# THE DEGREES OF REGULAR POLYTOPES OF TYPE $[4,4,4]$

MARIA ELISA FERNANDES AND CLAUDIO ALEXANDRE PIEDADE

Abstract. We give the list of all possible degrees of faithful transitive permutation representations, corresponding to the indexes of core-free subgroups, of the finite universal regular polytopes $\{\{4,4\}_{(t_1,t_2)},\{4,4\}_{(s_1,s_2)}\}$.

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## 1. Introduction

Grünbaum's problem, consisting in the classification of locally toroidal polytopes, was, during several years, one of the central problems in the theory of abstract polytopes. The classification of the known finite locally toroidal regular polytopes can be found in [16], in particular, the classification of the finite universal locally toroidal regular 4-polytopes whose facets and vertex-figures are maps $\{4,4\}_{(s,0)}$ or $\{4,4\}_{(s,s)}$, also known as polytopes of Euclidean type $[4,4,4]$ [12].

This classification is almost complete and listed in the following table, where all parameters, corresponding finite regular polytopes $\{\{4,4\}_{(t_1,t_2)},\{4,4\}_{(s_1,s_2)}\}$, and the respective automorphism groups $G$ are given. The classification of the universal

Table 1. The known finite universal regular polytopes $\{\{4,4\}_{(t_1,t_2)},\{4,4\}_{(s_1,s_2)}\}$.

| $(t_1,t_2)$ | $(s_1,s_2)$ | $\|G\|$ | $G$ |
|---|---|---|---|
| (2,0) | $(s,s)$, $s\geq 2$ | $64s^2$ | $(C_2\times C_2)\rtimes[4,4]_{(s,s)}$ |
| (2,0) | $(2s,0)$, $s\geq 1$ | $128s^2$ | $(C_2\times C_2)\rtimes[4,4]_{(2,0)}$, $s=1$<br>$((C_2\times C_2)\rtimes[4,4]_{(s,s)})\times C_2$, $s\geq 2$ |
| (3,0) | (3,0) | 1440 | $S_6\times C_2$ |
| (3,0) | (4,0) | 36864 | $C_2\wr[4,4]_{(3,0)}$ |
| (3,0) | (2,2) | 2304 | $(S_4\times S_4)\rtimes(C_2\times C_2)$ |
| (2,2) | (2,2) | 1024 | $C_2^4\rtimes[4,4]_{(2,2)}$ |
| (2,2) | (3,3) | 9216 | $C_2^6\rtimes[4,4]_{(3,3)}$ |
| (3,0) | (5,0) | 3916800 | $Sp_4(4)\times C_2\times C_2$ |

finite regular polytopes $\{\{4,4\}_{(t,0)},\{4,4\}_{(s,0)}\}$, for $s,t\geq 3$ and both odd, is still an open problem, being conjectured in [16] that those given in Table 1 are the only finite ones.

Given a group $G$ and a core-free subgroup $H$ of $G$. The action of $G$ on $G/H$ gives a faithful transitive permutation representation of degree $|G:H|$. Moreover, $H$ is the stabilizer of a point. On the other hand, the stabilizer of a point in a faithful transitive permutation representation is core-free. This gives a one-to-one correspondence between core-free subgroups and faithful transitive permutation representations. In the present paper, whenever we refer the degree of a polytope, we mean the degree of a faithful transitive permutation representation of its automorphism group, corresponding to the index of a core-free subgroup.

There are various atlases of abstract regular polytopes available [3, 14, 15], which give either group presentations of the polytopes or, alternatively, permutations that generate the group of automorphisms of the polytopes, corresponding to faithful permutation representations. Of course the latter are not uniquely determined.

Faithful permutation representations of the groups of abstract regular polytopes can be represented by *CPR graphs* [17]. The number of vertices of a CPR graph is the degree of the permutation representation and the edges are labeled with the elements of the set of types $I$ of the polytope. More precisely, $\{x, y\}$ is an $i$-edge of a CPR graph whenever $x\rho_i = y$, where $\rho_i$ is an involution of the generating set of the automorphism group of the polytope. Despite the fact that CPR graphs are not uniquely determined by the group of automorphisms of a polytope, they turn out to be an important tool in the classification of abstract regular polytopes. In particular in [2, 4, 5, 6, 7, 8], connected CPR graphs of degree $n$ were constructed to determine abstract regular polytopes of high rank for $A_n$ and $S_n$. For this reason the study of faithful transitive permutation representations of polytopes has emerged. Connected CPR graphs are in fact, *Schreier graphs*.

In [9] we gave the list of all possible degrees for toroidal regular maps (for the regular toroidal map of type $\{3, 6\}$ the degrees given in [9] were rectified in [10]). In [11] we completed the investigation on a surface of genus 1, and rank 3, considering the group of a regular toroidal hypermap of type $(3, 3, 3)$.

In this paper we continue this study considering the infinite families of lines 1 and 2 of Table 1. The degrees of the remaining polytopes of Table 1 can be determined computationally. Indeed we were able to compute them in GAP [13] (see Table 2). Using the same algorithm we found the degrees of the polytopes of lines 1 and 2 of Table 1 only up to $s = 79$ and $s = 47$, respectively.

This paper is organized as follows. In Section 2 we briefly give the definition of the abstract regular polytopes denoted by $\{\{4, 4\}_{(t_1,t_2)}, \{4, 4\}_{(s_1,s_2)}\}$. In Section 3 we give some results that were obtained in [9] which will be used in the following sections. In Section 4 we establish relations between $\{4, 4\}_{(s,s)}$, $\{\{4, 4\}_{(2,0)}, \{4, 4\}_{(s,s)}\}$ and $\{\{4, 4\}_{(2,0)}, \{4, 4\}_{(2s,0)}\}$. Finally, in Section 5, we determine the possible degrees of the faithful transitive permutation representations of the two infinite families of Table 1.

## 2. The finite universal regular polytopes $\{\{4, 4\}_{(t_1,t_2)}, \{4, 4\}_{(s_1,s_2)}\}$

The regular toroidal maps $\{4, 4\}_{(s,0)}$ and $\{4, 4\}_{(s,s)}$ are factorizations of the Coxeter group $[4, 4] = \langle \rho_0, \rho_1, \rho_2 \mid \rho_0^2, \rho_1^2, \rho_2^2, (\rho_0\rho_1)^4, (\rho_1\rho_2)^4, (\rho_0\rho_2)^2\rangle$, by

$$(\rho_0\rho_1\rho_2\rho_1)^s = 1 \text{ or } (\rho_0\rho_1\rho_2)^{2s} = 1,$$

respectively. The size of the automorphism group of $\{4, 4\}_{(s,0)}$ is $8s^2$ while the size of the automorphism group of $\{4, 4\}_{(s,s)}$ is $16s^2$. For the map $\{4, 4\}_{(s,0)}$, consider the translations $u = \rho_0\rho_1\rho_2\rho_1$ and $v = u^{\rho_1}$ (these correspond to two possible unitary translations, of a squared regular tiling, from a vertex to each of its adjacent vertices). We have the following equalities

$$u^{\rho_0} = u^{-1},\ u^{\rho_2} = u,\ v^{\rho_0} = v \text{ and } v^{\rho_2} = v^{-1}.$$

TABLE 2. The degrees of the known finite universal regular polytopes $\{\{4,4\}_{(t_1,t_2)}, \{4,4\}_{(s_1,s_2)}\}$.

| $(t_1, t_2)$ | $(s_1, s_2)$ | Set of Possible Degrees | Minimal degree Core-free subgroups |
|---|---|---|---|
| (2,0) | (2,0) | $\{m \mid m$ a divisor of $128 \wedge m \geq 32\}$ | $\langle \rho_0, \rho_2 \rangle$ |
| (2,0) | (4,0) | $\{m \mid m$ a divisor of $512 \wedge m \geq 32\}$ | $\langle \rho_0, \rho_3, (\rho_1\rho_2)^2 \rangle$ |
| (3,0) | (3,0) | $\{m \mid m$ a divisor of $1440 \wedge m \geq 60 \wedge$ $\wedge m \neq 96\} \cup \{40, 30, 24, 20, 12\}$ | $\langle \rho_0, \rho_2, \rho_1\rho_3 \rangle$ |
| (3,0) | (4,0) | $\{m \mid m$ a divisor of $36864 \wedge m \geq 72\}$ $\cup \{18, 36, 48\}$ | $\langle \rho_0, \rho_1, \rho_3^{\rho_2}, \rho_3^{\rho_2\rho_1\rho_2} \rangle$ |
| (3,0) | (2,2) | $\{m \mid m$ a divisor of $2304 \wedge m \geq 12\}$ | $\langle \rho_0, \rho_2, \rho_3, \rho_1\rho_2\rho_1 \rangle$ |
| (2,2) | (2,2) | $\{m \mid m$ a divisor of $1024 \wedge m \geq 16\}$ | $\langle \rho_0, \rho_1, \rho_2 \rangle$ |
| (2,2) | (3,3) | $\{m \mid m$ a divisor of $9216 \wedge m \geq 24\}$ | $\langle \rho_0, \rho_1\rho_0\rho_1, \rho_3, \rho_2\rho_3\rho_2, \rho_3^{\rho_2\rho_1\rho_2} \rangle$ |
| (3,0) | (5,0) | $\{2^i \cdot 255,\ 2^i \cdot 1275,$ $2^i \cdot 3825,\ 2^i \cdot 425 \mid 2 \leq i \leq 10\}$ $\cup \{2^i \cdot 765 \mid 3 \leq i \leq 10\}$ $\cup \{2^i \cdot 15,\ 2^i \cdot 17 \mid 5 \leq i \leq 6\}$ $\cup \{2^i \cdot 85 \mid i \in \{2,6,7,8\}\}$ $\cup \{2^i \cdot 225 \mid 8 \leq i \leq 10\}$ $\cup \{2^i \cdot 153 \mid 7 \leq i \leq 10\}$ $\cup \{2^i \cdot 51 \mid 7 \leq i \leq 8\}$ | $\langle (\rho_0\rho_1\rho_2)^2, (\rho_1\rho_2\rho_0)^2, [(\rho_1\rho_2)^2]^{\rho_3} \rangle$ |

In the case of the map $\{4,4\}_{(s,s)}$, consider $g := uv = (\rho_0\rho_1\rho_2)^2$ and $h := u^{-1}v = g^{\rho_0}$ (corresponding to unitary translations with the direction of the diagonal of a square). We have the following equalities

$$g^{\rho_1} = g,\ g^{\rho_2} = h^{-1} \text{ and } h^{\rho_1} = h^{-1}.$$

The universal regular polytope $\{\{4,4\}_{(t_1,t_2)}, \{4,4\}_{(s_1,s_2)}\}$ where $(t_1,t_2) \in \{(t,t),(t,0)\}$ and $(s_1,s_2) \in \{(s,s),(s,0)\}$ with $t, s \geq 2$, is the Coxeter group $[4,4,4] = \langle \rho_0, \ldots, \rho_3 \rangle$, factored out by two relations of the following set; one with parameter $t$ and the other with parameter $s$.

$$\{(\rho_0\rho_1\rho_2\rho_1)^t,\ (\rho_1\rho_2\rho_3\rho_2)^s,\ (\rho_0\rho_1\rho_2)^{2t},\ (\rho_1\rho_2\rho_3)^{2s}\}$$

The effect of this factorization is that the facets and vertex figures of the honeycomb $\{4,4,4\}$, which are planar infinite tilings $\{4,4\}$, collapse to a finite toroidal regular map, $\{4,4\}_{(t_1,t_2)}$ and $\{4,4\}_{(s_1,s_2)}$, respectively. That is, $G_3 = \langle \rho_0, \rho_1, \rho_2 \rangle$ and $G_0 = \langle \rho_1, \rho_2, \rho_3 \rangle$ are the automorphism groups of the toroidal maps $\{4,4\}_{(t_1,t_2)}$ and $\{4,4\}_{(s_1,s_2)}$, respectively.

This construction always gives a regular polytope of type $\{4,4,4\}$, but the known finite ones are those given in Table 1.

## 3. THE DEGREES OF THE TOROIDAL REGULAR MAPS OF TYPE [4, 4]

Let $M$ be the group of a toroidal map that is a quotient of an infinite irreducible Coxeter group of euclidean type [4, 4] by $\langle u^s \rangle$ (resp. $\langle g^s \rangle$), where $u$ (resp. $g$) is an unitary translation of the regular tessellation of the plane by squares, with the direction of an edge (resp. diagonal) of a square. Let $v$ (resp. $h$) be a conjugate of $u$ (resp. $g$) such that $v \notin \langle u \rangle$ (resp. $h \notin \langle g \rangle$) and $T = \langle u, v \rangle \cong C_s \times C_s$ (resp. $T = \langle g, h \rangle \cong C_s \times C_s$). Consider a faithful transitive action of $M$ on $n$ points.

In particular, if $M$ is the group of $\{4,4\}_{(s,s)}$, then $T$ is intransitive [9, Proposition 3.3]. As $T$ is a normal subgroup of $M$, the orbits of $T$ form a block system for the representation of $M$. Moreover, we have the following two results that give the size of a $T$-orbit.

**Lemma 3.1.** [9, Lemma 3.4] *Let $M$ be the group of a toroidal regular map $\{4,\,4\}_{(s,0)}$ or $\{4,\,4\}_{(s,s)}$ with a faithful transitive permutation representation of degree $n$. If $n \neq s^2$, then $M$ is embedded into $S_k \wr S_m$ with $n = km$ $(m,\,k > 1)$ and*

- (i) *$k = ab$ where $s = lcm(a,b)$ and,*
- (ii) *$m$ is a divisor of $\frac{|M|}{s^2}$.*

**Lemma 3.2.** *Let $K = \langle \alpha, \beta \rangle$ where $\alpha$ and $\beta$ are the actions of the generators of $T$ restricted to a block. If $B := |K : \langle \alpha \rangle|$ and $C := |K : \langle \beta \rangle|$, then the size of a $T$-orbit is $k = ds$ where $d = gcd(B,C)$.*

*Proof.* Consider that $\alpha$ and $\beta$ are the actions of the generators of $T$ on a block of size $k$. Let $K := \langle \alpha,\, \beta \rangle$, $A := |\alpha|$, $B := |K : \langle \alpha \rangle|$ and $C := |K : \langle \beta \rangle|$. The order of $K$ is $AB$ and $K$ acts regularly on the block, hence $k = AB$. As $\alpha$ and $\beta$ commute, we have the following

$$K/\langle \alpha \rangle = \{\langle \alpha \rangle,\, \langle \alpha \rangle\beta, \langle \alpha \rangle\beta^2, \ldots, \langle \alpha \rangle\beta^{B-1}\} \text{ and}$$

$$K/\langle \beta \rangle = \{\langle \beta \rangle,\, \langle \beta \rangle\alpha, \langle \beta \rangle\alpha^2, \ldots, \langle \beta \rangle\alpha^{C-1}\}.$$

Thus $B$ divides $|\beta|$ and $C$ divides $|\alpha| = A$. Let $D := A/C$. As $k = AB = |\beta|C$ we have $|\beta| = DB$. Now $s = lcm(|\alpha|, |\beta|) = lcm(CD,\, BD) = D\, lcm(C,B)$ and $k = AB = DCB = D\, lcm(C,B)\, gcd(C,B) = s\, gcd(C,B)$. To conclude the proof consider $d = gcd(C,B)$. □

In [9] we list all possible degrees of toroidal regular maps, in particular of the toroidal regular maps of type [4, 4], given by the following results. We replace the name used before, "CPR graph", by *Schreier graph*, as in this case the graphs are assumed to be connected.

**Theorem 3.3.** [9, Theorem 4.2] *Let $s > 2$. There exists a Schreier graph of a toroidal map $\{4,4\}_{(s,0)}$ with $n$ vertices if and only if $n = s^2$ or $n$ is either $2ab$, $4ab$ or $8ab$ where $a$ and $b$ are positive integers with $s = lcm(a,b)$.*

The list of degrees of the theorem above are in correspondence with the number $m$ of $T$-orbits, $n = mab$ for $m = 2$, 4, 8 and $n = s^2$ for $m = 1$.

For the toroidal map $\{4,4\}_{(2s,0)}$, Theorem 3.3 can be used to determine all the possible degrees.

**Corollary 3.4.** Let $s > 1$. There exists a Schreier graph of a toroidal map $\{4,4\}_{(2s,0)}$ with $n$ vertices if and only if $n = 4s^2$ or $n$ is either $2ab$, $4ab$ or $8ab$ where $a$ and $b$ are positive integers with $2s = lcm(a,b)$.

**Theorem 3.5.** [9, Theorem 4.4] *Let $s \geq 2$. There exists a Schreier graph of a toroidal map $\{4,4\}_{(s,s)}$ with $n$ vertices if and only if $n$ is $2s^2$, or either $4ab$, $8ab$ or $16ab$ where $a$ and $b$ are positive integers with $s = lcm(a,b)$.*

## 4. Relations between the degrees of types [4, 4] and [4, 4, 4]

As mentioned in Section 1, there is a correspondence between core-free subgroups and faithful transitive actions. Clearly if $H$ is a core-free subgroup of $G$, and $G$ is a subgroup of $K$ of index $\kappa$, then $H$ is also core-free in $K$ and $|K : H| = \kappa|G : H|$. Hence, if $G$ has a faithful transitive permutation representation of degree $n$, then $K$ has a faithful transitive permutation representation of degree $\kappa n$. Consequently, we have the following result.

**Corollary 4.1.** If $n$ is a degree of the toroidal map $\{4,4\}_{(s,s)}$ (resp. $\{4,4\}_{(2s,0)}$), then $4n$ is a degree of the locally toroidal polytope $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$ (resp. $\{\{4,4\}_{(2,0)},\{4,4\}_{(2s,0)}\}$).

This guarantees that $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$ has faithful transitive permutation representations with degrees

$$8s^2,\ 16ab,\ 32ab \text{ and } 64ab,$$

with $s = lcm(a,b)$; while $\{\{4,4\}_{(2,0)},\{4,4\}_{(2s,0)}\}$ has faithful transitive permutation representations with degrees

$$16s^2,\ 32ab,\ 64ab \text{ and } 128ab$$

with $s = lcm(a,b)$. We will prove that these lists are incomplete.

In what follows we give conditions under which there is a one-to-one correspondence between the degrees of $\{4,4\}_{(s,s)}$ and $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$. Before that, we prove the following result that can be used for any group having a central involution.

**Proposition 4.2.** Let $G$ be a transitive group of degree $n$ containing a central involution $\alpha$. Then $G$ is embedded into $S_2 \wr S_{\frac{n}{2}}$, where the blocks are the $\langle\alpha\rangle$-orbits. If $\langle\alpha\rangle$ is the kernel of this embedding, then $\frac{n}{2}$ is the degree of a faithful transitive permutation representation of $G/\langle\alpha\rangle$.

*Proof.* The orbits of $\alpha$, of size two, form a block system for $G$. Consider the group homomorphism $f : G \to S_{\frac{n}{2}}$ induced by the action of $G$ on these blocks. Therefore the isomorphism $G/\langle\alpha\rangle \cong f(G)$, determines a faithful transitive permutation representation of $G/\langle\alpha\rangle$ on $\frac{n}{2}$ points. □

**Proposition 4.3.** Let $s > 2$. Let $x \in \{1,\dots,n\}$ be a point of a faithful transitive permutation representation of $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$ whose group is $G = \langle\rho_0,\rho_1,\rho_2,\rho_3\rangle$. Let $G_0 = \langle\rho_1,\rho_2,\rho_3\rangle$ (the group of $\{4,4\}_{(s,s)}$). If $\rho_0$ is fixed-point free, then $G_0$ acts faithfully and transitively on the 4-sets $\{x,\ x\rho_0,\ x(\rho_0\rho_1)^2,\ x\rho_1\rho_0\rho_1\}$. In particular, $G_0$ has a faithful transitive permutation representation of degree $n/4$.

*Proof.* Let $\delta := (\rho_0\rho_1)^2$. Let $f : G \to S_{\frac{n}{2}}$ be the embedding of $G$ into $S_2 \wr S_{\frac{n}{2}}$ determined by the $\langle\delta\rangle$-orbits. Firstly let us prove that $Ker(f) = \langle\delta\rangle$.

Suppose that $\langle g, h\rangle \cap Ker(f)$ is nontrivial. As $Ker(f)$ is embedded into $C_2^{\frac{n}{2}}$, all the elements of the kernel are involutions. The only involutions of $\langle g, h\rangle$ are $g^{s/2}$, $h^{s/2}$ or $(gh)^{s/2}$ (in particular $s$ must be even). Any case implies that $(gh)^{s/2} \in Ker(f)$. As $(gh)^{s/2}$ is a central involution we get $(gh)^{s/2} = \delta$, a contradiction. Consequently $f(\rho_1)$, $f(\rho_2)$ and $f(\rho_3)$ are involutions and the group generated by these involutions satisfies all the defining relations of $\{4,4\}_{(s,s)}$. This implies that $H = f(G_0)$ must be the group of a toroidal map $\{4,4\}_{(s,s)}$. As $\rho_0$ is fixed-point-free, and $\langle\rho_0,(\rho_0\rho_1)^2\rangle$ is a normal subgroup of $G$, the orbits of $\langle\rho_0,(\rho_0\rho_1)^2\rangle$ must have the same size, i.e. they are 4-sets of the form $\{x,\, x\rho_0,\, x(\rho_0\rho_1)^2,\, x\rho_1\rho_0\rho_1\}$ with $x \in \{1,\ldots,n\}$. In particular $f(\rho_0)$ is nontrivial and commutes with $f(G_0)$. This shows that $G/Ker(f) \cong C_2 \times H$, which is precisely the Coxeter group with disconnected Coxeter diagram obtained factoring $G$ by $\langle\delta\rangle$. In particular, $Ker(f) = \langle\delta\rangle$.

By Proposition 4.2 $G/\langle\delta\rangle$ has a faithful transitive permutation representation of degree $\frac{n}{2}$. Furthermore $G/\langle\delta\rangle$ is isomorphic to $\langle\alpha\rangle \times H$, where $\alpha$ is an involution ($\rho_0$ acting on 2-sets $\{x, x(\rho_0\rho_1)^2\}$).

Now $\langle\alpha\rangle \times H$ is embedded into $S_2 \wr S_{\frac{n}{4}}$. Moreover we may use a similar argument to the one before, to conclude that the kernel of this embedding is $\langle\alpha\rangle$.

Factoring $\langle\alpha\rangle\times H$ by $\langle\alpha\rangle$ gives a group isomorphic to $G_0$. Thus by Proposition 4.2 $G_0$ has a faithful transitive permutation representation of degree $\frac{n}{4}$.

Moreover, the orbits of $\alpha$ are pairs of 2-sets $\{\{x,\, x(\rho_0\rho_1)^2\},\, \{x\rho_0,\, x\rho_0(\rho_0\rho_1)^2\}\}$ and the action of $G_0$ is faithful on the 4-sets $\{x,\, x(\rho_0\rho_1)^2,\, x\rho_0,\, x\rho_0(\rho_0\rho_1)^2\}$. □

The following corollary to Proposition 4.2 gives sufficient conditions that guarantees an one-to-one correspondence between the degrees of $\{\{4,4\}_{(2,0)}, \{4,4\}_{(s,s)}\}$ and $\{\{4,4\}_{(2,0)}, \{4,4\}_{(2s,0)}\}$ for $s \geq 2$.

**Corollary 4.4.** Let $G$ be the group of $\{\{4,4\}_{(2,0)}, \{4,4\}_{(2s,0)}\}$ $(s \geq 2)$ acting transitively and faithfully on $n$ points. Suppose that $f$ is the embedding of $G$ into $S_{\frac{n}{2}}$ determined by the connected components of $\delta := (\rho_1\rho_2\rho_3)^{2s}$. If $Ker(f) = \langle\delta\rangle$ then $n = 2n'$ where $n'$ is a faithful transitive permutation representation of the group of $\{\{4,4\}_{(2,0)}, \{4,4\}_{(s,s)}\}$.

*Proof.* Let $G = \langle\rho_0,\rho_1,\rho_2,\rho_3\rangle$ be the group of $\{\{4,4\}_{(2,0)}, \{4,4\}_{(2s,0)}\}$. The translation $\delta := (\rho_1\rho_2\rho_3)^{2s}$ is a central involution in $G$. Moreover $G/\langle\delta\rangle$ is the group of $\{\{4,4\}_{(2,0)}, \{4,4\}_{(s,s)}\}$. Thus by Proposition 4.2 we get the correspondence between the degrees of $\{\{4,4\}_{(2,0)}, \{4,4\}_{(s,s)}\}$ and $\{\{4,4\}_{(2,0)}, \{4,4\}_{(2s,0)}\}$ stated. □

## 5. The degrees of $\{\{4,4\}_{(2,0)}, \{4,4\}_{(s,s)}\}$ and $\{\{4,4\}_{(2,0)}, \{4,4\}_{(2s,0)}\}$

Let $n$ be the degree of a faithful transitive permutation representation of the group $G = \langle\rho_0,\rho_1,\rho_2,\rho_3\rangle$ of the polytope $\{\{4,4\}_{(2,0)}, \{4,4\}_{(s,s)}\}$ with $s \geq 2$. Let us denote by $G_0$ the maximal parabolic subgroup of $G$ generated by $\{\rho_1,\, \rho_2,\, \rho_3\}$, that is, the group of $\{4,4\}_{(s,s)}$. Consider the subgroup $T$ of $G_0$ generated by $g := (\rho_1\rho_2\rho_3)^2$ and $h := g^{\rho_1}$. We have the following relations.

$$h = h^{\rho_0},\; g = g^{\rho_0},\; g = g^{\rho_2},\; h^{-1} = h^{\rho_2} \text{ and } h^{-1} = g^{\rho_3}.$$

**Proposition 5.1.** The group $G$ has a faithful transitive permutation representation of degree $8s^2$, $16ab$, $32ab$ and $64ab$, where and $a$ and $b$ are positive integers such that $s = lcm(a,b)$.

*Proof.* This follows from Theorem 3.5 and Corollary 4.1. □

The degrees given above are in correspondence with the indexes of core-free subgroups of $\{4,4\}_{(s,s)}$. Let us now give other core-free subgroups corresponding to degrees that are not listed in Proposition 5.1.

**Proposition 5.2.** Let $a$ and $b$ be positive integers such that $s = lcm(a,b)$. The subgroups

(1) $\langle\rho_0\rangle \times \langle\rho_2,\rho_3\rangle$;

(2) $(\langle\rho_0\rangle \times \langle g^{a/2}, h^b\rangle) \rtimes \langle\rho_2, \rho_1\rho_2\rho_1\rangle$ if $a$ is even and $lcm(a/2, b) = s$ and $(\langle\rho_0\rangle \times \langle g^a, h^{b/2}\rangle) \rtimes \langle\rho_2, \rho_1\rho_2\rho_1\rangle$ if $b$ is even and $lcm(a, b/2) = s$;
(3) $(\langle\rho_0\rangle \times \langle g^a, h^b\rangle) \rtimes \langle\rho_2, \rho_1\rho_2\rho_1\rangle$

are core-free subgroups of $G$ with indexes $4s^2$, $4ab$ and $8ab$, respectively.

*Proof.* (1) Let $H = \langle\rho_0\rangle \times \langle\rho_2, \rho_3\rangle \cong C_2 \times D_8$. As $\langle\rho_0\rangle$ and $\langle\rho_0^{\rho_1}\rangle$ have a trivial intersection, we have

$$H \cap H^{\rho_1} = \langle\rho_2, \rho_3\rangle \cap \langle\rho_2^{\rho_1}, \rho_3\rangle = \langle\rho_3\rangle.$$

In addition,

$$H \cap H^{\rho_1\rho_2} = \langle\rho_2, \rho_3\rangle \cap \langle\rho_2^{\rho_1}, \rho_3^{\rho_1\rho_2}\rangle = \langle\rho_3^{\rho_2}\rangle,$$

hence $H \cap H^{\rho_1} \cap H^{\rho_1\rho_2}$ is trivial. Since $|H| = 16$, we have that $|G : H| = 4s^2$.

(2) As $lcm(a/2, b) = s$, $\langle g^{a/2}, h^b\rangle$ and $\langle h^{a/2}, g^b\rangle$ have trivial intersection. In addition the intersections $\langle\rho_2, \rho_2^{\rho_1}\rangle \cap \langle\rho_2^{\rho_3}, \rho_2^{\rho_3\rho_1}\rangle$ and $\langle\rho_0\rangle \cap \langle\rho_0^{\rho_1}\rangle$ are trivial. Therefore $H \cap H^{\rho_1} \cap H^{\rho_3}$ is trivial. Since $|H| = 8\frac{s^2}{\frac{a}{2}b}$, we have that $|G : H| = 4ab$.

The proofs for the other group given in (2) and for the group given in (3) follow similar arguments. □

**Lemma 5.3.** *The following two conditions are equivalent.*

(1) *$a$ and $b$ are even numbers.*
(2) *$a$ is even and $lcm(a/2, b) = lcm(a, b)$, or $b$ is even and $lcm(a, b/2) = lcm(a, b)$.*

*Proof.* Suppose that $a$ and $b$ are both even. Let $\alpha$ and $\beta$ be the maximal integers such that $2^\alpha$ divides $a$ and $2^\beta$ divides $b$. Then if $\alpha \leq \beta$, then $lcm(a/2, b) = lcm(a, b)$, otherwise $lcm(a, b/2) = lcm(a, b)$.

To prove that (2) implies (1), observe that if $a$ is even and $b$ is odd, then $lcm(a/2, b) < lcm(a, b)$. □

In what follows, it will be proven that the degrees given in Proposition 5.2 are the only ones missing in the list of degrees of $\{\{4, 4\}_{(2,0)}, \{4, 4\}_{(s,s)}\}$ obtained by Proposition 5.1.

Similarly to Lemma 3.1 we have the following result.

**Lemma 5.4.** *If $n \neq s^2$, then $G$ is embedded into $S_k \wr S_m$ with $n = km$ ($m$, $k > 1$) and*

(i) *$k = ab$ where $s = lcm(a, b)$ and,*
(ii) *$m$ is a divisor of $\frac{|G|}{s^2} = 64$.*

*Proof.* As $T = \langle g, h\rangle$ is a normal subgroup of $G$, in the proof of [9, Lemma 3.4] replace $M$, the group of a toroidal map, by $G$, the group of the locally toroidal polytope $\{\{4, 4\}_{(2,0)}, \{4, 4\}_{(s,s)}\}$. □

In this section, let $m$ be the number of $T$-orbits and $k$ be the size of a $T$-orbit (thus $n = km$). We will consider $m \in \{1, 2, 4\}$, as the existence of faithful transitive permutation representations of degrees $n = mab$ for $m \in \{8, 16, 32, 64\}$ (for any integers $a$ and $b$ with $lcm(a, b) = s$) is guaranteed by Propositions 5.1 and 5.2. Given a numbering on the $T$-orbits, let us denote by $g_i$ and $h_i$ the actions of $g$ and $h$ on block $i$ (or $T$-orbit $i$), respectively. Let $\Delta_i$ denote the block $i$.

We will consider cases $m \in \{1, 2\}$ and $m = 4$ separately, but before we proceed, let us prove a result that will be used later in both cases.

**Proposition 5.5.** Let $K$ be a transitive group containing the regular subgroup $H = \langle \alpha, \beta \mid \alpha^a = \beta^b = [\alpha, \beta] \rangle$ with $a, b \geq 1$. If $\delta \in K$ is an involution commuting with both $\alpha$ and $\beta$, then $\delta \in H$ and one of the following situations must occur:

(1) $\delta = \alpha^{a/2}$ and $a$ is even;
(2) $\delta = \beta^{b/2}$ and $b$ is even or;
(3) $\delta = \alpha^{a/2}\beta^{b/2}$ and both $a$ and $b$ are even.

*Proof.* If $\delta \in H$, then, as $\delta$ is an involution, there are at most the three possibilities for $\delta$ given in the statement of this proposition. Suppose that $\delta \notin H$. Then for some integers $i$ and $j$, $\delta\alpha^i\beta^j$ is in the stabilizer of a point $x$ [1, page 9]. As $H$ is regular, any point $y$ can be written as $xh$ with $h \in H$. But then, $y\delta\alpha^i\beta^j = xh\delta\alpha^i\beta^j = x\delta\alpha^i\beta^j h = xh = y$. This implies that $\delta\alpha^i\beta^j$ is trivial, a contradiction. □

In Proposition 5.5, when $a \neq 1$ and $b = 1$, the group $H$, is a cyclic group of order $a$. Then there is only one possibility for an involution commuting with the generator $\alpha$ of $H$, that is, $\alpha^{a/2}$.

5.1. **Case:** $m \in \{1, 2\}$**.**

**Proposition 5.6.** $m \neq 1$.

*Proof.* Suppose that $m = 1$, that is, $T$ is transitive. In this case $T$ is regular, hence $n = s^2$. If $\rho_0$ has a fixed point then, as $\rho_0$ commutes with $g$ and $h$, $\rho_0$ is trivial, a contradiction. Thus $\rho_0$ is fixed-point free. Hence, by Proposition 4.3, there exists a faithful transitive permutation representation of the group of the toroidal map $\{4, 4\}_{(s,s)}$ on 4-sets with $T$ being transitive. But $T$ cannot be transitive, as proven in [9, Proposition 3.3]. □

**Proposition 5.7.** If $m = 2$, then $k \neq s$.

*Proof.* Suppose that $k = s$. If $\rho_0$ is fixed-point free, then by Proposition 4.3, $\{4, 4\}_{(s,s)}$ has a faithful transitive permutation representation of degree $n = s/2$, contradicting Theorem 3.5. Hence, $\rho_0$ must have a fixed point. Consequently, $(\rho_0\rho_1)^2$ fixes the blocks and therefore, $s$ is even. Moreover, as $\rho_0$ commutes with both, $g$ and $h$, it fixes a block point-wise. In addition, $\rho_1$ must swap the two blocks, otherwise $\rho_0$ would be trivial.

As $k = s$, either the action of $g$ within a block, say $\Delta_1$, has order $s$ or $gcd(|g_1|, |g_2|) = 1$. Let us consider the two cases separately.

Firstly assume $g_1$ and $h_2$ are cycles of order $s$. Since $g$ and $h$ commute,

$$g = g_1 h_2^\alpha \text{ and } h = g_1^\beta h_2$$

for some integers $\alpha$ and $\beta$. As $(\rho_0\rho_1)^2$ is a fixed-point free central involution, by Proposition 5.5, we have $(\rho_0\rho_1)^2 = g_1^{s/2}h_2^{s/2}$. Assume without loss of generality that $\rho_0$ fixes a point in $\Delta_1$, so that $\rho_0^{\rho_1} = g_1^{s/2}$ and $\rho_0 = h_2^{s/2}$.

If $\alpha$ and $\beta$ are even, one gets $g^{s/2} = g_1^{s/2}$ and $h^{s/2} = h_2^{s/2}$, hence $(\rho_0\rho_1)^2 = (gh)^{s/2}$, a contradiction.

If $\alpha$ is odd and $\beta$ is even, one gets $g^{s/2} = g_1^{s/2}h_2^{s/2}$ and $h^{s/2} = h_2^{s/2}$, hence $(gh)^{s/2} = g_1^{s/2} = \rho_0^{\rho_1}$, a contradiction. Similarly if $\alpha$ is even and $\beta$ is odd one gets the contradiction $(gh)^{s/2} = h_2^{s/2} = \rho_0$.

If $\alpha$ and $\beta$ are both odd, one gets $g^{s/2} = h^{s/2}$, hence $(gh)^{s/2}$ is trivial, a contradiction.

Now consider the case $gcd(|g_1|, |g_2|) = 1$. Let $|g_1| = a$, $|g_2| = b$ with $b$ odd. Then $ab = s$. In this case, $(\rho_0\rho_1)^2 = g_1^{a/2}h_2^{a/2}$. But also, as $b$ is odd, $(gh)^{s/2} = (g_1h_2)^{a/2}$, a contradiction.

□

**Proposition 5.8.** $m \neq 2$.

*Proof.* Let $m = 2$. By Lemma 3.2 and Proposition 5.7, $k > s$. Then both $\langle g\rangle$ and $\langle h\rangle$ act intransitively within a block. If $\rho_0$ is fixed-point free, then the group of $\{4,4\}_{(s,s)}$ has a faithful transitive permutation representation on $n/4$ points by Proposition 4.3, with $T$ having either one or two orbits. We know $T$ cannot have one orbit by [9, Proposition 3.3] and if $T$ has two orbits, then $n/4 = 2s^2$ [9, Lemma 4.3], meaning that the size of a $T$-orbit acting on $n$ points is $k = (2s)^2$, a contradiction. Thus $\rho_0$ must have a fixed point and thus must fix an entire block point-wise.

The permutation $\rho_0$ cannot have a trivial action in both blocks, hence $\Delta_1\rho_1 = \Delta_2$ and $(\rho_0\rho_1)^2$ fixes the blocks. In particular, $s$ is even. In addition, neither $\rho_2$ nor $\rho_3$ can swap the blocks. Hence, since $\rho_3$ must fix a block, therefore $|g_i| = |g_i^{\rho_3}| = |h_i^{-1}|$. As in addition $|g_1| = |g_2|$ and $|h_1| = |h_2|$, we must have $|g_i| = |h_i|$ for $i = 1, 2$ (meaning that each cycle of the cyclic decomposition of $g$, and $h$, has order $s$).

Assume that $\rho_0$ acts trivially on block $\Delta_1$. As $(\rho_0\rho_1)^2$ is a central involution, Proposition 5.5 determines the possibilities for $(\rho_0\rho_1)^2$. The action of $(\rho_0\rho_1)^2$ on block $\Delta_1$ cannot be $(g_1h_1)^{s/2}$ otherwise $(\rho_0\rho_1)^2 = (gh)^{s/2}$. Thus either $(\rho_0\rho_1)^2 = (g_1h_2)^{s/2}$ or $(\rho_0\rho_1)^2 = (h_1g_2)^{s/2}$. Since $(\rho_0\rho_1)^2 = ((\rho_0\rho_1)^2)^{\rho_3}$, in any case one gets $g_i^{s/2} = h_i^{s/2}$ (for $i \in \{1, 2\}$). This gives $g^{s/2} = h^{s/2}$, a contradiction. □

## 5.2. **Case:** $m = 4$.

**Proposition 5.9.** Let $w = w_1w_2 \dots w_l$ with $w_j \in \{\rho_i \,|\, i = 0, 1, 2, 3\}$ for $j \in \{1, \dots, l\}$ and such that

$$|\{j \in \{1, \dots, l\} \,|\, w_j = \rho_1 \vee w_j = \rho_3\}|$$

is odd. If $w$ acts non-trivially within a $T$-orbit, then $k = s^2$.

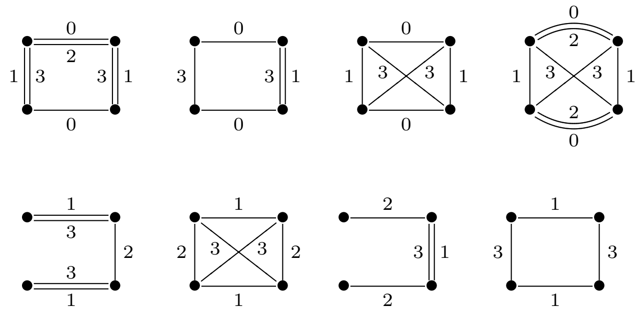
*Proof.* Suppose that $w$ acts non-trivially on $\Delta_1$ and let $K = \langle g_1, h_1\rangle$. As $g_1^w = h_1^{\pm 1}$, we have $|g_1| = |h_1|$. Moreover, by conjugation, we get $|g_i| = |h_i|$ for $i \in \{1, \dots, 4\}$. Hence $|g_1| = |h_1| = s$. Let $B = |K : \langle g_1\rangle| = |K : \langle h_1\rangle|$. We have $k = |K| = Bs$. Let us prove that $B = s$. There exists an integer $j$ such that $g_1^B = h_1^{Bj}$. Conjugating by $w$, this implies that $h_1^B = g_1^{Bj}$. Hence, $(g_1h_1)^B = (g_1h_1)^{Bj}$. As $|g_1h_1| = s$, $B \equiv Bj \mod s$. Now the equality $g_1^B = h_1^{Bj}$ can be rewritten as $g_1^B = h_1^B$, or equivalently $(g_1h_1^{-1})^B$ is trivial. As $|g_1h_1^{-1}| = s$, we have $B = s$. □

**Proposition 5.10.** The element $u := \rho_1\rho_2\rho_3\rho_2$ cannot fix all $T$-orbits.

*Proof.* Suppose that $u$ fixes $\Delta_i$ for some $i \in \{1, \dots, 4\}$. Then there exist a pair of integers $r$ and $t$ such that $ug^rh^t$ fixes a point $x \in \Delta_i$. Hence $u^s$ fixes $x$. Moreover, as $u^s$ commutes with both $g$ and $h$, it fixes every point in $\Delta_i$.

Thus, if $u$ fixes every block then $u^s$ is trivial, a contradiction. □

**Proposition 5.11.** If $m = 4$ and $k \neq s^2$ then the action of $G$ on the blocks is described by one of the following graphs.




*Proof.* The group $G$ acting on the 4 blocks is a group satisfying the defining relations of $G$ and such that

$$(\rho_0\rho_1\rho_2\rho_1)^2 \text{ and } (\rho_1\rho_2\rho_3)^2$$

are both trivial. Under these conditions, using GAP [13], we found the 51 block actions given in Table 3. By Propositions 5.9 and 5.10 this list can be reduced to only eight possibilities, those given in this proposition. □

**Proposition 5.12.** If $m = 4$ and $k \neq s^2$, then $k = ab$, with $a$ and $b$ being even divisors of $s$ such that $s = lcm(a, b)$.

*Proof.* Let us deal with two cases separately: (1) $\rho_0$ is fixed-point free; (2) $\rho_0$ has a fixed-point.

(1) If $\rho_0$ is fixed-point free, then the group of $\{4,4\}_{(s,s)}$ has a faithful transitive permutation representation on $n/4$ points by Theorem 4.3, with $T$ having either one, two or four orbits. The first possibility, $T$ having exactly one orbit, cannot happen by [9, Proposition 3.3]. This also excludes the second graph of the first row of Proposition 5.11 (note that $x$, $x\rho_0, x\rho_1\rho_0\rho_1$ and $x(\rho_0\rho_1)^2$ belong to different blocks). If $T$ has two orbits, then by [9, Lemma 4.3], $n/4 = 2s^2$, meaning that the size of a $T$-orbit acting on $n$ points is $k = 2s^2$, a contradiction. This excludes the remaining graphs of the first row of Proposition 5.11.

Finally, suppose that $T$ has four orbits when acting on the quadruples. Then the size $k$ of a $T$-orbit on the set of size $n$ must be divisible by 4. Thus, by Lemma 3.1, $k/4 = a'b'$ with $lcm(a', b') = s$ and, by Lemma 5.4, $k = ab$ with $lcm(a, b) = s$. Hence we have $k = 4gcd(a', b')s = gcd(a, b)s$, and therefore $gcd(a, b)$ is even, as desired.

(2) Suppose now that $\rho_0$ has a fixed point. Hence the action on the blocks cannot be given by the first four graphs given in Proposition 5.11, where $\rho_0$ is fixed-point free. Whenever $\rho_0$ has a fixed point in a block, say $\Delta_i$, then since it commutes with both $g$ and $h$, it must act trivially on $\Delta_i$.

Now consider the first three block actions described by the graphs given on the second row of Proposition 5.11. If $\rho_0$ is trivial on a block, then, as it commutes with $\rho_2$ and $\rho_3$, we get that $\rho_0$ acts as the identity, a contradiction. Thus the remaining possibility for the block action is described by the alternating $\{1, 3\}$-square, the one on the right side of the second row of Proposition 5.11.

Let $\Delta_2 = \Delta_1\rho_1$, $\Delta_3 = \Delta_2\rho_3$ and $\Delta_4 = \Delta_3\rho_1$. As $(\rho_0\rho_1)^2$ fixes the blocks, $k$ must be even and, consequently $s$ is even.

Let $K = \langle g_1, h_1\rangle$ be the action of $T$ on the block $\Delta_1$ and let $B := |K : \langle g_1\rangle|$ and $C := |K : \langle h_1\rangle|$. By Lemma 3.2 $k = lcm(C, B)s$ and, as seen in the proof of

Lemma 3.2, there exists some $D$ such that $|g_1| = DC$ and $|h_1| = DB$. One may consider $a = gcd(B, C)$ and $b = s$. Then it is sufficient to prove that both $B$ and $C$ are even numbers.

Let us first prove that both $|g_1|$ and $|h_1|$ are even. Note that $|g_1|$ and $|h_1|$ cannot both be odd, since $s = lcm(|g_1|, |h_1|)$. Hence, suppose $|g_1|$ is even and $|h_1|$ is odd. Since $|h_1|$ is odd, we must have $(\rho_0\rho_1)^2 = (g_1h_2h_3g_4)^{|g_1|/2}$. We have $s/2 \equiv 0 \mod |h_i|$ for $i \in \{1, 4\}$ and $s/2 \equiv 0 \mod |g_i|$ for $i \in \{2, 3\}$, hence $(h_1g_2g_3h_4)^{s/2}$ is trivial. In addition, note that $s/2 \equiv |g_1|/2 \mod |g_1|$. Consequently,

$$(gh)^{s/2} = (g_1h_2h_3g_4)^{s/2}(h_1g_2g_3h_4)^{s/2} = (g_1h_2h_3g_4)^{s/2} = (g_1h_2h_3g_4)^{|g_1|/2} = (\rho_0\rho_1)^2$$

a contradiction. The case where $|g_1|$ is odd and $|h_1|$ is even can be treated similarly. Then both $|g_1|$ and $|h_1|$ are even.

Suppose that $gcd(C, B)$ is odd. Assume that $C$ or $B$ is odd, but not both. Then, since the orders of both $g_1$ and $h_1$ are even, $D$ must be even. Suppose first that that $B$ is even and $C$ is odd. Let $i$ and $j$ be such that $h_1^B = g_1^{Ci}$ and $g_1^C = h_1^{Bj}$. As $|h_1^B| = |g_1^C| = D$ both $i$ and $j$ must be coprime with $D$. Hence $i$ and $j$ are odd numbers. Then $h_1^{\frac{Bs}{2}} = g_1^{\frac{Cis}{2}}$, implies that $\frac{Cis}{2} = 0 \mod s$, a contradiction, since $C$ and $i$ are odd. We get the same contradiction if we assume that $B$ is odd and $C$ is even. Thus $B$ and $C$ are both odd. Let $\alpha$ and $\beta$ be such that $lcm(B, C) = \alpha B = \beta C$. Then, both $\alpha$ and $\beta$ are odd. Thus, from the equalities below, we get that $(gh)^{s/2}$ is trivial:

$$\begin{aligned}
(gh)^{s/2} &= (g_1h_2h_3g_4)^{s/2}(h_1g_2g_3h_4)^{s/2} \\
&= (g_1h_2h_3g_4)^{\frac{D}{2}lcm(B,C)}(h_1g_2g_3h_4)^{\frac{D}{2}lcm(B,C)} \\
&= (g_1h_2h_3g_4)^{\frac{D}{2}\beta C}(h_1g_2g_3h_4)^{\frac{D}{2}\alpha B} \\
&= (g_1h_2h_3g_4)^{\frac{DC}{2}(\beta+\alpha i)} \\
&= id,
\end{aligned}$$

a contradiction. Hence $gcd(C, B)$ must be even. □

**Theorem 5.13.** *Let $a$ and $b$ be positive integers such that $s = lcm(a, b)$ and $s \geq 2$. Then the locally toroidal polytope $\{\{4, 4\}_{(2,0)}, \{4, 4\}_{(s,s)}\}$ has a faithful transitive permutation representation of degree $n$ if and only if*

$$n \in \{4s^2, 8ab, 16ab, 32ab, 64ab\} \textit{ or } n = 4ab \textit{ if } a \textit{ and } b \textit{ are both even.}$$

*Proof.* This result follows from Propositions 5.1, 5.2, 5.8 and 5.12 and Lemma 5.3. □

**Theorem 5.14.** *Let $a$ and $b$ be positive integers such that $s = lcm(a, b)$ and $s \geq 2$. Then the locally toroidal polytope $\{\{4, 4\}_{(2,0)}, \{4, 4\}_{(2s,0)}\}$ has a faithful transitive permutation representation of degree $n$ if and only if*

$$n \in \{8s^2, 16ab, 32ab, 64ab, 128ab\} \textit{ or } n = 8ab \textit{ if } a \textit{ and } b \textit{ are both even.}$$

*Proof.* Notice that the theorem holds when $s = 2$ (see Table 2). From now on assume that $s \geq 3$. Let $G$ be the group of $\{\{4, 4\}_{(2,0)}, \{4, 4\}_{(2s,0)}\}$ and let $n$ be a degree of a transitive faithful permutation representation of $G$. Consider the normal subgroup $T$ of $G$ generated by $u^2$ and $v^2$ where $u = \rho_1\rho_2\rho_3\rho_2$ and $v = u^{\rho_2}$. Let $\delta := (\rho_1\rho_2\rho_3)^{2s} = (uv)^s$ and $\beta := (\rho_0\rho_1)^2$. As $\delta$ is a central involution, it determines an embedding of $G$ into $S_2 \wr S_{\frac{n}{2}}$ where the blocks are the connected components of $\delta$. Let $f$ denote the homomorphism $G \to S_{\frac{n}{2}}$ determined by this embedding. If $Ker(f) = \langle \delta \rangle$ then, by Theorem 5.13 and Corollary 4.4, $n$ is one of the degrees listed in the statement of this theorem. In what follows we lead with the case $Ker(f) \neq \langle \delta \rangle$.

Firstly consider the case $s$ odd. In this case $T$ has no involutions. Hence, as all the elements of $Ker(f)$ are involutions, the intersection of $T$ with $Ker(f)$ is trivial, therefore $f(G_0)$ must be is the group of the map $\{4,4\}_{(s,s)}$. If $\rho_0 \in Ker(f)$ then $\beta \in Ker(f)$. But as $\beta$ is a central involution in $G$, we get $\beta = \delta$, a contradiction. Hence $|f(\rho_0)| = |f((\rho_0\rho_1)^2)| = 2$. This shows that $f(G)$ is the group of $\{\{4,4\}_{(2,0)}, \{4,4\}_{(s,s)}\}$, or equivalently, $Ker(f) = \langle \delta \rangle$, a contradiction.

Let us now lead with the case $s$ even. In this case $\delta \in T$. Suppose that $Ker(f) \cap T$ is not $\langle \delta \rangle$. Then $\langle u^s, v^s \rangle \leq Ker(f)$. Indeed $\langle u^s, v^s \rangle$ is the maximal subgroup of $T$ contained in $Ker(f)$. Consequently, $f(G_0)$ is, in this case, the group of the map $\{4,4\}_{(s,0)}$. Now suppose that $\rho_0 \in Ker(f)$ then $\beta := (\rho_0\rho_1)^2 \in Ker(f)$, as before one gets a contradiction. Hence $Ker(f) = \langle u^s, v^s \rangle$ and $G/Ker(f)$ is isomorphic to the group of $\{\{4,4\}_{(2,0)}, \{4,4\}_{(2s',0)}\}$ where $s' := s/2$.

We may assume by induction the degrees of $\{\{4,4\}_{(2,0)}, \{4,4\}_{(2s',0)}\}$ are precisely those of the following list where $lcm(a',b') = s'$.

$$8s'^2,\ 16a'b',\ 32a'b',\ 64a'b',\ 128a'b' \text{ or } n = 8a'b' \text{ if } a' \text{ and } b' \text{ are both even.}$$

Assume without loss of generality that $lcm(2a',b') = s$. Then the degrees of $G$ are contained in the following list.

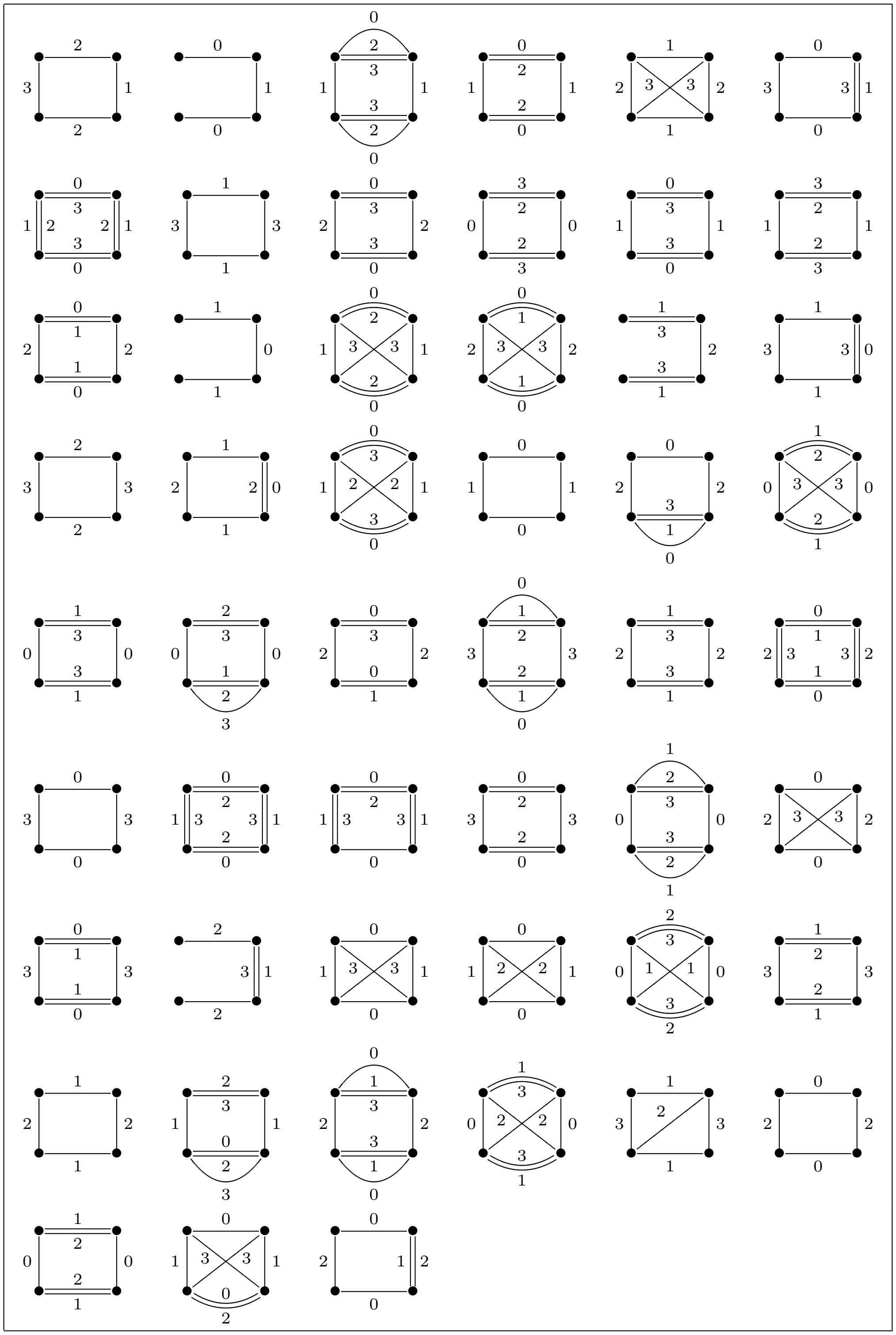
$$4s^2, 16(2a')b', 32(2a')b', 64(2a')b', 128(2a')b' \text{ or } n = 8(2a')b' \text{ if } a' \text{ and } b' \text{ are both even.}$$

All these degrees correspond to the ones given in the statement of this theorem with one exception, $n = 4s^2$. Let us now rule out this possibility.

Suppose that $n = 4s^2$ then, the number $m$ of $T$-orbits is at most 4. We remind that $T = \langle u^2, v^2 \rangle$, $\delta = u^s v^s$, $\beta = (\rho_0\rho_1)^2$ and, since $s$ is even, $Ker(f) = \langle u^s, v^s \rangle$. As $\delta$ is a fixed-point-free involution (swapping $n/2$ pairs of points), and $u^2$ and $v^2$ have the same cyclic decomposition, $u^s$ swaps exactly $\frac{n}{4}$ pairs of points while $v^s$ swaps the remaining $\frac{n}{4}$ pairs of points. As the orbits of $T$ have the same size and $T$ acts regularly on each orbit, there exist exactly two possible sizes, say $a$ and $b$, of a cycle of the cyclic decomposition of $u^2$ (and for $v^2$). Moreover, as $u^s$ has fixed points, $a$ and $b$ must be distinct. Let us see that this implies that $m = 2$. Firstly, as the case where $a = b = s$ cannot happen, $T$ cannot be transitive, thus $m \neq 1$. Secondly, again as $a = b = s$ cannot happen, $n/4 \neq s^2$, hence $m \neq 4$. Let $\Delta_1$ and $\Delta_2$ be the $T$-orbits. As $a \neq b$, $\Delta_1\rho_2 = \Delta_2$. Furthermore, $\rho_2$ is the unique permutation of the generating set of $G$, permuting the blocks. Indeed, as $\rho_0$ commutes with $u$ and $v$, $u^{\rho_1} = u^{-1}$, $u^{\rho_3} = u$, $v^{\rho_1} = v$ and $v^{\rho_3} = v^{-1}$, the other involutions, $\rho_0$, $\rho_1$ and $\rho_3$, cannot swap the blocks as this would force the cyclic decomposition of $u^2$ and $v^2$ to be the same on the blocks, and $a = b = s$. Let $u_1$, $v_1$, $u_2$ and $v_2$ denote the actions of $u$ and $v$ on $\Delta_1$ and $\Delta_2$, respectively. Let $a$ and $b$ be the orders of $u_1^2$ and $u_2^2$, respectively. Then $u^s = u_1^a$, $v^s = v_2^a$. The orbits of $\langle \beta, \delta \rangle$ have the same size, equal to 4. Thus, with no other possibilities, either $\beta = v_1^b u_2^b$ or $\beta = \delta v_1^b u_2^b$. In addition, as $\rho_0$ fixes the blocks and commutes with $u^2$ and $v^2$, we get $\rho_0 \in \langle \beta, \delta \rangle$, a contradiction.

□

TABLE 3. The possible actions on the blocks when $m = 4$



## 6. Acknowledgements

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Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Portugal
*Email address*: maria.elisa@ua.pt

Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Portugal
*Email address*: claudio.a.piedade@ua.pt