A_0$, where Δ is the Laplacian in the two-dimensional space (t,x) and $u : [0;T] \times \Omega \longrightarrow [0;1[$ is a considered control which permits to diminish the number of the infected individuals and to increase that of susceptible ones by devoting some special treatment to the most affected persons. The description of the parameters of model (2.6) is summarized in Table 2.5.

Table 2.5: Description of the parameters of the SICA model

Symbol	Description
Λ	Recruitment rate
μ	Natural death rate
β	HIV transmission rate
η_C	Modification parameter
η_A	Modification parameter
ϕ	HIV treatment rate for I individuals
ρ	Default treatment rate for I individuals
γ	AIDS treatment rate
ω	Default treatment rate for C individuals
d	AIDS induced death rate
d_S	Diffusion of susceptible individuals
d_I	Diffusion of infected individuals with no AIDS symptoms
d_C	Diffusion of chronic individuals
d_A	Diffusion of infected individuals with AIDS symptoms

In this stage, we have provided the current spatiotemporal SICA epidemic model.

2.5 Existence and uniqueness of a strong nonnegative solution

In order to prove existence and uniqueness of a strong solution to system (2.6), we define some tools. Consider the Hilbert spaces $H(\Omega) = (L_2(\Omega))^4$, $H^1(\Omega) = \{u \in L_2(\Omega) : \dfrac{\partial u}{\partial x} \in L_2(\Omega)$ and $\dfrac{\partial u}{\partial y} \in L_2(\Omega)\}$ and $H^2(\Omega) = \{u \in H^1(\Omega) : \dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial^2 u}{\partial y^2}, \dfrac{\partial^2 u}{\partial x \partial y}, \dfrac{\partial^2 u}{\partial y \partial x} \in L_2(\Omega)\}$

Let $L^2(0,T;H^2(\Omega))$ be the space of all strongly measurable functions $v : [0,T] \longmapsto H^2(\Omega)$ such that

$$\int_0^T \|v(t,x)\|_{H^2(\Omega)} \, \mathrm{d}t < \infty$$

and $L^\infty(0,T;H^1(\Omega))$ be the set of all functions $v : [0,T] \longmapsto H^1(\Omega)$ verifying

$$\sup_{t\in[0,T]} (\|v(t,x)\|_{H^1(\Omega)}) < \infty.$$

The norm in $L^\infty(0,T;H^1(\Omega))$ is defined by

$$\|v\|_{L^\infty(0,T;H^1(\Omega))} := \inf \left\{ c \in \mathbb{R}_+ : \|v(t,x)\|_{H^1(\Omega)} < c \right\}.$$

Our model is equivalent to

$$\frac{\partial z(t,x)}{dt} = Az(t,x) + g(t, z(t,x)), \quad z(0,x) = z_0, \tag{2.7}$$

where $z = (z_1, z_2, z_3, z_4) = (S, I, C, A)$ and $g = (g_1, g_2, g_3, g_4)$ is defined by

$$\begin{cases} g_1 = -\beta(z_2 + \eta_C z_3 + \eta_A z_1) z_1 - \mu z_1 + \Lambda + u z_2, \\ g_2 = \beta(z_2 + \eta_C z_3 + \eta_A z_1) z_1 - \xi_3 z_2 + \gamma z_4 + \omega z_3 - u z_2, \\ g_3 = \Phi z_2 - \xi_2 z_3, \\ g_4 = \rho z_2 - \xi_1 z_4. \end{cases}$$

For all $i \in \{1,2,3,4\}$,

$$\frac{\partial z_i}{\partial t} = d_i \Delta z_i + g_i(z(t,x)), \quad z_i(0,x) = z_{0_i}.$$

Let A denote the linear operator defined from $D(A) \subset H(\Omega)$ to $H(\Omega)$ by

$$Az = (d_S \triangle z_1, d_I \triangle z_2, d_C \triangle z_3, d_A \triangle z_4)$$

with

$$z \in D(A) = \left\{ z = (z_1, z_2, z_3, z_4) \in \left(H^2(\Omega)\right)^4 : \frac{\partial z_1}{\partial n} = \frac{\partial z_2}{\partial n} = \frac{\partial z_3}{\partial n} = \frac{\partial z_4}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \right\}$$

and U_{ad} be the admissible control set defined by

$$U_{ad} = \left\{ u \in L^2(Q), 0 \le u \le 1 \quad \text{a.e. on } Q \right\} \tag{2.8}$$

with $Q = [0,T] \times \Omega$ and Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$

Theorem 2.6. *[1] Let Ω be a bounded domain from $\mathbb{R}^2$ with a boundary of class $C^{2+\alpha}$, $\alpha > 0$. For nonnegative parameters of the spatiotemporal SICA model (2.6), $u \in U_{ad}$, $z^0 \in D(A)$ and $z_i^0 \geq 0$ on Ω, $i = 1, 2, 3, 4$, the system (2.6) has a unique (global) strong nonnegative solution $z \in W^{1,2}([0,T]; H(\Omega))$ such that*

$$z_1, z_2, z_3, z_4 \in L^2(0,T; H^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)) \cap L^\infty(Q).$$

Additionally, there exists $C > 0$, independent of u and of the corresponding solution z, such that for all $t \in [0,T]$ and all $i \in \{1, 2, 3, 4\}$ one has

$$\left\| \frac{\partial z_i}{\partial t} \right\|_{L^2(Q)} + \|z_i\|_{L^2(0,T,H^2(\Omega))} + \|z_i\|_{H^1(\Omega)} + \|z_i\|_{H^\infty(Q)} \leq C.$$

Proof. Because the Laplacian operator Δ is dissipating, self-adjoint, and generates a C_0- semigroup of contractions on $H(\Omega)$, it is clear that function $g = (g_1, g_2, g_3, g_4)$ becomes Lipschitz continuous in $z = (z_1, z_2, z_3, z_4)$ uniformly with respect to $t \in [0,T]$. Therefore, the problem admits a unique strong solution z. Let us now show that for all $i \in \{1, 2, 3, 4\}$, $z_i \in L^\infty(Q)$. Indeed, set $k = \max\left\{\|g_i\|_{L^\infty(Q)}, \|z_i^0\|_{L^\infty(\Omega)} \ : \ i \in \{1, 2, 3, 4\}\right\}$ and let

$$U_i(t,x) = z_i(t,x) - kt - \|z_i^0\|_{L^\infty(\Omega)}.$$

Then,

$$\begin{cases} \dfrac{\partial U_i(t,x)}{\partial t} = d_i \Delta U_i(t,x) + g_i(t, z(t,x)) - k, \quad t \in [0,T], \\ U_i(0,x,y) = z_i^0 - \|z_i^0\|_{L^\infty(\Omega)}. \end{cases}$$

Let $i \in \{1, 2, 3, 4\}$. There exists an infinitesimal semigroup $\Gamma(t)$ associated to the operator $d_i \Delta$ such that

$$U_i(t,x) = \Gamma(t) \left(z_i^0 - \|z_i^0\|_{L^\infty(\Omega)} \right) + \int_0^t \Gamma(t-s) \left(g_i(z(s)) - k \right) ds.$$

We deduce that $U_i(t,x) \leq 0$ and so $z_i \leq kt + \|z_i^0\|_{L^\infty(\Omega)}$.

Consider $V_i(t,x) = z_i(t,x) + kt + \|z_i^0\|_{L^\infty(\Omega)}$. Upon differentiation, we get

$$\begin{cases} \dfrac{\partial V_i(t,x)}{\partial t} = d_i \Delta V_i(t,x) + g_i(t, z(t,x)) + k, \quad t \in [0,T], \\ V_i(0,x,y) = z_i^0 + \|z_i^0\|_{L^\infty(\Omega)}. \end{cases}$$

The strong solution of the above equation is

$$V_i(t,x) = \Gamma(t) \left(z_i^0 + \|z_i^0\|_{L^\infty(\Omega)} \right) + \int_0^t \Gamma(t-s) \left(g_i(z(s)) + k \right) ds.$$

Then, $V_i(t,x) \geq 0$ and so $z_i \geq -kt - \|z_i^0\|_{L^\infty(\Omega)}$. Consequently, $|z_i(t,x,)| \leq kt + \|z_i^0\|_{L^\infty(\Omega)}$, which implies that $z_i \in L^\infty(Q)$.

Now, we proceed by proving that $z_i \in L^\infty\left(0,T; H^1(\Omega)\right)$ for all $i \in \{1, 2, 3, 4\}$. Indeed, let $i \in \{1, 2, 3, 4\}$. From equality

$$\frac{\partial z_i(t,x)}{\partial t} - d_i \Delta z_i(t,x) = g_i(t, z(t,x)) \quad (t,x) \in [0,T] \times \Omega$$

we obtain that

$$\int_0^t \int_\Omega \left(\frac{\partial z_i(s,x)}{\partial s} - d_i \Delta z_i(s,x) \right)^2 dxds = \int_0^t \int_\Omega \left(g_i(s, z(s,x)) \right)^2 dxds.$$

From Green's formula, we get

$$\int_0^t \int_\Omega \left(\frac{\partial z_i}{\partial s} \right)^2 dxds + d_i^2 \int_0^t \int_\Omega \left(\Delta z_i \right)^2 dxds = 2d_i \int_0^t \int_\Omega \frac{\partial z_i}{\partial s} \times \Delta z_i dxds$$
$$+ \int_0^t \int_\Omega \left(g_i(s, z_i) \right)^2 dxds = d_i \int_\Omega \left(z_i \right)^2 dx - d_i \int_\Omega \left(z_i^0 \right)^2 dx.$$

Since $g_i \in L^2(Q)$, $z_i^0 \in L^2(Q)$ and $z_i, z_i^0 \in L^\infty(Q)$, we obtain that $z_i \in L^\infty \left(0; T; H^1(\Omega)\right)$.

Finally, using the same arguments as for the Field–Noyes equations in [130, Example 4], we deduce that the solution (z_1, z_2, z_3, z_4) is nonnegative. Consider the set

$$\Sigma = \{(z_1, z_2, z_3, z_4) : 0 \leq z_i \leq C \text{ for } i \in \{1; 2; 3; 4\}\}$$

and the convex functions G_i defined on Σ by $G_i(z_1, z_2, z_3, z_4) = -z_i$. One can see that

$$\begin{aligned}
\nabla(G_1) \cdot g|_{z_1=0} &= \nabla(-z_1) \cdot g|_{z_1=0} = -\Lambda - u z_2 \leq 0, \\
\nabla(G_2) \cdot g|_{z_2=0} &= \nabla(-z_2) \cdot g|_{z_2=0} = -\beta \eta_C z_3 z_1 - \beta \eta_A z_4 z_1 - \gamma z_4 - \omega z_3 \leq 0, \\
\nabla(G_3) \cdot g|_{z_3=0} &= \nabla(-z_3) \cdot g|_{z_3=0} = -\phi z_1 - v_1 z_4 \leq 0, \\
\nabla(G_4) \cdot g|_{z_4=0} &= \nabla(-z_4) \cdot g|_{z_4=0} = -\rho z_2 \leq 0.
\end{aligned}$$

According to [130, Theorem 14.14], the region Σ is positively invariant.

□

In this achievement, we have employed a semi-group theory to demonstrate existence and uniqueness of the global nonnegative solution of the considered system.

2.6 Existence of an optimal control

To motivate the interest on optimal control, we begin by showing some numerical simulations of our spatiotemporal SICA model (2.6). Let us consider the following values for the parameters [125], the parameter d_S is assumed:

Table 2.6: Parameters values and units

Parameter	Value	Unit
μ	$\dfrac{1}{74.02}$	day^{-1}
Λ	2.19μ	day
β	0.755	$(people/km^2)^{-1} \times day^{-1}$
η_C	1.5	day^{-1}
η_A	0.2	day^{-1}
ϕ	1	day^{-1}
ρ	0.1	day^{-1}
γ	0.33	day^{-1}
ω	0.09	day^{-1}
d_S	0.9	km^2/day
d_I	0.1	km^2/day
d_C	0.1	km^2/day
d_A	0.1	km^2/day
ξ_1	$\gamma + \mu$	day^{-1}
ξ_2	$\omega + \mu$	day^{-1}
ξ_3	$\rho + \phi + \mu$	day^{-1}

Then, the dynamics without control, that is, with $u \equiv 0$ in (2.6), is given in Figure 2.14.

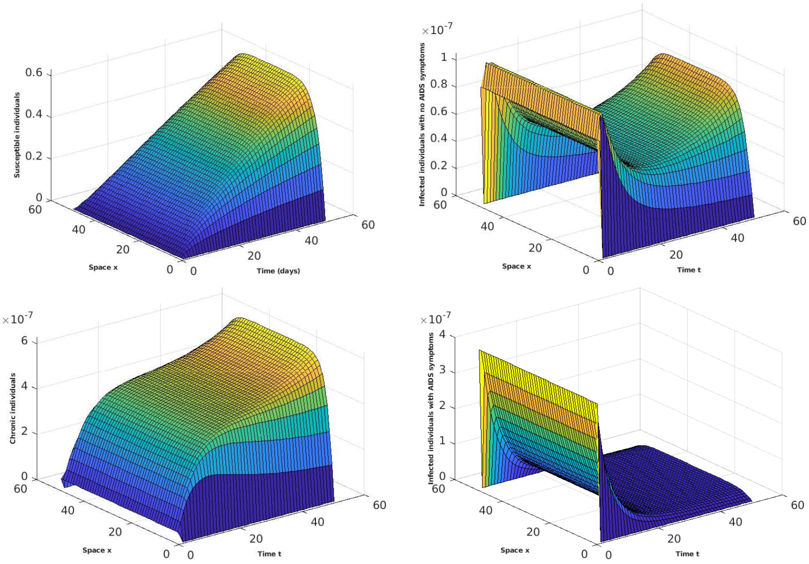

Figure 2.14: The behavior of the solution of the system (2.6) without control.

In contrast, dynamics in the presence of a control are given in Figures 2.15 and 2.16.

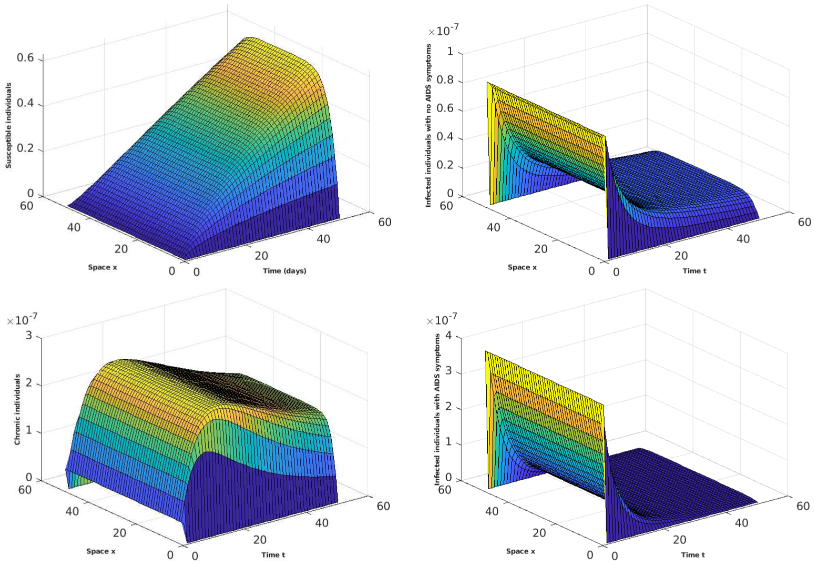

Figure 2.15: The behavior of the solution of the system (2.6) with the control $u \equiv 0.5$.

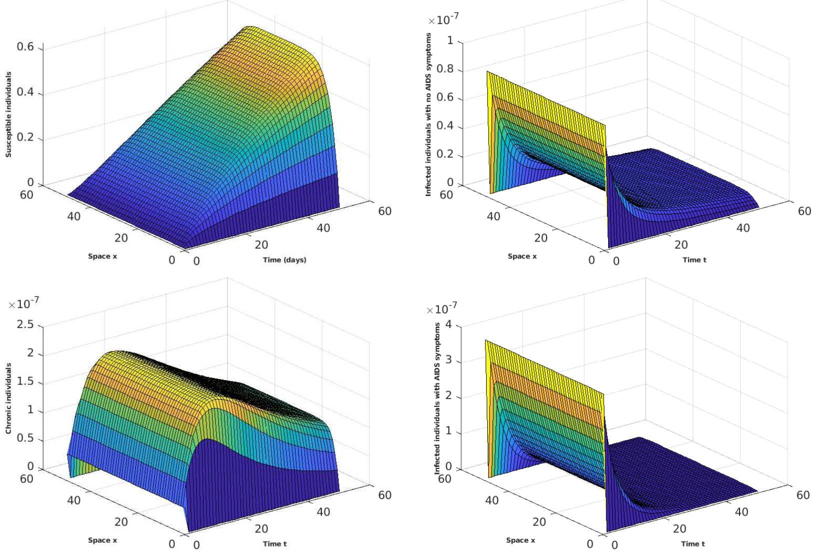

Figure 2.16: The behavior of the solution of the system (2.6) with the control $u \equiv 0.8$.

We conclude that the evolution of the system related with the absence of control differs totally to those related with the controls. Indeed, Figure 2.14 shows that in absence of the control the density of the infected individuals increases while in the presence of a control (Figures 2.15 and 2.16) it clearly decreases. The question of how to choose the control along time, in an optimal way, is therefore a natural one. Motivated by [126], our aim is to minimize the sum of the density of infected individuals and the cost of the treatment program. Mathematically, the problem we consider here is to minimize the objective functional

$$J(S,I,C,A,u)=\int_{\Omega}\int_{0}^{T} aI(t,x)dtdx+\frac{b}{2}\mid\mid u(t,x)\mid\mid^{2}_{L^{2}([0,T])} \tag{2.9}$$

subject to the control system (2.6) and where the admissible control set U_{ad} is defined as in (2.8).

Theorem 2.7. *Under the conditions of Theorem 2.6, our optimal control problem admits a solution* (z^*,u^*).

Proof. The proof is divided into three steps:

Step 1: The existence of a minimizing sequence (z^n,u_n):

The infimum of the objective function on the set of admissible controls is ensured by the positivity of J Assume that $J^*=\inf_{u\in U_{ad}} J(z,u)$. Let $\{u_n\}\subset U_{ad}$ be a minimizing sequence such that $\lim\limits_{n\to+\infty} J(z^n,u_n)=J^*$, where $(z_1^n,z_2^n,z_3^n,z_4^n)$ is the solution of the system corresponding to the control u_n. Subsequently,

$$\begin{cases}\frac{\partial z_1^n}{\partial t}=d_S\Delta z_1^n+\Lambda-\beta\left(z_2^n+\eta_C\cdot z_3^n+\eta_A\cdot z_4^n\right)z_1^n+u(t,x)\cdot z_2^n-\mu z_1^n,\\ \frac{\partial z_2^n}{\partial t}=d_I\Delta z_2^n+\beta\left(z_2^n+\eta_C z\cdot_3^n+\eta_A\cdot z_4^n\right)z_1^n-\xi_3 z_2^n+\gamma z_4^n+\omega z_3^n-u(t,x)\cdot z_2^n,\\ \frac{\partial z_3^n}{\partial t}=d_C\Delta z_3^n+\phi z_2^n-\xi_2 z_3^n,\\ \frac{\partial z_4^n}{\partial t}=d_A\Delta z_4^n+\rho z_2^n-\xi_1 z_4^n,\end{cases} \tag{2.10}$$

where $\frac{\partial z_1^n}{\partial n}=\frac{\partial z_2^n}{\partial n}=\frac{\partial z_3^n}{\partial n}=\frac{\partial z_4^n}{\partial n}=0$ on Q. Let $i\in\{1,2,3,4\}$.

Step 2: The convergence of the minimizing sequence (z^n,u_n) to (z^*,u^*):

Note that $z_i^n(t,x)$ is compact in $L^2(\Omega)$ from the fact that $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$. In order to apply the Ascoli–Arzela theorem, we need to demonstrate that $\{z_i^n(t,x),n\geq 1\}$ is equicontinuous in $C([0,T],L^2(\Omega))$. This is indeed true: because of the boundedness of $\frac{\partial z_i^n}{\partial t}$ in $L^2(Q)$, there exists a positive constant k such that

$$\left|\int_{\Omega}(z_i^n)^2(t,x)dx-\int_{\Omega}(z_i^n)^2(s,x)dx\right|\leq k\mid t-s\mid$$

for all $s,t\in[0,T]$. Hence, z_i^n is compact in $C([0,T],L^2(\Omega))$ and there exists a subsequence of $\{z_i^n\}$, denoted also $\{z_i^n\}$, converging uniformly to z_i^* in $L^2(\Omega)$ with respect to t. Since Δz_i^n is bounded in $L^2(Q)$, there exists a subsequence, denoted again Δz_i^n, converging weakly in $L^2(Q)$. For every distribution φ,

$$\int_Q\varphi\Delta z_i^n=\int_Q z_i^n\Delta\varphi\to\int_Q z_i^*\Delta\varphi=\int_Q\varphi\Delta z_i^*.$$

Thus, $\Delta z_i^n \rightharpoonup \Delta z_i^*$ in $L^2(Q)$. By the same argument,
$\dfrac{\partial z_i^n}{\partial t} \rightharpoonup \frac{\partial z_i^*}{\partial t}$ and $z_i^n \rightharpoonup z_i^*$ in $L^2(0,T;H^2(\Omega))$ and $z_i^n \rightharpoonup z_i^*$ in $L^\infty(0,T;H^1(\Omega))$. From $z_1^n z_2^n = (z_1^n - z_1^*)z_2^n + z_1^n(z_2^n - z_2^*)$, we deduce that $z_1^n z_2^n \rightharpoonup z_1^* z_2^*$ in $L^2(Q)$. Therefore, $u_n \rightharpoonup u^*$ in $L^2(Q)$. Since U_{ad} is closed, then $u^* \in U_{ad}$.

Step.3: We conclude that $u^n z_2^n \rightharpoonup u^* z_2^*$ in $L^2(Q)$. Letting $n \to \infty$ in (2.10), we obtain that z^* is a solution of equation (2.7) corresponding to u^*. Therefore,

$$\begin{aligned}
J(z^*, u^*) &= \int_0^T a z_2^*(t,x)dtdx + \frac{b}{2} \mid\mid u^*(t,x) \mid\mid^2_{L^2(Q])} \\
&\leq \liminf \int_0^T a z_2^n(t,x)dtdx + \frac{b}{2} \mid\mid u^n(t,x) \mid\mid^2_{L^2(Q)} \\
&\leq \lim \int_0^T a z_2^n(t,x)dtdx + \frac{b}{2} \mid\mid u^n(t,x) \mid\mid^2_{L^2(Q)} = J^*.
\end{aligned}$$

This shows that J attains its minimum at (z^*, u^*). □

The infimum of our objective function gives rise to the desired optimal control.

2.7 Necessary optimality conditions

Now we characterize the optimality that we proved to exist in Section 2.6. Let (z^*, u^*) be an optimal pair and $u^\epsilon = u^* + \epsilon u$, $\epsilon > 0$, be a control function such that $u \in L^2(Q)$ and $u \in U_{ad}$. We denote by $z^\epsilon = (z_1^\epsilon, z_2^\epsilon, z_3^\epsilon, z_4^\epsilon)$ and $z^* = (z_1^*, z_2^*, z_3^*, z_4^*)$ the corresponding trajectories associated with the controls u^ϵ and u^*, respectively.

In the following result we decompose the right-hand side of our control system into three quantities: M, related to the Laplacian part; R, linked to the control part; and F for the remaining terms.

Theorem 2.8. *For all $i \in \{1,2,3,4\}$, the mapping $u \longmapsto z_i(u)$ defined from U_{ad} to $W^{1,2}([0,T], H(\Omega))$ is Gateaux differentiable with respect to u^*. Moreover, for all $u \in U_{ad}$, set $z_i'(u^*)u = Z_i$. Then $Z = (Z_1, Z_2, Z_3, Z_4)$ is the unique solution of the problem*

$$\frac{\partial Z}{\partial t} = MZ + FZ + uR \quad \textit{subject to} \quad Z(0,x) = 0,$$

where

$$F = \begin{pmatrix} -\beta\left(z_2^* + \eta_C \cdot z_3^* + \eta_A \cdot z_4^*\right) - \mu & 0 & 0 & 0 \\ \beta\left(z_2^* + \eta_C \cdot z_3^* + \eta_A \cdot z_4^*\right) & -\xi_3 & \omega & \gamma \\ 0 & \phi & -\xi_2 & 0 \\ 0 & \rho & 0 & -\xi_1 \end{pmatrix} \quad \textit{and} \quad R = \begin{pmatrix} -z_2^* \\ z_2^* \\ 0 \\ 0 \end{pmatrix}.$$

Proof. Put $Z_i^\varepsilon = \frac{z_i^\varepsilon - z_i^*}{\varepsilon}$. By subtracting the two systems verified by z_i^ε and z_i^*, we get

$$\frac{\partial Z^\varepsilon}{\partial t} = MZ^\varepsilon + FZ^\varepsilon + uR \text{ subject to } Z^\varepsilon(0,x) = 0, \text{ for all } x \in \Omega.$$

Consider the semigroup $(\Gamma(t), t \geq 0)$ generated by M. Then the solution of this system is given by

$$Z^\varepsilon(t,x) = \int_0^t \Gamma(t-s) F Z^\varepsilon(s,x) ds + \int_0^t \Gamma(t-s) u R ds.$$

Since the elements of the matrix F^ε are uniformly bounded with respect to ε, according to Grönwall's inequality one has that Z_i^ε is bounded in $L^2(Q)$. Hence, $z_i^\varepsilon \to z_i^*$ in $L^2(Q)$. Letting $\varepsilon \to 0$, we have

$$\frac{\partial Z}{\partial t} = MZ + FZ + uR \quad \text{subject to} \quad Z(0,x) = 0, \ \text{ for all } x \in \Omega.$$

Adopting the same technique, we deduce that $Z_i^\varepsilon \to Z_i^*$ as $\varepsilon \to 0$. □

Let $p = (p_1, p_2, p_3, p_4)$ be the adjoint variable of Z and denote by F^* the adjoint of the Jacobian matrix F. We can write the dual system associated to our problem as

$$-\frac{\partial p}{\partial t} - Mp - F^* p = D^* D \psi \quad \text{subject to } p(T,x) = 0, \tag{2.11}$$

where

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} 0 \\ a \\ 0 \\ 0 \end{pmatrix}.$$

Lemma 2.9. *Under the hypothesis of Theorem 2.6, the system* (2.11) *of adjoint variables admits a unique solution $p \in W^{1,2}([0,T], H(\Omega))$ with $p_i \in G(T,\Omega)$.*

Proof. The result follows by the change of variables $s = T - t$ so as to apply the same method performed in the proof of Theorem 2.8. □

We are now in a position to obtain a necessary optimality condition for the optimal control u^*.

Theorem 2.10. *Let u^* be an optimal control and $z^* \in W^{1,2}([0,T]; H(\Omega))$ its corresponding solution. Then,*

$$u^* = \min\left(1, \max\left(0, \frac{z_2^*(p_2 - p_1)}{b}\right)\right). \tag{2.12}$$

Proof. Let u^* be an optimal control and let z^* be the corresponding optimal state. Set $u^\varepsilon = u^* + \varepsilon u \in U_{ad}$ and let z^ε be the corresponding state trajectory. We have

$$\begin{aligned} J'(u^*)(u) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(J(u^\varepsilon) - J(u^*) \right) \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(a \int_0^T \int_\Omega (z_2^\varepsilon - z_2^*) \, dx dt + \frac{b}{2} \int_0^1 \int_\Omega \left((u^\varepsilon)^2 - (u^*)^2 \right) dx dt \right) \\ &= \lim_{\varepsilon \to 0} \left(a \int_0^T \int_\Omega \left(\frac{z_2^\varepsilon - z_2^*}{\varepsilon} \right) dx dt + \frac{b}{2} \int_0^1 \int_\Omega \left(2uu^* + \varepsilon u^2 \right) dx dt \right). \end{aligned}$$

Since $\lim_{\varepsilon\to 0}\frac{z_2^\varepsilon - z_2^*}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{z_2(u^*+\varepsilon u) - z_2^*}{\varepsilon} = Z_2$, $\lim_{\varepsilon\to 0} z_2^\varepsilon = z_2^*$ and $z_2^\varepsilon, z_2^* \in L^\infty(Q)$, then J is Gateaux differentiable with respect to u^* with

$$\begin{aligned} J'(u^*)(u) &= \int_0^T \int_\Omega aZ_2 dxdt + b\int_0^T \int_\Omega uu^* dxdt \\ &= \int_0^T \langle D\psi, DZ\rangle dt + \int_0^1 \langle bu^*, u\rangle_{L^2(\Omega)} dt. \end{aligned}$$

If we take $u = v - u^*$, then we obtain

$$J'(u^*)(v-u^*) = \int_0^T \langle D\psi, DZ\rangle dt + \int_0^1 \langle bu^*, v-u^*\rangle_{L^2(\Omega)} dt.$$

Since

$$\begin{aligned} \int_0^T \langle D\psi, DZ\rangle dt &= \int_0^T \langle D^* D\psi, Z\rangle\, dt \\ &= \int_0^T \left\langle -\frac{\partial p}{\partial t} - Mp - F^* p, Z\right\rangle dt \\ &= \int_0^T \left\langle p, \frac{\partial Z}{\partial t} - MZ - FZ\right\rangle dt \\ &= \int_0^T \langle p, R(v-u^*)\rangle\, dt \\ &= \int_0^T \langle R^* p, v-u^*\rangle_{L^2(\Omega)}\, dt \end{aligned}$$

and U_{ad} is convex, then $J'(u^*)(v-u^*) \geq 0$ for all $v \in U_{ad}$, which is equivalent to

$$\int_0^T \langle R^* p + bu^*, v-u^*\rangle_{L^2(\Omega)} dt \geq 0 \text{ for all } v \in U_{ad}.$$

Thus, $bu^* = R^{*\intercal} p$ and, consequently, $u^* = \frac{z_2^*(p_2 - p_1)}{b}$. Since $u^* \in U_{ad}$, we have that (2.12) holds. $\square$

A constructed method is adopted here to give an explicit expression of our optimal control.

2.8 Conclusion and future work

We have extended the time deterministic epidemic SICA model due to Silva and Torres [125] to spatiotemporal dynamics, which take into account not only the local reaction of appearance of new infected individuals but also the global diffusion occurrence of the other infected individuals. This allows to incorporate an additional amount of arguments into the system. More precisely, firstly we have modeled the spatiotemporal behavior by incorporating the well-known Laplace operator, which has been employed in the literature, in different contexts, to better understand what happens during any possible displacement of different species and individuals. Here, we justify and interpret its use in the context of HIV/AIDS epidemics. Secondly, we have presented an optimal control problem to minimize the number of infected individuals through a suitable cost functional. Proved results include: existence and uniqueness of a strong global solution to the system, obtained using some adapted tools from semigroup theory; some characteristics of the existing solution; existence of an optimal control, investigated using an effective method based on some properties within the weak topology; and necessary optimality conditions to quantify explicitly the optimal control.

As future work, we plan to develop numerical methods for spatiotemporal optimal control problems, implementing the necessary optimality conditions we have proved here. This is under investigation and will be addressed elsewhere.

Chapter 3

Stochastic approach: Mathematical prerequisites and original results

tionMathematical prerequisites and original results

3.0.1 Existence and uniqueness of solutions of stochastic differential equations

Using [153], we recall the following:
Consider the stochastic differential equation

$$d\xi(t) = a(t, \xi(t))dt + \sigma(t, \xi(t))dW(t) \qquad (*),$$

whose solution, it is natural for us to expect, is a diffusion process with coefficient of diffusion $\sigma^2(t, x)$ and coefficient of transfer $a(t, x)$.

Let us assume that $\sigma^2(t, x)$ and $a(t, x)$ are Borel functions for $x \in \mathbb{R}$ and $t \in [t_0, T]$
Equation $(*)$ is equivalent to

$$\xi(t) = \xi(t_0) + \int_{t_0}^{T} a(s, \xi(s))ds + \int_{t_0}^{T} \sigma(s, \xi(s))dW(s) \qquad (**)$$

and it is solved under the condition that $\xi(t_0)$ is given.For the integrals in $(*)$ and hence the differentials in $(**)$ to be meaningful, we need to introduce the σ-algebras of events $\mathbb{F}_t$
In what follows, the quantity $\xi(t_0)$ is always assumed to be independent of the process $W(t) - W(t_0)$ and by the σ-algebra of $\mathbb{F}_t$ we shall understand the minimal σ-algebras with respect to which the variables $\xi(t_0)$ and $W(t) - W(t_0$ for $t_0 < s \leq t$ are measurable.
We shall consider $\xi(t)$ to be a solution of equation $(**)$ if $\xi(t)$ is $\mathbb{F}_t$ measurable, if the integrals in $(**)$ exist, and if $(**)$ holds for every $t \in [t_0, T]$ with probability 1.

Theorem 3.1. *Let $b(t, x)$ and $\sigma(t, x)$ for $t \in [t_0, T]$ denote two Borel functions satisfying the following conditions for some K:*

a. For all x and $y \in \mathbb{R}$,

$$| a(t, x) - a(t, y) | + | \sigma(t, x) - \sigma(t, y) | \leq K | x - y |$$

b. For all $x \in \mathbb{R}$,

$$| a(t,x) |^2 + | \sigma(t,x) |^2 \leq K^2(1+ | x |^2).$$

Then equation $(**)$ has a solution. If $\xi_1(t,x)$ and $\xi_2(t,x)$ ae two continuous solutions (for fixed $\xi_0(t)$ of equation $(**)$, then

$$P\{ \ sup_{t_0 \leq t \leq T} | \xi_1 - \xi_2 |> 0\} = 0$$

3.0.2 Statement of the Stochastic Maximum Principle

Let $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, P)$ be a given filtered probability space satisfying the usual conditions, on which an m dimensional standard Brownian motion $W(t)$ is given. We consider the following stochastic controlled system:

$$\begin{cases} dx(t) = b(t,x(t))dt + \sigma(t,x(t))dW(t), \quad t \in [0,T] \\ \hfill x(0) = x_0. \end{cases} \tag{3.1}$$

with the cost functional

$$J(u(.)) = \int_0^T f(t,x(t),u(t))dt + h(x(T)).$$

In the above, $b : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times m}, \quad \sigma : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times m}$,
$f : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}, \quad h : \mathbb{R}^n \to \mathbb{R}$.
We define
$b(t,x,u) = (b^j(t,x,u))_{1\leq j\leq n}, \qquad \sigma(t,x,u) = (\sigma^j(t,x,u))^{Tr}_{1\leq j\leq m}$,
$\sigma^j(t,x,u) = (\sigma^{i\times j}(t,x,u))_{1\leq i\leq n}, \qquad 1 \leq j \leq m$
Let us make the following assumptions:

(S_0) $\{\mathbb{F}\}_{t\geq 0}$ is the natural filtration generated by $W(t)$, augmented by all the P-nul sets in $\mathbb{F}$;

(S_1) (U,d) is a separable metric space and $T \geq 0$;

(S_2) The maps b, σ, f and h are mesurable, and there exist a constant L and a modulus of continuity $\overline{\omega} : [0,\infty) \to [0,\infty)$ such that for
$\phi(t,x,u) = b(t,x,u), \sigma(t,x,u), f(t,x,u), h(x)$, we have

$$\begin{cases} | \phi(t,x,u) - \phi(t,\hat{x},\hat{u}) |\leq L | x - \hat{x} | + \overline{\omega}(d(u,\hat{u})), \forall t \in [0,T], x, \hat{x} \in \mathbb{R}^n, u, \hat{u} \in U \\ | \phi(t,0,u) |\leq L, \forall t, u \in [0,T] \times U \end{cases} \tag{3.2}$$

(S_3) The maps b, σ, f and h are C^2, moreover, there exist a constant L and a modulus of continuity $\overline{\omega} : [0,\infty) \to [0,\infty)$ sush that for
$\phi(t,x,u) = b(t,x,u), \sigma(t,x,u), f(t,x,u), h(x)$, we have

$$\begin{cases} | \phi_x(t,x,u) - \phi_x(t,\hat{x},\hat{u}) |\leq L | x - \hat{x} | + \overline{\omega}(d(u,\hat{u})), \\ | \phi_{xx}(t,x,u) - \phi_{xx}(t,\hat{x},\hat{u}) |\leq L | x - \hat{x} | + \overline{\omega}(d(u,\hat{u})), \\ \hfill \forall t \in [0,T], x, \hat{x} \in \mathbb{R}^n, u, \hat{u} \in U \end{cases}$$

Define now,

$$\mathbb{U}_{[0,T]} := \{u : [0,T] \times \Omega \to U \mid u \quad is \quad \{\mathbb{F}_t\}_{t\geq 0} \quad adapted\}$$

Our optimal control problem can be stated as follows:
Any $\overline{u} \in \mathbb{U}_{[0,T]}$ satisfying

$$J(\overline{u}(.)) = \inf_{u(.)\in\mathbb{U}[0,T]} J(u(.)).$$

is called an optimal control, the corresponding state variable $\overline{x}(.))$ and $(\overline{x}(.)), \overline{u}(.))$ are called an optimal state process/trajectory and optimal pair respectively.
We introduce the following terminal value problem for a stochastic differential equation.

$$\begin{cases} dp(t) = -\{b_x(t,\overline{x}(t),\overline{u}(t))^T p(t) + \sum_{j=1}^m \sigma_x^j(t,\overline{x}(t),\overline{u}(t))^T q_j(t) \\ -f_x(t,\overline{x}(t),\overline{u}(t))\}dt + q(t)dW(t) & t \in [0,T]. \\ p(T) = -h_x(\overline{x}(T)). \end{cases}$$

One has to introduce another variable to reflect the uncertainty or the risk factor in the system. This is done by introducing an additional adjoint equation as follows:

$$\begin{cases} dP(t) = -\{b_x(t,\overline{x}(t),\overline{u}(t))^T P(t) + P(t) b_x(t,\overline{x}(t),\overline{u}(t)) \\ +\sum_{j=1}^m \sigma_x^j(t,\overline{x}(t),\overline{u}(t))^T P(t) \sigma_x^j(t,\overline{x}(t),\overline{u}(t)) \\ +\sum_{j=1}^m \left(\sigma_x^j(t,\overline{x}(t),\overline{u}(t))^T Q(t) + Q(t) \sum_{j=1}^m \sigma_x^j(t,\overline{x}(t),\overline{u}(t))\right) \\ +H_{xx}(t,\overline{x}(t),\overline{u}(t),p(t),q(t))\}dt \\ p(T) = -h_{xx(}\overline{x}(T)). \end{cases}$$

Where the Hamiltonian function is defined by:

$$\begin{aligned} H(t,x,u,p,q) &= (p \mid b(t,x,u)) + tr[q^T \sigma(t,x,u)] \\ &- f(t,x,u), \quad (t,x,u) \in [0,T] \times \mathbb{R}^n \times U. \end{aligned}$$

Theorem 3.2 (`Stochastic Maximum Princilpe`). *Let $(S_0)-(S_3)$ hold. Let $(x(.), u(.))$ be an optimal pair of the problem. Then there are a pair of processes*

$$\begin{cases} (p(.), q(.)) \in L^2_{\mathbb{F}}(0,T,\mathbb{R}^n) \times (L^2_{\mathbb{F}}(0,T,\mathbb{R}^n))^m \\ (P(.), Q(.)) \in L^2_{\mathbb{F}}(0,T,\mathbb{S}^n) \times (L^2_{\mathbb{F}}(0,T,\mathbb{S}^n))^m \end{cases}$$

verifying the above statements.

3.1 A stochastic time-delayed model for the effectiveness of Moroccan COVID-19 deconfinement strategy

The original results of this section are published in [158].

3.1.1 Introduction

Coronavirus disease 2019 (COVID-19), reclassified as a pandemic by the World Health Organization (WHO) on March 11, 2020 [107], is an infectious disease caused by a new type of virus belonging to the coronaviruses family and recently named severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2). All the countries affected by this disease have taken many preventive measures, including containment. The containment established by the Moroccan government and the public authorities at the right time made it possible to avoid the worst: according to the minister of Health, at least 6000 lives were saved thanks to the measures adopted to face the spread of this pandemic [96]. The resistance measures, regarded as necessary and urgent, cannot be sustainable.

Actually, the deconfinement is a new stage entered by the COVID-19 pandemic. Therefore, several countries strategically planned their deconfinement strategies. The extension of the state of emergency in Morocco until May 20, 2020 will no doubt have economic repercussions. If Morocco won the first round, or at least limited the consequences, especially in terms of limiting the pandemic and health management of the situation, the second seems difficult and complex. Indeed, it must not only be well thought out but also its axes of resistance have to be well-identified. In this context, all efforts should be focused on stabilizing the economy by intelligently relying on resources. Economic deconfinement is part of the solution and should be gradual and concerted. Indeed, it is absurd to think that the return to the normality is in the near months, because the unavailability of an effective vaccine implies that the virus will always be with us in the near future, which poses a risk for the population. This economic deconfinement should be prepared and accompanied by other related measures, in particular under health, security, education and social assistance. In this period of general crisis, the response must try to mitigate the impacts on priority sectors, such as agriculture, agrifood, transport and foreign trade, in relation to imports that are vital to the Moroccan economy. The challenge is to ensure resistance and a continuity of value creation while preventing a sector from being detached from the economic body. So to speak, priority must be given to vital sectors whose health directly affects all Moroccan activity, while protecting those bordering on chaos. According to the deconfinement strategy, which is applied by the Moroccan authorities, it is mandatory to study the occurrence of an eventual second wave and it's magnitude.

Mathematical modeling through dynamical systems plays an important role to predict the evolution of COVID-19 transmission [102]. However, while taking into account the deconfinement policies, the environmental effects and the social fluctuations should not be neglected in such a mathematical study in order to describe well the dynamics and consider an additional degree of realism [55, 79, 135, 150]. For these reasons, we describe here the dynamics of the deconfinement strategy by a new D-COVID-19 model, governed by delayed stochastic differential equations (DSDE), as follows:

$$
\begin{cases}
dS(t) = \left(\rho C(t) - \delta S(t) - \beta(1-u)\dfrac{S(t)I_s(t)}{N}\right) dt \\
\qquad -\sigma_1(1-u)\dfrac{S(t)I_s(t)}{N} dB_1(t) + \sigma_2(C(t)-S(t))dB_2(t), \\
dC(t) = (\delta S(t) - \rho C(t))\, dt + \sigma_2(S(t)-C(t))dB_2(t), \\
dI_s(t) = \left(\beta\epsilon(1-u)\dfrac{S(t-\tau_1)I_s(t-\tau_1)}{N} - \alpha I_s(t)\right. \\
\qquad \left. -(1-\alpha)(\mu_s+\eta_s)I_s(t)\right) dt \\
\qquad +\sigma_1\left(\epsilon(1-u)\dfrac{S(t-\tau_1)I_s(t-\tau_1)}{N}\right) dB_1(t) \\
\qquad +\sigma_3(\mu_s+\eta_s-1)I_s(t)dB_3(t), \\
dI_a(t) = \left(\beta(1-\epsilon)(1-u)\dfrac{S(t-\tau_1)I_s(t-\tau_1)}{N} - \eta_a I_a(t)\right) dt \\
\qquad +\sigma_1(1-\epsilon)(1-u)\dfrac{S(t-\tau_1)I_s(t-\tau_1)}{N} dB_1(t), \\
dF_b(t) = \Big(\alpha\gamma_b I_s(t-\tau_2) - \big(\mu_b + r_b\big)F_b(t)\Big) dt + \sigma_3\gamma_b I_s(t-\tau_2)dB_3(t), \\
dF_g(t) = \Big(\alpha\gamma_g I_s(t-\tau_2) - \big(\mu_g + r_g\big)F_g(t)\Big) dt + \sigma_3\gamma_g I_s(t-\tau_2)dB_3(t), \\
dF_c(t) = \Big(\alpha\gamma_c I_s(t-\tau_2) - \big(\mu_c + r_c\big)F_c(t)\Big) dt + \sigma_3\gamma_c I_s(t-\tau_2)dB_3(t), \\
dR(t) = \Big(\eta_s(1-\alpha)I_s(t-\tau_3) + \eta_a I_a(t-\tau_3) + r_b F_b(t-\tau_4) \\
\qquad + r_g F_g(t-\tau_4) + r_c F_c(t-\tau_4)\Big) dt - \sigma_3\eta_s I_s(t-\tau_3)dB_3(t), \\
dM(t) = \Big(\mu_s(1-\alpha)I_s(t-\tau_3) + \mu_b F_b(t-\tau_4) + \mu_g F_g(t-\tau_4) \\
\qquad + \mu_c F_c(t-\tau_4)\Big) dt - \sigma_3\mu_s I_s(t-\tau_3)dB_3(t),
\end{cases}
\tag{3.3}
$$

where S represents the susceptible sub-population, which is not infected and has not been infected before but is susceptible to develop the disease if exposed to the virus; C is the confined sub-population; I_s is the symptomatic infected sub-population, which has not yet been treated, it transmits the disease, and outside of proper support it can progress to spontaneous recovery or death; I_a is the asymptomatic infected sub-population who is infected but does not transmit the disease, is not known by the health system and progresses spontaneously to recovery; F_b, F_g and F_c are the patients diagnosed, supported by the Moroccan health system and under quarantine, and subdivided into three categories: benign, severe, critical forms, respectively. Finally, R and M are the recovered and died classes, respectively. At each instant of time, the equation

$$
\begin{cases}
D(t) := \mu_s(1-\alpha)I_s(t-\tau_3) + \mu_b F_b(t-\tau_4) + \mu_g F_g(t-\tau_4) \\
\qquad + \mu_c F_c(t-\tau_4) - \sigma_3\mu_s I_s(t-\tau_3)\dfrac{\Delta B_3(t)}{\Delta t} = \dfrac{\Delta M(t)}{\Delta t}
\end{cases}
$$

gives the number of the new dead due to disease. The parameter $1-u$ represents the level of measures undertaken on the susceptible population while δ is the confinement rate and ρ represents the deconfinement rate. We adopt the bilinear incidence rate to describe the infection of the disease and use the parameter β to denote the transmission rate. It is reasonable to assume that the infected individuals are subdivided into individuals with symptoms and others without symptoms, for which we employ the parameter ϵ to denote the proportion for the symptomatic individuals and $1-\epsilon$ for the asymptomatic ones. The parameter α measures the efficiency of public health administration for hospitalization. Diagnosed symptomatic infected population is completely distributed into one of the three forms F_b, F_g and F_c, by the rates γ_b, γ_g and γ_c, respectively. Then, $\gamma_b + \gamma_g + \gamma_c = 1$. The mean recovery period of these forms are denoted by $1/r_b$, $1/r_g$ and $1/r_c$, respectively. The latter forms die also with the rates μ_b, μ_g and μ_c, respectively. Symptomatic infected population, which is not diagnosed, moves to the recovery compartment with a rate η_s or dies with a rate μ_s. On the other hand, asymptomatic infected population moves to the recovery compartment with a rate η_a. The time delays τ_1 and τ_2 denote the incubation period and the period of time needed before the charge by the health system, respectively. The time delays τ_3 and τ_4 denote the time required before the death of individuals coming from the compartments I_s and the three forms F_b, F_g and F_c, respectively. Here, $B_1(t)$, $B_2(t)$ and $B_3(t)$ are independent standard Brownian motions defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ and satisfying the usual conditions, that is, they are increasing and right continuous while $\mathscr{F}_0$ contains all P-null sets and σ_i represents the intensity of B_i, $i = 1, 2, 3$. The schematic diagram of our extended model is illustrated in Figure 3.1.

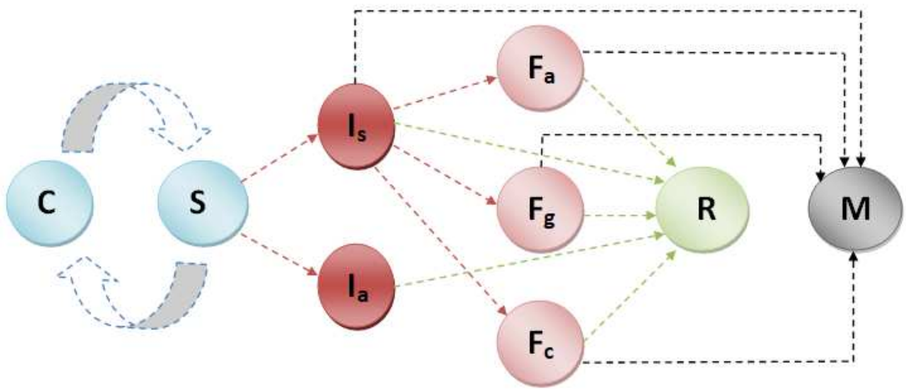

Figure 3.1: Schematic diagram of model (3.3).

Remark 3.3. *The stochasticity is introduced in model* (3.3) *by perturbing the most sensitive parameters: β, α, δ, and ρ.*

Remark 3.4. *For the sake of simplicity, we have assumed that the parameters δ and ρ are perturbed with the same intensities, that is, we assume that Moroccan individuals possess the same behaviors and reactions towards the authorities instructions.*

Remark 3.5. *Note that the multipliers of $I_s(t-\tau_2)$ terms are the same as $\sigma_3 dB_3(t)$ although they are premultiplied by different constants. Indeed, the portion of diagnosed symptomatic infected population is completely distributed into the three forms F_b, F_g, and F_c, by the rates γ_b, γ_g and γ_c, respectively. Then, $\gamma_b+\gamma_g+\gamma_c = 1$. In addition, we assume that the parameter α, which measures the efficiency of public health administration for hospitalization, undergoes random fluctuations.*

Remark 3.6. *For illustration and clarification purposes, let us suppose, as an example, that the disease progression is started from 15th of March and value of τ_1 is 5. Then a susceptible individual, after contact with an infected one at instant t, becomes himself infected at instant $t + \tau_1$. Suddenly, the compartment of the infected is fed at the instant t by the susceptible infected at the instant $t - \tau_1$. Therefore, in the considered situation, when the infection starts at March 15, the term $\beta\epsilon(1-u)S(T)I_s(T)/N$ of new infected is equal to zero on 12th, 13th and 14th March, due to the absence of the infection.*

Remark 3.7. *Temporarily asymptomatic individuals are included in the class I_s of symptomatic, while individuals in I_a, who are permanently asymptomatic, will remain asymptomatic until recovery and will not spread the virus, a fact which has been recently confirmed by the World Health Organization.*

For biological reasons, we assume that the initial conditions of system (3.3) satisfy:

$$\begin{array}{lll} S(\theta) = \phi_1(\theta) \geq 0, & C(\theta) = \phi_2(\theta) \geq 0, & I_a(\theta) = \phi_3(\theta) \geq 0, \\ I_s(\theta) = \phi_4(\theta) \geq 0, & F_b(\theta) = \phi_5(\theta) \geq 0, & F_g(\theta) = \phi_6(\theta) \geq 0, \\ F_c(\theta) = \phi_7(\theta) \geq 0, & R(\theta) = \phi_8(\theta) \geq 0, & M(\theta) = \phi_9(\theta) \geq 0, \end{array} \tag{3.4}$$

where $\theta \in [-\tau, 0]$ and $\tau = \max\{\tau_1, \tau_2 \tau_3, \tau_4\}$.

The rest of the section is organized as follows. The first part of the study deals with the existence and uniqueness of a positive global solution that ensures the well-posedness of the D-COVID-19 model (3.3). A sufficient condition for the extinction is established in the second phase. . Then, some numerical scenarios, to assess the effectiveness of the adopted deconfinement strategy, are presented in the next step. The study ends up with an adequate conclusion.

3.1.2 Existence and uniqueness of a positive global solution

Let us denote $\mathbb{R}^9_+ := \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \mid x_i > 0,\ i = 1, 2, \ldots, 9\}$. We begin by proving the following result.

Theorem 3.8. *For any initial value satisfying condition* (3.4)*, there is a unique solution*

$$x(t) = (S(t), C(t), I_s(t), I_a(t), F_b(t), F_g(t), F_c(t), R(t), M(t))$$

to the D-COVID-19 model (3.3) *that remains in $\mathbb{R}^9_+$ with probability one.*

Proof. Since the coefficients of the Stochastic Differential Equations with several delays (3.3) are locally Lipschitz continuous, it follows from [94] that for any square integrable initial value $x(0) \in \mathbb{R}^9_+$, which is independent of the considered standard Brownian motion B, there exists a unique local solution $x(t)$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. For showing that this solution is global, knowing that the linear growth condition is not verified, we need to prove that $\tau_e = \infty$. Let $k_0 > 0$ be sufficiently large for $\frac{1}{k_0} < x(0) < k_0$. For each integer $k \geq k_0$, we define the stopping time $\tau_k := \inf\left\{t \in [0, \tau_e)/x_i(t) \notin \left(\frac{1}{k}, k\right) \text{ for some } i = 1, 2, 3\right\}$, where $\inf \emptyset = \infty$. It is evident that $\tau_k \leq \tau_e$. Let $T > 0$, and define the twice differentiable function V on $\mathbb{R}^3_+ \to \mathbb{R}^+$ as follows:

$$V(x) := (x_1 + x_2 + x_3)^2 + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}.$$

By Itô's formula, for any $0 \leq t \leq \tau_k \wedge T$ and $k \geq 1$ we have

$$dV(x(t)) = LV(x(t))dt + \sigma(x(t))dB_t,$$

where L is the differential operator of function V:

$$\begin{aligned}
LV(x(t)) \;=\; & \left(2(S(t)+C(t)+I_s(t)) - \frac{1}{S^2(t)}\right)\left(\rho C(t) - \delta S(t) - \beta(1-u)\frac{S(t)I_s(t)}{N}\right) \\
& + \frac{1}{2}\left(2 + \tfrac{2}{S^3(t)}\right)\left(\left(-\sigma_1(1-u)\frac{S(t)I_s(t)}{N}\right)^2 + (\sigma_2(C(t)-S(t)))^2\right) \\
& + \left(2(S(t)+C(t)+I_s(t)) - \frac{1}{C^2(t)}\right)(\delta S(t) - \rho C(t)) \\
& + \frac{1}{2}\left(2 + \tfrac{2}{C^3(t)}\right)(-\sigma_2(C(t)-S(t)))^2 + \frac{1}{2}\left(2 + \tfrac{2}{C^3(t)}(-\sigma_2(C(t)-S(t))\right)^2 \\
& + \left(2(S(t)+C(t)+I_s(t)) - \frac{1}{I_s^2(t)}\right)\left(\beta\epsilon(1-u)\tfrac{S(t-\tau_1)I_s(t-\tau_1)}{N} - \alpha I_s(t)\right. \\
& \left. -(1-\alpha)(\mu_s+\eta_s)I_s(t)\right) + \left(\sigma_1\epsilon(1-u)\tfrac{S(t-\tau_1)I_s(t-\tau_1)}{N}\right)^2\Bigg).
\end{aligned}$$

Thus,

$$\begin{aligned}
LV(x(t)) \;\leq\; & 2(S(t)+C(t)+I_s(t))\rho C(t) + \tfrac{\delta}{S(t)} + \tfrac{\beta(1-u)S(t)I_s(t)}{NS^2(t)} \\
& + \left(1 + \frac{1}{S^3(t)}\right)\left(\left(\sigma_1(1-u)\frac{S(t)I_s(t)}{N}\right)^2 + (\sigma_2(C(t)-S(t)))^2\right) \\
& + 2(S(t)+C(t)+I_s(t))\delta S(t) + \frac{\rho}{S(t)} + \left(1 + \frac{1}{C^3(t)}\right)(\sigma_2(C(t)-S(t)))^2 \\
& + 2(S(t)+C(t)+I_s(t))\beta\epsilon(1-u)\tfrac{S(t-\tau_1)I_s(t-\tau_1)}{N} \\
& + \tfrac{\alpha}{I_s(t)} + (1-\alpha)(\mu_s+\eta_s)\tfrac{1}{I_s(t)} \\
& + \left(1 + \frac{1}{I_s^3(t)}\right)\left((-\sigma_3 I_s(t)(1-\mu_s-\eta_s)^2 + \left(\sigma_1\epsilon(1-u)\frac{S(t-\tau_1)I_s(t-\tau_1)}{N}\right)^2\right).
\end{aligned}$$

By applying the elementary inequality $2ab \leq a^2 + b^2$, we can easily increase the right-hand side of the previous inequality to obtain that

$$LV(x) \leq D(1 + V(x)),$$

where D is an adequate selected positive constant. By integrating both sides of the equality

$$dV(x(t)) = LV(x(t))dt + \sigma(x(t))dB_t$$

between t_0 and $t \wedge \tau_k$ and acting the expectation, which eliminates the martingale part, we get that

$$\begin{aligned}
E(V(x(t \wedge \tau_k)) &= E(V(x_0)) + E\int_{t_0}^{t\wedge\tau_k} LV(x_s))ds \\
&\leq E(V(x_0)) + E\int_{t_0}^{t\wedge\tau_k} D(1 + V(x_s))ds
\end{aligned}$$

$$\leq \quad E(V(x_0)) + DT + \int_{t_0}^{t\wedge\tau_k} EV(x_s))ds.$$

Gronwall's inequality implies that

$$E(V(x(t\wedge\tau_k)) \leq (EV(x_0) + DT)\exp(CT).$$

For $\omega \in \{\tau_k \leq T\}$, $x_i(\tau_k)$ equals k or $\dfrac{1}{k}$ for some $i = 1, 2, 3$. Hence,

$$V(x_i(\tau_k)) \geq \left(k^2 + \frac{1}{k}\right) \wedge \left(\frac{1}{k^2} + k\right).$$

It follows that

$$\begin{aligned}(EV(x_0) + DT)\exp(CT) \quad &\geq \quad E\left(\chi_{\{\tau_k \leq T\}}(\omega)V(x_{\tau_k})\right) \\ &\geq \quad \left(k^2 + \frac{1}{k}\right) \wedge \left(\frac{1}{k^2} + k\right) P\left(\tau_k \leq T\right).\end{aligned}$$

Letting $k \to \infty$, we get $P(\tau_e \leq T) = 0$. Since T is arbitrary, we obtain $P(\tau_e = \infty) = 1$. With the same technique, we also deduce that the rest of the variables of the system are positive on $[0, \infty)$. This concludes the proof. □

3.1.3 Extinction of the disease

In this section, we obtain a sufficient condition for the extinction of the disease.

Theorem 3.9. *Let* $(S(t), C(t), I_s(t), I_a(t), F_b(t), F_g(t), F_c(t), R(t), M(t))$ *be a solution of the D-COVID-19 model* (3.3) *with positive initial value defined in* (3.4). *Assume that*

$$\sigma_1^2 > \frac{\beta^2}{2(\alpha + (1-\alpha)(\mu_s + \eta_s))}.$$

Then,

$$\limsup_{t\to\infty} \ln \frac{I_s(t)}{t} < 0.$$

Namely, $I_s(t)$ *tends to zero exponentially a.s., that is, the disease dies out with probability* 1.

Proof. Let

$$\begin{aligned} d\ln I_s(t) \quad &= \Bigg[\frac{1}{I_s(t)}\left(\frac{\beta\epsilon(1-u)S(t-\tau_1)I_s(t-\tau_1)}{N} - \alpha I_s(t) - (1-\alpha)(\mu_s + \eta_s)I_s(t)\right) \\ &- \frac{1}{2I_s^2(t)}\left(\left(\sigma_1 \frac{\beta\epsilon(1-u)S(t-\tau_1)I_s(t-\tau_1)}{N}\right)^2 + (\sigma_3(\mu_s + \eta_s - 1)I_s(t))^2\right)\Bigg] dt \\ &+ \frac{1}{I_s(t)}\sigma_1 \frac{\beta\epsilon(1-u)S(t-\tau_1)I_s(t-\tau_1)}{N} dB_1 + \frac{1}{I_s(t)}\sigma_3(\mu_s + \eta_s - 1)I_s(t)dB_3. \end{aligned}$$

To simplify, we set

$$G(t) := \frac{\epsilon(1-u)S(t-\tau_1)I_s(t-\tau_1)}{N},$$

$$
\begin{aligned}
R_1(t) &:= \frac{\sigma_1}{I_s(t)} \frac{\beta\epsilon(1-u)S(t-\tau_1)I_s(t-\tau_1)}{N} = \frac{\beta\sigma_1}{I_s(t)}G, \\
R_3(t) &:= \frac{\sigma_3}{I_s(t)}(\mu_s + \eta_s - 1)I_s(t), \\
H &:= -\alpha - (1-\alpha)(\mu_s + \eta_s).
\end{aligned}
$$

We then get

$$
\begin{aligned}
d\ln I_s(t) = & \frac{\beta G(t)}{I_s(t)} - H(t) - \frac{1}{2}\left(\left(\frac{\sigma_1 G(t)}{I_s(t)}\right)^2 + (\sigma_3(\mu_s + \eta_s - 1)I_s(t))^2\right) \\
& + R_1(t)dB_1 + R_3(t)dB_3 \\
& = -\frac{\sigma_1^2}{2}\left[\left(\frac{G(t)}{I_s(t)}\right)^2 - \frac{2\beta}{\sigma_1^2}\frac{G(t)}{I_s(t)}\right] + H + R_1(t)dB_1 + R_3(t)dB_3 \\
& = -\frac{\sigma_1^2}{2}\left[\left(\frac{G(t)}{I_s(t)} - \frac{\beta}{\sigma_1^2}\right)^2 - \frac{\beta^2}{\sigma_1^4}\right] + H + R_1(t)dB_1 + R_3(t)dB_3 \\
& \leq \frac{\beta^2}{2\sigma_1^2} + H + R_1(t)dB_1 + R_3(t)dB_3.
\end{aligned}
$$

Hence,

$$
\frac{\ln I_s(t)}{t} \leq \frac{\ln I_s(0)}{t} + \frac{\beta^2}{2\sigma_1^2} + H + \frac{M_1(t)}{t} + \frac{M_3(t)}{t},
$$

where

$$
M_1(t) = \int_0^t R_1(s)dB_1, \quad M_3(t) = \int_0^t R_3(s)dB_3.
$$

We have

$$
\begin{aligned}
\langle M_1, M_1 \rangle_t &= \int_0^t \left(\frac{1}{I_s(s)}\sigma_1 G(s)\right)^2 ds \\
&= \int_0^t {\sigma_1}^2 \epsilon^2 (1-u)^2 \frac{S(t-\tau_1)^2 I_s(t-\tau_1)^2}{N^2 I_s^2} ds \\
&\leq \int_0^t {\sigma_1}^2 \epsilon^2 (1-u)^2 \frac{N^4}{N^2}\frac{1}{I_s^2} ds \\
&\leq \int_0^t {\sigma_1}^2 \epsilon^2 (1-u)^2 N^2 ds.
\end{aligned}
$$

Then,

$$
\limsup_{t\to\infty} \frac{< M_1, M_1 >_t}{t} \leq {\sigma_1}^2 \epsilon^2 (1-u)^2 N^2 < \infty.
$$

From the large number theorem for martingales [62], we deduce that

$$
\lim_{t\to\infty} \frac{M_1(t)}{t} = 0.
$$

We also have

$$
< M_2, M_2 >_t = \int_0^t \left(\frac{1}{I_s(s)}\sigma_3(\mu_s + \eta_s - 1)I_s(s)\right)^2 ds
$$

$$
\begin{aligned}
&= \int_0^t \left(\sigma_3^2(\mu_s + \eta_s - 1)^2\right) ds \\
&= \left(\sigma_3^2(\mu_s + \eta_s - 1)^2\right) t.
\end{aligned}
$$

Thus,

$$\limsup_{t\to\infty} \frac{< M_2, M_2 >_t}{t} \leq \sigma_3^2(\mu_s + \eta_s - 1) < \infty.$$

We deduce that

$$\lim_{t\to\infty} \frac{M_2(t)}{t} = 0.$$

Subsequently,

$$\limsup_{t\to\infty} \frac{\ln I_s(t)}{t} \leq \frac{\beta^2}{2\sigma_1^2} - \alpha - (1-\alpha)(\mu_s + \eta_s).$$

We conclude that if $\dfrac{\beta^2}{2\sigma_1^2} - \alpha - (1-\alpha)(\mu_s + \eta_s) < 0$, then $\lim_{t\to\infty} I(t) = 0$. The proof is complete. $\square$

3.1.4 Results and discussion

In this section, we simulate the forecasts of the D-COVID-19 model (3.3), relating the deconfinement strategy adopted by Moroccan authorities with two scenarios. We assume u defined as follows:

$$
u = \begin{cases}
u_0, & \text{on [March 2, March 10];} \\
u_1, & \text{on (March 10, March 16];} \\
u_2, & \text{on (March 16, March 20];} \\
u_3, & \text{on (March 20, April 6];} \\
u_4, & \text{on (April 6, April 25];} \\
u_5, & \text{from April 25 on;}
\end{cases}
$$

where $u_i \in (0, 1]$, for $i = 0, 1, 2, 3, 4, 5$, measures the effectiveness of applying the multiple preventive interventions imposed by the authorities and presented in Table 3.1.

Table 3.1: Summary of considered non-pharmaceutical interventions.

Policies	**Decisions**
With minimal social distancing measure	$u = 0.1$, on [March 2, March 16] $u = 0.2$, on (March 16, March 20]
With middle social distancing measure	$u = 0.1$, on [March 2, March 16], $u = 0.2$, on (March 16, March 20] $u = 0.4$, on (March 20, April 6]
With high social distancing measure	$u = 0.1$, on [March 2, March 16], $u = 0.2$, on (March 16, March 20], $u = 0.4$, on (March 20, April 6] $u = 0.6$, on (April 6, April 25]
With maximal social distancing measure	$u = 0.1$, on [March 2, March 16], $u = 0.2$, on (March 16, March 20], $u = 0.4$, on (March 20, April 6], $u = 0.6$, on (April 6, April 25] $u = 0.7$, from April 25 on.

COVID-19 is known as a highly contagious disease and its transmission rate, β, varies from country to country, according to the density of the country and movements of its population. Ozair et al. [111] assumed β to be $[0.198 - 0.594]$ per day for Romania, and $[0.097 - 0.291]$ per day for Pakistan. Further, Kuniya [79] estimated β as 0.26 ($95\% CI$, $2.4 - 2.8$). Observing the number of daily reported cases of COVID-19 in Morocco, we estimate β as 0.4517 ($95\% CI$, $0.4484 - 0.455$). After the infection, the patient remains in a latent period for 5.5 days [26], in average, before becoming symptomatic and infectious or asymptomatic with a percentage that varies from 20.6% of infected population to 39.9% [97], while the time needed before his hospitalization is estimated to be 7.5 days [73, 144]. All the parameter values chosen for the D-COVID-19 model 3.3 are summarized in Table 3.1.4.

Parameter	β	ϵ	γ_b	γ_g	γ_c	α	η_a	η_s	μ_s
Value	0.4517	0.794	0.8	0.15	0.05	0.06	1/21	0.8/21	0.01/21

Parameter	μ_b	μ_g	μ_c	r_b	r_g	r_c	τ_1	τ_2	τ_3	τ_4
Value	0	0	0.4/13.5	1/13.5	1/13.5	0.6/13.5	5.5	7.5	21	13.5

Remark 3.10. *From a biological point of view, the latency period is independent of the region or country under study, depending only on the structural nature of the SARS-CoV-2 coronavirus.*

We consider that all measures and the adopted confinement strategy previously discussed are conserved. The evolution on the number of diagnosed infected positive individuals given by the D-COVID-19 model (3.3) versus the daily reported confirmed cases of COVID-19 in Morocco, from March 2 to May 6, is presented in Figure 3.2. We see that the curve generated by the D-COVID-19 model (3.3) follows the trend of the daily reported cases in Morocco. So, we confirm that the implemented measures taken by the authorities have an explicit impact on the propagation of the virus in the population since the curve of the D-COVID-19 model (3.3) has been flattening from April 17 and tends to go towards the extinction of the disease from May 05. In Figure 3.3, we see that Morocco has spent almost 40% of the total duration of the epidemic at May 11 and will reach extinction after four months, in average, from the start of the epidemic on March 2, 2020 ($t = 0$).

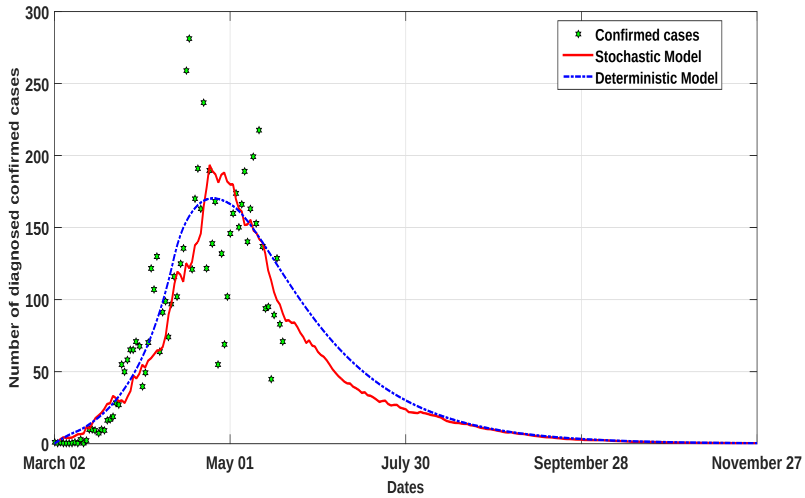

Figure 3.2: Evolution of COVID-19 confirmed cases in Morocco per day: curve predicted by our model (3.3) accordigly with Tables 3.1 and 3.1.4 versus real data.

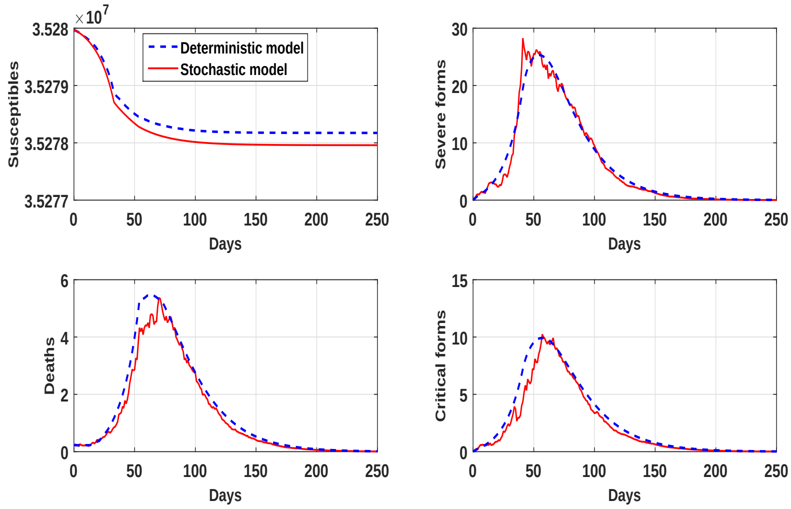

Figure 3.3: Evolution given by the D-COVID-19 model (3.3) without deconfinement ($\rho = 0$).

To prove the biological importance of delay parameters, we give the graphical results of Figure 3.4, which allow to compare the evolution of diagnosed positive cases with and without

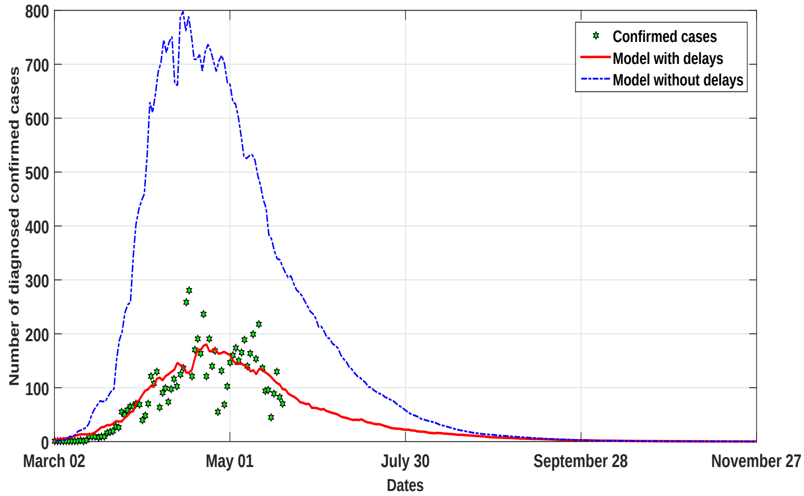

Figure 3.4: Effect of delays on the diagnosed confirmed cases versus clinical data.

delays. We observe in Figure 3.4 a high impact of delays on the number of diagnosed positive cases. Indeed, the plot of model (3.3) without delays ($\tau_i = 0,\ i = 1, 2, 3, 4$) is very far from the clinical data. Thus, we conclude that delays play an important role in the study of the dynamical behavior of COVID-19 worldwide, especially in Morocco, and allows to better understand the reality.

we consider the deconfinement of 30% of the population returning to work from May 20, and this proportion is immediately integrated into the susceptible population. Numerical simulations are presented for three possible scenarios. In the first, we consider that the whole population highly respects the majority of the measures announced by the authorities in relation with the deconfinement (Figure 3.5). The second and third scenarios show the direct impact on the curves when the population moderately respects the measures with different levels, $\sigma_2 = 0.10$ and $\sigma_2 = 0.15$, respectively. With the last two scenarios we observe the growth in the final number of infected, deaths, severe and critical forms, which are the most important to monitor, since the health system should not be saturated. It is also important to note the appearance of a second significant peak and the fact that the time required for extinction becomes longer, which relates to the value of σ_2 (Figures 3.6 and 3.7).

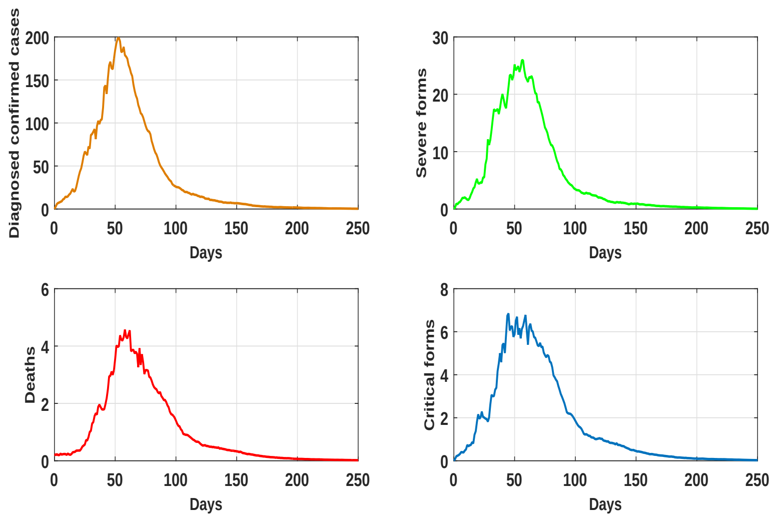

Figure 3.5: Evolution of the D-COVID-19 model (3.3) with deconfinement ($\rho = 0.3$) from May 20, 2020 and high effectiveness of the measures ($\sigma_2 = 0.01$).

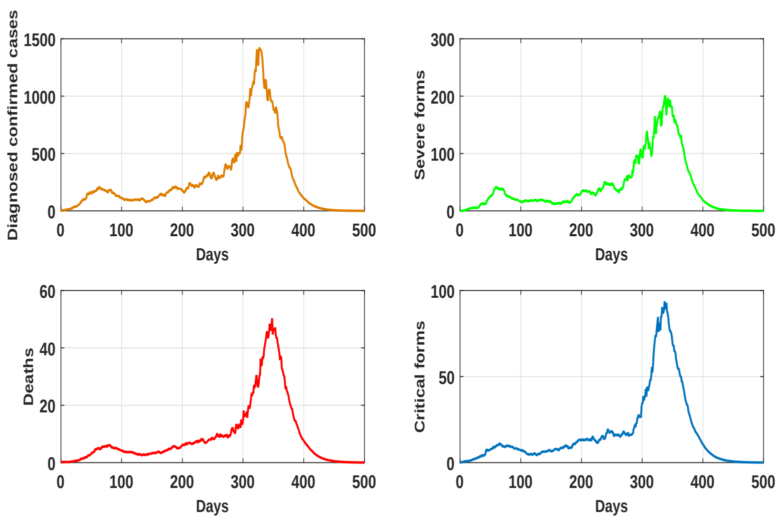

Figure 3.6: Evolution of the D-COVID-19 model (3.3) with deconfinement ($\rho = 0.3$) from May 20, 2020 and moderate effectiveness of the measures ($\sigma_2 = 0.10$).

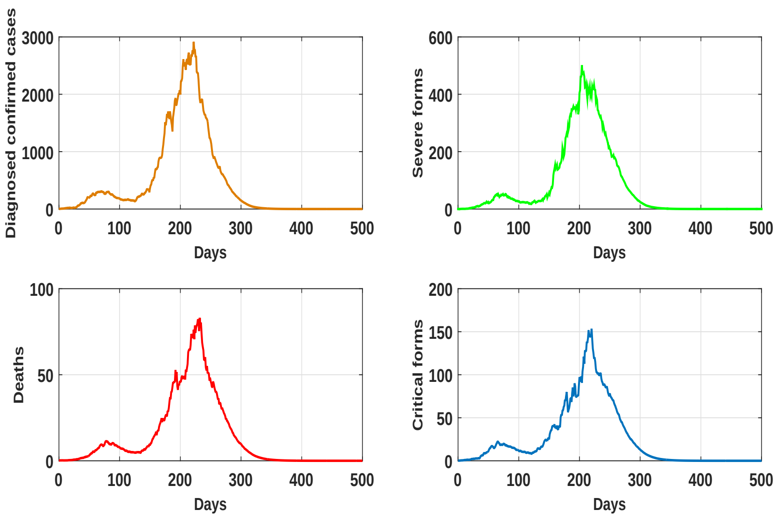

Figure 3.7: Evolution of the D-COVID-19 model (3.3) with deconfinement ($\rho = 0.3$) from May 20, 2020 and moderate effectiveness of the measures ($\sigma_2 = 0.15$).

It should be mentioned that all our plots were obtained using the Matlab numerical computing environment by discretizing system (3.3) by means of the higher order method of Milstein presented in [70] and used in [90].

3.1.5 Conclusions

In this work, we have proposed a delayed stochastic mathematical model to describe the dynamical spreading of COVID-19 in Morocco by considering all measures designed by authorities, such as confinement and deconfinement policies. More precisely, our model takes into account four types of delays: the first one is related to the incubation period, the second is the time needed to move from the symptomatic infected individuals to the three forms of diagnosed cases, the third is the time needed to move from the class of infected individuals to the recovered or dead class, while the last one is the time needed to pass from the three types of classes of individuals supported by the Moroccan health system, and under quarantine, to the recovered or dead compartments. Besides, to well describe reality, we have added a stochastic factor resulting from possible maladjustment of the population individuals to the measures.

To show that our model is mathematically and biologically well-posed, we have proved the global existence of a unique positive solution (see Theorem 3.8). Our result has shown a possible extinction of the disease when σ_1^2 is greater than a threshold parameter (see Theorem 3.9). In addition, numerical simulations have been performed to forecast the evolution of COVID-19. More precisely, we have shown that the evolution of our D-COVID-19 model follows the tendency of daily reported confirmed cases in Morocco (see Figure 3.2). Further, if Moroccan people would maintain, strictly, their confinement policy, we observe that the disease dies out around four months from March 2, 2020 (see Figure 3.3). On the other hand, in response to the decision of deconfinement represented by the liberation of the 30% of population, which took place at May 20, we simulate three scenarios corresponding to different values of the intensity σ_2. When $\sigma_2 = 0.01$ (Figure 3.5), the eradication of the disease from the population comes early compared to the cases when $\sigma_2 = 0.10$ (Figure 3.6) and $\sigma_2 = 0.15$ (Figure 3.7). Additionally, the number of diagnosed confirmed cases mainly changes because of the value of this intensity and a small perturbation leads to relevant quantitative changes and significant variations on the time needed for extinction. Thus, we observe that the value of this perturbation has a high impact on the evolution of COVID-19, which means the Moroccan population has a big interest to respect the governmental measures announced May 20, 2020, in order to have a successful and good deconfinement strategy.

Here we have compared the predictions of the proposed model (3.3) with real data until middle of May 2020. We leave the comparison of the real data in Morocco till the end of 2020 to a future work, where we also plan to incorporate the predictions of the evolution of our COVID-19 model with respect to preventive Moroccan measures by regions and cities.

3.2 Modeling and Forecasting of COVID-19 Spreading by Delayed Stochastic Differential Equations

The original results of this section are published in [88].

3.2.1 Introduction

Coronaviruses are a large family of viruses that cause illnesses, ranging from the common cold to more serious illnesses such as Middle Eastern Respiratory Syndrome (MERS-CoV) and Severe Acute Respiratory Syndrome (SARS-CoV). The new coronavirus COVID-19 corresponds to a new strain that has not previously been identified in humans. On 11 March 2020, COVID-19 was reclassified as a pandemic by the World Health Organization (WHO). The disease has spread rapidly from country to country, causing enormous economic damage and many deaths around the world, prompting governments to issue a dramatic decree, ordering the lockdown of entire countries.

Since the confirmation of the first case of COVID-19 in Morocco on 2 March 2020 in the city of Casablanca, numerous preventive measures and strategies to control the spread of diseases have been imposed by the Moroccan authorities. In addition, Morocco declared a health emergency during the period from 20 March to 20 April 2020 and gradually extended it until 10 June 2020 in order to control the spread of the disease. In this section, we report the assessment of the evolution of COVID-19 outbreak in Morocco. Besides shedding light on the dynamics of the pandemic, the practical intent of our analysis is to provide officials with the tendency of COVID-19 spreading, as well as gauge the effects of preventives measures using mathematical tools. Several other papers developed mathematical models for COVID-19 for particular regions in the globe and particular intervals of time, e.g., in [42] a Susceptible–Infectious–Quarantined–Recovered (SIQR) model to the analysis of data from the Brazilian Department of Health, obtained from 26 February 2020 to 25 March 2020 is proposed to better understand the early evolution of COVID-19 in Brazil; in [56], a new COVID-19 epidemic model with media coverage and quarantine is constructed on the basis of the total confirmed new cases in the UK from 1 February 2020 to 23 March 2020; while in [99] SEIR modelling to forecast the COVID-19 outbreak in Algeria is carried out by using available data from 1 March to 10 April, 2020.

Mathematical modeling, particularly in terms of differential equations, is a strong tool that attracts the attention of many scientists to study various problems arising from mechanics, biology, physics, and so on. For instance, in [133], a system of differential equations with density-dependent sublinear sensitivity and logistic source is proposed and blow up properties of solutions are investigated; paper [143] presents a mathematical model with application in civil engineering related to the equilibrium analysis of a membrane with rigid and cable boundaries; [81] studies nonnegative and classical solutions to porous medium problems; and [82] a two-dimensional boundary value problem under proper assumptions on the data. Herein, we will focus on the dynamic of COVID-19. Tang et al. [135] used a Susceptible–Exposed–Infectious–Recovered (SEIR) compartmental model to estimate the basic reproduction number of COVID-19 transmission, based on data of confirmed cases for the disease in mainland China. Wu et al. [150] provided an estimate of the size of the epidemic in Wuhan on the basis of the number of cases exported from Wuhan to cities outside mainland China by using a SEIR model. In [79], Kuniya applied the SEIR compartmental model for the prediction of the epidemic peak for COVID-19 in Japan, using real-time data from 15 January to 29 February, 2020. Fanelli and Piazza [55] analyzed and forecasted the COVID-19 spreading in China, Italy and France, by using a simple Susceptible–Infected–Recovered–Deaths (SIRD) model. A more elaborate model, which includes the transmissibility of super-spreader individuals, is proposed in Ndaïrou et al. [102]. The model we propose here is new and has completely different compartments: in the paper [102], they model susceptible, exposed, symptomatic and infectious, super-spreaders, infectious but asymptomatic, hospitalized, recovered and the fatality class, with the main contribution being the inclusion of super-spreader individuals; in contrast, here we consider susceptible individuals, symptomatic infected individuals, which have not yet been treated, the asymptomatic infected individuals who are infected but do not transmit the disease, patients diagnosed and under quarantine and subdivided into three categories—benign, severe and critical forms—recovered and dead individuals. Moreover, our model has delays, while the previous model [102] has no delays; our model is stochastic, while the previous model [102] is deterministic. In fact, all mentioned models are deterministic and neglect the effect of stochastic noises derived from environmental fluctuations. To the best of our knowledge, research works that predict the COVID-19 outbreak taking into account a stochastic component, are a rarity [24, 71, 128]. The novelty of our work is twofold: the extension of the models cited above to a more accurate model with time delay, suggested biologically in the first place; secondly, to combine between the deterministic and the stochastic approaches in order to well-describe reality. To do this, the formulation and the well-posedness of the model is performed. The second task is devoted to the qualitative analysis of the proposed model. Parameters estimation and forecast of COVID-19 spreading in Morocco is presented after that, however. The study ends with appropriate discussion and conclusions.

3.2.2 Models Formulation and Well-Posedness

Based on the epidemiological feature of COVID-19 and the several strategies imposed by the government, with different degrees, to fight against this pandemic, we extend the classical SIR model to describe the transmission of COVID-19 in the Kingdom of Morocco. In particular, we divide the population into eight classes, denoted by S, I_s, I_a, F_b, F_g, F_c, R and M, where S represents the susceptible individuals; I_s the symptomatic infected individuals, which have not yet been treated; I_a the asymptomatic infected individuals who are infected but do not transmit the disease; F_b, F_g and F_c denote the patients diagnosed, supported by the Moroccan health system and under quarantine, and subdivided into three categories: benign, severe and critical forms, respectively. Finally, R and M are the recovered and fatality classes.

This model satisfies the following assumptions:

(1) all coefficients involved in the model are positive constants;

(2) natural birth and death rate are not factors;

(3) true asymptomatic patients will stay asymptomatic until recovery and do not spread the virus;

(4) patients who are temporarily asymptomatic are included on symptomatic ones;

(5) the second infection is not considered in the model;

(6) the Moroccan health system is not overwhelmed.

According to the above assumptions and the actual strategies imposed by the Moroccan authorities, the spread of COVID-19 in the population is modeled by the following system of delayed differential equations (DDEs):

$$\begin{cases} \frac{dS(t)}{dt} &= -\beta(1-u)\frac{S(t)I_s(t)}{N}, \\ \frac{dI_s(t)}{dt} &= \beta\epsilon(1-u)\frac{S(t-\tau_1)I_s(t-\tau_1)}{N} - \alpha I_s(t) - (1-\alpha)(\mu_s+\eta_s)I_s(t), \\ \frac{dI_a(t)}{dt} &= \beta(1-\epsilon)(1-u)\frac{S(t-\tau_1)I_s(t-\tau_1)}{N} - \eta_a I_a(t), \\ \frac{dF_b(t)}{dt} &= \alpha\gamma_b I_s(t-\tau_2) - (\mu_b+r_b)F_b(t), \\ \frac{dF_g(t)}{dt} &= \alpha\gamma_g I_s(t-\tau_2) - (\mu_g+r_g)F_g(t), \\ \frac{dF_c(t)}{dt} &= \alpha\gamma_c I_s(t-\tau_2) - (\mu_c+r_c)F_c(t), \\ \frac{dR(t)}{dt} &= \eta_s(1-\alpha)I_s(t-\tau_3) + \eta_a I_a(t-\tau_3) + r_b F_b(t-\tau_4) + r_g F_g(t-\tau_4) + r_c F_c(t-\tau_4), \\ \frac{dM(t)}{dt} &= \mu_s(1-\alpha)I_s(t-\tau_3) + \mu_b F_b(t-\tau_4) + \mu_g F_g(t-\tau_4) + \mu_c F_c(t-\tau_4), \end{cases} \tag{3.5}$$

where $t \in \mathbb{R}_+$, N represents the total population size and $u \in [0,1]$ denotes the level of the preventive strategies on the susceptible population. The parameter β indicates the transmission rate and $\epsilon \in [0,1]$ is the proportion for the symptomatic individuals. The parameter α denotes the proportion of the diagnosed symptomatic infected population that moves to the three forms: F_b, F_g and F_c, by the rates γ_b, γ_g and γ_c, respectively. The mean recovery period of these forms are denoted by $1/r_b$, $1/r_g$ and $1/r_c$, respectively. The latter forms die also with the rates μ_b, μ_g and μ_c, respectively. Asymptomatic infected population, which are not diagnosed, recover with rate η_a and the symptomatic infected ones recover or die with rates η_s and μ_s, respectively. The time delays τ_1, τ_2, τ_3 and τ_4 denote the incubation period, the period of time needed before the charge by the health system, the time required before the death of individuals coming from the compartments I_s, F_b, F_g, and F_c, respectively. At each instant of time,

$$\mathscr{D}(t) =: \mu_s(1-\alpha)I_s(t-\tau_3) + \mu_b F_b(t-\tau_4) + \mu_g F_g(t-\tau_4) + \mu_c F_c(t-\tau_4) = \frac{dM(t)}{dt} \tag{3.6}$$

gives the number of new death due to the disease (cf. [102]).

Remark 3.11. *In system* (3.5)*, delays occur at the entrances, when the actions of infection take charge or the actions by the health system begin, and not at exits. Let us see an example. A susceptible individual, after contact with an infected person at instant t, becomes himself infected at instant $t + \tau_1$. Suddenly, the compartment of the infected is fed at the instant t by the susceptible infected at the instant $t - \tau_1$. The same operation occurs at the level of the other interactions between the compartments of the model.*

Remark 3.12. *We assume that the compartment of symptomatic infected I_s does not completely empty at any time t. For this reason, one has $\mu_s + \eta_s < 1$. Note also that the diagnosed symptomatic infected population is completely distributed into one of three possible forms: F_b, F_g and F_c, respectively by the rates γ_b, γ_g and γ_c. Then, $\gamma_b + \gamma_g + \gamma_c = 1$.*

Remark 3.13. *Biologically, $\tau_3 = 21$ days and $\tau_4 = 13.5$ days are the time periods needed before dying, deriving from I_s and the three forms F_b, F_g, F_c, respectively. That is why we inserted these delays in the last equation of system* (3.5)*.*

Remark 3.14. *We consider only a short time period in comparison to the demographic time-frame. From a biological point of view, this means that we can assume that there is neither entry (recruitment rate) nor exit (natural mortality rate), and vital parameters can be neglected. Note also that in our model, the individuals that die due to the disease are included in the population. Therefore, the total population is here assumed to be constant, that is, $N(t) \equiv N$ during the period under study. This assumption is also reinforced by the fact that the Moroccan authorities have closed geographic borders.*

The initial conditions of system (3.5) are

$$\begin{array}{llll} S(\theta) & = \varphi_1(\theta) \geq 0, & I_s(\theta) = \varphi_2(\theta) \geq 0, & I_a(\theta) = \varphi_3(\theta) \geq 0, \\ F_b(\theta) & = \varphi_4(\theta) \geq 0, & F_g(\theta) = \varphi_5(\theta) \geq 0, & F_c(\theta) = \varphi_6(\theta) \geq 0, \\ R(\theta) & = \varphi_7(\theta) \geq 0, & M(\theta) = \varphi_8(\theta) \geq 0, & \theta \in [-\tau, 0], \end{array} \tag{3.7}$$

where $\tau = \max\{\tau_1, \tau_2, \tau_3, \tau_4\}$. Let $\mathscr{C} = C([-\tau, 0], \mathbb{R}^8)$ be the Banach space of continuous functions from the interval $[-\tau, 0]$ into $\mathbb{R}^8$ equipped with the uniform topology. It follows from the theory of functional differential equations [64] that system (3.5) with initial conditions

$$(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6, \varphi_7, \varphi_8) \in \mathscr{C}$$

has a unique solution. On the other hand, due to continuous fluctuation in the environment, the parameters of the system are actually not absolute constants and always fluctuate randomly around some average value. Hence, using delayed stochastic differential equations (DSDEs) to model the epidemic provide some additional degree of realism compared to their deterministic counterparts. The parameters β and α play an important role in controlling and preventing COVID-19 spreading and they are not completely known, but subject to some random environmental effects. We introduce randomness into system (3.5) by applying the technique of parameter perturbation, which has been used by many researchers (see, e.g., [43, 69, 89]). In agreement, we replace the parameters β and α by $\beta \to \beta + \sigma_1 \dot{B}_1(t)$ and $\alpha \to \alpha + \sigma_2 \dot{B}_2(t)$, where $B_1(t)$ and $B_2(t)$ are independent standard Brownian motions defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a filtration $\{\mathscr{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathscr{F}_0$ contains all P-null sets) and σ_i represents the intensity of B_i for $i = 1, 2$. Therefore, we obtain the following model governed by delayed stochastic differential equations:

$$
\begin{cases}
dS(t) &= \left(-\beta(1-u)\dfrac{S(t)I_s(t)}{N}\right)dt - \sigma_1(1-u)\dfrac{S(t)I_s(t)}{N}dB_1(t),\\
dI_s(t) &= \left(\beta\epsilon(1-u)\dfrac{S(t-\tau_1)I_s(t-\tau_1)}{N} - \alpha I_s(t) - (1-\alpha)(\mu_s+\eta_s)I_s(t)\right)dt\\
&\quad +\sigma_1\left(\epsilon(1-u)\dfrac{S(t-\tau_1)I_s(t-\tau_1)}{N}\right)dB_1(t) + \sigma_2(\mu_s+\eta_s-1)I_s(t)dB_2(t),\\
dI_a(t) &= \left(\beta(1-\epsilon)(1-u)\dfrac{S(t-\tau_1)I_s(t-\tau_1)}{N} - \eta_a I_a(t)\right)d(t)\\
&\quad +\sigma_1(1-\epsilon)(1-u)\dfrac{S(t-\tau_1)I_s(t-\tau_1)}{N}dB_1(t),\\
dF_b(t) &= \left(\alpha\gamma_b I_s(t-\tau_2) - \left(\mu_b+r_b\right)F_b(t)\right)dt + \sigma_2\gamma_b I_s(t-\tau_2)dB_2(t),\\
dF_g(t) &= \left(\alpha\gamma_g I_s(t-\tau_2) - \left(\mu_g+r_g\right)F_g(t)\right)dt + \sigma_2\gamma_g I_s(t-\tau_2)dB_2(t),\\
dF_c(t) &= \left(\alpha\gamma_c I_s(t-\tau_2) - \left(\mu_c+r_c\right)F_c(t)\right)dt + \sigma_2\gamma_c I_s(t-\tau_2)dB_2(t),\\
dR(t) &= \left(\eta_s(1-\alpha)I_s(t-\tau_3) + \eta_a I_a(t-\tau_3) + r_b F_b(t-\tau_4) + r_g F_g(t-\tau_4) + r_c F_c(t-\tau_4)\right)dt\\
&\quad -\sigma_2\eta_s I_s(t-\tau_3)dB_2(t),\\
dM(t) &= \left(\mu_s(1-\alpha)I_s(t-\tau_3) + \mu_b F_b(t-\tau_4) + \mu_g F_g(t-\tau_4) + \mu_c F_c(t-\tau_4)\right)dt\\
&\quad -\sigma_2\mu_s I_s(t-\tau_3)dB_2(t),
\end{cases}
\tag{3.8}
$$

where the coefficients are locally Lipschitz with respect to all the variables, for all $t \in \mathscr{R}^+$.

We denote $\mathbb{R}^8_+ = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \mid x_i > 0,\ i = 1, 2, \dots, 8\}$. We have the following result.

Theorem 3.15. *For any initial value satisfying condition* (3.7)*, there is a unique solution*

$$x(t) = (S(t), I_s(t), I_a(t), F_b(t), F_g(t), F_c(t), R(t), M(t))$$

to the COVID-19 stochastic model (3.8) *that remains in* $\mathbb{R}^8_+$ *with a probability of one.*

Proof. Since the coefficients of the stochastic differential equations with several delays (3.8) are locally Lipschitz continuous, it follows from [94] that for any square integrable initial value $x(0) \in \mathbb{R}^8_+$, which is independent of the considered standard Brownian motion B, there exists a unique local solution $x(t)$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. For showing that this solution is global, knowing that the linear growth condition is not verified, we need to prove that $\tau_e = \infty$. Let $k_0 > 0$ be sufficiently large for $\dfrac{1}{k_0} < x(0) < k_0$. For each integer $k \geq k_0$, we define the stopping time $\tau_k = \inf\left\{t \in [0, \tau_e) \text{ s.t. } x_i(t) \notin \left(\dfrac{1}{k}, k\right) \text{ for some } i = 1, 2, 3\right\}$, where $\inf \emptyset = \infty$. It is clear that $\tau_k \leq \tau_e$. Let $T > 0$ be arbitrary. Define the twice differentiable function W on $\mathbb{R}^{*^3}_+ \to \mathbb{R}^+$ as follows:

$$W(x) = (x_1 + x_2 + x_3)^2 + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}.$$

By Itô's formula, for any $0 \le t \le \tau_k \wedge T$ and $k \ge 1$, we have

$$dW(x(t)) = LW(x(t))dt + \zeta(x(t))dB(t),$$

where ζ is a continuous functional defined on $[0, +\infty) \times C([-\tau, 0], \mathbb{R}^{3\times 2})$ by

$$\zeta(x(t)) = \begin{pmatrix} -\sigma_1(1-u)\dfrac{S(t)I_s(t)}{N} & 0 \\ \sigma_1\epsilon(1-u)\dfrac{S(t-\tau_1)I_s(t-\tau_1)}{N} & \sigma_2(\mu_s+\eta_s-1)I_s(t) \\ \sigma_1(1-\epsilon)(1-u)\dfrac{S(t-\tau_1)I_s(t-\tau_1)}{N} & 0 \end{pmatrix},$$

$B(t) = (B_1(t), B_2(t))^{\mathscr{T}}$ with the superscript "$\mathscr{T}$" representing transposition, and L is the differential operator of function W defined by

$$\begin{aligned}
LW(x(t)) &= \left(2(S(t)+I_s(t)+I_a(t)) - \frac{1}{S^2(t)}\right)\left(-\beta(1-u)\frac{S(t)I_s(t)}{N}\right) \\
&+ \left(1+\frac{1}{S^3(t)}\right)\left(-\sigma_1(1-u)\frac{S(t)I_s(t)}{N}\right)^2 \\
&+ \left(2(S(t)+I_s(t)+I_a(t)) - \frac{1}{I_s^2(t)}\right) \\
&\times \left[\beta\epsilon(1-u)\frac{S(t-\tau_1)I_s(t-\tau_1)}{N} - \alpha I_s(t) - (1-\alpha)(\mu_s+\eta_s)I_s(t)\right] \\
&+ \left(1+\frac{1}{I_s^3(t)}\right)\left[\left(\sigma_1\epsilon(1-u)\frac{S(t-\tau_1)I_s(t-\tau_1)}{N}\right)^2 + \big(\sigma_2(\mu_s+\eta_s-1)I_s(t)\big)^2\right] \\
&+ \left(2(S(t)+I_s(t)+I_a(t)) - \frac{1}{I_a^2(t)}\right)\left(\beta(1-\epsilon)(1-u)\frac{S(t-\tau_1)I_s(t-\tau_1)}{N} - \eta_a I_a(t)\right) \\
&+ \left(1+\frac{1}{I_a^3(t)}\right)\left(\sigma_1(1-\epsilon)(1-u)\frac{S(t-\tau_1)I_s(t-\tau_1)}{N}\right)^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
LW(x(t)) \le& \frac{\beta(1-u)S(t)I_s(t)}{NS^2(t)} + \left(1+\frac{1}{S^3(t)}\right)\left(\sigma_1(1-u)\frac{S(t)I_s(t)}{N}\right)^2 \\
&+ 2\beta\epsilon(1-u)\big(S(t)+I_s(t)+I_a(t)\big)\frac{S(t-\tau_1)I_s(t-\tau_1)}{N} + \frac{\alpha+(1-\alpha)(\mu_s+\eta_s)}{I_s(t)} \\
&+ \left(1+\frac{1}{I_s^3(t)}\right)\left[\left(\sigma_1\epsilon(1-u)\frac{S(t-\tau_1)I_s(t-\tau_1)}{N}\right)^2 + \big(\sigma_2(\mu_s+\eta_s-1)I_s(t)\big)^2\right] \\
&+ 2\beta(1-\epsilon)(1-u)\big(S(t)+I_s(t)+I_a(t)\big)\frac{S(t-\tau_1)I_s(t-\tau_1)}{N} \\
&+ \frac{\eta_a}{I_a(t)} + \left(1+\frac{1}{I_a^3(t)}\right)\left(\sigma_1(1-\epsilon)(1-u)\frac{S(t-\tau_1)I_s(t-\tau_1)}{N}\right)^2.
\end{aligned} \tag{3.9}$$

We now apply the elementary inequality $2xy \le x^2 + y^2$, valid for any $x, y \in \mathbb{R}$, by firstly taking $x = \beta\epsilon(1-u)$ and $y = S(t) + I_s(t) + I_a(t)$ and, secondly, $x = \beta(1-\epsilon)(1-u)$ and $y = S(t) + I_s(t) + I_a(t)$. In this way, we easily increase the right-hand side of inequality (3.9) to obtain that

$$\begin{aligned}
LW(x(t)) &\le b_1 + \psi\big(S(t)+I_s(t)+I_a(t)\big)^2 + \frac{b_2}{S(t)} + \frac{b_3}{I_s(t)} + \frac{b_4}{I_a(t)} \\
&\le D(1+W(x(t))),
\end{aligned}$$

where $\psi,\ b_1,\ b_2,\ b_3,$ and b_4 are positive constants and

$$D = \max\left(\psi,\ b_1,\ b_2,\ b_3,\ b_4\right)$$

By integrating both sides of equality

$$dW(x(t)) = LW(x(t))dt + \zeta(x(t))dB(t)$$

between t_0 and $t \wedge \tau_k$ and acting the expectation, which eliminates the martingale part, we get

$$\begin{aligned} E(W(x(t\wedge\tau_k)) &= E(W(x_0)) + E\int_{t_0}^{t\wedge\tau_k} LW(x(s)))ds \\ &\leq E(W(x_0)) + E\int_{t_0}^{t\wedge\tau_k} D(1+W(x(s)))ds \\ &\leq E(W(x_0)) + DT + \int_{t_0}^{t\wedge\tau_k} EW(x(s)))ds \end{aligned}$$

and Gronwall's inequality implies that

$$E(W(x(t\wedge\tau_k)) \leq (EW(x_0) + DT)\exp(CT).$$

For $\omega \in \{\tau_k \leq T\}$, $x_i(\tau_k)$ equals k or $\dfrac{1}{k}$ for some $i = 1, 2, 3$. Hence,

$$W(x_i(\tau_k)) \geq \left(k^2 + \frac{1}{k}\right) \wedge \left(\frac{1}{k^2} + k\right).$$

It follows that

$$\begin{aligned} (EW(x_0) + DT)\exp(CT) &\geq E\left(\chi_{\{\tau_k \leq T\}}(\omega) W(x_{\tau_k})\right) \\ &\geq \left(k^2 + \frac{1}{k}\right) \wedge \left(\frac{1}{k^2} + k\right) P(\tau_k \leq T). \end{aligned}$$

Since T is arbitrary, we obtain $P(\tau_e = \infty) = 1$.
By defining the stopping time:
$\tilde{\tau}_k = \inf\left\{t \in [0, \tau_e) \text{ s.t. } x_i(t) \notin \left(\frac{1}{k}, k\right) \text{ for some } i = 4, \ldots, 8\right\}$, and considering the twice differentiable function $\tilde{W}$ on $\mathbb{R}_+^{*^5} \to \mathbb{R}^+$ as

$$\tilde{W}(x) = \left(\sum_{i=4}^{8} x_i\right)^2 + \sum_{i=4}^{8} \frac{1}{x_i},$$

we deduce, with the same technique, that all the variables of the system are positive on $[0, \infty)$. □

3.2.3 Qualitative Analysis of the Models

The basic reproduction number, as a measure for disease spread in a population, plays an important role in the course and control of an ongoing outbreak [48]. This number is defined as the expected number of secondary cases produced, in a completely susceptible population, by a typical infective individual. Note that the calculation of the basic reproduction number R_0 does not depend on the variables of the system but depends on its parameters. In addition, the R_0 of our model does not depend on the time delays. For this reason, we use the next-generation matrix approach outlined in [141] to compute R_0. Precisely, the basic reproduction number $\mathscr{R}_0$ of system (3.5) is given by

$$\mathscr{R}_0 = \rho(FV^{-1}) = \frac{\beta\epsilon(1-u)}{(1-\alpha)(\eta_s+\mu_s)+\alpha}, \tag{3.10}$$

where ρ is the spectral radius of the next-generation matrix FV^{-1} with

$$F = \begin{pmatrix} \beta\epsilon(1-u) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} (1-\alpha)(\eta_s+\mu_s)+\alpha & 0 \\ 0 & \eta_a \end{pmatrix}.$$

Noting that the classes that are directly involved in the spread of disease are only I_s, I_a, F_b, F_g and F_c, we can reduce the local stability of system (3.5) to the local stability of

$$\begin{cases} \dfrac{dI_s(t)}{dt} &= \beta\epsilon(1-u)\dfrac{S(t-\tau_1)I_s(t-\tau_1)}{N} - \alpha I_s(t) - (1-\alpha)(\mu_s+\eta_s)I_s(t), \\ \dfrac{dI_a(t)}{dt} &= \beta(1-\epsilon)(1-u)\dfrac{S(t-\tau_1)I_s(t-\tau_1)}{N} - \eta_a I_a(t), \\ \dfrac{dF_b(t)}{dt} &= \alpha\gamma_b I_s(t-\tau_2) - \big(\mu_b + r_b\big)F_b(t), \\ \dfrac{dF_g(t)}{dt} &= \alpha\gamma_g I_s(t-\tau_2) - \big(\mu_g + r_g\big)F_g(t), \\ \dfrac{dF_c(t)}{dt} &= \alpha\gamma_c I_s(t-\tau_2) - \big(\mu_c + r_c\big)F_c(t). \end{cases} \tag{3.11}$$

The other classes are uncoupled to the equations of system (3.5) and the total population size N is constant. Then, we can easily obtain the following analytical results:

$$\begin{cases} S(t) &= N - \big(I_s(t) + I_a(t) + F_b(t) + F_g(t) + F_c(t) + R(t) + M(t)\big), \\ R(t) &= \int_0^t \Big[\eta_s(1-\alpha)I_s(\delta-\tau_3) + \eta_a I_a(\delta-\tau_3) + r_b F_b(\delta-\tau_4) \\ & \quad + r_g F_g(\delta-\tau_4) + r_c F_c(\delta-\tau_4)\Big] d\delta, \\ M(t) &= \int_0^t \big[\mu_s(1-\alpha)I_s(\delta-\tau_3) + \mu_a I_a(\delta-\tau_3) + \mu_b F_b(\delta-\tau_4) \\ & \quad + \mu_g F_g(\delta-\tau_4) + \mu_c F_c(\delta-\tau_4)\big] d\delta. \end{cases} \tag{3.12}$$

Let $\overline{E} = (\overline{I_s}, \overline{I_a}, \overline{F_b}, \overline{F_g}, \overline{F_c})$ be an arbitrary equilibrium, and consider into system (3.12), the following change of unknowns:

$$U_1(t) = I_s(t) - \overline{I_s},\ U_2(t) = I_a(t) - \overline{I_a},\ U_3(t) = F_b(t) - \overline{F_b}, U_4(t) = F_g(t) - \overline{F_g}$$
$$\text{and } U_5(t) = F_c(t) - \overline{F_c}.$$

By substituting $U_i(t)$, $i = 1, 2, \ldots, 5$, into system (3.12) and linearizing around the free equilibrium, we get a new system that is equivalent to

$$\frac{dX(t)}{dt} = AX(t) + BX(t-\tau_1) + CX(t-\tau_2), \tag{3.13}$$

where $X(t) = (U_1(t), U_2(t), U_3(t), U_4(t), U_5(t))^{\mathscr{T}}$ and A, B, C are the Jacobian matrix of (3.11) given by

$$A = \begin{pmatrix} -\alpha-(1-\alpha)(\mu_s+\eta_s) & 0 & 0 & 0 & 0 \\ 0 & -\eta_a & 0 & 0 & 0 \\ 0 & 0 & -(\mu_b+r_b) & 0 & 0 \\ 0 & 0 & 0 & -(\mu_g+r_g) & 0 \\ 0 & 0 & 0 & 0 & -(\mu_c+r_c) \end{pmatrix},$$

$$B = \begin{pmatrix} \beta\epsilon(1-u) & 0 & 0 & 0 & 0 \\ \beta(1-\epsilon)(1-u) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha\gamma_b & 0 & 0 & 0 & 0 \\ \alpha\gamma_g & 0 & 0 & 0 & 0 \\ \alpha\gamma_c & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic equation of system (3.12) is given by

$$P(\lambda) = (\lambda - a_1(\mathscr{R}_0 e^{-\lambda\tau_1} - 1))(\lambda+\eta_a)(\lambda+(\mu_b+r_b))(\lambda+(\mu_g+r_g))(\lambda+(\mu_c+r_c)), \tag{3.14}$$

where

$$a_1 = \alpha + (1-\alpha)(\mu_s+\eta_s).$$

Clearly, the characteristic Equation (3.14) has the roots $\lambda_1 = -\eta_a$, $\lambda_2 = -(\mu_b + r_b)$, $\lambda_3 = -(\mu_g + r_g)$, $\lambda_4 = -(\mu_c + r_c)$ and the root of the equation

$$\lambda - a_1(\mathscr{R}_0 e^{-\lambda\tau_1} - 1) = 0. \tag{3.15}$$

We suppose $Re(\lambda) \geq 0$. From (3.15), we get

$$Re(\lambda) = a_1(\mathscr{R}_0 e^{-Re(\lambda)\tau_1}\cos(Im\lambda\,\tau_1) - 1) < 0,$$

if $\mathscr{R}_0 < 1$, which contradicts $Re(\lambda) \geq 0$. On the other hand, we show that (3.15) has a real positive root when $\mathscr{R}_0 > 1$. Indeed, we put

$$\Phi(\lambda) = \lambda - a_1(\mathscr{R}_0 e^{-\lambda\tau_1} - 1).$$

We have that $\Phi(0) = -a_1(\mathscr{R}_0 - 1) < 0$, $\lim_{\lambda\to+\infty}\Phi(\lambda) = +\infty$ and function Φ is continuous on $(0, +\infty)$. Consequently, Φ has a positive root and the following result holds.

Theorem 3.16. *The disease free equilibrium of system (3.5), that is, $(N,0,0,0,0,0,0,0)$, is locally asymptotically stable if $\mathscr{R}_0 < 1$ and unstable if $\mathscr{R}_0 > 1$.*

Knowing the value of the deterministic threshold $\mathscr{R}_0$ characterizes the dynamical behavior of system (3.5) and guarantees persistence or extinction of the disease. Similarly, now we characterize the dynamical behavior of system (3.8) by a sufficient condition for extinction of the disease.

Theorem 3.17. *Let $x(t) = \big(S(t), I_s(t), I_a(t), F_b(t), F_g(t), F_c(t), R(t), M(t)\big)$ be the solution of the COVID-19 stochastic model (3.8) with initial value $x(0)$ defined in (3.7). Assume that*

$$\sigma_1^2 > \frac{\beta^2}{2(\alpha + (1-\alpha)(\mu_s + \eta_s))}.$$

Then,

$$\limsup_{t\to+\infty} \ln \frac{I_s(t)}{t} < 0. \tag{3.16}$$

Namely, $I_s(t)$ tends to zero exponentially almost surely, that is, the disease dies out with a probability of one.

Proof. Let

$$\begin{aligned}
d\ln I_s(t) = &\left[\frac{1}{I_s(t)}\left(\beta\epsilon(1-u)\frac{S(t-\tau_1)I_s(t-\tau_1)}{N} - \alpha I_s(t) - (1-\alpha)(\mu_s+\eta_s)I_s(t)\right)\right.\\
&\left.-\frac{1}{2I_s^2(t)}\left(\left(\sigma_1\frac{\beta\epsilon(1-u)S(t-\tau_1)I_s(t-\tau_1)}{N}\right)^2 + \big(\sigma_2(\mu_s+\eta_s-1)I_s(t)\big)^2\right)\right]dt\\
&+\sigma_1\beta\epsilon(1-u)\frac{S(t-\tau_1)I_s(t-\tau_1)}{NI_s(t)}dB_1(t) + \sigma_2(\mu_s+\eta_s-1)dB_2(t).
\end{aligned}$$

To simplify, we set

$$\begin{aligned}
G(t) = \epsilon(1-u)\frac{S(t-\tau_1)I_s(t-\tau_1)}{N}, \quad R_1(t) = \sigma_1\beta\frac{G(t)}{I_s(t)},\\
R_3 = \sigma_2(\mu_s+\eta_s-1), \quad H = -\alpha - (1-\alpha)(\mu_s+\eta_s).
\end{aligned}$$

$$\begin{aligned}
d\ln I_s(t) &= \left[\frac{\beta G(t)}{I_s(t)} + H - \frac{1}{2}\left(\left(\frac{\sigma_1 G(t)}{I_s(t)}\right)^2 + R_3^2\right)\right]dt + R_1(t)dB_1(t) + R_3dB_2(t)\\
&= \left[-\frac{\sigma_1^2}{2}\left[\left(\frac{G(t)}{I_s(t)}\right)^2 - \frac{2\beta}{\sigma_1^2}\frac{G(t)}{I_s(t)}\right] + H - \frac{R_3^2}{2}\right]dt + R_1(t)dB_1(t) + R_3dB_2(t)\\
&= \left[-\frac{\sigma_1^2}{2}\left[\left(\frac{G(t)}{I_s(t)} - \frac{\beta}{\sigma_1^2}\right)^2 - \frac{\beta^2}{\sigma_1^4}\right] + H - \frac{R_3^2}{2}\right]dt + R_1(t)dB_1(t) + R_3dB_2(t)\\
&\le \left[\frac{\beta^2}{2\sigma_1^2} + H\right]dt + R_1(t)dB_1(t) + R_3dB_2(t).
\end{aligned}$$

Integrating both sides of the above inequality between 0 and t, one has

$$\frac{\ln I_s(t)}{t} \leq \frac{\ln I_s(0)}{t} + \frac{\beta^2}{2\sigma_1^2} + H + \frac{M_1(t)}{t} + \frac{M_3(t)}{t},$$

where

$$M_1(t) = \int_0^t R_1(s)dB_1(s) \quad \text{and} \quad M_3(t) = \int_0^t R_3 dB_2(s).$$

We have

$$\begin{aligned} < M_1, M_1 >_t &= \int_0^t {\sigma_1}^2 \epsilon^2 (1-u)^2 \frac{S(s-\tau_1)^2 I_s(s-\tau_1)^2}{N^2 I_s^2(s)} ds \\ &\leq \int_0^t {\sigma_1}^2 \epsilon^2 (1-u)^2 \frac{N^4}{N^2} \frac{1}{I_s^2(s)} ds \\ &\leq \int_0^t {\sigma_1}^2 \epsilon^2 (1-u)^2 ds. \end{aligned}$$

Then,

$$\limsup_{t\to\infty} \frac{< M_1, M_1 >_t}{t} \leq {\sigma_1}^2 \epsilon^2 (1-u)^2 < +\infty.$$

From the large number theorem for martingales [62], we deduce that

$$\lim_{t\to\infty} \frac{M_1(t)}{t} = 0.$$

We also have

$$< M_3, M_3 >_t = \int_0^t \sigma_3^2 (\mu_s + \eta_s - 1)^2 ds = \sigma_3^2 (\mu_s + \eta_s - 1)^2 t.$$

Then,

$$\limsup_{t\to\infty} \frac{< M_3, M_3 >_t}{t} \leq \sigma_3^2 (\mu_s + \eta_s - 1) < +\infty$$

and

$$\lim_{t\to\infty} \frac{M_3(t)}{t} = 0.$$

Subsequently,

$$\limsup_{t\to+\infty} \ln \frac{I_s(t)}{t} \leq \frac{\beta^2}{2\sigma_1^2} - \alpha - (1-\alpha)(\mu_s + \eta_s).$$

We conclude that if $\dfrac{\beta^2}{2\sigma_1^2} - \alpha - (1-\alpha)(\mu_s + \eta_s) < 0$, then $\lim\limits_{t\to\infty} I(t) = 0$. This completes the proof. □

3.2.4 Assessment of Parameters

Estimating the model parameters poses a big challenge because the COVID-19 situation changes rapidly and from one country to another. The parameters are likely to vary over time as new policies are introduced on a day-to-day basis. For this reason, in order to simulate the COVID-19 models (3.5) and (3.8), we consider some parameter values from the literature, while the remaining ones are estimated or fitted.

As the transmission rate β is unknown, we carry out the least-square method [79] to estimate this parameter, based on the actual official reported confirmed cases from 2 March to 20 March, 2020 [96]. Through this method, we estimated β as 0.4517 (95%CI, 0.4484–0.455). Since the life expectancy for symptomatic individuals is 21 days on average and the crude mortality ratio is between 3% to 4% [148], we estimated $\mu_s = 0.01/21$ per day and $\eta_s = 0.8/21$ per day. Furthermore, since the hospitals are not yet saturated and the epidemic situation is under control, we assume that mortality comes mainly from critical forms with a percentage of 40% for an average period of 13.5 days [148]. Then, we choose $\mu_c = 0.4/13.5$ per day and $r_c = 0.6/13.5$ per day. According to [97], the proportion of asymptomatic individuals varies from 20.6% to 39.9% and of symptomatic individuals from 60.1% and 79.4% of the infected population. The progression rates γ_b, γ_g and γ_c, from symptomatic infected individuals to the three forms, are assumed to be 80% of diagnosed cases for benign form, 15% of diagnosed cases for severe form, and 5% of diagnosed cases for critical form, respectively [148]. The incubation period is estimated to be 5.5 days [26] while the time needed before hospitalization is to be 7.5 days [73, 144]. Following a clinical observation related to the situation of COVID-19 in Morocco, an evolution of symptomatic individuals is estimated towards recovery or death after 21 days without any clinical intervention. In the case when clinical intervention is applied, we estimate the evolution of the critical forms towards recovery or death after 13.3 days. The rest of the parameter values are shown in Table 3.2.

Table 3.2: Parameter values of models (3.5) and (3.8).

Parameter	**Value**	**Source**	**Parameter**	**Value**	**Source**
β	0.4517	Estimated	u	[0–1]	Varied
ϵ	0.794	[97]	γ_b	0.8	[148]
γ_g	0.15	[148]	γ_c	0.05	[148]
α	0.06	Assumed	η_a	1/21	Calculated
η_s	0.8/21	Calculated	μ_s	0.01/21	Calculated
μ_b	0	Assumed	μ_g	0	Assumed
μ_c	0.4/13.5	Calculated	r_b	1/13.5	Calculated
r_g	1/13.5	Calculated	r_c	0.6/13.5	Calculated
τ_1	5.5	[26]	τ_2	7.5	[73, 144]
τ_3	21	Assumed	τ_4	13.5	Assumed
σ_1	1.03	Calculated	σ_2	0.1	Assumed

3.2.5 Numerical Simulation of Moroccan COVID-19 Evolution

In this section, we present the forecasts of COVID-19 in Morocco related to different strategies implemented by Moroccan authorities.

Taking into account the four levels of measures attached to containment, the effectiveness level of the applied Moroccan preventive measures is estimated to be

$$u = \begin{cases} 0.2, & \text{on } (2 \text{ March}, 10 \text{ March}]; \\ 0.3, & \text{on } (10 \text{ March}, 20 \text{ March}]; \\ 0.4, & \text{on } (20 \text{ March}, 6 \text{ April}]; \\ 0.8, & \text{after 6 April.} \end{cases}$$

In Figure 3.8, we see that the plots and the clinical data are globally homogeneous. In

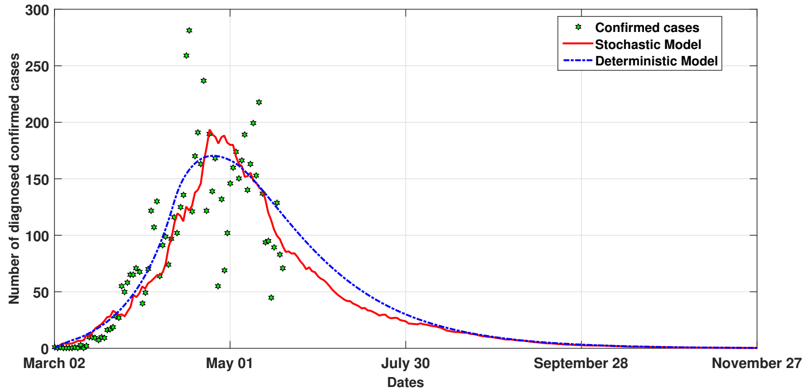

Figure 3.8: Comparison of the deterministic and the stochastic dynamical behavior with the daily reported cases of COVID-19 in Morocco.

addition, the last daily reported cases in Morocco [108], confirm the biological tendency of our model. Thus, our models are efficient to describe the spread of COVID-19 in Morocco. However, we note that some clinical data are far from the values of the models due to certain foci that appeared in some large areas or at the level of certain industrial areas. We conclude also that the stochastic behavior of COVID-19 presents certain particularities contrary to the deterministic one, namely the magnitude of its peak is higher and the convergence to eradication is faster. On the other hand, the conditions in Theorems 3.9 and 3.17 are verified. More precisely, the basic reproduction number $\mathscr{R}_0 = 0.5230$ is less than one from 12 May 2020 and $\sigma_1^2 = 1.0609 > 1.0598 = \dfrac{\beta^2}{2(\alpha + (1-\alpha)(\mu_s + \eta_s))}$, which means that the eradication of disease is ensured.

To prove the biological importance of delay parameters, we give the graphical results of Figure 3.9, which describe the evolution of diagnosed positive cases with and without delays. We observe in Figure 3.9, a high impact of delays on the number of diagnosed positive cases, thereby the plot of model (3.8) without delays ($\tau_i = 0,\ i = 1, 2, 3, 4$) is very different to that of the clinical data. Thus, we conclude that delays play an important role in the study of the dynamic behavior of COVID-19 worldwide, especially in Morocco, and allow us to better understand the reality.

In Figure 3.10, we present the forecast of susceptible, severe forms of deaths and critical forms, from which we deduce that COVID-19 will not attack the total population.

In addition, the number of hospitalization beds or artificial respiration apparatus required can be estimated by the number of different clinical forms. Moreover, we see that the number

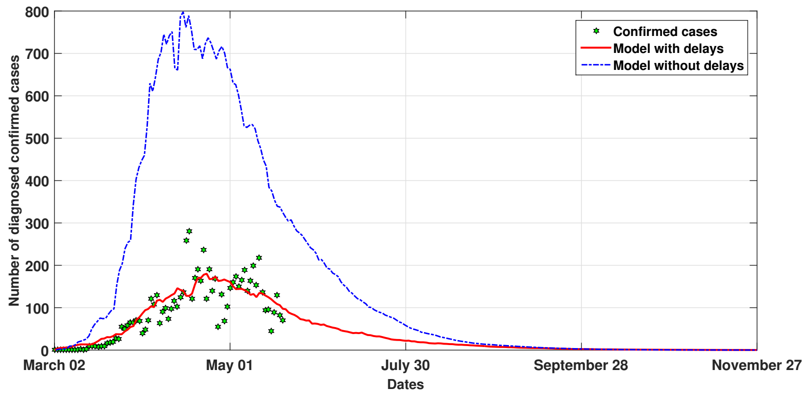

Figure 3.9: Effect of delays on the diagnosed confirmed cases.

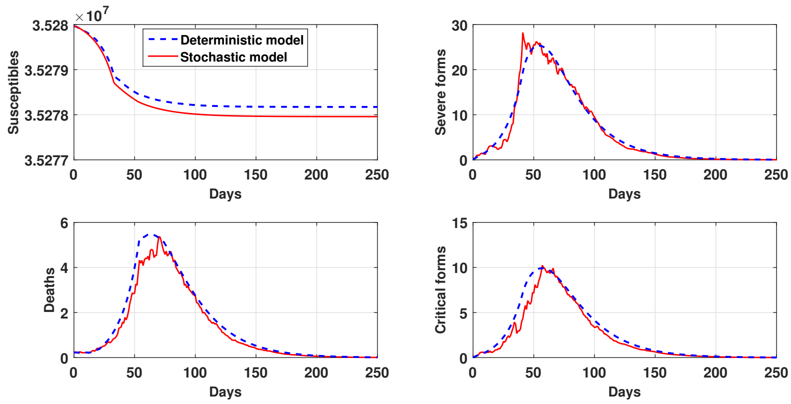

Figure 3.10: The evolution of susceptible, deaths, severe and critical forms from 2 March 2020.

of deaths given by the model is less than those declared in other countries [38], which shows that Morocco has avoided a dramatic epidemic situation by imposing the described strategies.

Finally, we present in Figure 3.11, the cumulative diagnosed cases, severe forms, deaths and critical forms 240 days from the start of the pandemic in Morocco. We summarize some important numbers in Table 3.3, which gives us some information about the future epidemic situation in Morocco.

Table 3.3: Estimated peaks and cumulative of diagnosed cases, severe forms, critical forms and deaths.

Compartments	**Peak**	**Cumulative**
Diagnosed	Around 190	18,890
Severe forms	Around 28	2233
Critical forms	Around 10	997
Deaths	Around 5	468

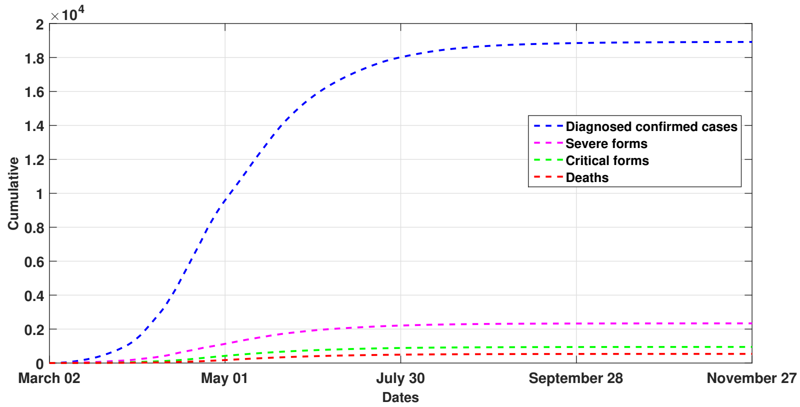

Figure 3.11: Cumulative diagnosed cases, severe forms, critical forms and deaths 240 days from the start of the COVID-19 pandemic in Morocco.

3.2.6 Conclusions

In this study, we proposed a new deterministic model with delay and its corresponding stochastic model to describe the dynamic behavior of COVID-19 in Morocco. These models provide us with the evolution and prediction of important categories of individuals to be monitored, namely, the positive diagnosed cases, which can help to examine the efficiency of the measures implemented in Morocco, and the different developed forms, which can quantify the capacity of the public health system as well as the number of new deaths. Firstly, we have shown that our models are mathematically and biologically well posed by proving global existence and uniqueness of positive solutions. Secondly, the extinction of the disease was established. By analyzing the characteristic equation, we proved that if $\mathscr{R}_0 < 1$, then the disease free equilibrium of the deterministic model is locally asymptotically stable (Theorem 3.9). Based on the Lyapunov analysis method, a sufficient condition for the extinction was obtained in the stochastic case (Theorem 3.17). Thirdly, and since there is a substantial interest in estimating the parameters, we applied the least square method to determine the confidence interval of the transmission rate β as 0.4517 (95%CI, 0.4484–0.455). In addition, the rest of the parameters were either assumed, based on some daily observations, or taken from the available literature. Finally, some numerical simulations were performed to gather information in order to be able to fight against the propagation of the new coronavirus. In 12 May 2020, the basic reproduction number was less than one ($\mathscr{R}_0 = 0.5230$), which means that the epidemic was tending toward eradication, which is conditional on strict compliance with the implemented measures. Currently, the consequences of the measures taken against COVID-19 in Morocco encourage their maintenance to control the spread of the epidemic and quickly move towards extinction.

As future work, we intend to study the regional evolution of COVID-19 in Morocco.

3.3 A Stochastic Capital-Labour Model with Logistic Growth Function

The original results of this section are published in [159].

3.3.1 Introduction

Labour supply and demand are the essential variables governing the labour market. They are influenced by demographic factors and the gross domestic product, which vary from household to household. In our context, the supply of labour is represented by the number of free jobs and the demand for labour, noting that the workforce or labour force is the total number of people eligible to work.

Motivated by the previous information, we propose to model the labour market by ordinary differential equations (ODEs) describing the different interactions between the essential components, that is, the free jobs and the labour force. The suggested model will take the following form:

$$\begin{cases} \dfrac{du}{dt}(t) = ru(t)\left(1 - \dfrac{u(t)}{K}\right) - mu(t)v(t), \\ \dfrac{dv}{dt}(t) = mu(t)v(t) - dv(t), \end{cases} \tag{3.17}$$

where u denotes the number of free jobs and v represents the total unemployed labour force. The positive constant r is the natural per capita growth of free jobs and K is the theoretical eventual maximum of the number of free jobs (related to the theoretical maximum of investment capital). The positive parameter d is the disappearance rate of labour force and muv is the rate by which the labour force fills in the free jobs.
We have adopted the bilinear form to pass from the labour force compartment to the free job one, while the recruitment of people depends progressively and proportionally to the considered employment policy.

It is well known that economies are subject to randomness in terms of natural perturbation processes. Therefore, stochastic models are more suitable than deterministic ones, because they can take into account not only the mean trend but also the variance structure around it. Moreover, deterministic models will always produce the same results for fixed initial conditions, whereas the stochastic ones may give different predicted values. Thus, in order to take into account all the previous arguments, in this work we propose the following stochastic capital-labour model with a logistic growth function:

$$\begin{cases} du(t) = \left[ru(t)\left(1 - \dfrac{u(t)}{K}\right) - mu(t)v(t)\right]dt - \sigma u(t)v(t)dB, \\ dv(t) = [mu(t)v(t) - dv(t)]\,dt + \sigma u(t)v(t)dB, \end{cases} \tag{3.18}$$

where $B(t)$ is a standard Brownian motion with intensity σ, defined on a complete filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ with the filtration $(\mathscr{F}_t)_{t\geq 0}$ satisfying the usual conditions.

Our work is organized as follows. first, we prove existence and uniqueness of a global positive solution to our stochastic model (3.18). Then, using the Lyapunov analysis method, we prove the extinction of the total labour force under an appropriate condition. Furthermore, we give sufficient conditions for the persistence in mean of the total labour force. Follows some numerical simulations to illustrate our analytical results. Lastly, we finish with suitable conclusions. Here all the equations and inequalities are intended almost surely (a.s.).

3.3.2 Existence and uniqueness of global economic solutions

To investigate the dynamical behaviour of a population model, the first concern is whether the solution of the model is positive and global. In order to get a stochastic differential equation for which a unique global solution exists, i.e., there is no explosion within a finite time, for any initial value, standard assumptions for existence and uniqueness of solutions are the linear growth condition and the local Lipschitz condition (cf. Mao [145]). However, the coefficients of system (3.18) do not satisfy the linear growth condition as the incidence is non-linear. Therefore, the solution of system (3.18) may explode at a finite time. In this section, using the Lyapunov analysis method [124, 126], we show that the solution of system (3.18) is positive and global.

Theorem 3.18. *For any given initial value* $(u(0), v(0)) \in \mathbb{R}^2_+$*, there exists a unique positive solution* $(u(t), v(t)) \in \mathbb{R}^2_+$ *of model* (3.18) *for all* $t \geq 0$ *a.s. (almost surely). Moreover,*

$$\limsup_{t\to\infty} u(t) \leq \frac{rK}{\mu} \ \ a.s., \quad \limsup_{t\to\infty} v(t) \leq \frac{rK}{\mu} \ \ a.s.,$$

where $\mu = \min\{r, d\}$.

Proof. Since the drift and the diffusion of (3.18) are locally Lipschitz, then for any given initial value $(u(0), v(0)) \in \mathbb{R}^2_+$, there exists a unique local solution for $t \in [0, \tau_e)$, where τ_e is the explosion time. To show that this solution is global, we need to show that $\tau_e = +\infty$. Define the stopping time τ^+ as

$$\tau^+ := \inf\{t \in [0, \tau_e) \ : u(t) \leq 0 \text{ or } v(t) \leq 0\}.$$

We suppose that $\tau^+ < +\infty$. For any $t \leq \tau^+$, we define the following function:

$$F(t) := \ln(u(t)v(t)).$$

By using Itô's formula and system (3.18), we obtain that

$$\begin{aligned} dF &= r\left(1 - \frac{u}{K}\right) - mv + mu - d - \frac{\sigma^2}{2}(u^2 + v^2) + \sigma(u - v)dB \\ &\geq -\frac{r}{K}u - mv - d - \frac{\sigma^2}{2}(u^2 + v^2) + \sigma(u - v)dB. \end{aligned}$$

Integrating both sides between 0 and t, we get that

$$F(t) \geq F(0) + \int_0^t H(s)ds + \sigma \int_0^t (u(s) - v(s))dB(s), \tag{3.19}$$

where $H(s) = -\frac{r}{K}u(s) - mv - d - \frac{\sigma^2}{2}(u^2(s) + v^2(s))$. At least one among $u(\tau^+)$ and $v(\tau^+)$ is equal to 0. Then, we get

$$\lim_{t\to\tau^+} F(t) = -\infty.$$

Letting $t \to \tau^+$ in (3.19) we obtain

$$-\infty \geq F(t) \geq F(0) + \int_0^{\tau^+} H(s)ds + \sigma \int_0^{\tau^+} (u(s) - v(s))dB(s) > -\infty,$$

which is a contradiction. Thereby, $\tau^+ = +\infty$, which means that the model has a unique global solution $(u(t), v(t)) \in \mathbb{R}_+^2$ a.s. We now prove the boundedness. If we sum the equations from system (3.18), then

$$dN(t) = \left(ru\left(1 - \frac{u}{K}\right) - dv\right) dt,$$

where $N(t) = u(t) + v(t)$. Thus,

$$\begin{aligned}
dN(t) &= \left(ru(1 - \frac{u}{K}) - dv\right) dt \\
&= \left(ru - dv - \frac{r}{K}(u-K)^2 - 2ur + rK\right) dt \\
&= \left(-ru - dv - \frac{r}{K}(u-K)^2 + rK\right) dt, \\
\frac{dN}{dt} &\leq -\mu N + rk,
\end{aligned}$$

where $\mu = \min\{r, d\}$, and so

$$\begin{aligned}
e^{\mu t}\frac{dN}{dt} &\leq e^{\mu t}\left(-\mu N + rk\right), \\
\int_0^t e^{\mu s}\frac{dN}{ds} ds &\leq \int_0^t e^{\mu s}\left(-\mu N(s) + rK\right) ds, \\
e^{\mu t}N(t) &\leq \frac{rK}{\mu}(e^{\mu t} - 1) + N(0), \\
N(t) &\leq \frac{rK}{\mu}(1 - e^{-\mu t}) + N(0)e^{-\mu t}, \\
\limsup_{t\to\infty} N(t) &\leq \frac{rK}{\mu} \text{ a.s.}
\end{aligned}$$

This fact implies that $\limsup_{t\to\infty} u(t) \leq \dfrac{rK}{\mu}$ a.s. and $\limsup_{t\to\infty} v(t) \leq \dfrac{rK}{\mu}$ a.s., which completes the proof. □

3.3.3 Extinction of total labour force

When studying dynamical systems, it is important to discuss the possibility of extinction or persistence of a population. Here we investigate extinction of the capital-labour.

Theorem 3.19. *For any initial data* $(u(0), v(0)) \in \mathbb{R}_+^2$, *if* $\dfrac{m^2}{2\sigma^2} - d < 0$, *then* $v(t) \to 0$ *when* $t \to +\infty$ *a.s.*

Proof. Let us define $G_1(t) := \log(v(t))$. Applying Itô's formula to G leads to

$$dG_1 = \left(mu(t) - d - \frac{\sigma^2}{2}u^2(t)\right) dt + \sigma u(t)dB,$$

$$dG_1 = \left(-\frac{\sigma^2}{2}\left(u(t) - \frac{m}{\sigma^2}\right)^2 + \frac{m^2}{2\sigma^2} - d\right)dt + \sigma u(t)dB,$$
$$dG_1 \leq \left(\frac{m^2}{2\sigma^2} - d\right)dt + \sigma u(t)dB.$$

Integrating from 0 to t and dividing both sides by t, we have

$$\begin{aligned}\frac{\log(v(t)}{t} &\leq \frac{\log(v_0)}{t} + \frac{1}{t}\int_0^t (\frac{m^2}{2\sigma^2} - d)ds + \frac{\sigma}{t}\int_0^t u(s)dB(s)\\ &\leq \frac{\log(v_0)}{t} + \frac{1}{t}\int_0^t (\frac{m^2}{2\sigma^2} - d)ds + \frac{\sigma}{t}\int_0^t u(s)dB(s)\\ &\leq \frac{\log(v_0)}{t} + \frac{m^2}{2\sigma^2} - d + \frac{\sigma}{t}\int_0^t u(s)dB(s).\end{aligned}$$

Let $M_t := \displaystyle\int_0^t \sigma u(s)dB_s$. Then,

$$\limsup_{t\to+\infty} \frac{< M_t, M_t >}{t} = \limsup_{t\to+\infty} \frac{\sigma^2}{t}\int_0^t u^2(s)ds \leq \sigma^2\left(\frac{rK}{\mu}\right)^2 < +\infty$$

and, by using the strong law of large numbers for martingales (see, e.g., [145]),

$$\limsup_{t\to+\infty} \frac{M_t}{t} = 0.$$

Therefore,

$$\limsup_{t\to+\infty} \frac{\log(v(t)}{t} \leq \frac{\log(v_0)}{t} + \frac{m^2}{2\sigma^2} - d.$$

Thus, if $\dfrac{m^2}{2\sigma^2} - d < 0$, then $v(t) \to 0$ when $t \to +\infty$ a.s. □

3.3.4 Persistence in the mean of total labour force

Now, we investigate the persistence property of $v(t)$ in the mean, that is, we give conditions for which

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t v(s)ds > 0.$$

For convenience, we introduce the following notation:

$$\langle x(t)\rangle := \frac{1}{t}\int_0^t x(s)ds.$$

Theorem 3.20. *Let $(u(t), v(t))$ be a solution of system (3.18) with initial value $(u(0), v(0)) \in \mathbb{R}_+^2$. If*

$$R_0^s = \frac{r}{d} - \frac{\sigma^2 K^2}{2d} > 1 \quad \textit{and} \quad m > \frac{r}{K}, \tag{3.20}$$

then the variable $v(t)$ satisfies the following expression:

$$\liminf_{t\to\infty}\langle v\rangle \geq \frac{d(R_0^s - 1)}{m + d} > 0. \tag{3.21}$$

Proof. Using the second equation of system (3.18), we have

$$\frac{v(t)-v(0)}{t} = m\langle uv\rangle - d\langle v\rangle + \frac{\sigma}{t}\int_0^t u(s)v(s)dB. \tag{3.22}$$

Applying Itô's formula on model (3.18) leads to

$$d\ln(u(t)) = \left[r\left(1-\frac{u}{K}\right) - mv - \frac{1}{2}\sigma^2 v^2\right]dt - \sigma v dB \tag{3.23}$$

and

$$d\ln(v(t)) = \left[mu - d - \frac{1}{2}\sigma^2 u^2\right]dt + \sigma u dB. \tag{3.24}$$

Integrating both sides of (3.23) and (3.24) from 0 to t, and dividing by t, leads to

$$\frac{\ln(u(t))-\ln(u(0))}{t} = r - \frac{r}{K}\langle u\rangle - m\langle v\rangle - \frac{\sigma^2}{2}\langle v^2\rangle - \frac{\sigma}{t}\int_0^t v(s)dB \tag{3.25}$$

and

$$\frac{\ln(v(t))-\ln(v(0))}{t} = m\langle u\rangle - d - \frac{\sigma^2}{2}\langle u^2\rangle + \frac{\sigma}{t}\int_0^t u(s)dB. \tag{3.26}$$

Combining (3.22), (3.25), and (3.26), we derive that

$$\begin{aligned}
&\frac{v(t)-v(0)}{t} + \frac{\ln(u(t))-\ln(u(0))}{t} + \frac{\ln(v(t))-\ln(v(0))}{t}\\
&= r - \frac{r}{K}\langle u\rangle + m\langle u\rangle + (m-\sigma^2)\langle uv\rangle - (d+m)\langle v\rangle - d - \frac{\sigma^2}{2}\langle (u+v)^2\rangle\\
&+ \frac{\sigma}{t}\int_0^t (u(s)-v(s)+u(s)v(s))dB_s\\
&\geq r - d - \frac{\sigma^2}{2}K^2 - (d+m)\langle v\rangle + \left(m - \frac{r}{K}\right)\langle u\rangle\\
&+ \frac{\sigma}{t}\int_0^t (u(s)-v(s)+u(s)v(s))dB_s.
\end{aligned}$$

Since $m - \dfrac{r}{K} > 0$, then

$$\begin{aligned}
\frac{v(t)-v(0)}{t} + \frac{\ln(u(t))-\ln(u(0))}{t} &+ \frac{\ln(v(t))-\ln(v(0))}{t}\\
&\geq r - d - \frac{\sigma^2}{2}K^2 - (d+m)\langle v\rangle\\
&+ \frac{\sigma}{t}\int_0^t (u(s)-v(s)+u(s)v(s))dB_s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(d+m)\langle v\rangle \geq & r - d - \frac{\sigma^2}{2}K^2 + \frac{\sigma}{t}\int_0^t (u(s)-v(s)+u(s)v(s))dB(s)\\
&- \frac{v(t)-v(0)}{t} - \frac{\ln(u(t))-\ln(u(0))}{t} - \frac{\ln(v(t))-\ln(v(0))}{t}.
\end{aligned}$$

Table 3.4: Parameter values used in the numerical simulations.

Parameters	Fig. 3.12	Fig. 3.13
r	1	1
d	0.2	0.2
m	0.001	0.1
K	100	100
σ	0.09	0.001

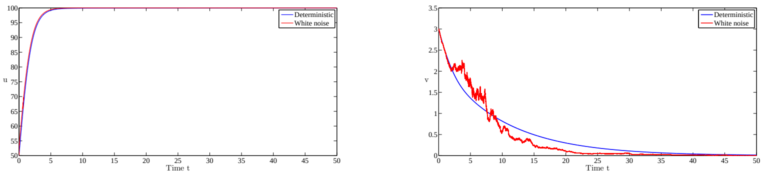

Figure 3.12: Extinction of the total labour force.

Let us denote

$$M_1(t) := \sigma \int_0^t (u(s) - v(s) + u(s)v(s))dB(s).$$

Using the strong law of large numbers for martingales, together with the fact that almost surely for every ε there exists T such that $0 < u(t), v(t) < \frac{rK}{\mu} + \varepsilon$ for every $t > T$, we can say that

$$\lim_{t\to\infty} \frac{v(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{u(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{M_1(t)}{t} = 0 \quad a.s.$$

Thus,

$$\liminf_{t\to\infty} \langle v\rangle \geq \frac{r - d - \dfrac{\sigma^2}{2}K^2}{(d+m)} = \frac{d(R_0^s - 1)}{m+d} > 0,$$

where $R_0^s = \dfrac{r}{d} - \dfrac{\sigma^2 K^2}{2d}$. The proof is complete. $\square$

In this section, we illustrate our mathematical results through numerical simulations. In the two considered examples, we apply the algorithm presented in [70] to solve system (3.18) and we use the parameter values from Table 3.4, inspired from [116].

Figure 3.12 shows the evolution of the free jobs and the total labour force during the period of observation. It can be seen that both curves of the total labour force, corresponding to the deterministic and to the stochastic models, converge toward zero. This indicates the extinction of the total labour force, which is consistent with our theoretical results. Indeed, for the used parameters (see Table 3.4), one has $\dfrac{m^2}{2\sigma^2} - d = -0.19 < 0$ and it follows, from Theorem 3.19, that $v(t) \to 0$ with probability one when $t \to +\infty$.

The evolution of the free jobs and the total labour force, for both deterministic and stochastic models, is also illustrated in Fig. 3.13. In this case, the key conditions (3.20) of our Theorem 3.20 are satisfied: $R_0^s = \dfrac{r}{d} - \dfrac{\sigma^2 K^2}{2d} = 4.99 > 1$ and $m - \dfrac{r}{K} = 0.09 > 0$. As predicted by Theorem 3.20, one can clearly see in Fig. 3.13 the persistence of the total labour force.

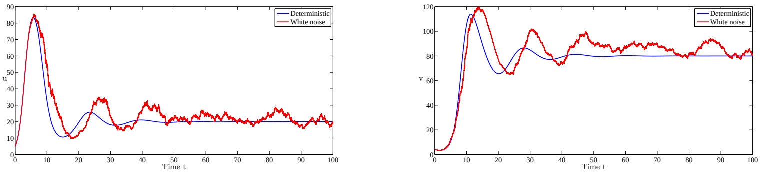

Figure 3.13: Persistence of the total labour force.

3.3.5 Conclusions

The labour force (workforce), which can be defined as the total number of people who are eligible to work, is a centred component of each modern economy, whereas free jobs are systematically supplied by companies. In this work, we have proposed and analysed a capital-labour model by means of an economic dynamical system describing the interaction between free jobs and labour force. Mathematically, our model is governed by stochastic differential equations, where the component of stochastic noise is considered for an additional degree of realism, intended to describe well reality. Furthermore, the transmission rate by which the labour force individuals are moving to the free jobs compartment is modelled by the logistic growth function with an appropriate carrying capacity K. Some relevant results were obtained. First of all, by proving existence and uniqueness of a global positive solution, as well as its boundedness, we have shown that the proposed model is mathematically and economically well-posed. Moreover, a sufficient condition for the extinction of labour force is obtained, via the strong law of large numbers for martingales, in addition to adequate sufficient conditions for the persistence in mean. In order to illustrate our theoretical results, we have implemented some numerical simulations where, for a good accuracy of the approximate numerical solutions, the Milstein scheme has been used.

3.4 A stochastic SICA Epidemic Model with Jump Lévy Processes

The results of this section are published in [160].

3.4.1 Introduction

Human immunodeficiency virus (HIV) is known as a pathogen causing the acquired immunodeficiency syndrome (AIDS), which is the end-stage of the infection. After that the immune system fails to play its life-sustaining role [30, 147]. On the other hand, according to the world health organization [148], 36.7 million people living with HIV, 1.8 million people becoming newly infected with HIV and more than 1 million deaths annually. Based on these alarming statistics, HIV becomes a major global public health issue. Mathematical modeling of HIV viral dynamics is a powerful tool for predicting the evolution of this disease [10, 77, 103, 129, 131].

On the other hand, stochastic quantification of several real life phenomena have been much helpful in understanding the random nature of their incidence or occurrence. This also helped in finding solutions to such problems arising from them either in form of minimization of their undesirability or maximizing their rewards. Besides, the infectious diseases are exposed to randomness and uncertainty in terms of normal infection progress. Therefore, the stochastic

Table 3.5: Parameters, their Meaning in the suggested SICA model

Parameters	Meaning
Λ	Recruitment rate
μ	Natural death rate
β	The transmission rate
ϕ	HIV treatment rate for I individuals
ρ	Default treatment rate for I individuals
α	AIDS treatment rate
ω	Default treatment rate for C individuals
d	AIDS induced death rate

modeling are more appropriate comparing to the deterministic models; considering the fact that the stochastic systems do not take into account only the variable mean but also the standard deviation behavior surround it. Moreover, the deterministic systems generate similar results for initial fixed values, but the stochastic ones can give different predicted results. Several stochastic infectious models describe the effect of white noise on viral dynamics have been deployed [156, 6, 91, 112, 115].

In this work, based on [125], we propose and analyze a mathematical model for the transmission dynamics of HIV and AIDS. Our aim is to show the effect of the lévy jump in the dynamics of the population, the Lévy noise is used to describe the contingency and the outburst. So will be interesting to consider the following stochastic model driven by white noise and the Lévy noise jointly:

$$\begin{cases} dS(t) = (\Lambda - \beta I(t)S(t) - \mu S(t))\,dt - \sigma I(t)S(t)dW_t \\ \qquad - \displaystyle\int_U J(u)I(t-)S(t-)\check{N}(dt,du), \\ dI(t) = (\beta I(t)S(t) - (\rho+\phi+\mu)I(t) + \alpha A(t) + \omega C(t))\,dt + \sigma I(t)S(t)dW_t \\ \qquad + \displaystyle\int_U J(u)I(t-)S(t-)\check{N}(dt,du), \\ dC(t) = (\phi I(t) - (\omega+\mu)C(t))\,dt, \\ dA(t) = (\rho I(t) - (\alpha+\mu+d)A(t))\,dt, \end{cases} \tag{3.27}$$

where susceptible individuals (S); HIV-infected individuals with no clinical symptoms of AIDS (the virus is living or developing in the individuals but without producing symptoms or only mild ones) but able to transmit HIV to other individuals (I); HIV-infected individuals under ART treatment (the so called chronic stage) with a viral load remaining low (C) and HIV-infected individuals with AIDS clinical symptoms (A). the parameters of the SICA model (3.27) is given in table 3.5. Moreover, W_t is a standard Brownian motion with intensity σ defined on a complete filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ with the filtration $(\mathscr{F}_t)_{t\geq 0}$ satisfying the usual conditions. We denote by $S(t-)$, $I(t-)$ and $z(t-)$ the left limits of $S(t)$, $I(t)$ and $z(t)$ respectively. $N(dt,du)$ is a Poisson counting measure with the stationary compensator $\nu(du)dt$, $\check{N}(dt,du) = N(dt,du) - \nu(du)dt$ where ν is defined on a measurable subset U of the non-negative half-line with $\nu(U) < \infty$. $J(u)$ represents the jumps intensity.

3.4.2 Existence and Uniqueness of the global positive solution

Define

$$\Omega = \{(S;I;C;A) \in \mathbb{R}^4_+ / \frac{\Lambda}{\mu+d} \leq S+I+C+A \leq \frac{\Lambda}{\mu}\}$$

Theorem 3.21. *For any given initial data $(S(0);I(0);C(0);A(0)) \in \Omega$, there exists a unique global positive solution $(S(t);I(t);C(t);A(t)) \in \Omega$ for every $t \geq 0$, a.s of the system 3.18. Moreover,*

$$\begin{aligned}
&\limsup_{t\to\infty} S(t) \leq \frac{\Lambda}{\mu} \ a.s., && \liminf_{t\to\infty} S(t) \geq \frac{\Lambda}{\mu+d} \ a.s.,\\
&\limsup_{t\to\infty} I(t) \leq \frac{\Lambda}{\mu} \ a.s., && \liminf_{t\to\infty} I(t) \geq \frac{\Lambda}{\mu+d} \ a.s.,\\
&\limsup_{t\to\infty} C(t) \leq \frac{\Lambda}{\mu} \ a.s., && \liminf_{t\to\infty} C(t) \geq \frac{\Lambda}{\mu+d} \ a.s.,\\
&\limsup_{t\to\infty} A(t) \leq \frac{\Lambda}{\mu} \ a.s., && \liminf_{t\to\infty} A(t) \geq \frac{\Lambda}{\mu+d} \ a.s..
\end{aligned}$$

Proof. By Assumption (H) J is a bounded function and $0 < J(u) \leq \frac{\mu}{\Lambda}$, $u \in U$. The local Lipschitzianity of the drift and the diffusion with the given initial data $(S(0);I(0);C(0);A(0)) \in \Omega$ enable us to confirm existence and uniqueness of the local solution $(S(t);I(t);C(t);A(t)) \in \Omega$ for $t \in [0,\tau_e)$, where τ_e is the explosion time.
To prove that the current solution is global, we need to define the stopping time

$$\tau = \{t \in [0,\tau_e)/S(t) \leq 0, I(t) \leq 0, C(t) \leq 0, A(t) \leq 0\}$$

Assuming that $\tau_e < \infty$, we have $\tau \leq \tau_e$ then there exist $T > 0$ and $\epsilon > 0$ such that $P(\tau \leq T) > \epsilon$. We consider the following function V on $\mathbb{R}^4_+$:
$V(x,y,z,t) = log(xyzt)$
Using the Itô's formula, we get:

$$\begin{aligned}
dV(t,X(t)) &= LV(t,X(t))dt + \partial_x V(t,X(t)).b(t,X(t))dW_t\\
&\quad + \int_U (V(X(t-)+J(u)) - V(X(t-)))\check{N}(dt,du)
\end{aligned}$$

where b reflects with abbreviation, the drift coefficient

$$\begin{aligned}
dV = LVdt + \left(\frac{\Lambda}{S} - \beta I + \beta S - rh - \phi - 2\mu + \alpha\frac{A}{I} + \omega\frac{C}{I}\right) dW_t\\
+ \int_U \log(1-JI)(1+JS)\check{N}(dt,du),
\end{aligned}$$

where L denotes the differential operator. We have,

$$\begin{aligned}
LV = (\Lambda - \beta IS - \mu S)\frac{1}{S} + (\beta IS - (\rho+\phi+\mu)I + \alpha A + \omega C)\frac{1}{I} - \frac{\sigma^2 I^2}{2} - \frac{\sigma^2 S^2}{2}\\
+ \int_U [\log(1-JI)(1+JS) - JI + JS]\nu(du).
\end{aligned}$$

Remark that $1 - JI > 0$ from Assumption (H). Consequently,

$$LV \geq -\frac{\beta\Lambda}{\mu} - 2\mu - \rho - \phi - \frac{\sigma^2\Lambda^2}{\mu^2} + \int_U [\log(1-JI) + JI]\nu(du) + \int_U [\log(1+JS) - JS]\nu(du) := K.$$

Observe that $x \longmapsto \log(1+x) - x$ and $x \longmapsto \log(1-x) + x$ are nonpositive functions.
So,
$dV \geq Kdt + \left(\frac{\Lambda}{S} - \beta I + \beta S - rh - \phi - 2\mu + \alpha\frac{A}{I} + \omega\frac{C}{I}\right) dW_t + \int_U \log(1-JI)(1+JS)\check{N}(dt, du)$,
Integrating the last equality from 0 to t, we get
$V(S(t), I(t), C(t), A(t)) \geq V(S(0), I(0), C(0), A(0)) + K(t)$
$+ \int_0^t \int_U \left(\frac{\Lambda}{S} - \beta I + \beta S - rh - \phi - 2\mu + \alpha\frac{A}{I} + \omega\frac{C}{I}\right) dW_s$
$+ \int_0^t \int_U \log(1 - JI)(1 + JS)\check{N}(ds, du)$.
Because of the continuity of the state variable, some components of

$$(S(\tau), I(\tau), C(\tau), A(\tau))$$

being equal to 0, we get therefore that $\lim_{t\to\tau} V(\tau) = -\infty$.
Letting $t \to \tau$, we deduce
$-\infty \geq V(S(0), I(0), C(0), A(0)) + K(t)$
$+ \int_0^t \int_U \left(\frac{\Lambda}{S} - \beta I + \beta S - rh - \phi - 2\mu + \alpha\frac{A}{I} + \omega\frac{C}{I}\right) dW_s$
$+ \int_0^t \int_U \log(1 - JI)(1 + JS)\check{N}(ds, du) > \infty$., which contradicts our assumption and achieves the proof.
We shall show the boundedness of the solution.
Summing up the equations from system (3.27) yields

$$\frac{dN(t)}{dt} = \Lambda - \mu N(t) - dA(t)$$

Adressing an upper and lower bounds, we get

$$\Lambda - (\mu + d)N(t) \leq \frac{dN(t)}{dt} \leq \Lambda - \mu N(t),$$

where $N(t) = S(t) + I(t) + C(t) + A(t)$. So,

$$\begin{aligned} e^{\mu t}\frac{dN(t)}{dt} &\leq e^{\mu t}\left(\Lambda - \mu N(t)\right), \\ \int_0^t e^{\mu s}\frac{dN(s)}{ds} ds &\leq \int_0^t e^{\mu s}\left(\Lambda - \mu N(s)\right) ds, \\ e^{\mu t}N(t) &\leq \frac{\Lambda}{\mu}(e^{\mu t} - 1) + N(0), \\ N(t) &\leq \frac{\Lambda}{\mu}(1 - e^{-\mu t}) + N(0)e^{-\mu t}, \\ \limsup_{t\to\infty} N(t) &\leq \frac{\Lambda}{\mu} \text{ a.s.} \end{aligned}$$

Adopting the same technique, we have also, $\liminf_{t\to\infty} N(t) \geq \dfrac{\Lambda}{\mu + d}$ a.s..
This confirms the requested boundedness. □

3.4.3 Extinction of $I(t)$

The extinction of $I(t)$ under some sufficient condition is investigated in this work.

Theorem 3.22. *Then, if* $\frac{\beta^2}{2\sigma^2} < (\rho + \phi + \mu) + (\alpha + \omega)\frac{\Lambda}{\mu}$, *so* $I(t) \to 0$ *when* $t \to +\infty$ *a.s.*

Proof. Let,

$$V(I) = \log(I)$$

Using Itô's formula corresponding to the Poissonian process, we get

$$\begin{aligned} dV(t, X(t)) &= LV(t, X(t))dt + \partial_x V(t, X(t)).\sigma I(t)S(t)dW_t \\ &+ \int_U (V(X(t-) + J(u)) - V(X(t-)))\, \check{N}(dt, du) \\ &= LV(t, X(t))dt + \sigma S(t)dW_t + \int_U (\log(1 + JS))\, \check{N}(dt, du), \end{aligned}$$

where,
$LV = (\beta IS - (\rho + \phi + \mu)I + \alpha A + \omega C)\frac{1}{I} - \frac{\sigma^2 S^2}{2} + \int_U \log(1 + JS) - JS]\nu(du)$.
Which implies that,
$LV \leq \frac{\beta^2}{2\sigma^2} - (\rho + \phi + \mu) + (\alpha + \omega)\frac{\Lambda}{\mu}$
$dV \leq (\frac{\beta^2}{2\sigma^2} - (\rho + \phi + \mu) + (\alpha + \omega)\frac{\Lambda}{\mu})dt + \sigma S(t)dW_t + \int_U \log(1 + JS)\check{N}(dt, du)$
Integrating this from 0 to t and dividing by t on both sides, we have

$$\begin{aligned} \frac{\log(I(t))}{t} &\leq \frac{\log(I_0)}{t} + \frac{1}{t}\int_0^t (\frac{\beta^2}{2\sigma^2} - (\rho + \phi + \mu) + (\alpha + \omega)\frac{\Lambda}{\mu})ds \\ &+ \frac{1}{t}\int_0^t \sigma S(s)dB(s) + \frac{1}{t}\int_0^t \int_U \log(1 + JS)\check{N}(ds, du) \\ &\leq \frac{\log(I_0)}{t} + \frac{\beta^2}{2\sigma^2} - (\rho + \phi + \mu) + (\alpha + \omega)\frac{\Lambda}{\mu} + \frac{\sigma}{t}\int_0^t S(s)dB(s) \\ &+ \frac{1}{t}\int_0^t \int_U \log(1 + JS)\check{N}(ds, du) \end{aligned}$$

We pose

$$M_t = \int_0^t \sigma S(s)dB_s,$$

so,

$$\limsup_{t\to+\infty} \frac{< M_t, M_t >}{t} = \limsup_{t\to+\infty} \frac{\sigma^2}{t}\int_0^t S^2(s)ds \leq \sigma^2 \left(\frac{\Lambda}{\mu}\right)^2 < \infty,$$

then by using the strong law of large numbers theorem for martingales (see [84]) and the fact that the solution of the principal system is bounded we get,

$$\limsup_{t\to+\infty} \frac{M_t}{t} = 0,$$

therefore,

$$\limsup_{t\to+\infty}\frac{\log(I(t)}{t}\leq\frac{\beta^2}{2\sigma^2}-(\rho+\phi+\mu)+(\alpha+\omega)\frac{\Lambda}{\mu}.$$

Then, if $\frac{\beta^2}{2\sigma^2}<(\rho+\phi+\mu)+(\alpha+\omega)\frac{\Lambda}{\mu}$, so $I(t)\to 0$ when $t\to+\infty$ a.s. □

3.4.4 Persistence of $I(t)$ and $S(t)$ in the mean

In this section, we shall investigate the persistence property of $S(t)$ and $I(t)$ in the mean. That's means that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds>0,$$

where, $x(t)\in\{S(t),I(t)\}$. For convenience, we define the following notation:

$$<x(t)>=\frac{1}{t}\int_0^t x(s)ds.$$

Theorem 3.23. *Let* $(S(t),I(t),C(t),A(t))$ *be a solution of system* (3.27) *with any initial value* $(S(0),I(0),C(0),A(0))$. *If*

$$\beta\frac{\Lambda}{\mu+d}>(\rho+\phi+\mu)+\frac{\sigma^2\Lambda^2}{2\mu^2}. \tag{3.28}$$

Then,

$$\liminf_{t\to\infty}<I(t)>\geq\frac{1}{(\rho+\phi+\mu)}\left(\frac{\beta\Lambda}{\mu+d}-(\rho+\phi+\mu)-\frac{\sigma^2\Lambda^2}{2\mu^2}\right)>0, \tag{3.29}$$

$$\liminf_{t\to\infty}<S(t)>\frac{\Lambda\mu}{\Lambda\beta+\mu^2}>0. \tag{3.30}$$

Proof. Using the first equation of the system (3.27), we have

$$\begin{aligned}
\frac{S(t)-S(0)}{t}&=\frac{1}{t}\int_0^t(\Lambda-\beta IS-\mu S)ds-\frac{1}{t}\int_0^t\sigma ISdW_s\\
&-\frac{1}{t}\int_0^t\int_U J(u)IS\check{N}(ds,du)\\
&\geq\frac{1}{t}\int_0^t(\Lambda-\beta\frac{\Lambda}{\mu}S-\mu S)ds-\frac{1}{t}\int_0^t\sigma ISdW_s\\
&-\frac{1}{t}\int_0^t\int_U J(u)IS\check{N}(ds,du)\\
(\frac{\Lambda\beta}{\mu}+\mu)<S>&\geq\Lambda-\frac{S(t)-S(0)}{t}\\
&-\frac{1}{t}\int_0^t\sigma ISdW_s\\
&-\frac{1}{t}\int_0^t\int_U J(u)IS\check{N}(ds,du)
\end{aligned}$$

Using the strong law of large numbers for martingales and the boundedness of solution, we get

$$\liminf_{t\to\infty} < S(t) >\geq \frac{\Lambda\mu}{\Lambda\beta+\mu^2} > 0.$$

Integrating the second equation of the system (3.27) from 0 to t and dividing the both sides by t, we obtain:

$$\begin{aligned}
\frac{I(t)-I(0)}{t} &= \frac{1}{t}\int_0^t ((\beta I(s)S(s) - (\rho+\phi+\mu)I(s) + \alpha A(s) + \omega C(s))\, ds \\
&+ \frac{1}{t}\int_0^t \sigma IS dW_s + \frac{1}{t}\int_0^t\int_U J(u)IS\check{N}(ds,du) \\
&\geq -(\rho+\phi+\mu) < I(t) > + \frac{1}{t}\int_0^t \sigma IS dW_s \\
&+ \frac{1}{t}\int_0^t\int_U J(u)IS\check{N}(ds,du)
\end{aligned}$$

Using again the Itô's formula on a function V with $V(I) = \log(I)$ and decreasing the right side part,
we get what follows:

$$\begin{aligned}
dV &= (\beta IS - (\rho+\phi+\mu)I + \alpha A + \omega C)\frac{1}{I} - \frac{\sigma^2 S^2}{2} \\
&+ \int_U \log(1+JS) - JS]\nu(du))dt + \sigma S(t)dW_t \\
&+ \int_U \log(1+JS)\check{N}(dt,du) \\
&\geq (\beta\frac{\Lambda}{\mu+d} - (\rho+\phi+\mu) + \alpha\frac{\mu}{\mu+d} + \omega\frac{\mu}{\mu+d}) \\
&- \frac{\sigma^2 S^2}{2} + \int_U \log(1+JS) - JS]\nu(du))dt \\
&+ \sigma S(t)dW_t + \int_U \log(1+JS)\check{N}(dt,du) \\
\frac{\log I(t) - logI(0)}{t} &\geq (\beta\frac{\Lambda}{\mu+d} - (\rho+\phi+\mu) + \alpha\frac{\mu}{\mu+d} + \omega\frac{\mu}{\mu+d}) - \\
&\frac{\sigma^2\Lambda^2}{2\mu^2} + \frac{1}{t}\int_U \log(1+JS) - JS]\nu(du))ds \\
&+ \frac{1}{t}\int_0^t \sigma S(s)dW_s + \frac{1}{t}\int_0^t\int_U \log(1+JS)\check{N}(ds,du)
\end{aligned}$$

By summing $\dfrac{I(t)-I(0)}{t}$ and $\frac{\log I(t)-\log I(0)}{t}$, applying yet the strong law of large numbers for martingales, positivity and boundedness of solution, we get

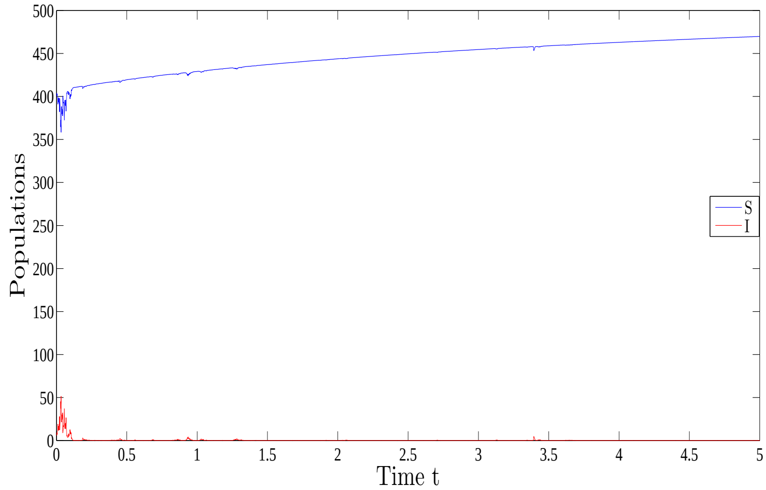

Figure 3.14: The Susceptible and infected population as function of time.

$$\liminf_{t\to\infty} < I(t) > \geq \frac{1}{(\rho+\phi+\mu)}\left(\frac{\beta\Lambda}{\mu+d} - (\rho+\phi+\mu) - \frac{\sigma^2\Lambda^2}{2\mu^2}\right) > 0.$$

Consequently, the requested persistence in mean of $I(t)$ holds.

□

3.4.5 Numerical results

This section is devoted to illustrate our mathematical findings by numerical simulations. In the following examples we apply the algorithm presented in [166] to solve system (3.27) and we take the parameter values given in Table 3.6.

Parameters	Fig. 3.14	Fig. 3.15
Λ	10	100
μ	0.0125	0.0013
β	0.0001	0.1
ϕ	1	1
ρ	0.1	0.1
α	0.33	0.33
ω	0.09	0.09
d	1	1

Table 3.6: Parameters values

Figure 3.14 shows the dynamics of susceptible and infected class during the period of observation for the case of the disease extinction. From this figure, we clearly observe that the curves representing the infected population converge to 0. It will be worthy to notice that in this case, the susceptible individuals increase to reach their maximum which means that the

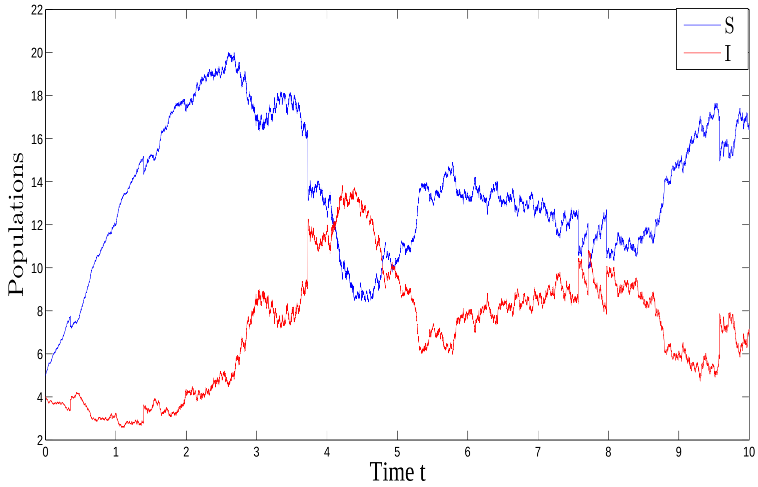

Figure 3.15: The Susceptible and infected population as function of time.

disease dies out. This is consistent with our theoretical findings concerning the extinction of SICA model.

The evolution of the infection for both the susceptible and infected population with Lévy jumps model is illustrated in Fig. 3.15 in the case of the disease persistence. We note that in this epidemic situation, all the four SICA compartments, i.e. the susceptible, the infected, HIV-infected individuals under ART treatment (the so called chronic stage) with a viral load remaining low, and HIV-infected individuals with AIDS clinical symptoms which means that the disease persists. This is consistent with our theoretical findings concerning the infection persistence.

3.4.6 Conclusion

In this work, we have considered and extended the Stochastic epidemic SICA model due to Silva and Torres , (see [125]) to a new model driven by white noise and Lévy noise jointly so as to better describe the sudden social fluctuations, and performed some adequate numerical simulation not only to support our theoretical results but also for predicting the asymptotic behavior of the solution of the corresponding system in this study. More precisely,first of all, existence and uniqueness of solutions of the current system are investigated using Lyapunov analysis method.
Secondly, w've demonstrated that the model is well posed biologically and mathematically by means of the establishment of the boundedness subject to: $\limsup_{t\to\infty} N(t) \leq \dfrac{\Lambda}{\mu}$ a.s. $\liminf_{t\to\infty} N(t) \geq \dfrac{\Lambda}{\mu+d}$ a.s., as well as the positivity of the solution which facilitated the rest of the study.
Thirdly, we have constructed an appropriate sufficient condition for having the extinction of the model showing that with an effective threshold of an eventual biggest magnitude of the volatility σ, subject to: $\frac{\beta^2}{2\sigma^2} < (\rho+\phi+\mu)+(\alpha+\omega)\frac{\Lambda}{\mu}$, the eradication of disease occurs.
Fourth, Novel significant sufficient condition, subject to:

$\liminf_{t\to\infty} < I(t) > \geq \frac{1}{(\rho+\phi+\mu)}\left(\frac{\beta\Lambda}{\mu+d} - (\rho+\phi+\mu) - \frac{\sigma^2\Lambda^2}{2\mu^2}\right) > 0$, and $\liminf_{t\to\infty} < S(t) > \frac{\Lambda\mu}{\Lambda\beta+\mu^2} > 0$ related to the occurrence of the corresponding persistence in mean is obtained, which mentions that with an adopted smallest magnitude of volatility σ, the model is persistent in mean.
Lastly, Some numerical simulations are implemented to confirm and illustrate our mathematical results linked to the considered model intending to give some supplementary information helping decision maker to select a good strategy controlling disease by means of the increasing or decreasing the intensity of volatility on one hand, on the other hand, influence of the Lévy noise on the evolution of variables of the system is taken into account.

3.5 Near-optimal control of a stochastic SICA model with imprecise parameters

3.5.1 Introduction

Nowadays various kinds of infectious diseases including hepatitis C, HIV/AIDS spread around the globe. Further, despite the advanced medical technology, infectious diseases remain growing threat to mankind. Therefore, the decision-maker of public health system build up different strategies to control the spread of diseases. Mathematical model have become important tools in analyzing the growth and controlling it. The infection by human immunodeficiency virus (HIV) goes on to be a major public issue. There is again no cure or vaccine to AIDS. Whereas, antiretroviral (ART) treatment improves health, prolongs life, and diminishes the risk of HIV infection. Several mathematical models have been proposed for HIV/AIDS transmission dynamics (see, e.g., Anderson, 1988; Anderson et al., 1989; Bhuni et al., 2009a, 2011; Cai et al., 2009; Granich et al., 2009; Greenhalgh et al., 2001; Hyman and Stanly, 1988. Joshi et al., 2008; May and Anderson, 1987; Musgrave and Watmough, 2009 and so on). The recent work is based on the work due to Djordjevic, Silva, and Torres [15] in which the authors consider a new stochastic SICA model by means of the presence of the white noise (Brownian motion with positive intensity) due to the environment fluctuations that perturb the coefficient rate of transmission $\beta \to \beta + \delta B_t$, in order to good describe reality as follows:
Stochastic SICA model (Djordjevic et al) [15] without control

$$\begin{cases} dS(t) = \left[\Lambda - \beta\left(I(t) + \eta_C.C(t) + \eta_A.A(t)\right).S(t) - \mu S(t)\right]dt \\ \qquad - \delta\left(I(t) + \eta_C.C(t) + \eta_A.A(t)\right).S(t)dB(t), \\ dI(t) = \left[\beta\left(I(t) + \eta_C.C\left(t\right) + \eta_A.A(t)\right).S(t) - \varepsilon_3 I(t) + \alpha A(t) + \omega C(t)\right]dt \\ \qquad + \delta\left(I(t) + \eta_C.C(t) + \eta_A.A(t)\right).S(t)dB(t), \\ dC(t) = \left[\phi I(t) - \varepsilon_2 C\left(t\right)\right]dt, \\ dA(t) = \left[eI(t) - \varepsilon_1 A\left(t\right)\right]dt, \end{cases} \tag{3.31}$$

The latter paper of Torres et al generalizes stochastically that made by Torres et al (2016), which is also based on that of Silva and Torres (2015) which subdivides the human population into four exhaustively and mutually exclusive compartments: susceptible individuals (S); HIV-infected individuals with no clinical symptoms of AIDS (the virus is living or developping in the individuals but without producing symptoms or only mild ones) but able to transmit HIV to other individuals (I); HIV-infected individuals under ART treatment (the so called chronic stage) with a viral load remaining low (C); and HIV-infected individuals with AIDS clinical symptoms (A). The total population at time t, denoted by $N(t)$ is given by

$$N(t) = S(t) + I(t) + C(t) + A(t).$$

The key issue in this study is to carry out some purposes, in the second section we have applied on the actual SICA model one of the most significant control associated to the proposed Hamiltonian function and ended up the use of Pontryagin's maximum principle, further, an other kind of control called: Near-optimal control in [88] which replaces the precise biological parameters by the imprecise ones is established to take into account all possibly environment perturbations, throught the constract of some estimates of the currently variables which are performed by the advanced mathematical techniques (Itô's formula, Convexity, Cauchy Scwhart'z inequality, Hôlder's inequality, Burkholder-Davis-Gundy inequality, and so on) in the third section, the fourth and fifth sections are devoted to the handling of the necessary and sufficient conditions for the near-optimal control.

3.5.2 Optimal control on the stochastic SICA model

We have inspired the form of the recent control from [88]

$$\begin{cases} dS(t) = \left[\Lambda - \beta\left(I(t) + \eta_C.C(t) + \eta_A.A(t)\right).S(t) - \mu S(t)\right]dt \\ \qquad - \delta\left(I(t) + \eta_C.C(t) + \eta_A.A(t)\right).S(t)dB(t), \\ dI(t) = \left[\beta\left(I(t) + \eta_C.C\left(t\right) + \eta_A.A(t)\right).S(t) - \varepsilon_3 I(t) + \alpha A(t) + \omega C(t) - \dfrac{mu(t).I(t)}{1+\gamma I(t)}\right]dt \\ \qquad + \delta\left(I(t) + \eta_C.C(t) + \eta_A.A(t)\right).S(t)dB(t), \\ dC(t) = \left[\phi I(t) - \varepsilon_2 C\left(t\right) + \dfrac{mu(t).I(t)}{1+\gamma I\left(t\right)}\right]dt, \\ dA(t) = \left[eI(t) - \varepsilon_1 A\left(t\right)\right]dt, \end{cases} \tag{3.32}$$

with

$$\varepsilon_1 = \alpha + \mu + d;\ \ \varepsilon_2 = \omega + \mu;\ \ \varepsilon_3 = e + \phi + \mu. \tag{3.33}$$

$$\begin{cases} dS(t) = dx_1(t) = f_1\left(x(t)\right)dt + \sigma_1\left(x(t)\right).dB_t, \\ dI(t) = dx_2(t) = f_2\left(x(t), u(t)\right)dt + \sigma_2\left(x(t)\right).dB_t, \\ dC(t) = dx_3(t) = f_3\left(x(t), u(t)\right)dt, \\ dA(t) = dx_4(t) = f_4\left(x(t)\right)dt. \end{cases} \tag{3.34}$$

with

$$\begin{cases} f_1\left(x(t)\right) = \Lambda - \beta\left(x_2 + \eta_C.x_3 + \eta_A.x_4\right).x_1 - \mu x_1, \\ f_2\left(x(t), u(t)\right) = \beta\left(x_2 + \eta_C.x_3 + \eta_A.x_4\right).x_1 - \varepsilon_3 x_2 + \alpha x_4 + \omega x_3 - \dfrac{mu(t).x_2}{1+\gamma x_2}, \\ f_3\left(x(t), u(t)\right) = \phi x_2 - \varepsilon_2 x_3 + \dfrac{mu(t).x_2}{1+\gamma x_2}, \\ f_4\left(x(t)\right) = ex_2 - \varepsilon_1 x_{4,} \\ \sigma_1\left(x(t)\right) = -\sigma_2\left(x(t)\right) = -\delta\left(x_2 + \eta_C.x_3 + \eta_A.x_4\right).x_1, \end{cases} \tag{3.35}$$

and $x(t) = \left(S(t), I(t), C\left(t\right), A(t)\right).$

Stochastic objective function

We introduce the cost function $J\left(u\right)$ such that

$$J\left(u\right) = E\left\{\int_0^T L\left(x(t), u\left(t\right)\right)dt + h\left(x(T)\right)\right\} \tag{3.36}$$

with $L : \mathbb{R}^4 \times \mathbb{R} \longrightarrow \mathbb{R};\ h : \mathbb{R}^4 \to \mathbb{R},\ L$ and h are assumed to be continuously differentiables . Let $U \subseteq \mathbb{R}$ be a bounded non empty closed set.

Control process $u\left(.\right)$ is a control process : $[0,T] \times \Omega \longrightarrow U$. That is called admissible if it is an F_t-adapted process with values in U. the set of all admissible controls is denoted by U_{ad}. The Control problem is formulated to find an admissible control that minimizes the objective function

$$J\left(0, x_0, u\left(.\right)\right) = J\left(u\right) \text{ over all } u\left(.\right) \in U_{ad}.$$

The value function is as follows.

$$v(0,x_0) = \min J(u).$$

Stochastic Hamiltonian function
We regard the Hamiltonian function associated to the stochastic optimal problem :

$$H(x,u,p,q) = \langle f(x,u), p\rangle + \langle \sigma, q\rangle - L(x,u), \tag{3.37}$$

with $\sigma = (\sigma_1, \sigma_2)$,
and $\langle ., .\rangle$ denotes an Euclidian inner product.

3.5.3 Stochastic Pontryagin's Maximum Principle

Related equations
It follows from the stochastic maximum principle [95] that:

$$\begin{cases} dx^*(t) = \dfrac{\partial}{\partial p} H(x^*, u^*, p, q)\, dt + \sigma(x^*(t))\, dB_t \\ dp_i(t) = -\dfrac{\partial}{\partial x_i} H(x^*, u^*, p, q)\, dt + q_i(t) dB_t \\ H(x^*, u^*, p, q) = \max\limits_{u \in U_{ad}} H(x^*, u, p, q) \end{cases} \tag{3.38}$$

Where $x^*(t)$ is an optimal trajectory of $x(t)$. The initial state and terminal costate conditions are

$$x^*(0) = x_0;\ p_i(T) = -\frac{\partial}{\partial x_i} h(x^*(T)). \tag{3.39}$$

Notice that the second order stochastic differential equation is omitted from the fact that the diffusion coefficient σ does not depend on the control u.

3.5.4 Near-optimal control with imprecise parameters

In the majority of the epidemic model, parameter values are quite often assumed to be precisely known, nevertheless, it's mandatory to make into account the influence of numerous uncertainties, this leads to the inclusion of the stochastic SICA model with imprecise parameters governed by an adequate near-optimal control which will be defined in the sequel.

Known definitions

(D1) An admissible control $u^*(.)$ is called optimal if $u^*(.)$ attains the minimum of $J(0, x_0, u^*(.))$
(D2) ($\epsilon-$*optimal control*) For a given $\epsilon > 0$, an admissible control $u^\epsilon(.)$ is called ($\epsilon-$*optimal control*), if

$$\mid J(0, x_0, u^*(.)) - V(0, x_0 \mid \leq \epsilon$$

(D3) (Near-optimal control) A family of admissible controls $u^\epsilon(.)$ parametrized by $\epsilon > 0$ and any element $u^\epsilon(.)$ in the family are called near-optimal, if

$$\mid J(0, x_0, u^*(.)) - V(0, x_0 \mid \leq \delta(\epsilon)$$

holds for a sufficient small $\epsilon > 0$, where δ is a function of ϵ satisfying $\delta(\epsilon) \to 0$ as $\epsilon \to 0$.
The estimate $\delta(\epsilon)$ is called an error bound. If $\delta(\epsilon) = c\epsilon^\omega$, for some $c, \omega > 0$, then $u^\epsilon(.)$ is called near-optimal with order ϵ^ω.
(D4) An interval number A is represented by closed interval $[a_l, a_u]$ and defined by
$A = [a_l, a_u] = \{x \in \mathbb{R}/a_l \leq x \leq a_u\}$,

where a_l and a_u are the left and the right limits of the interval number, respectively.
We can represent an interval $[a, b]$ by the so called an interval-valued function which is taken as: $h(k) = a^{1-k}b^k$ for $k \in [0, 1]$.
Notice that the sum, difference, product and the division of two interval numbers are interval numbers.

Lemma 3.24. *(Ekeland's principle [22]) Let (Q, d) be a complete metric space and $F(.) : Q \to \mathbb{R}$ be a lower-semicontinuous and bounded from below. For any $\epsilon > 0$, we assume that $u^\epsilon \in Q$ satisfies*

$$F(u^\epsilon) \leq \inf_{u \in Q} F(u) + \epsilon.$$

Then there is a $u^\lambda \in Q$ $(\lambda > 0)$ such that for all $u \in Q$

$$F(u^\lambda) \leq F(u^\epsilon), \quad d(u^\lambda, u^\epsilon) \leq \lambda, \quad F(u^\lambda) \leq F(u) + \frac{\epsilon}{\lambda} d(u^\lambda(.), u^\epsilon(.))$$

Setting

(S1) For all $0 \leq t \leq T$, the partial derivatives $l_{S(t)}, l_{I(t)}, l_{C(t)}, l_{A(t)}$ and $h_{S(t)}, h_{I(t)}$ are continuous, and there is an imprecise parameter C such that

$$\mid l_{S(t)} + l_{I(t)} + l_{C(t)} + l_{A(t)} \mid \leq C(1+ \mid S(t) \mid + \mid I(t) \mid + \mid C(t) \mid + \mid A(t) \mid).$$

And, $(1+ \mid S(t) \mid)^{-1} \mid h_{S(t)} \mid +(1+ \mid I(t) \mid)^{-1} \mid h_{I(t)} \mid \leq C.$

(S2) Let $x(t), x^{'}(t) \in \mathbb{R}^4_+$ and $u, u^{'} \in U_{ad}$. Then for any $0 \leq t \leq T$, the function $l(x(t), u(t))$ is differentiable on $u(.)$, and there exists an imprecise parameter C such that

$$\sum_{i=0}^{4} \mid h_{x_i}(x(t)) - h_{x_i^{'}}(x^{'}(t)) \mid \leq C. \sum_{i=0}^{4} \mid x_i(t) - x_i^{'}(t) \mid,$$

and

$$\mid l(x(t), u(t)) - l(x(t), u^{'}(t)) \mid + \mid l_{u(t)}(x(t), u(t)) - l_{u^{'}(t)}(x(t), u^{'}(t)) \mid \leq C \mid u(t) - u^{'}(t) \mid . \quad (3.40)$$

(S3) The control set U is convex.
Remark By the Mazur's theorem, we can obtain the existence of the optimal control under hypothesis $(S3)$.
We assume that the stochastic SICA model has some biological imprecise parameters through the interval numbers so as to better describe uncertain parameters.

Stochastic SICA model with control and imprecise parameters.

The system (3.32) becomes after replacing each parameter ζ with the imprecise one

$$\zeta_k = (\zeta_l)^{1-k}(\zeta_u)^k \in [\zeta_l, \zeta_u]$$

for $k \in [0,1]$.

$$\begin{cases} dS(t) = \left[\Lambda_k - \beta_k \left(I(t) + (\eta_C)_k.C(t) + (\eta_A)_k.A(t)\right).S(t) - \mu_k S(t)\right] dt \\ \qquad - \delta_k \left(I(t) + (\eta_C)_k.C(t) + (\eta_A)_k.A(t)\right).S(t)dB(t), \\ dI(t) = \left[\beta_k \left(I(t) + (\eta_C)_k.C(t) + (\eta_A)_k.A(t)\right).S(t) - (\varepsilon_3)_k.I(t)\right. \\ \qquad + \alpha_k.A(t) + \omega_k C(t) \dfrac{m_k.u(t).I(t)}{1+\gamma_k I(t)}]dt \\ \qquad + \delta_k \left(I(t) + (\eta_C)_k.C(t) + (\eta_A)_k.A(t)\right).S(t)dB(t), \\ dC(t) = \left[\phi_k.I(t) - (\varepsilon_2)_k.C(t) + \dfrac{m_k u(t).I(t)}{1+\gamma_k I(t)}\right] dt, \\ dA(t) = \left[e_k.I(t) - (\varepsilon_1)_k.A(t)\right] dt \end{cases} \tag{3.41}$$

3.5.5 The requested estimates of the state and costate variables.

Let u and $u^{'} \in U_{ad}$
Set the following metric on $U_{ad}[0,T]$ as: $d(u,u^{'}) = Emes\{t \in [0,T] : u(t) \neq u^{'}(t)\}$, where mes represents Lebesgue measure. Since U is closed, it can be shown that U_{ad} is a complete metric space under d.

Lemma 3.25. *For any $\theta \geq 0$ and $u \in U_{ad}$, we have*

$$E \sup_{0\leq t\leq r} (\mid S(t) \mid^\theta + \mid I(t) \mid^\theta + \mid C(t) \mid^\theta + \mid A(t) \mid^\theta) \leq C,$$

where C is an imprecise parameter depending only on θ.

Proof. Let $N(t) = S(t) + I(t) + C(t) + A(t)$
Adding member to member all equations of the system (10), we get

$$\frac{dN(t)}{dt} = \Lambda_k - \mu_k.N(t) - d_k.A(t),$$

then

$$\frac{dN(t)}{dt} \leq \Lambda_k - \mu_k.N(t),$$

since $\quad d_k.A(t) \geq 0$
Multiplying by $exp(u_k.t)$ the both sides of the previous inequality, we obtain

$$exp(u_k.t).\frac{dN(t)}{dt} \leq \Lambda_k.exp(u_k.t) - \mu_k.exp(u_k.t).N(t)$$

An integration by parts between 0 and t leads to

$$N(t).exp(u_k.t) - N(0) - \int_0^t N(s).exp(u_k.s)ds \leq \frac{\Lambda_k}{\mu_k}(exp(u_k.t) - 1) - \int_0^t \mu_k.exp(u_k.s).N(s)ds.$$

Equivalently, we have,

$$N(t) \leq \frac{\Lambda_k}{\mu_k}(1 - exp(-u_k.t)) + N(0).exp(-u_k.t).$$

Therefore,

$$\limsup_{t\to+\infty} N(t) \leq \frac{\Lambda_k}{\mu_k}.$$

We have also

$$\frac{dN(t)}{dt} \geq \Lambda_k - (\mu_k + d_k).N(t).$$

Applying the same foregoing manner, we get

$$\liminf_{t\to+\infty} N(t) \geq \frac{\Lambda_k}{\mu_k + d_k}$$

We conclude that

$$\frac{\Lambda_k}{\mu_k + d_k} \leq \liminf_{t\to+\infty} N(t) \leq \limsup_{t\to+\infty} N(t) \leq \frac{\Lambda_k}{\mu_k}.$$

Thus, all solutions $S(t), I(t), C(t)$, and $A(t)$ of the current model are bounded over the regarded almost surely positively invariant bounded set
$\Omega = \{(S(t), I(t), C(t), A(t)) \in \mathbb{R}^4_+ : \dfrac{\Lambda_k}{\mu_k + d_k} \leq N(t) \leq \dfrac{\Lambda_k}{\mu_k}\} \subset \mathbb{R}^4_+.$
Shortly, the trajectories of all solutions will enter and remain in Ω with probability 1.
This achieves the proof.

□

Lemma 3.26. *For all $\theta \geq 0$ and $0 < k < 1$ satisfying $k\theta < 1$ and $u, u^{'} \in U_{ad}$ along with the corresponding trajectories x and x', there exists an imprecise parameter $C = C(\theta, k)$ such that*

$$\sum_{i=1}^{4} E \sup_{0\leq t\leq T} \mid x_i(t) - x_i^{'}(t) \mid^{2\theta} \leq C d(u(t), u^{'}(t))^{k\theta} \tag{3.42}$$

Proof. Suppose firstly that $\theta \geq 1$. Set

$$x(t) = (x_1(t), x_2(t), x_3(t), x_4(t)) = (S(t), I(t), C(t), A(t)).$$

In view of the Hôlder's inequality for $\theta \geq 1$

$$\begin{aligned} &E \sup_{0\leq t\leq r} \mid x_1(t) - x_1^{'}(t) \mid^{2\theta} \\ &\leq C.E \int_0^r [(\beta_k^{2\theta} + \delta_k^{2\theta}) \mid (x_2 + (\eta_C)_k.x_3 + (\eta_A)_k.x_4).x_4 \\ &- ((x_2^{'} + (\eta_C)_k.x_3^{'} + (\eta_A)_k.x'_4)).x_4 \mid^{2\theta} + (\mu_k)^{2\theta} \mid x_1(t) - x_1{'}(t) \mid^{2\theta}]dt \\ &\leq C.E \int_0^r \sum_{i=1}^{4} \mid x_i - x'_i \mid^{2\theta} dt \end{aligned} \tag{3.43}$$

We have also,

$$
\begin{aligned}
& E \sup_{0\le t\le r} \mid x_2(t) - x_2^{'}(t) \mid^{2\theta} \\
& \le C.E \int_0^r [(\beta_k^{2\theta} + \delta^{2\theta}) \mid (x_2 + (\eta_C)_k.x_3 + (\eta_A)_k.x_4).x_4 - \\
& ((x_2^{'} + (\eta_C)_k.x_3^{'} + (\eta_A)_k.x'_4)).x_4 \mid^{2\theta} + \\
& (\epsilon_3)_k^{2\theta} \mid x_2(t) - x'_2(t) \mid^{2\theta} + \alpha_k^{2\theta} \mid x_4(t) - x_4^{'}(t) \mid^{2\theta} + \\
& \omega_k^{2\theta} \mid x_3(t) - x'_3(t) \mid^{2\theta} + \\
& m_k^{2\theta} \mid \frac{u(t).x_2(t)}{1+\eta_k.x_2} - \frac{u'(t).x'_2(t)}{1+\eta_k.x'_2} \mid^{2\theta}]dt \\
& \le C.E \int_0^r \sum_{i=1}^{4} \mid x_i - x'_i \mid^{2\theta} + CE \int_0^r \chi_{u(t)\neq u'(t)} dt \\
& \le C[E \int_0^r \sum_{i=1}^{4} \mid x_i - x'_i \mid^{2\theta} dt + d(u,u')^{k\theta}]
\end{aligned}
$$

since,

$$
\begin{aligned}
& E \int_0^r \chi_{u(t)\neq u'(t)} dt \\
& \le \left(E \int_0^r dt\right)^{1-k\theta} . \left(E \int_0^r \chi_{u(t)\neq u'(t)} dt\right)^{k\theta} \\
& \le C(\theta,k) \left(E mes\{t/u(t) \neq u'(t)\}\right)^{k\theta} \\
& \le C.d(u,u')^{k\theta}.
\end{aligned}
$$

Remark that the Hôlder's inequality occurred under the hypothesis $k\theta < 1$. We found also:

$$
E \sup_{0\le t\le r} \mid x_3(t) - x_3^{'}(t) \mid^{2\theta} \le C[E \int_0^r \sum_{i=1}^{4} \mid x_i - x'_i \mid^{2\theta} dt + d(u,u')^{k\theta}],
$$

and,

$$
E \sup_{0\le t\le r} \mid x_4(t) - x_4^{'}(t) \mid^{2\theta} \le C[E \int_0^r \sum_{i=1}^{4} \mid x_i - x'_i \mid^{2\theta} dt + d(u,u')^{k\theta}],
$$

Combining the last four inequalities, we get

$$
\sum_{i=1}^{4} E \sup_{0\le t\le r} \mid x_i(t) - x_i^{'}(t) \mid^{2\theta} \le C[E \int_0^r \sum_{i=1}^{4} \mid x_i - x'_i \mid^{2\theta} dt + d(u,u')^{k\theta}],
$$

Therefore, the lemma is true by using the Gronwall's inequality.
To proof the desired result in the case $0 \le \theta < 1$, we apply the Cauchy-Schwartz's inequality for obtaining,

$$
\begin{aligned}
&\sum_{i=1}^{4} E \sup_{0\leq t\leq r} \mid x_i(t) - x_i^{'}(t) \mid^{2\theta} \\
&\leq \sum_{i=1}^{4} \left[E \sup_{0\leq t\leq r} \mid x_i(t) - x_i^{'}(t) \mid^{2} \right]^{\theta} \\
&\leq \left[Cd(u,u')^{k} \right]^{\theta} \\
&\leq C^{\theta} d(u,u')^{k\theta}.
\end{aligned}
$$

Where C is an imprecise parameter because of its dependence of the considered imprecise ones. This completes the proof. $\square$

Lemma 3.27. *For any $u, u' \in U_{ad}$, we have*

$$
\sum_{i=1}^{4} E \sup_{0\leq t\leq T} \mid p_i(t) \mid^2 + \sum_{i=1}^{2} E \int_0^T \mid q_i(t) \mid^2 dt \leq C, \tag{3.44}
$$

where C is an imprecise.

Proof. We have,

$$
dp_1(t) = -\partial_{x_1} H(x^*, u^*, p, q)dt + q_1(t)dB_t = -b_1 dt + q_1 dB_t
$$

so,

$$
p_1(t) + \int_t^T q_1(s)dB_s = P_1(T) + \int_t^T b_1(s)ds
$$

For all $t \geq 0, x(t) \in \Omega$,

$$
\begin{aligned}
E \mid p_1(t) \mid^2 + E\{\int_0^T \mid q_1(s) \mid^2 ds\} &\leq CE \mid p_1(T) \mid^2 + C(T-t)E\{\int_0^T \mid b_1(s) \mid^2 ds\} \\
\leq CE \mid p_1(T) \mid^2 + C(T-t)\sum_{i=1}^{4} E\{\int_0^T \mid p_i(s) \mid^2 ds\} &+ C(T-t)\sum_{i=1}^{2} E\{\int_0^T \mid q_i(s) \mid^2 ds\},
\end{aligned}
$$

we obtain the same inequalities for $i \in \{2, 3, 4\}$.
By adding member to member, we get,

$$
\begin{aligned}
&\sum_{i=1}^{4} E \mid p_i(t) \mid^2 + \sum_{i=1}^{2} E \int_t^T \mid q_i(s) \mid^2 ds \\
&\leq CE \mid p_i(T) \mid^2 + C(T-t)\sum_{i=1}^{4} E\{\int_t^T \mid p_i(s) \mid^2 ds\} + C(T-t)\sum_{i=1}^{2} E\{\int_t^T \mid q_i(s) \mid^2 ds\}
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\sum_{i=1}^{4} E \mid p_i(t) \mid^2 + \frac{1}{2}\sum_{i=1}^{2} E\{\int_t^T \mid q_i(s) \mid^2 ds\} \\
&\leq \sum_{i=1}^{4} CE \mid p_i(T) \mid^2 + C(T-t)\sum_{i=1}^{4} E\{\int_t^T \mid p_i(s) \mid^2 ds\},
\end{aligned}
$$

where $t \in [T-\epsilon, T]$ with $\epsilon = \frac{1}{C}$,
Using Gronwall's inequality, we obtain

$$\sum_{i=1}^{4} \sup_{0 \le t \le T} E \mid p_i(t) \mid^2 \le C, \quad \sum_{i=1}^{4} E\{\int_t^T \mid q_i(s) \mid^2 ds\} \le C \tag{3.45}$$

One obtain the same result over $[T-2\epsilon, T]$, $[T-3\epsilon, T]$, and so on.
By repeating for a finite number of steps; we conclude that for any $t \in [0, T]$, the expected estimate emerges.
Further, from

$$p_1(t) = p_1(T) + \int_t^T b_1(s)ds - \int_0^T q_1(s)dB_s + \int_0^t q_1(s)dB_s$$

and the elementary inequality:

$$\mid m_1 + m_2 + m_3 + m_4 \mid^n \le 4^n(\mid m_1 \mid^n + \mid m_2 \mid^n + \mid m_3 \mid^n + \mid m_4 \mid^n),$$

for any $n > 0$.
we have,

$$\mid p_1(t) \mid^2 \le C[\mid p_1(T) \mid^2 + \int_0^T (\sum_{i=1}^{4} \mid p_i(s) \mid^2)ds + \int_0^T (\sum_{i=1}^{2} \mid q_i(s) \mid^2)ds$$
$$+ (\int_0^T q_1(s)dB)^2 + (\int_0^t q_1(s)dB)^2]$$

Analogous statements could be established similarly for p_2, p_3,and p_4,
next, by addition, we get

$$\sum_{i=1}^{4} \mid p_i(t) \mid^2 \le C[\sum_{i=1}^{4} \mid p_i(T) \mid^2 + \int_0^T (\sum_{i=1}^{4} \mid p_i(s) \mid^2)ds$$
$$+ \int_0^T (\sum_{i=1}^{2} \mid q_i(s) \mid^2)ds + \sum_{i=1}^{2} (\int_0^T q_i(s)dB)^2$$
$$+ \sum_{i=1}^{2} (\int_0^t q_i(s)dB)^2]$$

The Burkholder-Davis-Gundy inequality[23] leads to the following:

$$\sum_{i=1}^{4} E \sup_{0 \le t \le T} \mid p_i(t) \mid^2 \le C[\sum_{i=1}^{4} \mid p_i(T) \mid^2$$
$$+ \sum_{i=1}^{4} E\{\int_0^T \sup_{0 \le v \le s} \mid p_i(v) \mid^2)ds\}$$
$$+ \sum_{i=1}^{2} E\{\int_0^T \sup_{0 \le v \le s} \mid q_i(v) \mid^2)ds\}].$$

The establishment of our desired result is achieved by applying the Gronwall's inequality. □

Lemma 3.28. *Let (S_1) and (S_2) holds. For any $1 < \eta < 2$ and $0 < k < 1$ satisfying $(1+k)\eta < 2$, there exists a constant $C = C(\eta, k)$ such that for any $u, u' \in U_{ad}$ along with the corresponding trajectory*

x, x' and the solution $(p,q), (p',q')$ of the related adjoint equation, we have

$$\sum_{i=1}^{4} E\{\int_0^T \mid p_i(t) - p_i'(t) \mid^\eta dt\} + \sum_{i=1}^{4} E\{\int_0^T \mid q_i(t) - q_i'(t) \mid^\eta dt\} \leq Cd(u,u')^{\frac{k\eta}{2}} \tag{3.46}$$

Proof. Let $\bar{p_i}(t) = p_i(t) - p_i'(t), i \in \{1,2,3,4\}$ and $\bar{q_i}(t) = q_i(t) - q_i'(t), i \in \{1,2\}$.
It follows from the adjoint equation that:

$$\begin{cases} d\bar{p_1}(t) = -[(-\beta_k(x_2 + (\eta_C)_k x_3 + (\eta_A)_k x_4) - \mu_k)\bar{p_1} + (\beta_k(x_2 + (\eta_C)_k x_3 + (\eta_A)_k x_4) - \mu_k)\bar{p_2} \\ \qquad - \delta_k(x_2 + (\eta_C)_k x_3 + (\eta_A)_k x_4 - \mu_k)\bar{q_1} + \delta_k(x_2 + (\eta_C)_k x_3 + (\eta_A)_k x_4)\bar{q_2} + \bar{f_1}]dt + \bar{q_1}dB \\ d\bar{p_2}(t) = -[-\beta_k x_1 \bar{p_1} + (\beta_k x_1 - \epsilon_3 - \frac{m_k u}{(1+\delta_k x_2)^2})\bar{p_2} \\ \qquad + (\phi + \frac{m_k u}{(1+\gamma_k x_2)^2}\bar{p_3} + e\bar{p_4} - \delta x_1)\bar{q_1} \\ \qquad + \delta_k x_1 \bar{q_2} + \bar{f_2}]dt + \bar{q_2}dB \\ d\bar{p_3}(t) = -[-\beta_k(\eta_C)_k x_1 \bar{p_1} + \beta_k(\eta_C)_k x_1 \bar{p_2} - \epsilon_2 \bar{p_3} - \delta_k(\eta_C)_k x_1 \bar{q_1} + \delta_k \eta_C x_1 \bar{q_2} + \bar{f_3}]dt \\ d\bar{p_4}(t) = -[-\beta_k(\eta_A)_k x_1 \bar{p_1} + \beta_k(\eta_A)_k x_1 \bar{p_2} - (\epsilon_1)_k \bar{p_4} \\ \qquad - \delta_k(\eta_A)_k x_1 \bar{q_1} + \delta_k(\eta_A)_k x_1 \bar{q_2} + \bar{f_4}]dt, \end{cases} \tag{3.47}$$

where

$$\begin{cases} \bar{f_1} = \beta_k[(x_2 + (\eta_C)_k x_3 + (\eta_A)_k x_4) - (x_2' + (\eta_C)_k x_3' + (\eta_A)_k x_4')](p_2' - p_1') \\ \quad - (\sigma)_k[(x_2 + (\eta_C)_k x_3 + (\eta_A)_k x_4) - (x_2' + (\eta_C)_k x_3' + (\eta_A)_k x_4')](q_2' - q_1') \\ \quad - L_{x_1}(x,u) + L_{x_1}(x',u') \\ \bar{f_2} = (\beta)_k(x_1 - x_1')(p_2' - p_1') + [\frac{m_k u}{1 + (\eta)_k x_2} - \frac{(m_k u')_k}{1 + (\eta)_k x_2'}](p_3' - p_2') \\ \quad (\delta)_k(x_1 - x_1')(q_2' - q_1') - L_{x_2}(x,u) + L_{x_2}(x',u') \\ \bar{f_3} = (\beta)_k(\eta_C)_k(x_1 - x_1')(p_2' - p_1') + (\delta)_k(\eta_C)_k(x_1 - x_1')(q_2' - q_1') - L_{x_3}(x,u) + L_{x_3}(x',u') \\ \bar{f_4} = (\beta)_k(\eta_A)_k(x_1 - x_1')(p_2' - p_1') + (\delta)_k(\eta_A)_k(x_1 - x_1')(q_2' - q_1') - L_{x_4}(x,u) + L_{x_4}(x',u'). \end{cases} \tag{3.48}$$

We assume that

$$\phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t))$$

is the solution of the following linear stochastic differential equation

$$\begin{cases} d\phi_1(t) = [-(\beta)_k(x_2 + (\eta_C)_k x_3 + (\eta_A)_k x_4)\phi_1 - (\beta)_k x_1\phi_2 - (\beta)_k(\eta_C)_k x_1\phi_3 - \beta_k(\eta_A)_k x_1\phi_4 \\ \qquad + \mid \bar{p_1} \mid^{\eta-1} sgn(\bar{p_1})]dt + [-(\delta)_k(x_2 + (\eta_C)_k x_3 + (\eta_A)_k x_4)\phi_1 \\ \qquad - (\delta)_k x_1\phi_2 - (\sigma)_k(\eta_C)_k x_1\phi_3 - (\delta)_k(\eta_A)_k x_1\phi_4 + \mid \bar{q_1} \mid^{\eta-1} sgn(\bar{q_1})]dB \\ d\phi_2(t) = [(\beta)_k(x_2 + (\eta_C)_k x_3 + (\eta_A)_k x_4 - (\mu)_k)\phi_1 + ((\beta)_k x_1 - (\epsilon_3)_k - \dfrac{m_k u}{(1+(\gamma)_k x_2)^2}\phi_2 + \\ \qquad (\beta)_k(\eta_C)_k x_1\phi_3 + \\ \qquad \beta_k(\eta_A)_k x_1\phi_4 + \mid \bar{p_2} \mid^{\eta-1} sgn(\bar{p_2})]dt + [(\delta)_k(x_2 + (\eta_C)_k x_3 + \\ \qquad (\eta_A)_k x_4)\phi_1 + (\delta)_k x_1\phi_2 + (\delta)_k(\eta_C)_k x_1\phi_3 + (\delta)_k(\eta_A)_k x_1\phi_4 + \mid \bar{q_2} \mid^{\eta-1} sgn(\bar{q_2})]dB \\ d\phi_3(t) = [\phi_k + \dfrac{m_k u}{(1+(\gamma)_k x_2)^2} - (\epsilon_2)_k\phi_3 + \mid \bar{p_3} \mid^{\eta-1} sgn(\bar{p_3})]dt \\ d\phi_4(t) = [e_k\phi_2 - (\epsilon)_k\phi_4 + \mid \bar{p_4} \mid^{\eta-1} sgn(\bar{p_4})]dt, \end{cases} \tag{3.49}$$

where sgn(.) is a symbolic function. Using the previous Lemma and hypothesis (S1), we show the existence and uniqueness of the solution of the current equation. Using Cauchy-Schwartz's inequality, we obtain the following statement

$$\sum_{i=1}^{4} E \sup_{0\le t\le T} \mid \phi_i(t) \mid^{\eta_1} \le \sum_{i=1}^{4} E\int_0^T \mid \bar{p_i} \mid^{\eta} dt + \sum_{i=1}^{2} E\int_0^T \mid \bar{q_i} \mid^{\eta} dt,$$

where $\eta_1 > 2$ and $\frac{1}{\eta_1} + \frac{1}{\eta} = 1$. Set a function as follows

$$V(\bar{p}, \phi) = \sum_{i=1}^{4} \bar{p_i}\phi_i(t).$$

Using Ito's formula, we have

$$\begin{aligned} &\sum_{i=1}^{4} E\int_0^T \mid \bar{p_i} \mid^{\eta} dt + \sum_{i=1}^{2} E\int_0^T \mid \bar{q_i} \mid^{\eta} dt \\ &= -\sum_{i=1}^{4} E\int_0^T \phi_i \bar{f_i} dt + \sum_{i=1}^{4} E\phi_i(T)\left[h_{x_i}(x(T)) - h_{x_i}(x'(T))\right] \\ &\le C\sum_{i=1}^{4}\left(E\int_0^T \mid \bar{f_i} \mid^{\eta} dt\right)^{\frac{1}{\eta}}\left(E\int_0^T \mid \phi_i \mid^{\eta_1} dt\right)^{\frac{1}{\eta_1}} \\ &+ C\sum_{i=1}^{4}\left(E\left[h_{x_i}(x(T)) - h_{x_i}(x'(T))\right]^{\eta}\right)^{\frac{1}{\eta}} \times \left(E \mid \phi_i(T) \mid^{\eta_1}\right)^{\frac{1}{\eta_1}} \\ &\le C\left(\sum_{i=1}^{4} E\int_0^T \mid \bar{p_i} \mid^{\eta} dt + \sum_{i=1}^{2} E\int_0^T \mid \bar{q_i} \mid^{\eta} dt\right)^{\frac{1}{\eta_1}} \\ &\times \left(\sum_{i=1}^{4}\left(E\int_0^T \mid \bar{f_i} \mid^{\eta} dt\right)^{\frac{1}{\eta}} + \sum_{i=1}^{4}\left(E \mid h_{x_i}(x(T) - h_{x_i}(x'(T)) \mid^{\eta})^{\frac{1}{\eta}}\right)\right) \end{aligned}$$

By the elementary inequality previously mentioned, we have

$$\sum_{i=1}^{4} E\int_0^T |\bar{p_i}|^\eta\, dt + \sum_{i=1}^{2} E\int_0^T |\bar{q_i}|^\eta\, dt$$
$$\leq C\left[\sum_{i=1}^{4} E\int_0^T |\bar{f_i}|^\eta\, dt + \sum_{i=1}^{4} E\,|\, h_{x_i}(x(T)) - h_{x_i}(x'(T))\,|^\eta\right] \tag{3.50}$$

Using (S2) and previous Lemma , we obtain

$$\sum_{i=1}^{4} E\,|\, h_{x_i}(x(T)) - h_{x_i}(x'(T))\,|^\eta \leq C^\eta \sum_{i=1}^{4} E\,|\, x_i(t) - x_i'(t)\,|^\eta] \leq Cd(u,u')^{\frac{k\eta}{2}}.$$

Using Cauchy-Schwartz's inequality, we have

$$E\int_0^T |\bar{f_i}|^\eta\, dt$$
$$\leq C\left[E\int_0^T |\, x_2 - x_2'\,|^\eta \left(\sum_{i=1}^{4} |\, p_i'\,|^\eta + \sum_{i=1}^{2} |\, q_i'\,|^\eta\right) dt + E\int_0^T |\, L_{x_1}(x,u) - L_{x_1}(x',u')\,|^\eta\, dt\right] \tag{3.51}$$
$$\leq C\left(E\int_0^T |\, x_2 - x_2'\,|^{\frac{2\eta}{2-\eta}}\, dt\right)^{1-\frac{\eta}{2}} \left[\left(\sum_{i=1}^{2}\int_0^T |\, p_i'\,|^2\, dt\right)^{\frac{\eta}{2}} + \left(\sum_{i=1}^{2}\int_0^T |\, q_i'\,|^2\, dt\right)^{\frac{\eta}{2}}\right] + Cd(u,u')^{\frac{k\eta}{2}}.$$

Observe that $\frac{2\eta}{1-\eta} < 1, 1 - \frac{\eta}{2} > \frac{k\eta}{2}$ and $d(u,u') < 1$. It comes from Lemmas 3.27 and 3.28 that

$$E[\int_0^T |\bar{f_1}|^\eta\, dt] \leq d(u,u')^{\frac{k\eta}{2}}.$$

Using a similar manner and omitting the details, we obtain

$$\sum_{i=1}^{4} E\int_0^T |\bar{f_i}|^\eta\, dt \leq Cd(u,u')^{\frac{k\eta}{2}}.$$

Next,

$$\sum_{i=1}^{4} E\int_0^T |\bar{p_i}|^\eta\, dt + \sum_{i=1}^{2} E\int_0^T |\bar{q_i}|^\eta\, dt \leq Cd(u,u')^{\frac{k\eta}{2}}.$$

This achieves the proof. □

3.5.6 Necessary condition for near-optimal control

The following theorem provides a necessary condition for the near-optimal control of our system.

Theorem 3.29. *Let (S_1) and (S_2) hold, h and L are convex, and $(p^\epsilon(t), q^\epsilon(t))$ be the solution of the associated adjoint equation under control u^ϵ. There exists an imprecise parameter C such that for any $\eta \in [0,1], \epsilon > 0$ and any $\epsilon -$ optimal pair $(x^\epsilon(t), u^\epsilon(t))$, the following condition emerges*

$$\underset{u\in U_{ad}}{inf}\, E\{\int_0^T (\frac{m_k u(t) I^\epsilon(t)}{1+\gamma_k I^\epsilon(t)}(p_3^\epsilon(t) - p_2^\epsilon(t)) + L(x^\epsilon(t), u(t)))dt\}$$
$$\geq E\{\int_0^T (\frac{m_k u^\epsilon(t) I^\epsilon(t)}{1+\gamma_k I^\epsilon(t)}(p_3^\epsilon(t) - p_2^\epsilon(t)) + L(x^\epsilon(t), u^\epsilon(t)))dt\} - C\epsilon^{\frac{\eta}{3}}. \tag{3.52}$$

Proof. The function $J(x_0, .) : U_{ad} \to \mathbb{R}$ is continuous under the metric d.
Applying previous Lemma and taking $\lambda = \epsilon^{\frac{2}{3}}$ there exists an admissible pair $x^\epsilon(t), u_2^\epsilon(t)$ such that

$$d(u^\epsilon(t), \bar{u}^\epsilon(t)) \leq \epsilon^{\frac{2}{3}}, \tag{3.53}$$

$$\bar{J}(0, x_0; \bar{u}^\epsilon(t)) \leq \bar{J}(0, x_0; u(t)) \tag{3.54}$$

for all $u \in U_{ad}[0, T]$, where

$$\bar{J}(0, x_0; u(t)) = J(0, x_0; u(t)) + \epsilon^{\frac{1}{3}} d(u^\epsilon, \bar{u}^\epsilon(t)). \tag{3.55}$$

It's equivalent to the fact that $(\bar{x}^\epsilon(t), \bar{u}^\epsilon(t))$ is an optimal pair for the principal system under the considered cost function. Moreover, a necessary condition for $(\bar{x}^\epsilon(t), \bar{u}^\epsilon(t))$ will be extracted by the following setting.
Let $\bar{t} \in [0, T]$, $\rho > 0$ and $u \in U_{ad}[0, T]$ by
$u^\rho = u(t)$ if $t \in [\bar{t}, \bar{t} + \rho]$,
and $u^\rho = \bar{u}^\epsilon t$ if $t \in [0, T] \backslash [\bar{t}, \bar{t} + \rho]$
We deduce from that

$$\bar{J}(0, x_0, \bar{u}^\epsilon(t)) \leq \bar{J}(0, x_0, u^\rho(t), \tag{3.56}$$

and

$$d(u^\epsilon(t), \bar{u}^\epsilon t) \leq \epsilon^{\frac{2}{3}}. \tag{3.57}$$

It comes from the Taylor's expansion that

$$\begin{aligned}
&-\rho\epsilon^{\frac{1}{3}} \\
&\quad \leq J(0, x_0; u^\rho(t)) - J(0, x_0; , \bar{u}^\epsilon(t)) \\
&\quad = E\int_0^T \left[L(x^\rho, u^\rho) - L(\bar{x}^\epsilon, \bar{u}^\epsilon)\right] dt + E[h(x^\rho(T)) - h(\bar{x}^\epsilon(T))] \\
&\quad \leq \sum_{i=1}^4 E\{\int_0^T L_{x_i}(\bar{x}^\epsilon, u^\rho)(x_i^\rho - \bar{x}_i^\epsilon) dt\} \\
&\quad + E\{\int_{\bar{t}}^{\bar{t}+\epsilon} [L(\bar{x}^\rho, u) - L(\bar{x}^\epsilon, \bar{u}^\epsilon)] dt\} \\
&\quad + \sum_{i=1}^4 E[h_{x_i}(x^\rho(T))(x_i^\rho(T) - \bar{x}_i^\epsilon(t)))] + o(\rho].
\end{aligned} \tag{3.58}$$

The Itô's formula applied on $\sum_{i=1}^4 \bar{p}_i^\epsilon(x_i^\rho - \bar{x}_i^\epsilon)$ yields to
$\sum_{i=1}^4 Eh_{x_i}(x^\rho(T))(x_i^\rho(T) - \bar{x}_i^\epsilon(T))] \leq \{E\int_{\bar{t}}^{\bar{t}+\rho}[(u^\rho - \bar{u}^\epsilon)(\bar{p}_3^\epsilon) - \bar{p}_1^\epsilon) + (u^\rho - \bar{u}^\epsilon)\bar{p}_3^\epsilon] dt\}$.
Subsequently,

$$\begin{aligned}
-\rho\epsilon^{\frac{1}{3}} &\leq J(0, x_0; u^\rho(t)) - J(0, x_0; , \bar{u}^\epsilon(t)) \\
&+ E\{\int_{\bar{t}}^{\bar{t}+\rho} [L(\bar{x}^\rho, u) - L(\bar{x}^\epsilon, \bar{u}^\epsilon)] dt\} \\
&+ E\{\int_{\bar{t}}^{\bar{t}+\rho} [(u(t) - \bar{u}^\epsilon)(t))((\bar{p}_3(t) - \bar{p}_1(t)) + (u(t) - \bar{u}^\epsilon))\bar{p}_3^\epsilon] dt\} + o(\rho).
\end{aligned} \tag{3.59}$$

Dividing by ρ and letting $\rho \to 0$, we have

$$-\epsilon^{\frac{1}{3}} \leq E[L(\bar{x}^{\epsilon}(t), u(\bar{t})) - L(\bar{x}^{\epsilon}(t), \bar{u}^{\epsilon}(t))] + E[(u(\bar{t}) \\ -\bar{u}^{\epsilon}(t))((\bar{p}_3(t) - \bar{p}_1(t)) + (u(t) - \bar{u}^{\epsilon}))\bar{p}_3^{\epsilon}(t)].$$

We estimate here the following variation

$$\begin{aligned}
&E\{\int_0^T [(u^{\rho}(t) - \bar{u}^{\epsilon}(t)))\bar{p}_3^{\epsilon}(t) - (u^{\rho}(t) - u^{\epsilon}(t)))p_3^{\epsilon}(t)]dt\} \\
&= E\{\int_0^T (\bar{p}_3^{\epsilon}(t) - p_3^{\epsilon}))(u^{\rho}(t) - u^{\epsilon}(t))dt\} + \{\int_0^T p_3^{\epsilon}(t)(u^{\epsilon}(t) - \bar{u}^{\epsilon})(t))dt\} \\
&= I_1 + I_2.
\end{aligned}$$

We easily conclude that for $0 < k < 1$ and $1 < \eta < 2$ verifying $(1+k]\eta < 2$,

$$\begin{aligned}
I_1 &\leq (E\int_0^T \mid \bar{p}_3^{\epsilon}(t) - p_3^{\epsilon}(t) \mid^{\eta} dt)^{\frac{1}{\eta}}(E\int_0^T \mid u_3^{\rho}(t) - u^{\epsilon}(t) \mid^{\frac{\eta}{\eta-1}} dt)^{\frac{\eta-1}{\eta}} \\
&\leq C(d(u,u')^{\frac{k\eta}{2}})^{\frac{1}{\eta}}(E\int_0^T (\mid u^{\rho}(t) \mid^{\frac{\eta}{\eta-1}} + \mid u^{\epsilon}(t) \mid^{\frac{\eta}{\eta-1}})dt)^{\frac{\eta-1}{\eta}} \\
&\leq C\epsilon^{\frac{k}{3}},
\end{aligned}$$

and

$$\begin{aligned}
I_2 &\leq C(E\int_0^T \mid p_3^{\epsilon}(t) \mid^2 dt)^{\frac{1}{2}}(E\int_0^T \mid u^{\epsilon}(t) - \bar{u}^{\epsilon}(t)) \mid^2 \chi_{u^{\epsilon}(t)\neq \bar{u}^{\epsilon}(t)}(t)dt)^{\frac{1}{2}} \\
&\leq C(E\int_0^T (\mid u^{\epsilon}(t) \mid^4 + \mid \bar{u}^{\epsilon}(t) \mid^4)dt)^{\frac{1}{4}}(E\int_0^T \chi_{u^{\epsilon}(t)\neq \bar{u}^{\epsilon}(t)}(t)dt)^{\frac{1}{4}}
\end{aligned}$$

Thus,

$$E\int_0^T [(u^{\rho}(t) - \bar{u}^{\epsilon}(t))(\bar{p}_3^{\epsilon}(t) - (\bar{p}_1^{\epsilon}(t)) - (u^{\rho}(t) - u^{\epsilon})(t))p_3^{\epsilon}(t)]dt \leq \epsilon^{\frac{k}{3}}. \tag{3.60}$$

With a similar argument, we obtain

$$\begin{aligned}
&E\{\int_0^T [(u^{\rho}(t) - \bar{u}^{\epsilon}(t))(\bar{p}_3^{\epsilon}(t)\bar{p}_1^{\epsilon}(t)) - u^{\rho}(t) - u^{\epsilon}(t))(p_3^{\epsilon}(t) - p_1^{\epsilon}(t))]dt\} \\
&+ E\{\int_0^T [L(\bar{x}^{\epsilon}(t), u^{\rho}(t)) - L(\bar{x}^{\epsilon}(t), \bar{u}^{\epsilon}(t))] - [L(x^{\epsilon}(t), u^{\rho}(t)) - L(x^{\epsilon}(t), u^{\epsilon}(t))]dt\} \\
&\leq \epsilon^{\frac{k}{3}}.
\end{aligned} \tag{3.61}$$

We can also obtain the desired result from the above inequalities and the definition of the Hamiltonian function. □

3.5.7 Sufficient condition for near-optimal control

Theorem 3.30. *. Suppose that hypothesis* $(S1)$, $(S2)$ *and* $(S3)$ *hold.*
Let $(x^{\epsilon}(t), u^{\epsilon}(t))$ *be an admissible pair and* $(p^{\epsilon}(t), q^{\epsilon}(t))$ *be the solution of adjoint Equation corresponding to* $(x^{\epsilon}(t), u^{\epsilon}(t))$. *Assume that* H *and* h *are convex, a.s. If for some* $\epsilon > 0$,

$$\begin{aligned}
&E\int_0^T \left(\frac{m_k u^{\epsilon}(t)x_2^{\epsilon}(t)}{1+\gamma_k x_2^{\epsilon}(t)}(p_3^{\epsilon} - p_2^{\epsilon}) + L\left(x^{\epsilon}(t), u^{\epsilon}(t)\right)\right)dt \\
&\leq \inf_{u\in U_{ad}} E\int_0^T \frac{m_k u(t)x_2^{\epsilon}(t)}{1+\gamma_k x_2^{\epsilon}(t)}(p_3^{\epsilon} - p_2^{\epsilon}) + L\left(x^{\epsilon}(t), u(t)\right)dt + \epsilon,
\end{aligned} \tag{3.62}$$

Then,

$$J(0, x_0, u^\epsilon(t)) \leq \underset{u \in U_{ad}}{inf} J(0, x_0, u(t)) + C\epsilon^{\frac{1}{3}}$$

.

Proof. From the definition of the Hamiltonian function H, we have

$$J(0, x_0, u^\epsilon(t)) - J(0, x_0, u(t)) = I_1(t) + I_2(t) + I_3(t), \tag{3.63}$$

With

$$\begin{aligned} I_1 =& E \int_0^T [H(t, x^\epsilon(t), u^\epsilon(t), p^\epsilon(t), q^\epsilon(t)) - H(t, x(t), u(t), p^\epsilon(t), q^\epsilon(t))]dt; \\ I_2 =& E[h(x^\epsilon(T)) - h(x(T))] \\ I_3 =& E \int_0^T [(f^\top(x^\epsilon(t), u^\epsilon(t)) - f^\top(x(t), u(t)))p^\epsilon(t) \\ &+ (\sigma^\top(x^\epsilon(t), u^\epsilon(t)) - \sigma^\top(x(t), u(t)))q^\epsilon(t)]dt \end{aligned} \tag{3.64}$$

Using the convexity of H, we obtain

$$\begin{aligned} I_1 \leq& \sum_{i=1}^4 E \int_0^T H_{x_i}(t, x^\epsilon(t), u^\epsilon(t), p^\epsilon(t), q^\epsilon(t))(x_i^\epsilon(t) - x_i(t))dt \\ &+ E \int_0^T H_u(t, x(t), u(t), p^\epsilon(t), q^\epsilon(t))(u^\epsilon(t) - u(t))dt \end{aligned} \tag{3.65}$$

Similarly,

$$I_2 \leq \sum_{i=1}^4 E(h_{x_i}(x^\epsilon(T) - x_i(T))). \tag{3.66}$$

Itô's formula on $\sum_{i=1}^4 p_i^\epsilon(t)(x_i^\epsilon(t) - x_i(t))$ yields to

$$\begin{aligned} &\sum_{i=1}^4 E(h_{x_i}(x^\epsilon(T) - x_i(T)))) \\ &= -\sum_{i=1}^4 E \int_0^T (x_i^\epsilon(t) - x_i(t))H_{x_i}(t, x^\epsilon(t), u^\epsilon(t), p^\epsilon(t), q^\epsilon(t))dt \\ &+ \sum_{i=1}^4 E \int_0^T p_i^\epsilon(t) \mid f_i(x^\epsilon(t), u^\epsilon(t)) - f_i(x(t), u(t)) \mid dt \\ &+ \sum_{i=1}^4 E \int_0^T q_i^\epsilon(t) \mid \sigma_i(x^\epsilon(t)) - \sigma(x(t)) \mid dt. \end{aligned}$$

Hence,

$$\begin{aligned} I_3 =& \sum_{i=1}^4 E(h_{x_i}(x^\epsilon(T))(x_i^\epsilon(T) - x_i(T))) \\ &+ \sum_{i=1}^4 E \int_0^T (x_i^\epsilon(t) - x_i(t)).(H_{x_i}(t, x^\epsilon(t), u^\epsilon(t), p^\epsilon(t), q^\epsilon(t))dt \end{aligned} \tag{3.67}$$

We arrive then at,

$$\begin{aligned} & J(0,x_0,u^\epsilon(t)) - J(0,x_0,u(t))dt \\ & \leq E\int_0^T (H_u(t,x^\epsilon(t),u^\epsilon(t),p^\epsilon(t),q^\epsilon(t))(u^\epsilon(t)-u(t))dt. \end{aligned} \tag{3.68}$$

To finish this proof, we need to estimate the quantity

$$H_u(t,x^\epsilon(t),u^\epsilon(t),p^\epsilon(t),q^\epsilon(t))(u^\epsilon(t)-u(t))dt.$$

Consider then, the following metric $\bar{d}$ on U_{ad} defining by:

$$\bar{d}(u,u') = E\int_0^T y^\epsilon(t) \mid u(t)-u'(t) \mid dt \tag{3.69}$$

where

$$y^\epsilon(t) = 1 + \sum_{i=1}^4 \mid p_i^\epsilon(t) \mid + \sum_{i=1}^2 \mid q_i^\epsilon(t) \mid . \tag{3.70}$$

It is straightforward to state that $\bar{d}$ is a complete metric as a weighted L^1 norm.
We might get from the definition of the Hamiltonian function

$$E\int_0^T H(t,x^\epsilon(t),u^\epsilon(t),p^\epsilon(t),q^\epsilon(t))dt \geq \underset{u\in U_{ad}}{sup}\, E\int_0^T H(t,x^\epsilon(t),u(t),p^\epsilon(t),q^\epsilon(t))dt - \epsilon \tag{3.71}$$

By setting a functional $F(.); U_{ad} \to \mathbb{R}$

$$F(u(t)) = E\int_0^T H(t,x^\epsilon(t),u^\epsilon(t),p^\epsilon(t),q^\epsilon(t))dt \tag{3.72}$$

Using (S2) we can see that $F(.)$ is continuous with respect to the metric $\bar{d}$.
Thus, there exists a $\bar{u}^\epsilon(t) \in U_{ad}$ such that

$$\begin{aligned} & \bar{d}(u^\epsilon(t),\bar{u}^\epsilon(t)) \leq \epsilon^{\frac{1}{2}} \\ & F(\bar{u}^\epsilon(t)) \leq F(u(t)) + \epsilon^{\frac{1}{2}}\bar{d}(u^\epsilon(t),\bar{u}^\epsilon(t)), \end{aligned} \tag{3.73}$$

for any $u \in Uad$.
It follows that

$$H(t,x^\epsilon(t),u^\epsilon(t),p^\epsilon(t),q^\epsilon(t)) = \underset{u\in U_{ad}}{min}\,[H(t,x^\epsilon(t),u(t),p^\epsilon(t),q^\epsilon(t)) + \epsilon^{\frac{1}{2}} y^\epsilon(t) \mid u(t)-\bar{u}^\epsilon(t)) \mid] \tag{3.74}$$

Using [40], we get

$$0 \in \partial_u(t)H(t,x^\epsilon(t),\bar{u}^\epsilon(t)),p^\epsilon(t),q^\epsilon(t)) \subset \partial_u(t,x^\epsilon(t),u^\epsilon(t)),p^\epsilon(t),q^\epsilon(t)) + [-\epsilon^{\frac{1}{2}}y^\epsilon(t),\epsilon^{\frac{1}{2}}y^\epsilon(t)]. \tag{3.75}$$

From the differentiability of the function H with respect to u, (S1), there exists

$$\lambda_1^\epsilon(t) \in [-\epsilon^{\frac{1}{2}}y^\epsilon(t),\epsilon^{\frac{1}{2}}y^\epsilon(t)]. \tag{3.76}$$

such that

$$H_u(t, x^\epsilon(t), u^\epsilon(t), p^\epsilon(t), q^\epsilon(t)) + \lambda_1^\epsilon(t) = 0. \tag{3.77}$$

We might conclude from (3.77) and (S2) that

$$\begin{aligned} &\mid H_u(t, x^\epsilon(t), \bar{u}^\epsilon(t)), p^\epsilon(t), q^\epsilon(t)) \mid \\ &\leq \mid \lambda_1^\epsilon(t) \mid \\ &\leq 2\epsilon^{\frac{1}{2}} y^\epsilon(t). \end{aligned} \tag{3.78}$$

The previous statements and Hôlder's inequality lead to the desired result. □

Chapter 4

Fractional approach: Mathematical prerequisites and original results

4.1 Mathematical prerequisites

4.1.1 Some definitions and properties within the fractional calculus

In this subsection, we recall some definitions and properties of fractional operators that will be useful in the sequel. For more details, see [16, 21, 39, 74, 75, 85, 114].

Definition 4.1. *Let $f \in L^1(t_0, +\infty)$ and $0 < \alpha \leq 1$. The Riemann–Liouville (RL) fractional integral of function f is defined by*

$$^{RL}I_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-x)^{\alpha-1} f(x) dx, \tag{4.1}$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 4.2. *Let $f \in H^1(t_0, +\infty)$ and $0 < \alpha \leq 1$. The Caputo (C) fractional derivative of function f is given by*

$$^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{f^{'}(x)}{(t-x)^{\alpha}} dx. \tag{4.2}$$

Definition 4.3. *Let $f \in H^1(t_0, +\infty)$ and $0 < \alpha \leq 1$. The Caputo–Fabrizio (CF) fractional derivative of function f is given by*

$$^{CF}D_t^{\alpha} f(t) = \frac{1}{2} \frac{B(\alpha)(2-\alpha)}{1-\alpha} \int_{t_0}^{t} f^{'}(x) \exp\left[-\frac{\alpha}{1-\alpha}(t-x)\right] dx, \tag{4.3}$$

where $B(\alpha)$ denotes a normalization function obeying $B(0) = B(1) = 1$. The fractional integral associated with the CF fractional derivative is defined by

$$^{CF}_{t}I_{t_0}^{\alpha} f(t) = \frac{2(1-\alpha)}{B(\alpha)(2-\alpha)} f(t) + \frac{2\alpha}{B(\alpha)(2-\alpha)}\, ^{RL}_{t}I_{t_0}^{1} f(t). \tag{4.4}$$

Definition 4.4. *Let $f \in H^1(t_0, +\infty)$ and $0 < \alpha \leq 1$. The Atangana–Baleanu–Caputo (ABC) fractional derivative of function f is given by*

$$^{ABC}_{t_0}D_t^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_{t_0}^{t} f^{'}(x) E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-x)^{\alpha}\right] dx. \tag{4.5}$$

The fractional integral associated with the ABC fractional derivative is defined by

$$ {}^{ABC}_{\ t}I^{\alpha}_{t_0}f(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}\,{}^{RL}_{\ t}I^{\alpha}_{t_0}f(t). \tag{4.6} $$

Definition 4.5. *Let $\alpha > 0$ and $\beta > 0$. The Mittag-Leffler function of two parameters α and β is defined by*

$$ E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j+\beta)},\ z \in \mathbb{C}. $$

Remark 4.6. *If $\beta = 1$, then we have*

$$ E_{\alpha,1}(z) = E_{\alpha}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j+1)}, $$

which is called the Mittag-Leffler function of one parameter α; if $\alpha = \beta = 1$, then one gets

$$ E_{1,1}(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!} = \exp(z). $$

Theorem 4.7. *The derivative of the Mittag-Leffler function satisfies:*

$$ \frac{dE_{\alpha,\beta}}{dz}(z) = E^2_{\alpha,\alpha+\beta}(z). $$

Definition 4.8 (See [67])**.** *Let $0 \le \alpha < 1$ and $\beta > 0$. The left-weighted generalized fractional derivative of order α of function f, in the Riemann–Liouville sense, is defined by*

$$ {}^{R}_{a,w}D^{\alpha,\beta}f(x) = \frac{1}{\phi(\alpha)}\frac{1}{w(x)}\frac{d}{dx}\int_a^x (wf)(s)E_{\beta}\left[-\mu_{\alpha}(x-s)^{\beta}\right]ds, \tag{4.7} $$

where E_{β} denotes the Mittag–Leffler function of parameter β defined by

$$ E_{\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\beta j+1)}, \quad z \in \mathbb{C}, \tag{4.8} $$

and $w \in C^1([a,b])$ with $w, w' > 0$. The corresponding fractional integral is defined by

$$ {}_{a,w}I^{\alpha,\beta}f(x) = \phi(\alpha)f(x) + \psi(\alpha)\,{}^{RL}_{a,w}I^{\beta}f(x), \tag{4.9} $$

where ${}^{RL}_{a,w}I^{\beta}$ is the standard weighted Riemann–Liouville fractional integral of order β given by

$$ {}^{RL}_{a,w}I^{\beta}f(x) = \frac{1}{\Gamma(\beta)}\frac{1}{w(x)}\int_a^x (x-s)^{\beta-1}w(s)f(s)ds, \quad x > a. \tag{4.10} $$

In [5, 120, 134], the authors prove the following inequalities for estimating the fractional derivative of certain functions.

Lemma 4.9. *Let $u(t)$ be a real continuous and differentiable function. Then, for any $t \ge t_0$ and $0 < \alpha \le 1$, we have*

$$ {}^{ABC}_{\ t_0}D^{\alpha}_{t}\left(u^2(t)\right) \le 2u(t)\,{}^{ABC}_{\ t_0}D^{\alpha}_{t}u(t), \tag{4.11} $$

$$ {}^{CF}_{\ t_0}D^{\alpha}_{t}\left(u^2(t)\right) \le 2u(t)\,{}^{CF}_{\ t_0}D^{\alpha}_{t}u(t), \tag{4.12} $$

$$ {}^{C}_{t_0}D^{\alpha}_{t}\left(u^2(t)\right) \le 2u(t)\,{}^{C}_{t_0}D^{\alpha}_{t}u(t). \tag{4.13} $$

Lemma 4.10. *Let $u(t)$ be a positive real continuous and differentiable function. Then, for any $t \geq t_0$, $0 < \alpha \leq 1$, and $u^* > 0$, one has*

$$ {}^{ABC}_{t_0}D_t^\alpha \left[u(t) - u^* - u^* \ln \frac{u(t)}{u^*} \right] \leq \left(1 - \frac{u^*}{u(t)}\right) {}^{ABC}_{t_0}D_t^\alpha u(t), \tag{4.14}$$

$$ {}^{C}_{t_0}D_t^\alpha \left[u(t) - u^* - u^* \ln \frac{u(t)}{u^*} \right] \leq \left(1 - \frac{u^*}{u(t)}\right) {}^{C}_{t_0}D_t^\alpha u(t). \tag{4.15}$$

In this subsection, we present some definitions and properties from the fractional calculus literature, which will help us to prove our main results. Along the text, $f \in H^1(a,b)$ is a sufficiently smooth function on $[a,b]$ with $a, b \in \mathbb{R}$. In addition, we adopt the following notations:

$$\phi(\alpha) := \frac{1-\alpha}{B(\alpha)}, \quad \psi(\alpha) := \frac{\alpha}{B(\alpha)},$$

where $0 \leq \alpha < 1$ and $B(\alpha)$ is a normalization function obeying $B(0) = B(1) = 1$. Along the text, we denote

$$\mu_\alpha := \frac{\alpha}{1-\alpha}.$$

Lemma 4.11 (See [118]). *Let $\alpha > 0$, $p \geq 1$, $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$). If $f \in L_p(a,b)$ and $g \in L_q(a,b)$, then*

$$\int_a^b f(x)\, {}^{RL}_{a,1}I^\alpha g(x) dx = \int_a^b g(x)\, {}^{RL}I^\alpha_{b,1} f(x) dx,$$

where ${}^{RL}_{a,1}I^\beta$ is the left standard Riemann–Liouville fractional integral of order α given by

$$ {}^{RL}_{a,1}I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s) ds, \quad x > a, \tag{4.16}$$

and ${}^{RL}I^\alpha_{b,1}$ is the right standard Riemann–Liouville fractional integral of order α given by

$$ {}^{RL}I^\alpha_{b,1} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} f(s) ds, \quad x < b. \tag{4.17}$$

4.2 Lyapunov Functions and Stability Analysis of Fractional-Order Systems

The original results of this section are published in [33].

4.2.1 Introduction

In the last few years, the application of fractional differential equations (FDEs) has increased and gained much attention from researchers due to their ability in modeling and describing anomalous dynamics of real-world processes with memory and hereditary properties. Due to these properties, FDEs have been widely and successfully applied in various fields of science and engineering, such as viscoelasticity, signal and image processing, physics, mechanics, control, biology, and economy and finance [44, 132].

Fractional calculus (FC) literature assists to remarkable development of the fractional notions of differentiation. Several types of fractional derivatives were proposed, such as the Riemann–Liouville (RL), Caputo (C), Caputo–Fabrizio (CF) and Atangana–Baleanu–Caputo (ABC) operators. The standard RL and C derivatives [75] have certain disadvantages, being classified as fractional derivatives with singular kernels. Caputo and Fabrizio [39] suggested a new fractional derivative in which the memory is represented by an exponential kernel. Few years later, another fractional derivative was proposed by Atangana and Baleanu [74], where the memory kernel is modeled by the Mittag–Leffler function. These operators are extensively used by different researchers to describe the dynamics of various nonlinear systems [25, 34, 72, 98, 122, 139].

Stability is one of the powerful tools for analyzing the qualitative properties of non-linear dynamical systems. Lyapunov's direct method, also called the second Lyapunov's method, represents an effective way to examine the global behavior of a system without resolving it explicitly. This technique is based on constructing appropriate functionals, called the Lyapunov functionals, that should satisfy some conditions. In physics, these functionals can be either energy, potential, or other, but generally there is no precise technique to determine them. Recently, many scholars have focused on the stability analysis of fractional-order systems and some others have proposed specific Lyapunov functionals candidates, such as Volterra-type and quadratic functions [5, 46, 83, 120, 134, 142]. Nevertheless, these functions remain inadequate and incompatible with certain classes of fractional-order systems.

Motivated by the aforementioned works and observations, our main contribution here is to propose general Lyapunov functionals as candidates for fractional-order systems. We first develop new inequalities to estimate the fractional-order derivative of specific functions that generalize some works existing in the literature. These estimates allow us to construct suitable Lyapunov's functionals for fractional-order systems and, therefore, to establish the global stability of their steady-states.

The rest of the work is structured as follows. First, some necessary definitions and properties related to the fractional calculus are recalled. Second, useful estimations for fractional derivatives are proved. Further, as an application of our results, the global stability of a SEIR fractional-order model with a general incidence rate is studied. Finally, we end with an adequate conclusions.

4.2.2 Useful fractional derivative estimates

The aim of this section is to establish some new estimates for the fractional derivative of function Ψ defined by

$$\begin{aligned}\Psi(u) &= \int_{u^*}^{u} \frac{g(s)-g(u^*)}{g(s)}ds \\ &= u - u^* - \int_{u^*}^{u} \frac{g(u^*)}{g(s)}ds,\end{aligned} \tag{4.18}$$

where g is a non-negative, differentiable, and strictly increasing function on $\mathbb{R}^+$. Our estimates will allow us to extend the classical Lyapunov functions to fractional-order systems.

Note that Ψ is positive in $\mathbb{R}^+ \setminus \{u^*\}$ with $\Psi(u^*) = 0$. In fact, Ψ is differentiable and

$$\frac{d\Psi}{du} = 1 - \frac{g(u^*)}{g(u)}.$$

Since g is a strictly increasing function, then Ψ is strictly decreasing if $u < u^*$ and strictly increasing if $u > u^*$, with u^* its global minimum.

Theorem 4.12. *Let $u(t)$ be a real positive differentiable function. Then, for any $t \geq t_0$, $0 < \alpha \leq 1$, and $u^* > 0$, we have*

$${}^{ABC}_{t_0}D^{\alpha}_{t}\Psi(u(t)) \leq \left(1 - \frac{g(u^*)}{g(u(t))}\right) {}^{ABC}_{t_0}D^{\alpha}_{t}u(t), \tag{4.19}$$

$${}^{CF}_{t_0}D^{\alpha}_{t}\Psi(u(t)) \leq \left(1 - \frac{g(u^*)}{g(u(t))}\right) {}^{CF}_{t_0}D^{\alpha}_{t}u(t). \tag{4.20}$$

Proof. We start by reformulating inequality (4.19). By the linearity of the ABC fractional derivative, we obtain that

$${}^{ABC}_{t_0}D^{\alpha}_{t}\Psi(u(t)) = {}^{ABC}_{t_0}D^{\alpha}_{t}u(t) - {}^{ABC}_{t_0}D^{\alpha}_{t}\left[\int_{u^*}^{u(t)} \frac{g(u^*)}{g(s)}ds\right].$$

Hence, the inequality (4.19) becomes

$$ {}_{t_0}^{ABC}D_t^{\alpha}u(t) - {}_{t_0}^{ABC}D_t^{\alpha}\left[\int_{u^*}^{u(t)} \frac{g(u^*)}{g(s)}ds\right] \leq \left(1 - \frac{g(u^*)}{g(u)}\right) {}_{t_0}^{ABC}D_t^{\alpha}u(t). $$

Because g is a non-negative function, we get

$$ g(u(t))\left[{}_{t_0}^{ABC}D_t^{\alpha}u(t) - {}_{t_0}^{ABC}D_t^{\alpha}\left(\int_{u^*}^{u(t)} \frac{g(u^*)}{g(s)}ds\right)\right] \leq (g(u(t)) - g(u^*))\, {}_{t_0}^{ABC}D_t^{\alpha}u(t). $$

Thus,

$$ {}_{t_0}^{ABC}D_t^{\alpha}u(t) - g(u(t)){}_{t_0}^{ABC}D_t^{\alpha}\left[\int_{u^*}^{u(t)} \frac{1}{g(s)}ds\right] \leq 0. \tag{4.21} $$

Using the definition of ABC fractional derivative (4.5), we have

$$ {}_{t_0}^{ABC}D_t^{\alpha}u(t) = \frac{B(\alpha)}{1-\alpha}\int_{t_0}^{t} u'(x)E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-x)^{\alpha}\right]dx $$

and

$$ {}_{t_0}^{ABC}D_t^{\alpha}\left[\int_{u^*}^{u(t)} \frac{1}{g(s)}ds\right] = \frac{B(\alpha)}{1-\alpha}\int_{t_0}^{t} \frac{u'(x)}{g(u(x))}E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-x)^{\alpha}\right]dx. $$

Consequently, the inequality (4.21) can be written as

$$ \frac{B(\alpha)}{1-\alpha}\int_{t_0}^{t} u'(x)\left(1 - \frac{g(u(t))}{g(u(x))}\right)E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-x)^{\alpha}\right]dx \leq 0. \tag{4.22} $$

Now, we show that the inequality (4.22) is verified. For this, we denote

$$ H(t) = \int_{t_0}^{t} u'(x)\left(1 - \frac{g(u(t))}{g(u(x))}\right)E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-x)^{\alpha}\right]dx $$

and set

$$ v(x,t) = E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-x)^{\alpha}\right]; \quad \frac{dv(x,t)}{dx} = \frac{\alpha^2(t-x)^{\alpha-1}}{1-\alpha}E^2_{\alpha,\alpha+1}\left[-\frac{\alpha}{1-\alpha}(t-x)^{\alpha}\right]; $$

$$ w(x,t) = u(x) - u(t) - \int_{u(t)}^{u(x)} \frac{g(u(t))}{g(s)}ds; \quad \frac{dw(x,t)}{dx} = u'(x)\left(1 - \frac{g(u(t))}{g(u(x))}\right). $$

Integrating by parts the integral $H(t)$, we obtain that

$$ \begin{aligned} H(t) = &\left[E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-x)^{\alpha}\right]w(x,t)\right]_{x=t_0}^{x=t} \\ &- \int_{t_0}^{t} \frac{\alpha^2(t-x)^{\alpha-1}}{1-\alpha}E^2_{\alpha,\alpha+1}\left[-\frac{\alpha}{1-\alpha}(t-x)^{\alpha}\right]w(x,t)dx. \end{aligned} \tag{4.23} $$

Since $w(x,t) \geq 0$ and

$$ \lim_{x\to t} E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-x)^{\alpha}\right]w(x,t) = 0, $$

it follows that

$$ \begin{aligned} H(t) = &-E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-t_0)^{\alpha}\right]w(t_0,t) \\ &- \int_{t_0}^{t} \frac{\alpha^2(t-x)^{\alpha-1}}{1-\alpha}E^2_{\alpha,\alpha+1}\left[-\frac{\alpha}{1-\alpha}(t-x)^{\alpha}\right]w(x,t)dx \leq 0. \end{aligned} $$

As a result, the inequality (4.22) is satisfied and (4.19) holds true. □

Remark 4.13. *Inequality* (4.20) *is obtained by replacing*

$$E_\alpha\left[-\frac{\alpha}{1-\alpha}(t-x)^\alpha\right]$$

with

$$\exp\left[-\frac{\alpha}{1-\alpha}(t-x)\right]$$

and following the same steps as given in the proof of Theorem 4.12.

Remark 4.14. *A similar inequality also holds for the Caputo fractional derivative as follows [36]:*

$$ {}_{t_0}^{C}D_t^\alpha \Psi(u(t)) \leq \left(1-\frac{g(u^*)}{g(u(t))}\right) {}_{t_0}^{C}D_t^\alpha u(t). \tag{4.24}$$

If $g(s)=s$, then we obtain $\Psi(u(t))=u(t)-u^*-u^*\ln\dfrac{u(t)}{u^*}$. We obtain from Theorem 4.12 the following corollary.

Corollary 4.15. *Let $u(t)$ be a positive differentiable function. For any $t \geq t_0$, $0<\alpha\leq 1$, and $u^*>0$, we have*

$$\begin{aligned} {}_{t_0}^{ABC}D_t^\alpha\left[u(t)-u^*-u^*\ln\frac{u(t)}{u^*}\right] &\leq \left(1-\frac{u^*}{u(t)}\right){}_{t_0}^{ABC}D_t^\alpha u(t),\\ {}_{t_0}^{CF}D_t^\alpha\left[u(t)-u^*-u^*\ln\frac{u(t)}{u^*}\right] &\leq \left(1-\frac{u^*}{u(t)}\right){}_{t_0}^{CF}D_t^\alpha u(t). \end{aligned} \tag{4.25}$$

4.2.3 An application

In [151], Yang and Xu proposed a SEIR model with Caputo fractional derivative and general incidence rate as follows:

$$\begin{cases} {}_0^CD_t^\alpha S(t) = \Lambda^\alpha - d^\alpha S(t) - \beta^\alpha F\left(S(t)\right)G\left(I(t)\right),\\ {}_0^CD_t^\alpha E(t) = \beta^\alpha F\left(S(t)\right)G\left(I(t)\right) - (\sigma^\alpha + d^\alpha)E(t),\\ {}_0^CD_t^\alpha I(t) = \sigma^\alpha E(t) - (\gamma^\alpha + d^\alpha)I(t),\\ {}_0^CD_t^\alpha R(t) = \gamma^\alpha I(t) - d^\alpha R(t), \end{cases} \tag{4.26}$$

where $\alpha \in (0,1]$. The variables $S(t)$, $E(t)$, $I(t)$ and $R(t)$ represent the number of susceptible, exposed, infective and recovered individuals at time t, respectively. All the other parameters are assumed to be positive constants.

The authors of [151] first analyzed the global stability of the disease-free equilibrium and discussed the stability of the endemic equilibrium but only when $F(S)=S$. However, they mentioned that they can not use the estimation (4.15) in Lemma 4.10 to establish global stability in the general case and kept this problem as an open question, for future work.

In addition, we note that function $F\left(S(t)\right)G\left(I(t)\right)$ does not cover all the incidence functions existing in the literature, e.g., $\dfrac{SI}{1+\alpha_1 S+\alpha_2 I+\alpha_3 SI}$, $\alpha_1,\alpha_2,\alpha_3 \geq 0$ [86, 90], where we can not separate the variables S and I. Here, we generalize the SEIR model (4.26) and apply our results to give a rigorous proof of the stability for both equilibrium points.

Let us consider the general fractional-order SEIR model

$$\begin{cases} {}_0D_t^\alpha S(t) = \Lambda^\alpha - d^\alpha S(t) - F\left(S(t),I(t)\right),\\ {}_0D_t^\alpha E(t) = F\left(S(t),I(t)\right) - (\sigma^\alpha + d^\alpha)E(t),\\ {}_0D_t^\alpha I(t) = \sigma^\alpha E(t) - (\gamma^\alpha + d^\alpha)I(t),\\ {}_0D_t^\alpha R(t) = \gamma^\alpha I(t) - d^\alpha R(t), \end{cases} \tag{4.27}$$

where ${}_0D_t^\alpha$ denotes any fractional-order derivative mentioned above. The general incidence function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is assumed to be continuously differentiable and to satisfy the following hypotheses:

$$\begin{aligned} F(S,0) = F(0,I) = 0 \ \text{ and } \ F(S,I) = IF_1(S,I) \quad \text{for all } S, I \geq 0,\\ \frac{\partial F_1}{\partial S}(S,I) > 0 \ \text{ and } \ \frac{\partial F_1}{\partial I}(S,I) \leq 0 \quad \text{for all } S \geq 0 \ \text{ and } \ I \geq 0,\\ \frac{\partial F}{\partial I}(S,I) \geq 0 \quad \text{for all } S \geq 0 \ \text{ and } \ I \geq 0. \end{aligned} \tag{H}$$

Since $R(t)$ does not appear in the first three equations of system (4.27), without loss of generality we discuss the following system:

$$\begin{cases} {}_0D_t^\alpha S(t) = \Lambda^\alpha - d^\alpha S(t) - F\left(S(t), I(t)\right),\\ {}_0D_t^\alpha E(t) = F\left(S(t), I(t)\right) - m_1 E(t),\\ {}_0D_t^\alpha I(t) = \sigma^\alpha E(t) - m_2 I(t), \end{cases} \tag{4.28}$$

where $m_1 = \sigma^\alpha + d^\alpha$ and $m_2 = \gamma^\alpha + d^\alpha$.

System (4.28) has a disease-free equilibrium $P_f = (S_0, 0, 0)$ with $S_0 = \dfrac{\Lambda^\alpha}{d^\alpha}$ and an endemic equilibrium $P^* = (S^*, E^*, I^*)$ when $R_0 > 1$, where

$$R_0 = \frac{\sigma^\alpha}{m_1 m_2} \frac{\partial F(S_0, 0)}{\partial I}$$

and $E^* \in \left[0, \dfrac{\Lambda^\alpha}{d^\alpha}\right]$, $S^* = \dfrac{\Lambda^\alpha - m_1 E^*}{d^\alpha}$ and $I^* = \dfrac{\sigma^\alpha E^*}{m_2}$.

Next, we prove the global stability of both equilibriums by constructing appropriate Lyapunov functionals and using our results of Section 4.2.2.

Theorem 4.16. *The disease-free equilibrium P_f is asymptotically stable when $R_0 \leq 1$. The endemic equilibrium P^* is asymptotically stable whenever $R_0 > 1$.*

Proof. For the disease-free equilibrium we define the following Lyapunov functional:

$$V_0(t) = \int_{S_0}^{S(t)} \frac{F_1(x,0) - F_1(S_0,0)}{F_1(x,0)} dx + E(t) + \frac{m_1}{\sigma^\alpha} I.$$

Applying our results, we estimate the fractional time derivative of function V_0 as

$${}_0D_t^\alpha V_0(t) \leq \left(1 - \frac{F_1(S_0,0)}{F_1(S(t,0)}\right) {}_0D_t^\alpha S(t) + {}_0D_t^\alpha E(t) + \frac{m_1}{\sigma^\alpha}\, {}_0D_t^\alpha I(t).$$

Using the fact that $\Lambda^\alpha = d^\alpha S_0$, we get

$$\begin{aligned} {}_0D_t^\alpha V_0(t) &\leq \left(1 - \frac{F_1(S_0,0)}{F_1(S(t),0)}\right) \left(d^\alpha (S_0 - S(t)\right) + I(t) F_1(S_0,0) \frac{F_1(S(t), I(t))}{F_1(S(t),0)}\\ &\quad - \frac{m_1 m_2}{\sigma^\alpha} I(t)\\ &\leq \left(1 - \frac{F_1(S_0,0)}{F_1(S(t),0)}\right) \left(d^\alpha (S_0 - S(t)\right) + I(t) F_1(S_0,0) - \frac{m_1 m_2}{\sigma^\alpha} I(t)\\ &= \left(1 - \frac{F_1(S_0,0)}{F_1(S(t),0)}\right) \left(d^\alpha (S_0 - S(t)\right) + \left(\frac{\partial F(S_0,0)}{\partial I} - \frac{m_1 m_2}{\sigma^\alpha}\right) I(t)\\ &= \left(1 - \frac{F_1(S_0,0)}{F_1(S(t),0)}\right) \left(d^\alpha (S_0 - S(t)\right) + \frac{m_1 m_2}{\sigma^\alpha} \left(R_0 - 1\right) I(t). \end{aligned}$$

Since F_1 is an increasing function with respect to S, one has

$$1-\frac{F_1(S_0,0)}{F_1(S,0)}\geq 0 \quad \text{for} \quad S\geq S_0,$$
$$1-\frac{F_1(S_0,0)}{f(S,0)}< 0 \quad \text{for} \quad S< S_0.$$

Then, we get

$$\left(1-\frac{F_1(S_0,0)}{F_1(S,0)}\right)(S_0-S)\leq 0.$$

It follows that ${}_0D_t^{\alpha}V_0(t)\leq 0$ for $R_0\leq 1$ with ${}_0D_t^{\alpha}V_0(t)=0$ if $S=S_0$ and $I=0$. Substituting $(S,I)=(S_0,0)$ in (4.28) shows that $E\to 0$ as $t\to\infty$. We conclude that the disease-free equilibrium P_f is asymptotically stable when $R_0\leq 1$.

Next, we assume that $R_0>1$ and we propose the following Lyapunov functional V_1 for the endemic equilibrium:

$$V_1(t)=\int_{S^*}^{S(t)}\frac{F(x,I^*)-F(S^*,I^*)}{F(x,I^*)}dx+\int_{E^*}^{E(t)}\frac{x-E^*}{x}dx+\frac{m_1}{\sigma^{\alpha}}\left(\int_{I^*}^{I(t)}\frac{x-I^*}{x}dx\right).$$

Computing the time fractional derivative of V_1, we get

$$\begin{aligned}{}_0D_t^{\alpha}V_1(t)\leq&\left(1-\frac{F(S^*,I^*)}{F(S(t),I^*)}\right){}_0D_t^{\alpha}S(t)+\left(1-\frac{E^*}{E}\right){}_0D_t^{\alpha}E(t)\\&+\frac{m_1}{\sigma^{\alpha}}\left(1-\frac{I^*}{I}\right){}_0D_t^{\alpha}I(t).\end{aligned}$$

Using the fact that $\Lambda^{\alpha}=d^{\alpha}S^*+F(S^*,I^*)$, $F(S^*,I^*)=m_1E^*$ and $\sigma^{\alpha}E^*=m_2I^*$, we obtain

$$\begin{aligned}{}_0D_t^{\alpha}V_1(t)\leq&\left(1-\frac{F(S^*,I^*)}{F(S(t),I^*)}\right)d^{\alpha}(S^*-S(t))\\&+F(S^*,I^*)\left[3-\frac{F(S^*,I^*)}{F(S,I^*)}+\frac{F(S,I)}{F(S,I^*)}-\frac{E^*F(S,I)}{EF(S^*,I^*)}-\frac{I}{I^*}-\frac{I^*E}{IE^*}\right]\\=&\left(1-\frac{F_1(S^*,I^*)}{F_1(S(t),I^*)}\right)d^{\alpha}(S^*-S(t))-F(S^*,I^*)\left[G\left(\frac{I}{I^*}\right)-G\left(\frac{F(S,I)}{F(S,I^*)}\right)\right.\\&\left.+G\left(\frac{F(S^*,I^*)}{F(S,I^*)}\right)+G\left(\frac{E^*F(S,I)}{EF(S^*,I^*)}\right)+G\left(\frac{I^*E}{IE^*}\right)\right],\end{aligned}$$

where $G(x)=x-1-\ln(x)$. Now, we show that $G\left(\frac{I}{I^*}\right)-G\left(\frac{F(S,I)}{F(S,I^*)}\right)\geq 0$. For this, we set

$$H(I)=G\left(\frac{F(S,I)}{F(S,I^*)}\right)-G\left(\frac{I}{I^*}\right).$$

Computing the derivative of H with respect to I, we obtain

$$\frac{dH}{dI}=\frac{F(S,I)-F(S,I^*)}{F(S,I)F(S,I^*)}\frac{\partial F(S,I)}{\partial I}-\frac{I-I^*}{II^*}.$$

We discuss two cases:

Case 1. If $I\geq I^*$, then $F(S,I)\geq F(S,I^*)$. Because

$$\frac{\partial F(S,I)}{\partial I}=F_1(S,I)+I\frac{\partial F_1(S,I)}{\partial I}\leq F_1(S,I)$$

it follows that

$$\begin{aligned}\frac{dH}{dI} &\leq \frac{F(S,I) - F(S,I^*)}{F(S,I)F(S,I^*)} F_1(S,I) - \frac{I - I^*}{II^*} \\ &= \frac{1}{F(S,I^*)} \left[F_1(S,I) - F_1(S,I^*)\right] \leq 0.\end{aligned}$$

Hence $H(I) \leq H(I^*) = 0$.

Case 2. If $I \leq I^*$, then $F(S,I) \leq F(S,I^*)$. Therefore,

$$\begin{aligned}\frac{dH}{dI} &\geq \frac{F(S,I) - F(S,I^*)}{F(S,I)F(S,I^*)} F_1(S,I) - \frac{I - I^*}{II^*} \\ &= \frac{1}{F(S,I^*)} \left[F_1(S,I) - F_1(S,I^*)\right] \geq 0.\end{aligned}$$

Hence, $H(I) \leq H(I^*) = 0$.

We conclude that ${}_0D_t^\alpha V_1(t)$ is negative definite. Consequently, the endemic equilibrium P^* is asymptotically stable whenever $R_0 > 1$. □

4.2.4 Conclusions

In this section, we have developed some estimates for fractional derivatives without a singular kernel and applied it to establish the stability of fractional-order systems. To illustrate the efficacy of the obtained results, we have employed them to solve an open problem posed by Yang and Xu in [151] and prove the stability of a SEIR fractional-order system with a general incidence rate. We construct suitable Lyapunov functionals and proved the globally asymptotically stability of the disease-free and endemic equilibriums in terms of the basic reproduction number R_0. Our results generalize and improve those of [11, 61, 119].

4.3 Taylor's Formula for Generalized Weighted Fractional Derivatives with Nonsingular Kernels

This work is published in [163]

4.3.1 Introduction

Taylor's formulas play a crucial role in mathematical analysis, e.g., in asymptotic methods, nonlinear programming, the calculus of variations and optimal control [41, 101, 154]. In the literature, one can find different forms of Taylor's formulas, both on the classical and smooth one-dimensional case as well as multi-dimensional, nonsmooth, and non-Newtonian cases [136, 8, 50] The appearance of fractional-order theories requires the establishment of corresponding Taylor's formulas [104, 149]. For this reason, Taylor's theorems have been immediately proved, in different forms, for the Riemann–Liouville fractional calculus [65, 146, 138] as well as for Caputo fractional derivatives [105]. The literature on fractional Taylor theorems is now vast: see, e.g., [29, 45, 47] and references therein. However, all such fractional-order Taylor's formulas are valid for fractional derivatives with a singular kernel only. More recently, several researchers have been trying to use fractional calculus in the treatment of dynamics of complex systems, which have complicated dynamics that cannot be properly described with classical/singular-kernel fractional models [17, 32, 100, 157]. That gave rise to the appearance of fractional derivatives with nonsingular kernels and, as consequence, to the need of obtaining Taylor's formulas for such kind of operators [75]. In particular, Fernandez and Baleanu have established in [57] analogues of Taylor's theorem for fractional differential operators defined using a Mittag–Leffler kernel and a mean value theorem for the Atangana–Baleanu–Caputo (ABC) fractional derivative, introduced in [74] and now under strong current investigations [27, 66, 76]. Here we consider the generalized weighted fractional derivative in Caputo sense as introduced in 2020 by Hattaf [67, 68]. Our main results, formulated for this generalized weighted fractional calculus, allows one to extend, in a natural and direct way, the 2020 results of Al-Refai [7] and the 2018 results of Fernandez and Baleanu [57] as simple corollaries.

The work is organized as follows. First, for completeness and for fixing notations, we recall necessary definitions and properties needed in the sequel to prove our results. The elaboration of new tools, enabling us to obtain a general and rich Taylor's formula (cf. Theorem 4.25), are given in the sequel of main results. As an application, we obtain after that several new mean value theorems. We end lastly an appropriate conclusion.

4.3.2 Preliminaries

In this section, we present some definitions and properties from the fractional calculus literature, which will help us to prove our main results. Along the text, $f \in H^1(a,b)$ is a sufficiently smooth function on $[a,b]$ with $a,b \in \mathbb{R}$.

Definition 4.17 (See, e.g., [93]). *The Riemann–Liouville (RL) fractional integral operator of order $\alpha > 0$ with $a \geq 0$ is defined by*

$$ {}^{RL}I_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s)ds, \quad x > a, \tag{4.29}$$

where $\Gamma(\cdot)$ is the Gamma function.

For the sake of simplicity, we adopt the following notations:

$$\begin{aligned} \phi(\alpha) &= \frac{1-\alpha}{B(\alpha)}, \\ \psi(\alpha) &= \frac{\alpha}{B(\alpha)}, \end{aligned}$$

where $B(\alpha)$ denotes a normalization function obeying $B(0) = B(1) = 1$.

Definition 4.18 (See [39]). *The Caputo–Fabrizio (CF) fractional derivative of order $0 \leq \alpha \leq 1$ of function f is given by*

$$ {}_a^{CF}D^{\alpha} f(x) = \frac{1}{\phi(\alpha)} \int_a^x f'(s) \exp\left[-\mu_\alpha (x-s)\right] ds \tag{4.30}$$

with $\mu_\alpha = \dfrac{\alpha}{1-\alpha}$. The fractional integral associated with the CF fractional derivative is defined by

$$ {}^{CF}I_a^{\alpha} f(x) = \phi(\alpha) f(x) + \psi(\alpha)\, {}^{RL}I_a^{1} f(x). \tag{4.31}$$

Definition 4.19 (See [74]). *The Atangana–Baleanu–Caputo (ABC) fractional derivative of order $0 \leq \alpha \leq 1$ of function f is given by*

$$ {}_a^{ABC}D^{\alpha} f(x) = \frac{1}{\phi(\alpha)} \int_a^x f'(s) E_\alpha\left[-\mu_\alpha (x-s)^\alpha\right] ds, \tag{4.32}$$

where E_α denotes the Mittag–Leffler function of parameter α defined by

$$ E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}, \quad z \in \mathbb{C}.$$

The fractional integral associated with the ABC fractional derivative is given by

$$ {}^{ABC}I_a^{\alpha} f(x) = \phi(\alpha) f(x) + \psi(\alpha)\, {}^{RL}I_a^{\alpha} f(x). \tag{4.33}$$

Definition 4.20 (See [7]). *The weighted ABC fractional derivative of order $0 \leq \alpha \leq 1$ of function f with respect to the weight function w is given by*

$$ {}_a^C D_w^\alpha f(x) = \frac{1}{\phi(\alpha)} \frac{1}{w(x)} \int_a^x (wf)'(s) E_\alpha \left[-\mu_\alpha (x-s)^\alpha \right] ds, \tag{4.34} $$

where $w \in C^1([a,b])$ with $w, w' > 0$. The corresponding fractional integral is defined by

$$ {}^C I_{a,w}^\alpha f(x) = \phi(\alpha) f(x) + \psi(\alpha)\, {}^{RL} I_{a,w}^\alpha f(x), \tag{4.35} $$

where ${}^{RL} I_{a,w}^\alpha$ is the standard weighted Riemann–Liouville fractional integral of order α given by

$$ {}^{RL} I_{a,w}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \frac{1}{w(x)} \int_a^x (x-s)^{\alpha-1} w(s) f(s) ds, \quad x > a. \tag{4.36} $$

Definition 4.21 (See [67]). *Let $\beta > 0$. The generalized fractional derivative of order $0 \leq \alpha \leq 1$ of function f with respect to the weight function w is given by*

$$ {}_a^C D_w^{\alpha,\beta} f(x) = \frac{1}{\phi(\alpha)} \frac{1}{w(x)} \int_a^x (wf)'(s) E_\beta \left[-\mu_\alpha (x-s)^\beta \right] ds, \tag{4.37} $$

where $w \in C^1([a,b])$ with $w, w' > 0$. The corresponding fractional integral is defined by

$$ {}^C I_{a,w}^{\alpha,\beta} f(x) = \phi(\alpha) f(x) + \psi(\alpha)\, {}^{RL} I_{a,w}^\beta f(x), \tag{4.38} $$

where ${}^{RL} I_{a,w}^\beta$ is the standard weighted Riemann–Liouville fractional integral of order β.

Theorem 4.22 (See [67]). *Let $\alpha \in [0,1)$, $\beta > 0$. Then,*

$$ {}^C I_{a,w}^{\alpha,\beta} \left({}_a^C D_w^{\alpha,\beta} f(x) \right) = f(x) - \left(\frac{w(a)}{w(x)} f(a) \right). $$

To simplify the writing, we denote by $\mathfrak{D}_a^{[\alpha,\beta]}$ the generalized fractional derivative (4.37) and by $\mathfrak{I}_a^{[\alpha,\beta]}$ its associated fractional integral (4.38).

4.3.3 Main results

We begin by proving an important result that has a crucial role in the proof of our Taylor's formula for weighted generalized fractional derivatives with a nonsingular kernel (cf. proofs of Lemma 4.24 and Theorem 4.25).

Theorem 4.23. *Suppose that $f \in C([a,b])$ and $n \in \mathbb{N}$. Then,*

$$ \mathfrak{I}_a^{n[\alpha,\beta]} f(x) = \sum_{k=0}^{n} C_n^k \phi(\alpha)^{n-k} \psi(\alpha)^k \left({}^{RL} I_{a,w}^{k\beta} f(x) \right) $$

with $x \in [a,b]$ and $\alpha \in [0,1]$, where $\mathfrak{I}_a^{n[\alpha,\beta]} = \mathfrak{I}_a^{[\alpha,\beta]} \cdot \mathfrak{I}_a^{[\alpha,\beta]} \cdots \mathfrak{I}_a^{[\alpha,\beta]}$ (n-times).

Proof. We proceed by induction. Firstly, note that the equality of Theorem 4.23 is true for $n = 0$. Supposing that the equality of Theorem 4.23 is true, we show that

$$ \mathfrak{I}_a^{(n+1)[\alpha,\beta]} f(x) = \sum_{k=0}^{n+1} C_{n+1}^k \phi(\alpha)^{n+1-k} \psi(\alpha)^k \left({}^{RL} I_{a,w}^{k\beta} f(x) \right), \quad x \in [a,b], $$

holds. Indeed,

$$ \mathfrak{I}_a^{(n+1)[\alpha,\beta]} f(x) = \mathfrak{I}_a^{\alpha,\beta} \left(\mathfrak{I}_a^{n[\alpha,\beta]} f(x) \right) $$

$$
\begin{aligned}
&= \phi(\alpha)\big(\mathfrak{I}_a^{n[\alpha,\beta]} f(x)\big) + \psi(\alpha)\, {}^{RL}I_{a,w}^{\beta} \left(\mathfrak{I}_a^{n[\alpha,\beta]} f(x)\right)\\
&= \phi(\alpha)\left[\sum_{k=0}^{n} C_n^k \phi(\alpha)^{n-k}\psi(\alpha)^k \left({}^{RL}I_{a,w}^{k\beta} f(x)\right)\right]\\
&\quad + \psi(\alpha)^{RL}I_{a,w}^{\beta}\left[\sum_{k=0}^{n} C_n^k \phi(\alpha)^{n-k}\psi(\alpha)^k \left({}^{RL}I_{a,w}^{k\beta} f(x)\right)\right]\\
&= \sum_{k=0}^{n} C_n^k \phi(\alpha)^{n+1-k}\psi(\alpha)^k \left({}^{RL}I_{a,w}^{k\beta} f(x)\right)\\
&\quad + \sum_{k=0}^{n} C_n^k \phi(\alpha)^{n-k}\psi(\alpha)^{k+1} \left({}^{RL}I_{a,w}^{(k+1)\beta} f(x)\right)\\
&= \phi(\alpha)^{n+1} f(x) + \sum_{k=1}^{n} C_n^k \phi(\alpha)^{n+1-k}\psi(\alpha)^k \left({}^{RL}I_{a,w}^{k\beta} f(x)\right)\\
&\quad + \sum_{k=1}^{n} C_n^{k-1} \phi(\alpha)^{n+1-k}\psi(\alpha)^k ({}^{RL}I_{a,w}^{k\beta} f(x))\\
&\quad + \psi(\alpha)^{n+1}\left({}^{RL}I_{a,w}^{(n+1)\beta} f(x)\right)\\
&= \sum_{k=0}^{n+1} C_{n+1}^k \phi(\alpha)^{n+1-k}\psi(\alpha)^k \left({}^{RL}I_{a,w}^{k\beta} f(x)\right), \quad x \in [a,b],
\end{aligned}
$$

which completes the proof. □

The following lemma will allow us to construct our weighted Taylor's formula for weighted generalized fractional derivatives with a nonsingular kernel.

Lemma 4.24. *Suppose that* $\mathfrak{D}_a^{n[\alpha,\beta]} f$, $\mathfrak{D}_a^{(n+1)[\alpha,\beta]} f \in C([a,b])$ *for* $0 \le \alpha \le 1$. *Then,*

$$
\begin{aligned}
&\mathfrak{I}_a^{n[\alpha,\beta]}\mathfrak{D}_a^{n[\alpha,\beta]} f(x) - \mathfrak{I}_a^{(n+1)[\alpha,\beta]}\mathfrak{D}_a^{(n+1)[\alpha,\beta]} f(x)\\
&\qquad = \frac{w(a)}{w(x)}\left(\mathfrak{D}_a^{n[\alpha,\beta]} f(a)\right)\sum_{k=0}^{n} C_n^k \phi(\alpha)^{n-k}\psi(\alpha)^k \left(\frac{(x-a)^{k\beta}}{\Gamma(k\beta+1)}\right),
\end{aligned}
$$

where $\mathfrak{D}_a^{n[\alpha,\beta]} = \mathfrak{D}_a^{[\alpha,\beta]} \cdot \mathfrak{D}_a^{[\alpha,\beta]} \cdots \mathfrak{D}_a^{[\alpha,\beta]}$ *n-times.*

Proof. From the fact that $\mathfrak{I}_a^{r[\alpha,\beta]}\mathfrak{I}_a^{l[\alpha,\beta]} f = \mathfrak{I}_a^{(r+l)[\alpha,\beta]} f$, one has

$$
\begin{aligned}
&\mathfrak{I}_a^{n[\alpha,\beta]}\mathfrak{D}_a^{n[\alpha,\beta]} f(x) - \mathfrak{I}_a^{(n+1)[\alpha,\beta]}\mathfrak{D}_a^{(n+1)[\alpha,\beta]} f(x)\\
&\quad = \mathfrak{I}_a^{n[\alpha,\beta]}\left(\mathfrak{D}_a^{n[\alpha,\beta]} f(x) - \mathfrak{I}_a^{[\alpha,\beta]}\mathfrak{D}_a^{(n+1)[\alpha,\beta]} f(x)\right)\\
&\quad = \mathfrak{I}_a^{n[\alpha,\beta]}\left(\mathfrak{D}_a^{n[\alpha,\beta]} f(x) - \mathfrak{I}_a^{[\alpha,\beta]}\mathfrak{D}_a^{[\alpha,\beta]}(\mathfrak{D}_a^{n[\alpha,\beta]} f(x))\right)\\
&\quad = \mathfrak{I}_a^{n[\alpha,\beta]}\left(\frac{w(a)\mathfrak{D}_a^{n[\alpha,\beta]} f(a)}{w(x)}\right)\\
&\quad = w(a)\mathfrak{D}_a^{n[\alpha,\beta]} f(a)\mathfrak{I}_a^{n[\alpha,\beta]}\frac{1}{w(x)}.
\end{aligned}
$$

Using Theorem 4.23, we get that

$$\begin{aligned}
&\mathfrak{I}_a^{n[\alpha,\beta]}\mathfrak{D}_a^{n[\alpha,\beta]}f(x) - \mathfrak{I}_a^{(n+1)[\alpha,\beta]}\mathfrak{D}_a^{(n+1)[\alpha,\beta]}f(x)\\
&\qquad = w(a)\left(\mathfrak{D}_a^{n[\alpha,\beta]}f(a)\right)\sum_{k=0}^{n} C_n^k \phi(\alpha)^{n-k}\psi(\alpha)^k \left({}^{RL}I_{a,w}^{k\beta}\left(\frac{1}{w(x)}\right)\right)\\
&\qquad = \frac{w(a)}{w(x)}\left(\mathfrak{D}_a^{n[\alpha,\beta]}f(a)\right)\sum_{k=0}^{n} C_n^k \phi(\alpha)^{n-k}\psi(\alpha)^k \frac{(x-a)^{k\beta}}{\Gamma(k\beta+1)}
\end{aligned}$$

and the proof is complete. □

Follows the main result of our section.

Theorem 4.25 (Taylor's formula for weighted generalized fractional derivatives with a nonsingular kernel)**.** *Suppose that* $\mathfrak{D}_a^{k[\alpha,\beta]} \in C([a,b])$ *for* $k = 0, 1, \ldots, n+1$ *and* $0 \le \alpha \le 1$. *Then,*

$$\begin{aligned}
f(x) = \frac{1}{w(x)}\Bigg[&w(a)\sum_{i=0}^{n}\mathfrak{D}_a^{i[\alpha,\beta]}f(a)\sum_{k=0}^{i} C_i^k \phi(\alpha)^{i-k}\psi(\alpha)^k \frac{(x-a)^{k\beta}}{\Gamma(k\beta+1)}\\
&+ w(\xi)\mathfrak{D}_a^{(n+1)[\alpha,\beta]}f(\xi)\sum_{k=0}^{n+1} C_{n+1}^k \phi(\alpha)^{n+1-k}\psi(\alpha)^k \frac{(x-a)^{k\beta}}{\Gamma(k\beta+1)}\Bigg]
\end{aligned}\tag{4.39}$$

with $a \le \xi \le x$, $x \in [a,b]$, *where*

$$\mathfrak{D}_a^{i[\alpha,\beta]} = \mathfrak{D}_a^{[\alpha,\beta]} \cdot \mathfrak{D}_a^{[\alpha,\beta]} \cdots \mathfrak{D}_a^{[\alpha,\beta]} \quad (i\text{-times}).$$

Proof. From Lemma 4.24, we have

$$\begin{aligned}
&\sum_{i=0}^{n}\left(\mathfrak{I}_a^{i[\alpha,\beta]}\mathfrak{D}_a^{i[\alpha,\beta]}f(x) - \mathfrak{I}_a^{(i+1)[\alpha,\beta]}\mathfrak{D}_a^{(i+1)[\alpha,\beta]}f(x)\right)\\
&\qquad = \frac{w(a)}{w(x)}\sum_{i=0}^{n}\left(\mathfrak{D}_a^{i[\alpha,\beta]}f(a)\right)\sum_{k=0}^{i} C_i^k \phi(\alpha)^{i-k}\psi(\alpha)^k \frac{(x-a)^{k\beta}}{\Gamma(k\beta+1)},
\end{aligned}$$

that is,

$$f(x) - \mathfrak{I}_a^{(n+1)[\alpha,\beta]}\mathfrak{D}_a^{(n+1)[\alpha,\beta]}f(x) = \frac{w(a)}{w(x)}\sum_{i=0}^{n}\left(\mathfrak{D}_a^{i[\alpha,\beta]}f(a)\right)\sum_{k=0}^{i} C_i^k \phi(\alpha)^{i-k}\psi(\alpha)^k \frac{(x-a)^{k\beta}}{\Gamma(k\beta+1)}.$$

Using Theorem 4.23, we get

$$\begin{aligned}
f(x) &= \frac{w(a)}{w(x)}\sum_{i=0}^{n}\left(\mathfrak{D}_a^{i[\alpha,\beta]}f(a)\right)\sum_{k=0}^{i} C_i^k \phi(\alpha)^{i-k}\psi(\alpha)^k \frac{(x-a)^{k\beta}}{\Gamma(k\beta+1)}\\
&\quad + \sum_{k=0}^{n+1} C_{n+1}^k \phi(\alpha)^{n+1-k}\psi(\alpha)^k \left({}^{RL}I_a^{k\beta}\mathfrak{D}_a^{(n+1)[\alpha,\beta]}f(x)\right).
\end{aligned}$$

Applying the integral mean value theorem yields

$$\begin{aligned}
f(x) &= \frac{1}{w(x)}\Bigg[w(a)\sum_{i=0}^{n}\mathfrak{D}_a^{i[\alpha,\beta]}f(a)\sum_{k=0}^{i} C_i^k \phi(\alpha)^{i-k}\psi(\alpha)^k \frac{(x-a)^{k\beta}}{\Gamma(k\beta+1)}\\
&\quad + w(\xi)\mathfrak{D}_a^{(n+1)[\alpha,\beta]}f(\xi)\sum_{k=0}^{n+1} C_{n+1}^k \phi(\alpha)^{n+1-k}\psi(\alpha)^k \frac{(x-a)^{k\beta}}{\Gamma(k\beta+1)}\Bigg]
\end{aligned}$$

and the proof is complete. □

As immediate consequences of our Taylor's theorem for generalized weighted fractional derivatives with a nonsingular kernel (Theorem 4.25), we obtain most fractional-order Taylor's formulas existing in the literature.

Corollary 4.26 (Taylor's formula for the weighted ABC derivative). *Suppose that* ${}_a^C D_w^{k\alpha} f \in C([a,b])$, *where* $0 \leq \alpha \leq 1$ *and* $k = 0, 1, \ldots, n+1$. *Then,*

$$\begin{aligned} f(x) &= \frac{1}{w(x)}\Bigg[w(a)\sum_{i=0}^{n} {}_a^C D_w^{i\alpha} f(a) \sum_{k=0}^{i} C_i^k \phi(\alpha)^{i-k}\psi(\alpha)^k \frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} \\ &\quad + w(\xi){}_a^C D_w^{(n+1)\alpha} f(\xi) \sum_{k=0}^{n+1} C_{n+1}^k \phi(\alpha)^{n+1-k}\psi(\alpha)^k \frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)}\Bigg] \end{aligned}$$

with $a \leq \xi \leq x$ *and* $x \in [a,b]$, *where* ${}_a^C D_w^{i\alpha} = {}_a^C D_w^{\alpha} \cdot {}_a^C D_w^{\alpha} \cdots {}_a^C D_w^{\alpha}$ *i-times.*

Proof. Choose $\alpha = \beta$ in Theorem 4.25. □

Corollary 4.27 (Taylor's formula for the ABC derivative). *Let* ${}_a^{ABC} D^{k\alpha} f \in C([a,b])$ *with* $0 \leq \alpha \leq 1$ *and* $k = 0, 1, \ldots, n+1$. *Then,*

$$\begin{aligned} f(x) &= \sum_{i=0}^{n} \left({}_a^{ABC} D^{i\alpha} f(a)\right) \sum_{k=0}^{i} C_i^k \phi(\alpha)^{i-k}\psi(\alpha)^k \frac{(x-a)^k}{\Gamma(k+1)} \\ &\quad + \left({}_a^{ABC} D^{i\alpha} f(\xi)\right) \sum_{k=0}^{n+1} C_{n+1}^k \phi(\alpha)^{n+1-k}\psi(\alpha)^k \frac{(x-a)^k}{\Gamma(k+1)} \end{aligned}$$

with $a \leq \xi \leq x$ *and* $x \in [a,b]$, *where* ${}_a^{ABC} D^{i\alpha} = {}_a^{ABC} D^{\alpha} \cdot {}_a^{ABC} D^{\alpha} \cdots {}_a^{ABC} D^{\alpha}$ *i-times.*

Proof. Choose $\alpha = \beta$ and $w(x) = 1$ in Theorem 4.25. □

Corollary 4.28 (Taylor's formula for the CF derivative). *Let* ${}_a^{CF} D^{k\alpha} f \in C([a,b])$ *with* $0 \leq \alpha \leq 1$ *and* $k = 0, 1, \ldots, n+1$. *Then,*

$$\begin{aligned} f(x) &= \sum_{i=0}^{n} \left({}_a^{CF} D^{i\alpha} f(a)\right) \sum_{k=0}^{i} C_i^k \phi(\alpha)^{i-k}\psi(\alpha)^k \frac{(x-a)^k}{\Gamma(k+1)} \\ &\quad + \left({}_a^{CF} D^{i\alpha} f(\xi)\right) \sum_{k=0}^{n+1} C_{n+1}^k \phi(\alpha)^{n+1-k}\psi(\alpha)^k \frac{(x-a)^k}{\Gamma(k+1)} \end{aligned}$$

with $a \leq \xi \leq x$ *and* $x \in [a,b]$, *where* ${}_a^{CF} D^{i\alpha} = {}_a^{CF} D^{\alpha} \cdot {}_a^{CF} D^{\alpha} \cdots {}_a^{CF} D^{\alpha}$ *i-times.*

Proof. Choose $\alpha = \beta$, $w(x) \equiv 1$, and the RL fractional integral of order one in Theorem 4.25. □

4.3.4 An Application

As an application, we employ the obtained weighted Taylor's formula to establish an appropriate generalized mean value theorem for weighted generalized derivatives.

Theorem 4.29 (Generalized mean value theorem for the weighted generalized derivative). *Suppose that* $f \in C([a,b])$ *and* $\mathfrak{D}_a^{[\alpha,\beta]} f \in C([a,b])$ *for* $0 \leq \alpha \leq 1$. *Then,*

$$f(x) = \frac{1}{w(x)}\left(w(a)f(a) + w(\xi)\mathfrak{D}_a^{[\alpha,\beta]} f(\xi)\left(\phi(\alpha) + \psi(\alpha)\frac{(x-a)^\beta}{\Gamma(\beta+1)}\right)\right)$$

for all $x \in [a,b]$ *with* $a \leq \xi \leq x$.

Proof. The result follows by taking $n = 0$ in Theorem 4.25 and performing some direct calculations. $\square$

As straight corollaries of our Theorem 4.29, we obtain mean value theorems for weighted ABC, ABC, and CF derivatives.

Corollary 4.30 (Generalized mean value theorem for the weighted ABC derivative)**.** *Suppose that $f \in C([a,b])$ and ${}^{C}_{a}D^{\alpha}_{w} f \in C([a,b])$ for $0 \leq \alpha \leq 1$. Then,*

$$f(x) = \frac{1}{w(x)} \left(w(a) f(a) + w(\xi) {}^{C}_{a}D^{\alpha}_{w} f(\xi) \left(\phi(\alpha) + \psi(\alpha) \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} \right) \right)$$

for all $x \in [a,b]$ with $a \leq \xi \leq x$.

Corollary 4.31 (Generalized mean value theorem for the ABC derivative)**.** *Suppose that $f \in C([a,b])$ and ${}^{ABC}_{a}D^{\alpha} f \in C([a,b])$ for $0 \leq \alpha \leq 1$. Then,*

$$f(x) = f(a) + {}^{ABC}_{a} D^{\alpha} f(\xi) \left(\phi(\alpha) + \psi(\alpha) \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} \right)$$

for all $x \in [a,b]$ with $a \leq \xi \leq x$.

Corollary 4.32 (Generalized mean value theorem for the CF derivative)**.** *Suppose that $f \in C([a,b])$ and ${}^{CF}_{a}D^{\alpha} f \in C([a,b])$ for $0 \leq \alpha \leq 1$. Then,*

$$f(x) = f(a) + {}^{CF}_{a} D^{\alpha} f(\xi) \left(\phi(\alpha) + \psi(\alpha)(x-a) \right)$$

for all $x \in [a,b]$ with $a \leq \xi \leq x$.

Note that the classical mean value theorem is obtained from Theorem 4.29 by choosing $w(x) = 1$ and $\alpha = \beta = 1$; from Corollary 4.30 by choosing $w(x) \equiv 1$ and $\alpha = 1$; and from Corollaries 4.31 and 4.32 by choosing $\alpha = 1$.

4.3.5 Conclusion

Nobody can deny the theoretical and practical interests of a Taylor formula in mathematical analysis, not only to solve simple and complex real problems, but also to establish new mathematical results within theories in development, like the ones found in the fractional frameworks in which researchers have proved and used extensively generalized Taylor's formulas for Riemann–Liouville, Caputo and ABC fractional derivatives. In this work, a weighted Taylor's formula for nonsingular kernels, valid for weighted generalized fractional derivatives under some justified prerequisites, was proved. As a result, we obtained various theoretical consequences, one of them being several generalized mean value theorems, extending those available in the literature. We claim that our generalized Taylor's formula (4.39) has a great potential for the development of mathematical modeling with fractional nonsingular kernel derivatives.

4.4 Weighted generalized fractional integration by parts and the Euler–Lagrange equation

The original results of this section are published in [164].

4.4.1 Introduction

In the last decade, fractional calculus took an important part in the theoretical study of dynamical systems by showing significant results in many natural fields and engineering domains [35, 88]. For this reason, mathematicians pay now more attention to the generalization of several important formulas in integral theory of Mathematical Analysis, namely the Newton–Leibniz formula, the Green formula, and the Gauss and Stokes formulas [123]. Moreover, some are central tools that enable mathematicians to extend other theories, good examples of that being the integration by parts formula, Taylor's formula, the Euler–Lagrange equation, Grönwall's inequality, Lyapunov theorems and LaSalle's invariance principle [36, 80, 87].

Often, memory effects are modeled fractionally with Riemann–Liouville and Caputo derivatives [93]. However, the fact that the Mittag–Leffer function is a generalization of the exponential function gives naturally rise to new definitions of fractional operators. In 2020, Hattaf [67] has proposed a new left-weighted generalized fractional derivative for both Caputo and Riemann–Liouville senses and their associated integral operator. Motivated by applications in mechanics, where the introduction of a right operator is needed [137], we introduce here the right-weighted generalized fractional derivative and its associated integral operator, proving their main properties and, in particular, an integration by parts formula.

It is worth to emphasize that integration by parts is of great interest in integral calculus and mathematical analysis. For example, it represents a strong tool to develop the calculus of variations through the so-called Euler–Lagrange equation, which is the central result of dynamic optimization [15, 13, 92]. Motivated by the works [3, 12, 2, 165], and with the help of our weighted generalized fundamental integration by parts formula, we extend here the Euler–Lagrange equation.

This section is organized as follows. In Section 4.4.2, we introduce the right-weighted generalized fractional derivative and its associated integral, studying their well-posedness. Integration by parts is investigated in Section 4.4.3, followed by Section 4.4.4 where the weighted generalized fractional Euler–Lagrange equation is rigorously proved. We end with Section 4.4.5, illustrating the obtained theoretical results with an application in the quantum mechanics framework.

4.4.2 Well-posedness of the right-weighted fractional operators

We denote the right-weighted generalized fractional derivative of order α in the Riemann–Liouville sense by ${}^{R}D^{\alpha,\beta}_{b,w}$ and we define it so that the following identity occurs:

$$Q\left({}^{R}_{a,w}D^{\alpha,\beta}f\right)(x) = \left({}^{R}D^{\alpha,\beta}_{b,w}Qf\right)(x)$$

with Q being the *reflection operator*, that is, $(Qf)(x) = f(a+b-x)$ with function f defined on the interval $[a,b]$.

Definition 4.33 (right-weighted generalized fractional derivative)**.** *Let $0 \leq \alpha < 1$ and $\beta > 0$. The right-weighted generalized fractional derivative of order α of function f, in the Riemann–Liouville sense, is defined by*

$${}^{R}D^{\alpha,\beta}_{b,w}f(x) = \frac{-1}{\phi(\alpha)}\frac{1}{w(x)}\frac{d}{dx}\int_x^b (wf)(s)E_\beta\left[-\mu_\alpha(s-x)^\beta\right]ds, \tag{4.40}$$

where $w \in C^1([a,b])$ with $w, w' > 0$.

To properly define the new right-weighted fractional integral, we need to solve the equation ${}^{R}D^{\alpha,\beta}_{b,w}f(x) = u(x)$. We have

$${}^{R}D^{\alpha,\beta}_{b,w}f(x) = {}^{R}D^{\alpha,\beta}_{b,w}QQf(x) = Q\,{}^{R}_{a,w}D^{\alpha,\beta}Qf(x) = u(x).$$

Then,

$${}^{R}_{a,w}D^{\alpha,\beta}Qf(x) = Qu(x)$$

and thus

$$Qf(x) = \phi(\alpha)Qu(x) + \psi(\alpha)\,{}^{RL}_{a,w}I^{\beta}Qu(x) = \phi(\alpha)Qu(x) + \psi(\alpha)\,Q^{RL}I^{\beta}_{b,w}u(x),$$

where ${}^{RL}I^{\beta}_{b,w}$ is the right-weighted standard Riemann–Liouville fractional integral of order β given by

$$^{RL}I^{\beta}_{b,w}f(x) = \frac{1}{\Gamma(\beta)}\frac{1}{w(x)}\int_x^b (s-x)^{\beta-1}w(s)f(s)ds, \quad x < b. \tag{4.41}$$

Applying Q to both sides of (4.41), we get

$$f(t) = \phi(\alpha)u(x) + \psi(\alpha)\,{}^{RL}I^{\beta}_{b,w}u(x).$$

Moreover,

$$\begin{aligned} {}_{a,w}I^{\alpha,\beta}Qf(x) &= \phi(\alpha)Qf(x) + \psi(\alpha)\,{}^{RL}_{a,w}I^{\beta}Qf(x) \\ &= \phi(\alpha)Qf(x) + \psi(\alpha)\,Q^{RL}I^{\beta}_{b,w}f(x) \\ &= Q\big[\phi(\alpha)f(x) + \psi(\alpha)\,{}^{RL}I^{\beta}_{b,w}f(x)\big]. \end{aligned}$$

We are now in conditions to introduce the concept of right-weighted generalized fractional integral.

Definition 4.34 (right-weighted generalized fractional integral)**.** *Let $0 \le \alpha < 1$ and $\beta > 0$. The right-weighted generalized fractional integral of order α of function f is given by*

$$I^{\alpha,\beta}_{b,w}f(x) = \phi(\alpha)f(x) + \psi(\alpha)\,{}^{RL}I^{\beta}_{b,w}f(x), \tag{4.42}$$

where $w \in C^1([a,b])$ with $w, w' > 0$.

Our next result provides a series representation to the left- and right-weighted generalized fractional derivatives.

Theorem 4.35. *Let $0 \le \alpha < 1$ and $\beta > 0$. The left- and right-weighted generalized fractional derivatives of order α of function f can be written, respectively, as*

$${}^{R}_{a,w}D^{\alpha,\beta}f(x) = \frac{1}{\phi(\alpha)}\sum_{j=0}^{\infty}(-\mu_\alpha)^j\,{}^{RL}_{a,w}I^{\beta j}f(x) \tag{4.43}$$

and

$${}^{R}D^{\alpha,\beta}_{b,w}f(x) = \frac{-1}{\phi(\alpha)}\sum_{j=0}^{\infty}(-\mu_\alpha)^j\,{}^{RL}_{b,w}I^{\beta j}f(x). \tag{4.44}$$

Proof. The Mittag–Leffler function $E_\beta(x)$ is an entire series of x. Since the series (4.8) is locally converging uniformly in the whole complex plane, then the left-weighted generalized fractional derivative can be rewritten as

$$\begin{aligned} {}^{R}_{a,w}D^{\alpha,\beta}f(x) &= \frac{1}{\phi(\alpha)}\frac{1}{w(x)}\frac{d}{dx}\int_a^x (wf)(s)\sum_{j=0}^{\infty}(-\mu_\alpha)^j\frac{(x-s)^{\beta j}}{\Gamma(\beta j+1)}ds \\ &= \frac{1}{\phi(\alpha)}\frac{1}{w(x)}\sum_{j=0}^{\infty}(-\mu_\alpha)^j\frac{1}{\Gamma(\beta j+1)}\frac{d}{dx}\int_a^x (wf)(s)(x-s)^{\beta j}ds \\ &= \frac{1}{\phi(\alpha)}\frac{1}{w(x)}\sum_{j=0}^{\infty}(-\mu_\alpha)^j\frac{1}{\Gamma(\beta j)}\int_a^x (wf)(s)(x-s)^{\beta j-1}ds \\ &= \frac{1}{\phi(\alpha)}\sum_{j=0}^{\infty}(-\mu_\alpha)^j\big({}^{RL}_{a,w}I^{\beta j}f(x)\big). \end{aligned}$$

From Definition 4.33, and the same steps used before, one can easily rewrite the new right-weighted generalized fractional derivative as equality (4.44). The proof of (4.43) is similar. □

Theorem 4.36. *Let $0 \leq \alpha < 1$ and $\beta > 0$. The left- and right-weighted generalized fractional derivative and their associated integrals satisfy the following formulas:*

$$ {}_{a,w}I^{\alpha,\beta}\big({}^{R}_{a,w}D^{\alpha,\beta}f\big)(x) = {}^{R}_{a,w}D^{\alpha,\beta}\big({}_{a,w}I^{\alpha,\beta}f\big)(x) = f(x) \tag{4.45} $$

and

$$ I^{\alpha,\beta}_{b,w}\big({}^{R}D^{\alpha,\beta}_{b,w}f\big)(x) = {}^{R}D^{\alpha,\beta}_{b,w}\big({}^{\alpha,\beta}I_{b,w}f\big)(x) = -f(x). \tag{4.46} $$

Proof. We note that

$$ \begin{aligned} {}_{a,w}I^{\alpha,\beta}\big({}^{R}_{a,w}D^{\alpha,\beta}f\big)(x) &= \phi(\alpha)\big({}^{R}_{a,w}D^{\alpha,\beta}f\big)(x) + \psi(\alpha){}^{RL}_{a,w}I^{\beta}\big({}^{R}_{a,w}D^{\alpha,\beta}f\big)(x) \\ &= \sum_{j=0}^{\infty}(-\mu_\alpha)^j\,{}^{RL}_{a,w}I^{\beta j}f(x) + \mu_\alpha\,{}^{RL}_{a,w}I^{\beta}\big(\sum_{j=0}^{\infty}(-\mu_\alpha)^j\,{}^{RL}_{a,w}I^{\beta j}f\big)(x) \\ &= \sum_{j=0}^{\infty}(-\mu_\alpha)^j\,{}^{RL}_{a,w}I^{\beta j}f(x) - \sum_{j=0}^{\infty}(-\mu_\alpha)^{j+1}\,{}^{RL}_{a,w}I^{\beta+\beta j}f(x) \\ &= f(t). \end{aligned} $$

Then,

$$ \begin{aligned} {}^{R}_{a,w}D^{\alpha,\beta}\big({}_{a,w}I^{\alpha,\beta}f\big)(x) &= \frac{1}{\phi(\alpha)}\sum_{j=0}^{\infty}(-\mu_\alpha)^j\,{}^{RL}_{a,w}I^{\beta j}\big({}_{a,w}I^{\alpha,\beta}f\big)(x), \\ &= \frac{1}{\phi(\alpha)}\sum_{j=0}^{\infty}(-\mu_\alpha)^j\,{}^{RL}_{a,w}I^{\beta j}\Big[\phi(\alpha)f(x) + \psi(\alpha)\,{}^{RL}_{a,w}I^{\beta}f(x)\Big] \\ &= \sum_{j=0}^{\infty}(-\mu_\alpha)^j\,{}^{RL}_{a,w}I^{\beta j}f(x) + \mu_\alpha\sum_{j=0}^{\infty}(-\mu_\alpha)^j\,{}^{RL}_{a,w}I^{\beta j+\beta}f(x) \\ &= \sum_{j=0}^{\infty}(-\mu_\alpha)^j\,{}^{RL}_{a,w}I^{\beta j}f(x) - \sum_{j=0}^{\infty}(-\mu_\alpha)^{j+1}\,{}^{RL}_{a,w}I^{\beta j+\beta}f(x) \\ &= f(x) \end{aligned} $$

and equality (4.45) holds true. The proof of equality (4.46) is similar. □

4.4.3 Integration by parts

Our formulas of integration by parts are proved in suitable function spaces.

Definition 4.37. [75] For $\alpha > 0$, $\beta > 0$ and $1 \leq p \leq \infty$, the following function spaces are defined:

$$ {}_{a,w}I^{\alpha,\beta}(L_p) := \big\{f : f = {}_{a,w}I^{\alpha,\beta}(\eta), \eta \in L_p(a,b)\big\} $$

and

$$ I^{\alpha,\beta}_{b,w}(L_p) := \Big\{f : f = I^{\alpha,\beta}_{b,w}(\theta), \theta \in L_p(a,b)\Big\}. $$

Theorem 4.38 (integration by parts without the weighted function)**.** *Let $0 \leq \alpha < 1$, $\beta > 0$, $p \geq 1$, $q \geq 1$ and $\dfrac{1}{p} + \dfrac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case $\dfrac{1}{p} + \dfrac{1}{q} = 1 + \alpha$.*

- *If $f \in L_p(a,b)$ and $g \in L_q(a,b)$, then*

$$ \int_a^b f(x)({}_{a,1}I^{\alpha,\beta}g)(x)dx = \int_a^b g(x)(I^{\alpha,\beta}_{b,1}f)(x)dx. \tag{4.47} $$

- *If $f \in I^{\alpha,\beta}_{b,w}(L_p)$ and $g \in {}_{a,w}I^{\alpha,\beta}(L_q)$, then*

$$\int_a^b f(x)({}^{R}_{a,1}D^{\alpha,\beta}g)(x)dx = \int_a^b g(x)({}^{R}D^{\alpha,\beta}_{b,1}f)(x)dx. \tag{4.48}$$

Proof. First, we prove equality (4.47). Since

$$\begin{aligned}\int_a^b f(x)({}_{a,1}I^{\alpha,\beta}g)(x)dx &= \int_a^b f(x)\left[\phi(\alpha)g(x) + \psi(\alpha)\,{}^{RL}_{a,1}I^{\beta}g(x)\right]\\ &= \phi(\alpha)\int_a^b f(x)g(x)dx + \psi(\alpha)\int_a^b f(x)\,{}^{RL}_{a,1}I^{\beta}g(x)dx,\end{aligned}$$

it follows from Lemma 4.11 that

$$\begin{aligned}\int_a^b f(x)({}_{a,1}I^{\alpha,\beta}g)(x)dx &= \phi(\alpha)\int_a^b f(x)g(x)dx + \psi(\alpha)\int_a^b g(x)\,{}^{RL}I^{\beta}_{b,1}f(x)dx\\ &= \int_a^b g(x)\left[\phi(\alpha)f(x) + \psi(\alpha)\,{}^{RL}I^{\beta}_{b,1}f(x)\right]\\ &= \int_a^b g(x)(I^{\alpha,\beta}_{b,1}f)(x)dx.\end{aligned}$$

Now, we prove (4.48):

$$\begin{aligned}\int_a^b f(x)({}^{R}_{a,1}D^{\alpha,\beta}g)(x)dx &= \int_a^b I^{\alpha,\beta}_{b,1}\theta(x)\left({}^{R}_{a,1}D^{\alpha,\beta}({}_{a,1}I^{\alpha,\beta}\eta)\right)(x)dx\\ &= \int_a^b \eta(x)I^{\alpha,\beta}_{b,1}\theta(x)dx \quad \big(\text{from Theorem 4.36}\big)\\ &= \int_a^b \theta(x)\,{}_{a,1}I^{\alpha,\beta}\eta(x)dx \quad \big(\text{from equality (4.47)}\big)\\ &= \int_a^b g(x)({}^{R}D^{\alpha,\beta}_{b,1}f)(x)dx \quad \big(\text{from Theorem 4.36}\big).\end{aligned}$$

The proof is complete. □

Theorem 4.39 (weighted generalized integration by parts)**.** *Let $0 \le \alpha < 1$, $\beta > 0$, $p \ge 1$, $q \ge 1$ and $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$ ($p \ne 1$ and $q \ne 1$ in the case $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$). If $f \in L_p(a,b)$ and $g \in L_q(a,b)$, then*

$$\int_a^b f(x)({}_{a,w}I^{\alpha,\beta}g)(x)dx = \int_a^b w(x)^2 g(x)\left(I^{\alpha,\beta}_{b,w}\left(\frac{f}{w^2}\right)\right)(x)dx, \tag{4.49}$$

$$\int_a^b f(x)({}^{R}_{a,w}D^{\alpha,\beta}g)(x)dx = \int_a^b w(x)^2 g(x)\left({}^{R}D^{\alpha,\beta}_{b,w}\left(\frac{f}{w^2}\right)\right)(x)dx. \tag{4.50}$$

Proof. We have

$$\begin{aligned}\int_a^b f(x)({}_{a,w}I^{\alpha,\beta}g)(x)dx &= \int_a^b w(x)\frac{f(x)}{w(x)}\left({}_{a,w}I^{\alpha,\beta}\left(\frac{gw}{w}\right)\right)(x)dx\\ &= \int_a^b \frac{f(x)}{w(x)}\left({}_{a,1}I^{\alpha,\beta}(gw)\right)(x)dx\\ &= \int_a^b w(x)g(x)\left(I^{\alpha,\beta}_{b,1}\left(\frac{f}{w}\right)\right)(x)dx \;\big(\text{from Theorem 4.38}\big)\\ &= \int_a^b g(x)w(x)^2\left(I^{\alpha,\beta}_{b,w}\left(\frac{f}{w^2}\right)\right)(x)dx.\end{aligned}$$

Therefore, equality (4.49) is true. Similarly, we have

$$\begin{aligned}
\int_a^b f(x)({}^R_{a,w}D^{\alpha,\beta}g)(x)dx &= \int_a^b w(x)\frac{f(x)}{w(x)}\left({}^R_{a,w}D^{\alpha,\beta}\left(\frac{gw}{w}\right)\right)(x)dx \\
&= \int_a^b \frac{f(x)}{w(x)}\left({}^R_{a,1}D^{\alpha,\beta}(gw)\right)(x)dx \\
&= \int_a^b w(x)g(x)\left(D^{\alpha,\beta}_{b,1}\left(\frac{f}{w}\right)\right)(x)dx \ \big(\text{from Theorem 4.38}\big) \\
&= \int_a^b g(x)w(x)^2\left(D^{\alpha,\beta}_{b,w}\left(\frac{f}{w^2}\right)\right)(x)dx
\end{aligned}$$

and equality (4.50) holds. □

From (4.49) and (4.50) we get the following consequence.

Corollary 4.40. *Let* $0 \le \alpha < 1$, $\beta > 0$, $p \ge 1$, $q \ge 1$ *and* $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$ *(*$p \ne 1$ *and* $q \ne 1$ *in the case* $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$*). If* $f \in L_p(a,b)$ *and* $g \in L_q(a,b)$*, then*

$$\int_a^b f(x)\left(I^{\alpha,\beta}_{b,w}g\right)(x)dx = \int_a^b w(x)^2 g(x)\left({}_{a,w}I^{\alpha,\beta}\left(\frac{f}{w^2}\right)\right)(x)dx,$$

$$\int_a^b f(x)\left({}^R D^{\alpha,\beta}_{b,w}g\right)(x)dx = \int_a^b w(x)^2 g(x)\left({}^R_{a,w}D^{\alpha,\beta}\left(\frac{f}{w^2}\right)\right)(x)dx.$$

For a symmetric view of weighted generalized integration by parts, we propose the following corollary of Theorem 4.39.

Corollary 4.41. *Let* $0 \le \alpha < 1$, $\beta > 0$, $p \ge 1$, $q \ge 1$ *and* $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$ *(*$p \ne 1$ *and* $q \ne 1$ *in the case* $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$*). If* $f \in L_p(a,b)$ *and* $g \in L_q(a,b)$*, then*

$$\int_a^b w(x)f(x)\left({}_{a,w}I^{\alpha,\beta}\frac{g}{w}\right)(x)dx = \int_a^b w(x)g(x)\left(I^{\alpha,\beta}_{b,w}\frac{f}{w}\right)(x)dx, \tag{4.51}$$

$$\int_a^b w(x)f(x)\left({}^R_{a,w}D^{\alpha,\beta}\frac{g}{w}\right)(x)dx = \int_a^b w(x)g(x)\left({}^R D^{\alpha,\beta}_{b,w}\frac{f}{w}\right)(x)dx. \tag{4.52}$$

4.4.4 The weighted generalized fractional Euler–Lagrange equation

Let us denote by $AC(I \to \mathbb{R})$ the set of absolutely continuous functions X, where $I = [a,b]$, such that the left and right Riemann–Liouville weighted generalized fractional derivatives of X exist, endowed with the norm

$$\|X\| = \sup_{t\in I}\left(|\ X(t)\ | + |\ {}^{RL}_{a,w}D^{\alpha,\beta}X(t)\ | + |{}^{RL}\ D^{\alpha,\beta}_{b,w}X(t)\ |\right).$$

Let $L \in C^1(I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R})$ and consider the following minimization problem:

$$J[X] = \left(\int_a^b L\left(t, X(t), {}^{RL}\ D^{\alpha,\beta}_{a,w}X(t), {}^{RL}\ D^{\alpha,\beta}_{b,w}X(t)\right)dt\right) \longrightarrow \min \tag{4.53}$$

subject to the boundary conditions

$$X(a) = X_a, \quad X(b) = X_b. \tag{4.54}$$

With the help of weighted generalized fractional integration by parts, given by our Theorem 4.39, we obtain the following necessary optimality condition for the fundamental weighted generalized fractional problem of the calculus of variations (4.53)–(4.54).

Theorem 4.42 (the weighted generalized fractional Euler–Lagrange equation). *If $L \in C^1(I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R})$ and $X \in AC([a,b] \to \mathbb{R})$ is a minimizer of* (4.53) *subject to the fixed end points* (4.54), *then X satisfies the following weighted generalized fractional Euler–Lagrange equation:*

$$\partial_2 L + w(x)^2\, {}^R D^{\alpha,\beta}_{b,w} \left(\frac{\partial_3 L}{w(x)^2} \right) + w(x)^2\, {}^R_{a,w} D^{\alpha,\beta} \left(\frac{\partial_4 L}{w(x)^2} \right) = 0.$$

Proof. Let $J[X] = \displaystyle\int_a^b L\left(t, X(t), {}^R_{a,w} D^{\alpha,\beta} X(t), {}^R D^{\alpha,\beta}_{b,w} X(t)\right) dt$ and assume that X^* is the optimal solution of problem (4.53)–(4.54). Set

$$X = X^* + \varepsilon\eta,$$

where $\eta \in AC([a,b] \to \mathbb{R})$ and ε is a small real parameter. By linearity of the weighted generalized fractional derivative, we get

$${}^R_{a,w} D^{\alpha,\beta} X(t) = {}^R_{a,w} D^{\alpha,\beta} X^* + \varepsilon \left({}^R_{a,w} D^{\alpha,\beta} \eta(t) \right)$$

and

$${}^R D^{\alpha,\beta}_{b,w} X(t) = {}^R D^{\alpha,\beta}_{b,w} X^* + \varepsilon \left({}^R D^{\alpha,\beta}_{b,w} \eta(t) \right).$$

Consider now the following function:

$$\begin{aligned} J(\varepsilon) = \int_a^b L\big(t, X^*(t) + \varepsilon\eta(t),\ & {}^R_{a,w} D^{\alpha,\beta} X^*(t) + \varepsilon \left({}^R_{a,w} D^{\alpha,\beta} \eta(t) \right), \\ & {}^R D^{\alpha,\beta}_{b,w} X^*(t) + \varepsilon \left({}^R D^{\alpha,\beta}_{b,w} \eta(t) \right)\big)\, dt. \end{aligned}$$

Fermat's theorem asserts that $\left.\dfrac{d}{d\varepsilon} J(\varepsilon)\right|_{\varepsilon=0} = 0$ and we deduce, by the chain rule, that

$$\int_a^b \left(\partial_2 L \cdot \eta + \partial_3 L \cdot {}^R_{a,w} D^{\alpha,\beta} \eta + \partial_4 L \cdot {}^R D^{\alpha,\beta}_{b,w} \eta \right) dt = 0,$$

where $\partial_i L$ denotes the partial derivative of the Lagrangian L with respect to its ith argument. Using Theorem 4.39 of weighted fractional integration by parts, we obtain that

$$\int_a^b \left(\partial_2 L \cdot \eta + w(x)^2 \cdot \eta \cdot {}^R D^{\alpha,\beta}_{b,w} \left(\frac{\partial_3 L}{w(x)^2} \right) + w(x)^2 \cdot \eta \cdot {}^R_{a,w} D^{\alpha,\beta} \left(\frac{\partial_4 L}{w(x)^2} \right) \right) dt = 0.$$

The result follows by the fundamental theorem of the calculus of variations. □

4.4.5 An application

Let us consider the weighted generalized fractional variational problem (4.53)–(4.54) with

$$\begin{aligned} L&\left(t, X(t), {}^R_{a,w} D^{\alpha,\beta} X(t), {}^R D^{\alpha,\beta}_{b,w} X(t)\right) \\ &= \frac{1}{2} \left(\frac{1}{2} m \left| {}^R_{a,w} D^{\alpha,\beta} X(t) \right|^2 + \frac{1}{2} m \left| {}^R D^{\alpha,\beta}_{b,w} X(t) \right|^2 \right) - V(X(t)), \end{aligned}$$

where X is an absolutely continuous function on $[a,b]$ and V maps $C^1(I \to \mathbb{R})$ to $\mathbb{R}$. Note that

$$\frac{1}{2}\left(\frac{1}{2}m \mid_{a,w}^{R} D^{\alpha,\beta} X(t) \mid^2 + \frac{1}{2}m \mid^{R} D_{b,w}^{\alpha,\beta} X(t) \mid^2\right)$$

can be viewed as a weighted generalized kinetic energy in the quantum mechanics framework. By applying our Theorem 1.10 to the current variational problem, we get that

$$\frac{1}{2}m\left[w(x)^2 \, {}^{R}D_{b,w}^{\alpha,\beta}\left(\frac{{}_{a,w}^{R}D^{\alpha,\beta}X(t)}{w(x)^2}\right) + w(x)^2 \, {}_{a,w}^{R}D^{\alpha,\beta}\left(\frac{{}^{R}D_{b,w}^{\alpha,\beta}X(t)}{w(x)^2}\right)\right] = V'(X(t)),$$

where V' is the derivative of the potential energy of the system. We observe that relation (1.3) generalizes Newton's dynamical law $m\ddot{X}(t) = V'(X(t))$.

Chapter 5

Conclusion and future work

5.1 Conclusion

In this PhD thesis, we introduced new definitions related to the stochastic fractional operators with their special properties, which were used to investigate the stochastic fractional integration by parts formulas. Based on the last results, a Stochastic Fractional Euler-Lagrange equations are obtained in the both Stochastic fractional Riemann-Liouville/Caputo sense. An application on the quantum mechanics is established, see our paper [165].

The outbreak of COVID-19 pandemic has called the scientific community to fight against its spreed, to help the Moroccan health authorities, we have contributed through some works driven by the models of delayed deterministic and stochastic differential equations, equipped with different controls providing many corresponding scenarios of the possibly expected behavioral situation of COVID-19, allowing to well describe the evolution of the disease and to correctly predict the right trajectory of the virus so that the Moroccan government takes the right decisions [88, 158, 162]. Inspired by some existing works in the literature, we constructed an extended controlled spatio-temporal epidemic SICA model, with a view to converting it as a perspective into a stochastic fractional spatio-temporal one. Moreover, to deeply apply the stochasticity component, which is not limited to Brownian motion as the only source of randomness, but also considers Jump Lévy noise, we have performed a new work titled Stochastic SICA Epidemic Model with Jump Lévy Processes, see [160], in order to take into account the continuous and discontinuous factors of the model. Further, a new tendency of the controlled optimization problem driven by the elaboration of a large number of controls called quasi-optimal controls with imprecise parameters is expressed in our established work: Quasi-optimal control on the stochastic SICA model, in which we applied a suitable stochastic Pontryagin's maximum principle. In order to transform some advanced stochastic mathematical tools to the economic framework, we have implemented our following work: A Stochastic Capital-Labour Model with Logistic Growth Function, see [159]. My different works varies along my PHD thesis between a number of applications in diverse areas, and certain theoretical published papers, see [165], and our second recently published paper called Lyapunov functions for fractional order systems [33]. To bridge the gap in the fractional calculus theory, we have proposed two works, the first one is related to the establishment of the Taylor's Formula for Generalized Weighted Fractional Derivatives with non singular Kernels, the second achievement is about the release of the weighted generalized fractional integration by parts and the Euler–Lagrange equation.

5.2 Published works

1. A Stochastic Fractional Calculus with Applications to Variational Principles [165];
2. Modeling the spread of COVID-19 pandemic in Morocco [162];

3. A stochastic time-delayed model for the effectiveness of Moroccan COVID-19 deconfinement strategy [158];
4. Modeling and Forecasting of COVID-19 Spreading by Delayed Stochastic Differential Equations [88];
5. A Stochastic Capital-Labour Model with Logistic Growth Function [159];
6. A stochastic SICA Epidemic Model with Jump Lévy Processes [160];
7. Lyapunov Functions and Stability Analysis of Fractional-Order Systems [33];
8. Weighted generalized fractional integration by parts and the Euler–Lagrange equation [164].

5.3 Submitted work

1. Taylor's Formula for Generalized Weighted Fractional Derivatives with Nonsingular Kernels

5.4 Work in progress

1. Near-optimal control on the stochastic SICA model;
2. The Spatiotemporal transmission of the epidemic SICA model with control strategy.

5.5 Future works

As future work, we will focus on a new construction of a new weighted generalized fractional calculus with non-singular kernel extending those available in the literature, aiming to expand the possibilities of selecting the appropriate derivative with its corresponding integral with respect to the phenomenon under development. Further, numerous eventual works linked to the previous announced notion might be established, namely, the extended Cauchy-Lipshitz theorem, the extended mean value integral, the extended Taylor's formula, the extended integration by parts theorem, the extended Euler-Lagrange equation and so on. Time-scale theory plays a pivotal role in the combination between the discrete and the continuous processes, therefore, thinking about the conversion of some existing results within the Time-scale area with the mentioned expected fractional operators is more significant both for the development of mathematics and for its applications in the vital domains.

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