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Taylor's Formula for Generalized Weighted Fractional Derivatives with Nonsingular Kernels

Houssine Zine [1], El Mehdi Lotfi [2], Delfim F. M. Torres [1,*] and Noura Yousfi [2]

[1] Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal; zinehoussine@ua.pt
[2] Laboratory of Analysis, Modeling and Simulation (LAMS), Faculty of Sciences Ben M'sik, Hassan II University of Casablanca, P.O. Box 7955, Sidi Othman, Casablanca 20000, Morocco; lotfiimehdi@gmail.com (E.M.L.); nourayousfi.fsb@gmail.com (N.Y.)
* Correspondence: delfim@ua.pt

Abstract: We prove a new Taylor's theorem for generalized weighted fractional calculus with nonsingular kernels. The proof is based on the establishment of new relations for nth-weighted generalized fractional integrals and derivatives. As an application, new mean value theorems for generalized weighted fractional operators are obtained. Direct corollaries allow one to obtain the recent Taylor's and mean value theorems for Caputo–Fabrizio, Atangana–Baleanu–Caputo (ABC) and weighted ABC derivatives.

Keywords: generalized weighted fractional derivatives; nonsingular kernels; Taylor's formula; mean value theorems

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1. Introduction

Among the numerous achievements and visionary discoveries of Leonhard Euler in the 18th century is the generalization of the factorial by the gamma function, which allowed him to evaluate fractional-order (i.e., not necessarily integer order) derivatives of x^n by

$$D^\alpha x^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

and thus generalize the integer-order derivatives of x^n. Moreover, Euler also wrote the particular case $\alpha = 1/2$ and $n = 1$, presenting us with the beautiful formula $D^{1/2}x = 2\sqrt{\frac{x}{\pi}}$ for the half-order derivative of x [1].

In the 20th century, fractional calculus, by regarding the historical values of the considered functions according to their order, was adopted as an important tool to model memory effects [2,3]. This resulted in significant and useful real-word applications of wave equations [4], chemical kinetics [5], optimal control of biochemical reactions [6], among many others [7].

Different fractional-order calculi theories are nowadays addressed, in a wide range of scientific areas, in order to accurately better describe real-world problems with memory effects [8,9]. In particular, fractional calculus has also recently shown its efficiency in modeling uncertain financial markets [10] and reaction–diffusion epidemics [11].

On the other hand, Taylor's formulas play a crucial role in mathematical analysis, e.g., in asymptotic methods, nonlinear programming, and the calculus of variations and optimal control [12–14]. Different forms of Taylor's formulas can be found in the literature, covering both classical and smooth one-dimensional cases as well as multi-dimensional, non-smooth, and non-Newtonian cases [15–17].

The appearance of fractional-order theories requires the establishment of corresponding Taylor's formulas [18,19]. For this reason, Taylor's theorems have been immediately proved, in different forms, for Riemann–Liouville fractional calculus [20–22] as well as for Caputo fractional derivatives [23]. The literature on fractional Taylor theorems is now vast: see, e.g., [24–26] and references therein. However, all such fractional-order Taylor's formulas are valid for fractional derivatives with a singular kernel only.

More recently, several researchers have been trying to use fractional calculus in the treatment of dynamics of complex systems, which have complicated dynamics that cannot be properly described with classical/singular-kernel fractional models [27–30]. That gave rise to the appearance of fractional derivatives with nonsingular kernels [31,32] and, as a consequence, to the need to obtain Taylor's formulas for such kinds of operators [33]. In particular, in [34], Fernandez and Baleanu established analogues of Taylor's theorem for fractional differential operators defined using a Mittag–Leffler kernel and a mean value theorem for the Atangana–Baleanu–Caputo (ABC) fractional derivative, introduced in [35] and now under strong current investigations [36–38]. Here, we consider the generalized weighted fractional derivative in Caputo sense, as introduced in 2020 by Hattaf [39,40]. Our main results, formulated for this generalized weighted fractional calculus, allows one to extend, in a natural and direct way, the 2020 results of Al-Refai [23] and the 2018 results of Fernandez and Baleanu [34], which are now obtained as simple corollaries.

The paper is organized as follows. In Section 2, for completeness and to fix notations, we recall necessary definitions and properties needed to prove our results in the sequel. The elaboration of new tools, enabling us to obtain a general and rich Taylor's formula (cf. Theorem 3), is given in Section 3 of main results. An example to clarify the main Theorem 3 is given in Section 4. We proceed with Section 5, obtaining several new mean value theorems. In our results, if one considers the particular case $w(t) \equiv 1$ and $\alpha = \beta = 1$, then we obtain well-known classical results. We end with Section 6 with a conclusion and some possible future directions for research.

2. Preliminaries

In this section, we present some definitions and properties from the fractional calculus literature, which will help us to prove our main results. Along the text, $f \in H^1(a,b)$ is a sufficiently smooth function on $[a,b]$ with $a,b \in \mathbb{R}$.

Definition 1 (See, e.g., [41])**.** *The Riemann–Liouville (RL) fractional integral operator of order $\alpha > 0$ with $a \geq 0$ is defined by*

$$^{RL}I_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s)ds, \quad x > a, \tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function.

For the sake of simplicity, we adopt the following notations:

$$\phi(\alpha) := \frac{1-\alpha}{B(\alpha)}, \quad \psi(\alpha) := \frac{\alpha}{B(\alpha)},$$

where $B(\alpha)$ denotes a normalization function obeying $B(0) = B(1) = 1$.

Definition 2 (See [42])**.** *The Caputo–Fabrizio (CF) fractional derivative of order $0 \leq \alpha \leq 1$ of function f is given by*

$$^{CF}_{a}D^{\alpha} f(x) = \frac{1}{\phi(\alpha)} \int_a^x f'(s) \exp[-\mu_\alpha (x-s)]ds \tag{2}$$

with

$$\mu_\alpha = \frac{\alpha}{1-\alpha}. \tag{3}$$

The fractional integral associated with the CF fractional derivative is defined by

$$^{CF}I_a^{\alpha} f(x) = \phi(\alpha) f(x) + \psi(\alpha)\, ^{RL}I_a^1 f(x). \tag{4}$$

Definition 3 (See [35]). *The Atangana–Baleanu–Caputo (ABC) fractional derivative of order α, $0 \le \alpha \le 1$, of function f, is given by*

$$^{ABC}_{a}D^{\alpha} f(x) = \frac{1}{\phi(\alpha)} \int_a^x f'(s) E_\alpha[-\mu_\alpha (x-s)^\alpha] ds, \tag{5}$$

where E_α denotes the Mittag–Leffler function of parameter α defined by

$$E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}, \quad z \in \mathbb{C}.$$

The fractional integral associated with the ABC fractional derivative is given by

$$^{AB}I_a^{\alpha} f(x) = \phi(\alpha) f(x) + \psi(\alpha)\, ^{RL}I_a^{\alpha} f(x). \tag{6}$$

Definition 4 (See [43]). *The weighted ABC fractional derivative of order $0 \le \alpha \le 1$ of function f with respect to the weight function w is given by*

$$^{C}_{a}D_w^{\alpha} f(x) = \frac{1}{\phi(\alpha)} \frac{1}{w(x)} \int_a^x (wf)'(s) E_\alpha[-\mu_\alpha (x-s)^\alpha] ds, \tag{7}$$

where $w \in C^1([a,b])$ with $w, w' > 0$. The corresponding fractional integral is defined by

$$^{C}I_{a,w}^{\alpha} f(x) = \phi(\alpha) f(x) + \psi(\alpha)\, ^{RL}I_{a,w}^{\alpha} f(x), \tag{8}$$

where $^{RL}I_{a,w}^{\alpha}$ is the standard weighted Riemann–Liouville fractional integral of order α given by

$$^{RL}I_{a,w}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \frac{1}{w(x)} \int_a^x (x-s)^{\alpha-1} w(s) f(s) ds, \quad x > a. \tag{9}$$

Definition 5 (See [39]). *Let $\beta > 0$. The weighted generalized fractional derivative of order $0 \le \alpha \le 1$ of function f with respect to the weight function w is given by*

$$^{C}_{a}D_w^{\alpha,\beta} f(x) = \frac{1}{\phi(\alpha)} \frac{1}{w(x)} \int_a^x (wf)'(s) E_\beta\left[-\mu_\alpha (x-s)^\beta\right] ds, \tag{10}$$

where $w \in C^1([a,b])$ with $w, w' > 0$. The corresponding fractional integral is defined by

$$^{C}I_{a,w}^{\alpha,\beta} f(x) = \phi(\alpha) f(x) + \psi(\alpha)\, ^{RL}I_{a,w}^{\beta} f(x), \tag{11}$$

where $^{RL}I_{a,w}^{\beta}$ is the standard weighted Riemann–Liouville fractional integral of order β.

Theorem 1 (See [39]). *Let $\alpha \in [0,1)$, $\beta > 0$. Then, $^{C}I_{a,w}^{\alpha,\beta}\left(^{C}_{a}D_w^{\alpha,\beta} f(x)\right) = f(x) - \left(\frac{w(a)}{w(x)} f(a)\right)$.*

To simplify the writing, we denote by $\mathfrak{D}_a^{[\alpha,\beta]}$ the generalized fractional derivative (10) and by $\mathfrak{I}_a^{[\alpha,\beta]}$ its associated fractional integral (11).

3. Main Results

We begin by proving an important result that has a crucial role in the proof of our Taylor's formula for weighted generalized fractional derivatives with a nonsingular kernel (cf. proofs of Lemma 1 and Theorem 3).

Theorem 2. *Suppose that $f \in C([a,b])$ and $n \in \mathbb{N}$. Then,*

$$\mathfrak{I}_a^{n[\alpha,\beta]} f(x) = \sum_{k=0}^{n} C_n^k \phi(\alpha)^{n-k} \psi(\alpha)^k \left({}^{RL}I_{a,w}^{k\beta} f(x) \right)$$

with $x \in [a,b]$ and $\alpha \in [0,1]$, where $\mathfrak{I}_a^{n[\alpha,\beta]} = \mathfrak{I}_a^{[\alpha,\beta]} \cdots \mathfrak{I}_a^{[\alpha,\beta]}$, n-times.

Proof. We proceed by induction. Firstly, note that the equality of Theorem 2 is true for $n = 0$: from Definition 2 in [23], $\mathfrak{I}_a^0 f(x) = f(x)$ and

$$\sum_{k=0}^{0} C_0^k \phi(\alpha)^{0-k} \psi(\alpha)^k \left({}^{RL}I_{a,w}^{k\beta} f(x) \right) = {}^{RL}I_{a,w}^{0\times\beta} f(x) = f(x).$$

Supposing that the equality of Theorem 2 is true, we show that

$$\mathfrak{I}_a^{(n+1)[\alpha,\beta]} f(x) = \sum_{k=0}^{n+1} C_{n+1}^k \phi(\alpha)^{n+1-k} \psi(\alpha)^k \left({}^{RL}I_{a,w}^{k\beta} f(x) \right), \quad x \in [a,b],$$

holds. Indeed,

$$\begin{aligned}
\mathfrak{I}_a^{(n+1)[\alpha,\beta]} f(x) &= \mathfrak{I}_a^{\alpha,\beta} \left(\mathfrak{I}_a^{n[\alpha,\beta]} f(x) \right) = \phi(\alpha) (\mathfrak{I}_a^{n[\alpha,\beta]} f(x)) + \psi(\alpha)\, {}^{RL}I_{a,w}^{\beta} \left(\mathfrak{I}_a^{n[\alpha,\beta]} f(x) \right) \\
&= \phi(\alpha) \left[\sum_{k=0}^{n} C_n^k \phi(\alpha)^{n-k} \psi(\alpha)^k \left({}^{RL}I_{a,w}^{k\beta} f(x) \right) \right] \\
&\quad + \psi(\alpha)^{RL}I_{a,w}^{\beta} \left[\sum_{k=0}^{n} C_n^k \phi(\alpha)^{n-k} \psi(\alpha)^k \left({}^{RL}I_{a,w}^{k\beta} f(x) \right) \right] \\
&= \sum_{k=0}^{n} C_n^k \phi(\alpha)^{n+1-k} \psi(\alpha)^k \left({}^{RL}I_{a,w}^{k\beta} f(x) \right) + \sum_{k=0}^{n} C_n^k \phi(\alpha)^{n-k} \psi(\alpha)^{k+1} \left({}^{RL}I_{a,w}^{(k+1)\beta} f(x) \right) \\
&= \phi(\alpha)^{n+1} f(x) + \sum_{k=1}^{n} C_n^k \phi(\alpha)^{n+1-k} \psi(\alpha)^k \left({}^{RL}I_{a,w}^{k\beta} f(x) \right) \\
&\quad + \sum_{k=1}^{n} C_n^{k-1} \phi(\alpha)^{n+1-k} \psi(\alpha)^k ({}^{RL}I_{a,w}^{k\beta} f(x)) + \psi(\alpha)^{n+1} \left({}^{RL}I_{a,w}^{(n+1)\beta} f(x) \right) \\
&= \sum_{k=0}^{n+1} C_{n+1}^k \phi(\alpha)^{n+1-k} \psi(\alpha)^k \left({}^{RL}I_{a,w}^{k\beta} f(x) \right), \quad x \in [a,b],
\end{aligned}$$

which completes the proof. □

The following lemma will allow us to construct our weighted Taylor's formula for weighted generalized fractional derivatives with a nonsingular kernel.

Lemma 1. *Suppose that $\mathfrak{D}_a^{n[\alpha,\beta]} f$, $\mathfrak{D}_a^{(n+1)[\alpha,\beta]} f \in C([a,b])$ for $0 \le \alpha \le 1$. Then,*

$$\begin{aligned}
&\mathfrak{I}_a^{n[\alpha,\beta]} \mathfrak{D}_a^{n[\alpha,\beta]} f(x) - \mathfrak{I}_a^{(n+1)[\alpha,\beta]} \mathfrak{D}_a^{(n+1)[\alpha,\beta]} f(x) \\
&\qquad = \frac{w(a)}{w(x)} \left(\mathfrak{D}_a^{n[\alpha,\beta]} f(a) \right) \sum_{k=0}^{n} C_n^k \phi(\alpha)^{n-k} \psi(\alpha)^k \left(\frac{(x-a)^{k\beta}}{\Gamma(k\beta+1)} \right),
\end{aligned}$$

where $\mathfrak{D}_a^{n[\alpha,\beta]} = \mathfrak{D}_a^{[\alpha,\beta]} \cdots \mathfrak{D}_a^{[\alpha,\beta]}$, n-times.

Proof. From the fact that $\mathfrak{I}_a^{r[\alpha,\beta]}\mathfrak{I}_a^{l[\alpha,\beta]}f = \mathfrak{I}_a^{(r+l)[\alpha,\beta]}f$, one has

$$\begin{aligned}
&\mathfrak{I}_a^{n[\alpha,\beta]}\mathfrak{D}_a^{n[\alpha,\beta]}f(x) - \mathfrak{I}_a^{(n+1)[\alpha,\beta]}\mathfrak{D}_a^{(n+1)[\alpha,\beta]}f(x) \\
&\quad = \mathfrak{I}_a^{n[\alpha,\beta]}\left(\mathfrak{D}_a^{n[\alpha,\beta]}f(x) - \mathfrak{I}_a^{[\alpha,\beta]}\mathfrak{D}_a^{(n+1)[\alpha,\beta]}f(x)\right) \\
&\quad = \mathfrak{I}_a^{n[\alpha,\beta]}\left(\mathfrak{D}_a^{n[\alpha,\beta]}f(x) - \mathfrak{I}_a^{[\alpha,\beta]}\mathfrak{D}_a^{[\alpha,\beta]}(\mathfrak{D}_a^{n[\alpha,\beta]}f(x))\right) \\
&\quad = \mathfrak{I}_a^{n[\alpha,\beta]}\left(\frac{w(a)\mathfrak{D}_a^{n[\alpha,\beta]}f(a)}{w(x)}\right) = w(a)\mathfrak{D}_a^{n[\alpha,\beta]}f(a)\mathfrak{I}_a^{n[\alpha,\beta]}\frac{1}{w(x)}.
\end{aligned}$$

Using Theorem 2, we get that

$$\begin{aligned}
&\mathfrak{I}_a^{n[\alpha,\beta]}\mathfrak{D}_a^{n[\alpha,\beta]}f(x) - \mathfrak{I}_a^{(n+1)[\alpha,\beta]}\mathfrak{D}_a^{(n+1)[\alpha,\beta]}f(x) \\
&\quad = w(a)\left(\mathfrak{D}_a^{n[\alpha,\beta]}f(a)\right)\sum_{k=0}^{n} C_n^k \phi(\alpha)^{n-k}\psi(\alpha)^k\left({}^{RL}I_{a,w}^{k\beta}\left(\frac{1}{w(x)}\right)\right) \\
&\quad = \frac{w(a)}{w(x)}\left(\mathfrak{D}_a^{n[\alpha,\beta]}f(a)\right)\sum_{k=0}^{n} C_n^k \phi(\alpha)^{n-k}\psi(\alpha)^k\frac{(x-a)^{k\beta}}{\Gamma(k\beta+1)}
\end{aligned}$$

and the proof is complete. □

Follows the main result of our paper.

Theorem 3 (Taylor's formula for weighted generalized fractional derivatives with a non-singular kernel)**.** *Suppose that $\mathfrak{D}_a^{k[\alpha,\beta]} \in C([a,b])$ for $k = 0,1,\ldots,n+1$ and $0 \le \alpha \le 1$. Then,*

$$\begin{aligned}
f(x) = \frac{1}{w(x)}&\Bigg[w(a)\sum_{i=0}^{n}\mathfrak{D}_a^{i[\alpha,\beta]}f(a)\sum_{k=0}^{i} C_i^k\phi(\alpha)^{i-k}\psi(\alpha)^k\frac{(x-a)^{k\beta}}{\Gamma(k\beta+1)} \\
&+ w(\xi)\mathfrak{D}_a^{(n+1)[\alpha,\beta]}f(\xi)\sum_{k=0}^{n+1} C_{n+1}^k\phi(\alpha)^{n+1-k}\psi(\alpha)^k\frac{(x-a)^{k\beta}}{\Gamma(k\beta+1)}\Bigg] \qquad (12)
\end{aligned}$$

with $a \le \xi \le x$, $x \in [a,b]$, where $\mathfrak{D}_a^{i[\alpha,\beta]} = \mathfrak{D}_a^{[\alpha,\beta]}\cdots\mathfrak{D}_a^{[\alpha,\beta]}$, i-times.

Proof. From Lemma 1, we have

$$\begin{aligned}
&\sum_{i=0}^{n}\left(\mathfrak{I}_a^{i[\alpha,\beta]}\mathfrak{D}_a^{i[\alpha,\beta]}f(x) - \mathfrak{I}_a^{(i+1)[\alpha,\beta]}\mathfrak{D}_a^{(i+1)[\alpha,\beta]}f(x)\right) \\
&\qquad = \frac{w(a)}{w(x)}\sum_{i=0}^{n}\left(\mathfrak{D}_a^{i[\alpha,\beta]}f(a)\right)\sum_{k=0}^{i} C_i^k\phi(\alpha)^{i-k}\psi(\alpha)^k\frac{(x-a)^{k\beta}}{\Gamma(k\beta+1)},
\end{aligned}$$

that is,

$$f(x) - \mathfrak{I}_a^{(n+1)[\alpha,\beta]}\mathfrak{D}_a^{(n+1)[\alpha,\beta]}f(x) = \frac{w(a)}{w(x)}\sum_{i=0}^{n}\left(\mathfrak{D}_a^{n[\alpha,\beta]}f(a)\right)\sum_{k=0}^{i} C_i^k\phi(\alpha)^{i-k}\psi(\alpha)^k\frac{(x-a)^{k\beta}}{\Gamma(k\beta+1)}.$$

Using Theorem 2, we get

$$\begin{aligned}
f(x) \quad = \quad &\frac{w(a)}{w(x)}\sum_{i=0}^{n}\left(\mathfrak{D}_a^{i[\alpha,\beta]}f(a)\right)\sum_{k=0}^{i} C_i^k\phi(\alpha)^{i-k}\psi(\alpha)^k\frac{(x-a)^{k\beta}}{\Gamma(k\beta+1)} \\
&+ \sum_{k=0}^{n+1} C_{n+1}^k\phi(\alpha)^{n+1-k}\psi(\alpha)^k\left({}^{RL}I_a^{k\beta}\mathfrak{D}_a^{(n+1)[\alpha,\beta]}f(x)\right).
\end{aligned}$$

Applying the integral mean value theorem yields

$$\begin{aligned} f(x) &= \frac{1}{w(x)}\Bigg[w(a)\sum_{i=0}^{n}\mathfrak{D}_a^{i[\alpha,\beta]}f(a)\sum_{k=0}^{i}C_i^k\phi(\alpha)^{i-k}\psi(\alpha)^k\frac{(x-a)^{k\beta}}{\Gamma(k\beta+1)} \\ &\quad +w(\xi)\mathfrak{D}_a^{(n+1)[\alpha,\beta]}f(\xi)\sum_{k=0}^{n+1}C_{n+1}^k\phi(\alpha)^{n+1-k}\psi(\alpha)^k\frac{(x-a)^{k\beta}}{\Gamma(k\beta+1)}\Bigg] \end{aligned}$$

and the proof is complete. □

As immediate consequences of our Taylor's theorem for generalized weighted fractional derivatives with a nonsingular kernel (Theorem 3), we obtain most fractional-order Taylor's formulas that exist in the literature.

Corollary 1 (Taylor's formula for the weighted ABC derivative [43])**.** *Let* ${}_a^C D_w^{k\alpha} f \in C([a,b])$, *where* $0 \le \alpha \le 1$ *and* $k = 0,1,\ldots,n+1$. *Then,*

$$\begin{aligned} f(x) &= \frac{1}{w(x)}\Bigg[w(a)\sum_{i=0}^{n}{}_a^C D_w^{i\alpha}f(a)\sum_{k=0}^{i}C_i^k\phi(\alpha)^{i-k}\psi(\alpha)^k\frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} \\ &\quad +w(\xi){}_a^C D_w^{(n+1)\alpha}f(\xi)\sum_{k=0}^{n+1}C_{n+1}^k\phi(\alpha)^{n+1-k}\psi(\alpha)^k\frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)}\Bigg] \end{aligned}$$

with $a \le \xi \le x$ *and* $x \in [a,b]$, *where* ${}_a^C D_w^{i\alpha} = {}_a^C D_w^{\alpha} \cdot {}_a^C D_w^{\alpha} \cdots {}_a^C D_w^{\alpha}$, *i-times.*

Proof. Choose $\alpha = \beta$ in Theorem 3. □

Corollary 2 (Taylor's formula for the ABC derivative [34])**.** *Let* ${}_a^{ABC}D^{k\alpha} f \in C([a,b])$ *with* $0 \le \alpha \le 1$ *and* $k = 0,1,\ldots,n+1$. *Then,*

$$\begin{aligned} f(x) = &\sum_{i=0}^{n}\left({}_a^{ABC}D^{i\alpha}f(a)\right)\sum_{k=0}^{i}C_i^k\phi(\alpha)^{i-k}\psi(\alpha)^k\frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} \\ &+\left({}_a^{ABC}D^{(n+1)\alpha}f(\xi)\right)\sum_{k=0}^{n+1}C_{n+1}^k\phi(\alpha)^{n+1-k}\psi(\alpha)^k\frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} \end{aligned}$$

with $a \le \xi \le x$ *and* $x \in [a,b]$, *where* ${}_a^{ABC}D^{i\alpha} = {}_a^{ABC}D^{\alpha} \cdot {}_a^{ABC}D^{\alpha} \cdots {}_a^{ABC}D^{\alpha}$, *i-times.*

Proof. Choose $\alpha = \beta$ and $w(x) \equiv 1$ in Theorem 3. □

Corollary 3 (Taylor's formula for the CF derivative [23])**.** *Let* ${}_a^{CF}D^{k\alpha} f \in C([a,b])$ *with* $0 \le \alpha \le 1$ *and* $k = 0,1,\ldots,n+1$. *Then,*

$$\begin{aligned} f(x) &= \sum_{i=0}^{n}\left({}_a^{CF}D^{i\alpha}f(a)\right)\sum_{k=0}^{i}C_i^k\phi(\alpha)^{i-k}\psi(\alpha)^k\frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} \\ &\quad +\left({}_a^{CF}D^{(n+1)\alpha}f(\xi)\right)\sum_{k=0}^{n+1}C_{n+1}^k\phi(\alpha)^{n+1-k}\psi(\alpha)^k\frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} \end{aligned}$$

with $a \le \xi \le x$ *and* $x \in [a,b]$, *where* ${}_a^{CF}D^{i\alpha} = {}_a^{CF}D^{\alpha} \cdots {}_a^{CF}D^{\alpha}$, *i-times.*

Proof. Choose $\alpha = \beta$, $w(x) \equiv 1$, and the RL fractional integral of order one in Theorem 3. □

Remark 1. *From the geometrical point of view, a Taylor approximation with two terms is a straight-line approximation, which is the tangent at the given point; once with three terms, Taylor's approximation is a parabola whose tangent and curvature are in accordance with the given function at the given point; etc. The same geometric interpretation is conserved in our case.*

4. An Illustrative Example

To illustrate our main result, we will choose function $f(x) = (x-1)^\gamma$ with γ a positive real number. Before that, we prove two useful technical lemmas.

Lemma 2. *The weighted generalized fractional derivative* $\mathfrak{D}_a^{[\alpha,\beta]} f(x)$ *can be expressed as*

$$\mathfrak{D}_a^{[\alpha,\beta]} f(x) = \frac{1}{\phi(\alpha)} \frac{1}{w(x)} \sum_{j=0}^{+\infty} (-\mu_\alpha)^j \left({}^{RL}I_{a,w}^{\beta j+1} (wf)'(x) \right). \tag{13}$$

Proof. Beginning with Definition 5, one has

$$\begin{aligned} \mathfrak{D}_a^{[\alpha,\beta]} f(x) &= \frac{1}{\phi(\alpha)} \frac{1}{w(x)} \int_a^x (wf)'(s) E_\beta[-\mu_\alpha (x-s)^\beta] ds \\ &= \frac{1}{\phi(\alpha)} \frac{1}{w(x)} \int_a^x (wf)'(s) \sum_{j=0}^{+\infty} (-\mu_\alpha)^j \frac{(x-s)^{\beta j}}{\Gamma(\beta j+1)} ds \\ &= \frac{1}{\phi(\alpha)} \frac{1}{w(x)} \sum_{j=0}^{+\infty} (-\mu_\alpha)^j \frac{1}{\Gamma(\beta j+1)} \int_a^x (wf)'(x-s)^{\beta j} ds \end{aligned}$$

and the intended relation (13) follows. □

The following lemma is given to handle our example adequately.

Lemma 3. *Let* $w(x) \equiv 1$*. The ith generalized fractional derivative* $\mathfrak{D}_a^{i[\alpha,\beta]} f(x)$*, where*

$$\mathfrak{D}_a^{i[\alpha,\beta]} = \mathfrak{D}_a^{[\alpha,\beta]} \cdots \mathfrak{D}_a^{[\alpha,\beta]}, \quad i\text{-times},$$

can be expressed as

$$\mathfrak{D}_a^{i[\alpha,\beta]} f(x) = \frac{1}{\phi(\alpha)^i} \sum_{r=0}^{+\infty} C_{r+i-1}^{i-1} (-\mu_\alpha)^r \left({}^{RL}I_{a,1}^{\beta r+1} f'(x) \right). \tag{14}$$

Proof. Using Lemma 2,

$$\begin{aligned} \mathfrak{D}_a^{i[\alpha,\beta]} f(x) &= \left[\frac{1}{\phi(\alpha)} \sum_{j=0}^{+\infty} (-\mu_\alpha)^j \left({}^{RL}I_{a,1}^{\beta j+1} \frac{d}{dx} \right) \right]^{(i)} f(x) \\ &= \frac{1}{\phi(\alpha)^i} \sum_{j_1,\ldots,j_i} (-\mu_\alpha)^{\sum j_k} \left({}^{RL}I_{a,1}^{\beta \sum j_k + 1} \frac{d}{dx} f(x) \right), \end{aligned}$$

which proves equality (14). □

We now apply our Theorem 3 in the case $\beta = \alpha$ and $f(x) = (x-1)^\gamma$, where γ is a positive real number. Using (14), we obtain that

$$\begin{aligned} (x-1)^\gamma = \sum_{i=0}^{n} \frac{1}{\phi(\alpha)^i} \sum_{r=0}^{+\infty} C_{r+i-1}^{i-1} (-\mu_\alpha)^r \left(\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha r+1)} (a-1)^{\gamma+\alpha r} \right) \\ \times \sum_{k=0}^{i} C_i^k \phi(\alpha)^{i-k} \psi(\alpha)^k \frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} \\ + \frac{1}{\phi(\alpha)^{n+1}} \sum_{r=0}^{+\infty} C_{r+n}^{n} (-\mu_\alpha)^r \left(\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha r+1)} (\xi-1)^{\gamma+\alpha r} \right) \\ \times \sum_{k=0}^{n+1} C_{n+1}^k \phi(\alpha)^{n+1-k} \psi(\alpha)^k \frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)}. \end{aligned} \tag{15}$$

5. Mean Value Theorems

As an application, we employ the obtained weighted Taylor's formula to establish an appropriate generalized mean value theorem for weighted generalized derivatives.

Theorem 4 (Generalized mean value theorem for the weighted generalized derivative)**.** *Suppose that $f \in \mathcal{C}([a,b])$ and $\mathfrak{D}_a^{[\alpha,\beta]} f \in \mathcal{C}([a,b])$ for $0 \le \alpha \le 1$. Then,*

$$f(x) = \frac{1}{w(x)}\left(w(a)f(a) + w(\xi)\mathfrak{D}_a^{[\alpha,\beta]} f(\xi)\left(\phi(\alpha) + \psi(\alpha)\frac{(x-a)^{\beta}}{\Gamma(\beta+1)}\right)\right)$$

for all $x \in [a,b]$ with $a \le \xi \le x$.

Proof. It follows by taking $n = 0$ in Theorem 3 and performing some direct calculations. □

As straight corollaries of our Theorem 4, we obtain mean value theorems for weighted ABC, ABC, and CF derivatives.

Corollary 4 (Generalized mean value theorems)**.** *Let $f \in \mathcal{C}([a,b])$.*

- *For the weighted ABC derivative: if ${}_a^C D_w^{\alpha} f \in \mathcal{C}([a,b])$ for $0 \le \alpha \le 1$, then*

$$f(x) = \frac{1}{w(x)}\left(w(a)f(a) + w(\xi){}_a^C D_w^{\alpha} f(\xi)\left(\phi(\alpha) + \psi(\alpha)\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}\right)\right)$$

  for all $x \in [a,b]$ with $a \le \xi \le x$.
- *For the ABC derivative: if ${}_a^{ABC} D^{\alpha} f \in \mathcal{C}([a,b])$ for $0 \le \alpha \le 1$, then*

$$f(x) = f(a) + {}_a^{ABC} D^{\alpha} f(\xi)\left(\phi(\alpha) + \psi(\alpha)\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}\right)$$

  for all $x \in [a,b]$ with $a \le \xi \le x$.
- *For the CF derivative: if ${}_a^{CF} D^{\alpha} f \in \mathcal{C}([a,b])$ for $0 \le \alpha \le 1$, then*

$$f(x) = f(a) + {}_a^{CF} D^{\alpha} f(\xi)(\phi(\alpha) + \psi(\alpha)(x-a))$$

  for all $x \in [a,b]$ with $a \le \xi \le x$.

Remark 2. *Note that the classical mean value theorem is obtained from Theorem 4 by choosing $w(x) \equiv 1$ and $\alpha = \beta = 1$; from Corollary 4 by choosing $w(x) \equiv 1$ and $\alpha = 1$ for the weighted ABC derivative; and by choosing $\alpha = 1$ for the ABC and CF derivatives.*

6. Conclusions

In this work, a weighted Taylor's formula for nonsingular kernels, valid for weighted generalized fractional derivatives under some justified prerequisites, was proven. As a result, we obtained various theoretical consequences, one of them being generalized mean value theorems which extended those available in the literature. We claim that our generalized Taylor's Formula (12) has great potential for the development of mathematical modeling with fractional nonsingular kernel derivatives. As a perspective, we plan to use our results to linearize some nonlinear weighted generalized fractional dynamical systems. This is under investigation and will be addressed elsewhere.

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