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Representações fiéis de C-grupos por permutações

Faithful permutation representations of C-groups

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Faithful permutation representations of C-groups

Tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutoramento em Matemática Aplicada, realizada sob orientação científica de Maria Elisa Carrancho Fernandes, Professora Auxiliar do Departamento de Matemática da Universidade de Aveiro.

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An expert problem solver must be endowed with two incompatible qualities
- a restless imagination and a patient pertinacity.

- Howard W. Eves

o júri

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agradecimentos

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palavras-chave

Polítopos; Hipertopos; Polítopos Abstratos Regulares; Mapas Toroidais; Hipermapas; Representações Fiéis por Permutações; Teoria de Grupos; Simetrias.

resumo

O estudo de objectos simétricos, como o caso dos polítopos, e das suas simetrias é um assunto que atrai a atenção de diferentes subáreas da Matemática, tal como Geometria e Álgebra, mas também noutras áreas do conhecimento científico, nomeadamente na Química, pela existência de um elevado grau de simetria nas moléculas. Os polítopos clássicos, a generalização dos polígonos e poliédros para outras dimensões, podem ser analisados do pontos de vista combinatório, dando origem aos polítopos abstratos. Os polítopos abstratos regulares podem ser descritos de diferentes formas, tais como um conjunto parcialmente ordenado, uma geometria de incidência ou ainda pelo seu grupo de simetrias, que é um C-grupo com diagrama linear. Os hipertopos foram introduzidos como uma estrutura semelhante aos polítopos, mas em que o seu grupo de simetrias é um C-grupo com diagrama não necessariamente linear.

O problema de Grünbaum, um dos problemas clássicos da teoria dos politopos abstratos, ainda não totalmente resolvido, consiste na classificação de politopos localmente toroidais. Este problema é extensível a hipertopos de dimensão 4 com resíduos toroidais de dimensão 3. Hipertopos localmente toroidais são construidos a partir de hipermapas regulares toroidais $\{4,4\}$, $\{6,3\}$, $\{3,6\}$ ou $(3,3,3)$. Pelo Teorema de Cayley, sabemos que qualquer grupo pode ser representado fielmente por um grupo de permutações. Assim podemos construir representações fiéis por grupos de permutações dos grupos destes hipermapas, que podem ser usados tanto para classificar politopos localmente toroidais como para construir novos politopos/hipertopos com resíduos toroidais.

Nesta tese determinam-se todos os possíveis graus de representações fiéis por permutações dos hipermapas toroidais regulares e dos polítopos localmente toroidais do tipo $\{4,4,4\}$. A partir destas representações fiéis por permutações, famílias de hipertopos localmente toroidais de tipo $\{6, {}^{3}_{3}\}$ e $\{3, {}^{4}_{4}\}$ são construídas. Adicionalmente, usando a operação de *halving* em politopos não degenerados $2^{\mathcal{K},G(s)}$, foi possível construir exemplos de famílias infinitas de hipertopos regulares localmente toroidais, euclidianos e hiperbólicos.

keywords

Polytopes; Hypertopes; Abstract Regular Polytopes; Toroidal Maps; Hypermaps; Faithful Permutation Representations; Group Theory; Symmetries.

abstract

The study of regular objects, such as polytopes, and their symmetries is a subject that attracts researchers from different areas of mathematics, such as geometers and algebraists, but also researchers from other areas of knowledge such as chemistry, thanks to the high symmetry of the molecules. An abstract polytope is a structure that combinatoricaly describes a classical polytope (a generalization of polygons and polyhedra to higher dimensions). Abstract regular polytopes can be described as a poset, as an incidence geometry or as C-group with linear diagram. A hypertope was introduced as a polytope-like structure where its group of symmetries is a C-group however it does not need to have a linear diagram.

Grünbaum's problem, one of the classical problems of the theory of abstract polytopes, not yet completely solved, consists in the classification of locally toroidal polytopes. The problem is extensible to hypertopes of rank 4 with toroidal rank 3 residues. Locally toroidal hypertopes are constructed from toroidal regular hypermaps $\{4,4\}$, $\{6,3\}$, $\{3,6\}$ or $(3,3,3)$. The groups of these toroidal regular hypermaps can be represented as faithful transitive permutation representation graphs, which can be then used either to classify locally toroidal polytopes or to construct new polytopes/hypertopes with toroidal residues.

In this thesis, a classification of all the possible degrees of faithful transitive permutation representations of the toroidal regular hypermaps and of the locally toroidal regular polytopes of type $\{4,4,4\}$ is given. With these faithful transitive permutation representations, families of locally toroidal hypertopes of types $\{6,\substack{3\\3}\}$ and $\{3,\substack{4\\4}\}$ are constructed. Additionally, using the halving operation on the non-degenerate polytopes $2^{\mathcal{K},G(s)}$, examples of infinite families of regular hypertopes of locally toroidal, euclidean or hyperbolic type are obtained.

Contents

List of Symbols

$\lvert G\rvert$	Order of the group G
$o(g)$	Order of the element g
$\langle X\rangle$	(Sub)Group generated by the set X
$\langle X\lvert R\rangle$	Group presentation
G/H	Set of (right/left) cosets of H in G
$\lvert G:H\rvert$	Index of H in G
$Stab_G(x)$	Stabilizer of the element x by G
$Orb_G(x)$	Orbit of the element x by G
$Stab_G(X)$	Stabilizer of the set X in G
G_i	Subgroup of $G = \langle \rho_0, \ldots, \rho_n\rangle$ generated by $\{\rho_k \lvert k \neq i\}$
id , id_G	Identity (in group G)
x^y	The element $y^{-1}xy$
H^y	The set $y^{-1}Hy$
$\rightarrow$	Mapping
$\mapsto$	Effect of mapping on an element
S_n	Symmetric group of degree n
S_Ω	Symmetric group on the set Ω
A_n	Alternating group of degree n
D_n	Dihedral group of order $2n$
C_n	Cyclic group of order n
$Aut(\mathcal{P})$	Automorphism group of $\mathcal{P}$
$\times$	Direct product
$\rtimes$	Semidirect product
$\wr$	Wreath product

$\lhd$, $\unlhd$	(Proper) Normal subgroup of
$\overline{N}$	Normal closure of subgroup N in parent group
$\cong$	Isomorphic to
$lcm(a,b)$	Least common multiple between a and b
$gcd(a,b)$	Greatest common divisor between a and b
$a \equiv b \text{ mod } n$	$a - b$ is a multiple of n
$a \mid b$	b is a multiple of a
$:=$	Equal by definition
$\mathcal{G}_J$	Spanning subgraph of a graph $\mathcal{G}$ whose edges have labels in J
$\square$	End of proof

List of Acronyms

FTPR	Faithful Transitive Permutation Representation
ggi	group generated by distinguished involutions
sggi	string group generated by distinguished involutions
poset	partially ordered set

Chapter 1

Introduction

1.1 Overview and Motivation

The classification of the finite simple groups was one of the biggest milestones achieved in group theory in the twentieth century. Groups can be seen in a more abstract fashion or as a set of permutations on a set of points, particularly the groups of symmetries of geometric objects. Classifying highly symmetric structures in which groups act has also been an active research topic of mathematics, including geometry. Cayley's theorem states that any group is isomorphic to a subgroup of the symmetric group. This allows us to see a group as a permutation group of a certain degree n, i.e. there exists always a faithful permutation representation of a group acting on a set of size n. However, this faithful permutation representation is not uniquely determined. For instance, the group of automorphisms of the cube, which is isomorphic to $S_4 \times C_2$, has a faithful transitive permutation representation on 8 points, with generators $\rho_0 = (1,2)(3,4)(5,6)(7,8)$, $\rho_1 = (2,3)(6,7)$ and $\rho_2 = (3,5)(4,6)$; and a faithful intransitive permutation representation on 6 points, with generators $\rho_0 = (1,2)(3,4)(5,6)$, $\rho_1 = (2,3)(5,6)$ and $\rho_2 = (3,4)(5,6)$. The number of points in which a faithful intransitive permutation representations acts on can always be increased indefinitely by repeating the actions on copies of the transitive parts. Libraries of finite groups, particularly of automorphism groups of highly symmetric structures such as polytopes, regular maps and hypermaps, usually give one faithful permutation representation of the group [LV06; Har06] which is transitive and usually of minimal degree.

Abstract Polytopes and their theory is a well studied subject of mathematics and intimately connected with geometry and group theory, specifically Coxeter groups. These groups can be represented by a diagram, where the set of vertices are the generators of the group and with an edge between two generators if the order of the product of these is greater or equal to 3. Smooth quotients of Coxeter groups, that is, factorizations that preserve the diagram are designated by C-groups ("C" stands for Coxeter). Finite Coxeter groups and C-groups with linear diagram are the automorphism groups of finite abstract regular polytopes. A classical example of abstract regular polytopes are the toroidal regular polytopes (or toroidal regular maps), which are regular tilings on the surface of a torus by either squares, triangles or hexagons.

As refered before, some of the atlas of regular maps, hypermaps and polytopes use faithful permutation representations of their automorphism groups. As observed above, these representations are not unique. This thesis is the starting point to the study of faithful transitive permutation representations of the automorphism groups of maps, hypermaps, polytopes and hypertopes. Some faithful transitive permutation representations were of great importance to the classification of polytopes with the automorphism group being the symmetric groups [FL11] or the alternating groups [FL19; FLM12a; FLM12b; CFLM17]. The knowledge of faithful transitive permutation representations of C-groups associated with the toroidal maps seemed like a promissing way to contribute to the classification of locally toroidal polytopes, known as Grünbaum's problem.

We can tile the surface of the torus using a regular *hyper*graph, usually called a toroidal *hyper*map. The automorphism group of this tiling is a Coxeter group however it does not have a linear diagram, contrarily to the group of an abstract regular polytope. Of course most Coxeter groups do not have a linear diagram. Some of these Coxeter groups had already been studied associated with structures such as hypermaps and semi-regular polytopes. Hence, it makes sense not to restrict our study to symmetric structures with Coxeter group having linear diagram. Having this idea in mind, a new theory of highly symmetric structures, called hypertopes, started recently. As you will see, these structures are not as well-behaved as abstract regular polytopes. Nevertheless, this generalization allows to study all the Coxeter groups (and their quotients) associated with symmetric structures not yet known.

The theory of hypertopes is still in its starting stage, having a lot of unanswered questions and being a fertile area of research. As in the case of abstract regular polytopes, we are interested in the classifications of regular hypertopes when a Coxeter group is specified. Specifically we can extend Grünbaum's problem to these structures. Using faithful transitive permutation representations of the C-groups of toroidal maps, we are able to build locally toroidal hypertopes. In fact, this idea was first used in [FLW15], where families of regular hypertopes were constructed. More recently, the regular hypertopes of locally spherical type were characterized in [FLW20], generalizing the concept of spherical, euclidean and hyperbolic type. In [FLW20] some computational examples of hypertopes of hyperbolic and euclidean type were given and later some families of these types were characterized [MW20; MW21; Ens18].

The main results of this thesis are the classification of all the possible degrees of faithful transitive permutation representations of the toroidal regular hypermaps and of the locally toroidal regular polytopes $\{4, 4, 4\}$. Another result is the characterization and construction of new families of hypertopes, for which previous faithful transitive permutation representations were used.

1.2 Organization of the Thesis

This thesis is organized into ten chapters, including this introductory chapter, where an overview of the thematic of this thesis was provided, and a conclusion, which indicates a future research based on the topics presented here.

In Chapter 2, a theoretical introduction to the concepts of abstract regular polytopes, C-groups, Coxeter groups and faithful transitive permutation representations is given, which are the base to all chapters in this thesis. Moreover, the concept of diagonals and centrally symmetric polytopes is provided, which will come to great importance in Chapter 9, when building polytopes $2^{\mathcal{K},G(s)}$.

Chapter 3 is dedicated to toroidal regular maps $\{4, 4\}$, $\{3, 6\}$ and $\{6, 3\}$, which are abstract regular polytopes, and to toroidal regular hypermap $(3,3,3)$. After introducing these toroidal regular maps and hypermaps, in Chapter 4 we characterize all possible degrees of faithful transitive permutation representations of the automorphism groups of these maps and hypermaps, giving examples of faithful transitive permutation representations graphs.

Having in mind the results of Chapter 4, we move to the faithful transitive permutation representations of locally toroidal regular polytopes of rank 4 with toroidal residues of type $\{4,4\}$. The list of all degrees of these faithful transitive permutation representations will be given in Chapter 5 for the known finite universal locally toroidal regular polytopes of type $\{4,4,4\}$.

The concept of a hypertope is introduced in Chapter 6, standing as a combinatorial structure for which its automorphism group does not have necessarily a linear Coxeter diagram. The definition of incidence systems, geometries and coset geometries will be needed to introduce hypertopes. In addition, an extension of the definitions of (locally) spherical and toroidal polytopes will be done for regular hypertopes. With the previous information, we will be able to construct families of regular hypertopes of locally toroidal (Chapters 7, 8 and 9), euclidean type and hyperbolic type (both Chapter 9).

To read this thesis, the order presented is advised, serving the following chart as a guide for each chapter's dependencies.

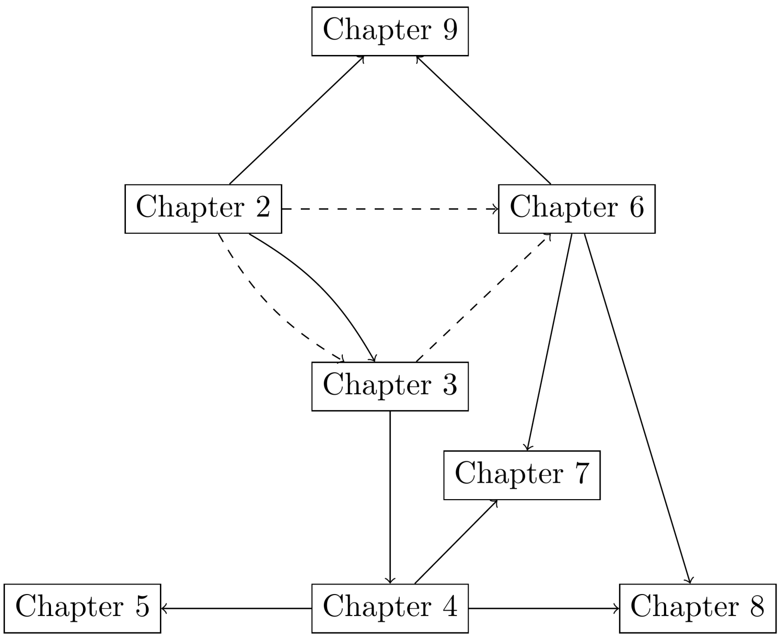


Figure 1.1: Chapter's dependencies. The dashed arrows represent the dependencies of the theoretical chapters.

Chapter 2

Abstract Regular Polytopes

The theory of *abstract regular polytopes* as a combinatorial description of a classical polytope was introduced by McMullen and Schulte on several works both published and collected on a book of the same name [MS02]. In this chapter, the definition of an abstract (regular) polytope will be given, focussing on the automorphism group (which is a C-group) and on its faithful permutation representations which can be described by graphs.

2.1 Abstract Polytopes

An *abstract n-polytope* $\mathcal{P}$ (or, for short in this thesis, a n-polytope) is a ranked partially ordered set (*poset*) of *faces* (a face $F \in \mathcal{P}$ with rank i, rank(F)= i is called an i-face) and follows four defining properties:

(P1) $\mathcal{P}$ contains two improper faces, a least facet F_{-1} of rank -1, and a greatest facet F_n of rank n;

(P2) Each *flag* (i.e. a maximal totally ordered subset) of $\mathcal{P}$ contains $n+2$ faces (including the two improper faces);

(P3) $\mathcal{P}$ is strongly connected (see below Definition 2.1.1);

(P4) $\mathcal{P}$ must satisfy the diamond condition, i.e. for any $F, G \in \mathcal{P}$, whenever $F < G$ with rank(F)=$i-1$ and rank(G)=$i+1$, there is *exactly two* i-faces H, such that $F < H < G$.

Let $F, G \in \mathcal{P}$ be faces of $\mathcal{P}$. We say that F and G are *incident* if $F \leq G$ or $G \leq F$. Two faces F and G with $F \leq G$ determine a *section* of $\mathcal{P}$, defined by $G/F := \{H | H \in \mathcal{P}, F \leq H \leq G\}$.

Definition 2.1.1. [MS02, Connected and Strongly Connected] Let $\mathcal{P}$ be a partially ordered set with properties (P1) and (P2). We say $\mathcal{P}$ is *connected* if either $n \leq 1$, or $n \geq 2$ and for any two *proper* faces F and G of $\mathcal{P}$ there is a sequence of proper faces $F = H_0, H_1, \ldots, H_{k-1}, H_k = G$ such that H_{i-1} and H_i are incident for $i = 1, \ldots, k$.

We say that $\mathcal{P}$ is *strongly connected* if for each section of $\mathcal{P}$ (including the polytope $\mathcal{P}$ itself), that section is connected.

Two flags of a n-polytope $\mathcal{P}$ are said to be *adjacent* if they differ in exactly one face. If Φ is a flag of $\mathcal{P}$, the diamond condition tells us that for $i = 0, 1, \ldots, n-1$ there is exactly one flag that differ from Φ only in the i-face. This flag is denoted as Φ^i and is said to be *i-adjacent* to Φ. Note that $(\Phi^i)^i = \Phi$ for each i and $(\Phi^i)^j = (\Phi^j)^i$ if $|i-j| \geq 2$.

We say that $\mathcal{P}$ is *flag-connected* if, for any two distinct flags Φ and Ψ of $\mathcal{P}$, there is a sequence of flags

$$\Phi = \Phi_0, \Phi_1, \ldots, \Phi_{k-1}, \Phi_k = \Psi$$

from Φ to Ψ such that, for $j \in \{1, \ldots, k\}$, Φ_{j-1} and Φ_j are adjacent. Similarly to the Definition 2.1.1, we say that $\mathcal{P}$ is *strongly flag-connected* if each section of $\mathcal{P}$ (including the polytope $\mathcal{P}$ itself) is flag-connected. This leads to the following result.

Proposition 2.1.2. *[MS02, Proposition 2A1] Let $\mathcal{P}$ be a poset with properties (P1) and (P2). Then $\mathcal{P}$ is strongly connected if and only if it is strongly flag-connected.*

This allows us to change the property (P3) to an equivalent one, using flag-connectedness instead

(P3*) $\mathcal{P}$ is strongly flag-connected.

We say a n-polytope $\mathcal{P}$ (with $n \geq 2$) is *equivelar* if for each $i = 1, 2, \ldots, n-1$ there is an integer p_i such that any section G/F defined by an $(i-2)$-face F and an $(i+1)$-face G is a p_i-gon. Then the *(Schläfli) type* of $\mathcal{P}$ is $\{p_1, p_2, \ldots, p_{n-1}\}$.

We say a poset is a *lattice* if, for every two faces $F, G \in \mathcal{P}$, there is a least upper bound and a greatest lower bound for $\{F, G\}$. Whenever the partial order induces a lattice, we will say that $\mathcal{P}$ is *non-degenerate*, otherwise it is *degenerate* [Sch85; Dan84].

As an example, consider the cube, with labelled vertices $\{1, \ldots, 8\}$ as in Figure 2.1. Let an edge be denoted by $[i, j]$, where i and j are distinct pair of vertices that are

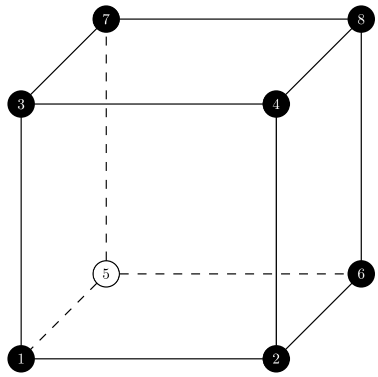


Figure 2.1: A cube with labelled vertices.

incident to it; and consider the faces, as in the classical sense, denoted as $[i, j, k, l]$ where i, j, k and l are distinct vertices incident to it. Then, we can build the following poset in Figure 2.2. It is easy to see that we have a rank -1 element ($\emptyset$) and a rank 3 element $[1, 2, 3, 4, 5, 6, 7, 8]$, i.e. the empty set and the whole cube, respectively. All flags of the poset have 5 elements. All sections of the poset are connected (as defined in Definition 2.1.1), including the whole poset. Finally, any two faces[1] of the poset with

[1]Not to be confused with faces of the cube.

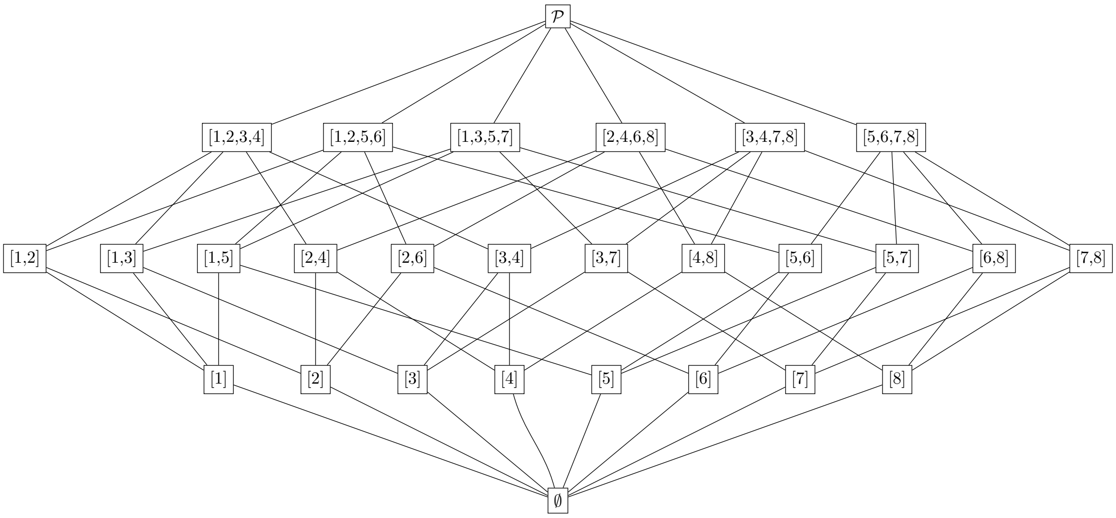


Figure 2.2: Poset of a cube.

rank $i-1$ and $i+1$ (for $i \in \{0,1,2\}$), results in a section with exactly two i faces. Hence, we can clearly see that the cube is an abstract polytope[2].

2.2 Abstract Regular Polytopes

Consider the mapping $\psi : \mathcal{P} \to \mathcal{Q}$ between two polytopes $\mathcal{P}$ and $\mathcal{Q}$. We say that ψ is a *homomorphism* if it *preserves incidence*, i.e. for all $F, G \in \mathcal{P}$ such that $F \leq G$, we have that $F\psi \leq G\psi$ in $\mathcal{Q}$. Moreover, ψ is an *isomorphism* between $\mathcal{P}$ and $\mathcal{Q}$ if ψ is a bijection where both ψ and ψ^{-1} are homomorphisms. If there is an isomorphism between $\mathcal{P}$ and $\mathcal{Q}$, we say that these polytopes are *isomorphic* and we write $\mathcal{P} \cong \mathcal{Q}$. Furthermore, if ψ is an isomorphism then it preserves adjacency, i.e. $\Phi^j\psi = (\Phi\psi)^j$, for any flag Φ of $\mathcal{P}$ and $j \in \{0, \ldots, n-1\}$.

An *automorphism* of $\mathcal{P}$ is an isomorphism from $\mathcal{P}$ to itself and the set of all automorphisms of $\mathcal{P}$ form a group, the *automorphism group* of $\mathcal{P}$ (or simply the *group* of $\mathcal{P}$), denoted by $\Gamma(\mathcal{P})$. Moreover, it is easy to see that if $\mathcal{P}$ is a finite n-polytope, then $\Gamma(\mathcal{P})$ is also finite.

The *dual polytope* $\mathcal{P}^*$ of $\mathcal{P}$is the one obtained by inverting the partial order of $\mathcal{P}$, and if the latter has type $\{p_1, p_2, \ldots, p_{n-2}, p_{n-1}\}$, then its dual $\mathcal{P}^*$ has type $\{p_{n-1}, p_{n-2}, \ldots, p_2, p_1\}$. A bijection $\phi : \mathcal{P} \to \mathcal{P}^*$ that invertes the partial order is called a *duality*.

Consider the set of all flags of $\mathcal{P}$, denoted as $\mathcal{F}(\mathcal{P})$. The group $\Gamma(\mathcal{P})$ acts freely on $\mathcal{F}(\mathcal{P})$, i.e. for any $\psi, \phi \in \Gamma(\mathcal{P})$ if for a flag $\Phi \in \mathcal{F}(\mathcal{P})$ we have $\psi(\Phi) = \phi(\Phi)$, then $\psi = \phi$. Moreover, we have that the order of $\Gamma(\mathcal{P})$ divides $|\mathcal{F}(\mathcal{P})|$ and, if $|\Gamma(\mathcal{P})| = |\mathcal{F}(\mathcal{P})|$, then $\Gamma(\mathcal{P})$ also acts transitively on the flags (for all two flags $\Phi, \tilde{\Phi} \in \mathcal{F}(\mathcal{P})$ there is a automorphism $\psi \in \Gamma(\mathcal{P})$ such that $\psi(\Phi) = \tilde{\Phi}$) , i.e. $\Gamma(\mathcal{P})$ is regular. In this circumstances, $\mathcal{P}$ is said to be *regular*. Consider the following proposition, where a different definition of an abstract regular polytope is given.

[2]The poset presented is usually called the face-lattice of the cube since this poset is actually a lattice.

Proposition 2.2.1. *[MS02, Proposition 2B4] An n-polytope $\mathcal{P}$ is regular if and only if, for some flag $\Phi = \{F_{-1}, F_0, F_1, \ldots, F_{n-1}, F_n\}$ of $\mathcal{P}$ and each $j \in \{0, \ldots, n-1\}$, there exists a (unique) involutory automorphism ρ_j of $\mathcal{P}$ such that $\Phi\rho_j = \Phi^j$.*

Consider a flag Φ and the involutory automorphism ρ_j such that $\Phi\rho_j = \Phi^j$. Then there is an automorphism $\varphi \in \Gamma(\mathcal{P})$ such that, for another flag $\tilde{\Phi}$, the involutory automorphism $\tilde{\rho}_j = \varphi^{-1}\rho_j\varphi$ is such that $\tilde{\Phi}\tilde{\rho}_j = \tilde{\Phi}^j$. That is, the involutory automorphisms corresponding to other flags (not Φ) are conjugates of the ones of flag Φ. Hence, we can fix on one flag

$$\Phi := \{F_{-1}, F_0, \ldots, F_n\}$$

which we will call a *base flag*. So, whenever the elements $\rho_0, \rho_1, \ldots, \rho_{n-1}$ (called *distinguished generators* of $\Gamma(\mathcal{P})$) are used, a base flag has been implicitly been chosen. Moreover, the group $\Gamma(\mathcal{P}) = \langle \rho_0, \rho_1, \ldots, \rho_{n-1} \rangle$ and, if $\mathcal{P}$ is a polytope of type $\{p_1, p_2, \ldots, p_{n-1}\}$, for all $j \in \{1, \ldots, n-1\}$ we have that the order of $\rho_{j-1}\rho_j$ is equal to p_j of the polytope's type. Furthermore, for $|j-k| \geq 2$ the order of $\rho_j\rho_k$ is 2, meaning that generators ρ_j and ρ_k commute with each other. This is a consequence of $(\Phi^j)^k = (\Phi^k)^j$) if $|j-k| \geq 2$.

The *Coxeter diagram* is a graph whose nodes represent the distinguished generators of a group generated by involutions and two generators ρ_j and ρ_k are connected if $o(\rho_j\rho_k) \geq 3$. Moreover, we write a label on the edges if $o(\rho_j\rho_k) \geq 4$. For polytopes, this diagram is a string, like the one represented below.

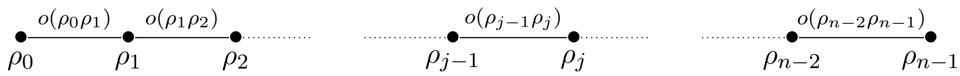


Figure 2.3: Coxeter diagram of an abstract n-regular polytope.

Since $\Gamma(\mathcal{P})$ is generated by involutions and has the property of two non-consecutive generators commuting with each other, it is said to be a *string group generated by involutions* (or *sggi* for short).

A *C-group* is a group generated by involutions (ggi) that satisfies the *intersection property*, i.e. for a group $\langle \rho_0, \rho_1, \ldots, \rho_{n-1} \rangle$ and $I, J \subseteq \{0, 1, \ldots, n-1\}$ we have that

$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle. \tag{2.1}$$

The automorphism group of $\mathcal{P}$ satisfies the intersection property, hence it is a C-group. In addition, it has a string Coxeter diagram, thus it is a called a *string C-group*. Later we will consider C-group without a string Coxeter diagram when studying hypertopes (see Chapter 6)

2.3 From a string C-group to an Abstract Regular Polytope

Let Γ be a string C-group. Let us define for each $k \in \{0, \ldots, n-1\}$ the subgroups of Γ in the following way.

$$\Gamma_k := \langle \rho_j \mid j \neq k \rangle, \tag{2.2}$$

usually designated as the *maximal parabolic subgroups* of Γ. We will also denote as $\Gamma_{k,m}$ the subgroup $\langle \rho_j \,|\, j \notin \{k, m\} \rangle$. Since for any $i, j \in \{0, \ldots, n-1\}$ such that $|i-j| \geq 2$ we have that generators ρ_i and ρ_j commute, then $\Gamma_k \cong \langle \rho_0, \ldots, \rho_{k-1} \rangle \times \langle \rho_{k+1}, \ldots, \rho_{n-1} \rangle$

for $k \in \{1, \ldots, n-2\}$. As Γ satisfies the intersection property, $\{\rho_i \mid i \in \{0, \ldots, n-1\}\}$ is an independent generating set, which means that $\rho_j \notin \Gamma_j$, for all $j \in \{0, \ldots, n-1\}$. Particularly, all Γ_j are distinct from one another and distinct from Γ itself.

We will construct an abstract regular polytope from the string C-group Γ. For each $j \in \{0, \ldots, n-1\}$, consider the set of right cosets of Γ_j, which we will call the j-faces. Moreover, consider $\Gamma_{-1} = \Gamma_n = \Gamma$. Furthermore, let us define a partial order between cosets. We say two cosets $\Gamma_j\phi$ and $\Gamma_k\psi$ (for $\phi, \psi \in \Gamma$) are incident $\Gamma_j\phi \leq \Gamma_k\psi$ if and only if $-1 \leq j \leq k \leq n$ and $\Gamma_j\phi \cap \Gamma_k\psi \neq \emptyset$. We will denote a poset built using this construction as $\mathcal{P}(\Gamma)$.

Theorem 2.3.1. *[MS02, Theorem 2E11] Let $n \geq 1$, and let $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ be a string C-group and $\mathcal{P} := \mathcal{P}(\Gamma)$ the corresponding poset. Then $\mathcal{P}$ is a regular n-polytope such that $\Gamma(\mathcal{P}) = \Gamma$.*

It is important to note that string C-groups and abstract regular polytopes are in one-to-one correspondence, as expressed in the following corollary.

Corollary 2.3.2. *[MS02, Corollary 2E13] The string C-groups are precisely the groups of abstract regular polytopes.*

With this correspondence, we will be working oftenly with the string C-groups instead of considering the poset that was firstly introduced for an abstract regular polytope. Hence, it will be important to prove in some situations that the group we are working with is a (string) C-group. The following result will help with this.

Proposition 2.3.3. *[FL18, Proposition 6.1] Let Γ be a group generated by n involutions $\rho_0, \ldots, \rho_{n-1}$. Suppose that Γ_i is a C-group for every $i \in \{0, \ldots, n-1\}$. Then Γ is a C-group if and only if $\Gamma_i \cap \Gamma_j = \Gamma_{i,j}$ for all $0 \leq i, j \leq n-1$.*

2.4 Coxeter groups

A *Coxeter group* W is a group with the presentation

$$W := \langle \sigma_1, \sigma_2, \ldots, \sigma_n \mid i, j \in \{1, \ldots, n\} \ : \ (\sigma_i\sigma_j)^{m_{ij}} = id_W \rangle,$$

where $m_{ii} = 1$ and $m_{ij} = m_{ji} \geq 2$, for $i \neq j$. This imposes that for all $i \in \{1, \ldots, n\}$ we have $(\sigma_i\sigma_i)^1 = \sigma_i^2 = id_W$, meaning all generators of the group are involutions. Moreover, if for some $i, j \in \{1, \ldots, n\}$ we have $m_{ij} = 2$, then σ_i commutes with σ_j. These groups satisfy the intersection property (2.1) and have associated the Coxeter diagram as defined previously. The relations corresponding to the definition of the Coxeter diagram are precisely the ones defining the group. We say a Coxeter group is *irreducible* or *reducible* depending on whether its Coxeter diagram is connected or disconnected, respectively. If a Coxeter group is reducible, then it is isomorphic to the direct product of its connected components.

In fact, C-groups are factorizations of Coxeter groups preserving the Coxeter diagram structure and the intersection property. Whenever we say the Schläfli type of a C-group, we are in fact referring to the parent Coxeter group, which we factorized by extra relations. Moreover, Coxeter groups with string Coxeter diagram are in one-to-one correspondence with an abstract regular polytope. Coxeter groups associated with abstract regular polytopes of Schläfli type $\{p_1, \ldots, p_{n-1}\}$ are usually denoted as $[p_1, \ldots, p_{n-1}]$.

Let W be a irreducible Coxeter group and $A := (\alpha_{ij})$ be the $n \times n$ symmetric matrix where its entries are $\alpha_{ij} = -cos(\pi/m_{ij})$. Whenever $det(A) > 0$, we have that W is finite. All possible finite Coxeter groups are listed in Table 6.1 and are said to be of *spherical type*. Among these, only those with string diagram are automorphism groups of abstract regular polytopes, which are precisely the groups of the convex regular polytopes.

When we have $det(A) = 0$, then W is not finite, however these groups have a normal abelian subgroup, which acts as a translation subgroup of W, such that the quotient of W by these subgroups give a group which is finite. These Coxeter groups W are said to be *affine*, or alternatively, of *euclidean type* since they are the automorphism groups of tesselations of the Euclidean space. The list of all Coxeter groups of Euclidean type is given in Table 6.2, where those with string diagram are automorphism groups of infinite regular polytopes corresponding with tesselations of the Euclidean space.

Finally, when $det(A) < 0$, there is a larger variety of Coxeter groups, known as Coxeter groups of *hyperbolic type*. For these, we will introduce the notion of locally spherical/toroidal. We say that W is *locally spherical* (or *locally finite*) if all its maximal parabolic subgroups have Coxeter diagrams whose connected components are of spherical type. We say that W is *locally toroidal* if all its maximal parabolic subgroups are either of spherical type or euclidean type, but with at least one of them of euclidean type. Notice that all Coxeter groups of euclidean type are locally spherical and not locally toroidal. Locally toroidal polytopes (and hypertopes) will encompass a great deal of this thesis.

2.5 Diagonals and Central Symmetry

Let $\mathcal{P}$ be an abstract regular n-polytope and $F_A, F_B \in \mathcal{P}$ be distinct vertices (0-faces) of $\mathcal{P}$. The unordered pair of vertices $\{F_A, F_B\}$ is called a *diagonal* of $\mathcal{P}$. Two diagonals $\{F_A, F_B\}$ and $\{F_C, F_D\}$, with $F_C, F_D \in \mathcal{P}$, are said to be *equivalent* if there is some $\sigma \in \Gamma(\mathcal{P})$ such that $\{F_C, F_D\} = \{F_A\sigma, F_B\sigma\}$. Thus, the diagonals of $\mathcal{P}$ form equivalence classes, called *diagonal classes*. Since these vertices can be represented as right cosets of Γ_0, we can write a diagonal as $\{\Gamma_0\phi, \Gamma_0\psi\}$, where $\Gamma_0\phi = F_0\phi = F_A$, $\Gamma_0\psi = F_0\psi = F_B$, $F_A \neq F_B$ and $\phi, \psi \in \Gamma(\mathcal{P})$. Moreover, due to the transitivity of $\Gamma(\mathcal{P})$, we can think of the diagonal classes by their representative $\{\Gamma_0, \Gamma_0\sigma\}$ for $\sigma \notin \Gamma_0$, where we fix one of the vertices as Γ_0. In this case, two diagonals $\{F_0, F_A\} = \{\Gamma_0, \Gamma_0\phi\}$ and $\{F_0, F_B\} = \{\Gamma_0, \Gamma_0\psi\}$ are equivalent under $\Gamma(\mathcal{P})$ if and only if

$$\psi \in \Gamma_0\phi\Gamma_0 \cup \Gamma_0\phi^{-1}\Gamma_0, \tag{2.3}$$

for $\phi, \psi \notin \Gamma_0$. If the polytope is realizable in an Euclidean space, the diagonal classes can be ordered by the distance between their representative vertices. For instance, the edges (the 1-faces) of the polytope $\mathcal{P}$ form a diagonal class.

An abstract regular polytope $\mathcal{P}$ is said to be *centrally symmetric* if its automorphism group $\Gamma(\mathcal{P})$ has a proper central involution α which is fixed-point free on its vertices. A pair of vertices of a centrally symmetric polytope is *antipodal* if they are permuted by this central involution. In the diagonal classes of a centrally symmetric polytope, there will be a diagonal class of all the pairs of antipodal points, with representative $\{\Gamma_0, \Gamma_0\alpha\}$.

2.6 Faithful Transitive Permutation Representation Graphs

Let Γ be a group and let X be a set. A *permutation representation* of Γ on X is a homomorphism π from Γ to the symmetric group of X, i.e.

$$\pi : \Gamma \to Sym(X).$$

The image of the representation $\pi(\Gamma)$, which is isomorphic to Γ, is a permutation group hence the elements of Γ can be represented as permutations of elements of X. This gives a natural action of Γ on the set X

$$\Gamma \times X \to X.$$

If $|X| = d$, we say that Γ has a *permutation representation of degree d.*

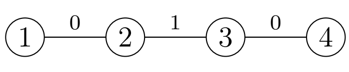
A permutation representation is *faithful* if Γ acts faithfully on X, i.e. the identity is the unique element of Γ fixing all elements of X. We say that a permutation representation is *transitive* if the group action is transitive on X. Let $\Lambda \leq \Gamma$ and consider the permutation representation given by the action of Γ on the coset space of Λ. This action is transitive and, if Λ is a core-free subgroup of Γ, this action is also faithful. In addiction, given a faithful transitive action of Γ on a set X, the stabilizer of any point is a core-free subgroup of Γ, giving a one-to-one correspondence between core-free subgroups and faithful transitive permutation representations (FTPR).

A permutation representation of a group Γ with generating set $\{\alpha_i,\ i \in I\}$ acting on a set X can be represented as a graph $\mathcal{G}$, with a set of vertices X and with a directed edge (x, y) with label α_i if there is an element α_i of the generating set such that $\alpha_i x = y$. When α_i is an involution, we substitute the two directed edges (x, y) and (y, x) by a single undirected edge $\{x, y\}$ with label α_i. Consider then a C-group $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ and let π be a faithful transitive permutation representation of Γ into the symmetric group S_d, for some d. A *FTPR graph* $\mathcal{G}$ *of* Γ *given by* π is a graph with d vertices, such that vertices i and j of the vertex set $V(\mathcal{G})$ are connected with an edge of label k if and only if $(\pi(\rho_k))i = j$.

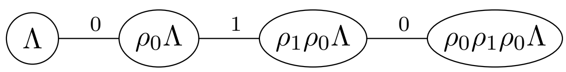
Let us consider an example. Let $\Gamma := \langle \rho_0, \rho_1 \rangle$ be the string C-group of type $[4] \cong D_4$, the automorphism group of a square, and consider the correspondence

$$\rho_0 \overset{\pi}{\longmapsto} (1, 2)(3, 4)$$
$$\rho_1 \overset{\pi}{\longmapsto} (2, 3).$$

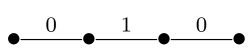
The FTPR graph associated with π is the following



where the labels on the edges are the indices of the involutions that act on those vertices. It is a FTPR since it can be seen as the action of Γ on the core-free subgroup $\Lambda = \langle \rho_1 \rangle$.



The stabilizer of each vertex of the graph is a different conjugate of Λ. Usually, the numbering on the vertices is not important, and so we will use more commonly the following presentation of the graph.



When Γ is a string C-group, some authors call these *(string) C-group Permutation Representation Graph* or *CPR Graph*, following the designation used by Pellicer in [Pel08]. Any transitive permutation representation of a group Γ gives a connected graph designated a *Schreier coset graph*, corresponding to the action of the group on the coset space of a subgroup (not necessarily core-free). CPR graphs do not need to be connected while Schreier coset graphs do. Since we will be working with graphs which are not necessarily transitive, and we want to consider C-groups that are not necessarily string C-groups, we will not use any of these two designations.

Let $\mathcal{G}_J$ denote a spanning subgraph of $\mathcal{G}$ whose edge-set has label-set $J \subseteq \{0, \dots, n-1\}$. It can be easily seen, by the definition, that $\mathcal{G}_{\{i\}}$, for any $i \in \{0, \dots, n-1\}$, is a matching. Moreover, if Γ is a string C-group and $|i-j| \geq 2$, ρ_i and ρ_j commute and all connected components of $\mathcal{G}_{\{i,j\}}$ are either single edges, double edges or alternating $\{i,j\}$-squares (squares that alternate labels i and j).

Faithful transitive permutation representation graphs have been extensively used to characterize abstract regular polytopes [FL11; FL18; FL19; FLM12a; FLM12b; CFLM17; Pel08] and also to build regular hypertopes [FLW15] . If Γ is the automorphism groups of an abstract regular polytope or of a regular hypertope, and $\mathcal{G}$ is a FTPR graph of Γ with d vertices, we will say that d is a *degree of the regular polytope/hypertope.* In chapters 4 and 5 we will use core-free subgroups to study the possible degrees of the toroidal (hyper)maps and locally toroidal polytopes $\{4, 4, 4\}$. Moreover, in chapters 7 and 8 FTPR graphs will be used to build and prove that certain C-groups are automorphism groups of hypertopes.

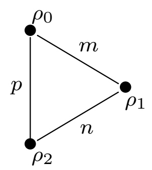
Chapter 3

Toroidal Regular Maps and Hypermaps

3.1 Regular Maps

A *map* M is an embedding of a connected (multi)graph X into a compact surface S in which when removing X from S, the components obtained (the faces of the map) are homeomorphic to unitary disks [Con09; Širl06]. We can consider the vertices and edges of the graph are the 0-elements and 1-elements of this map, while the faces defined before are the 2-elements. Maps can be seen as abstract regular polytopes, as long as their automorphism groups are string C-groups. If so, the flags can be seen as triples of incident vertex-edge-face. When the surface in which the graph is embedded is a sphere, this map is a *spherical map*, while if the surface is a torus, the map is a *toroidal map*. In our case, we are interested in toroidal maps.

We say a map is *regular* if the automorphism group of the map M, denoted as $Aut(M)$, acts regularly on the set of flags of M (freely and transitively). The automorphism group of a regular toroidal map $Aut(M)$ is a quotient of a triangle group. A *triangle group* is a Coxeter group [MS02] with a triangular Coxeter diagram



and group presentation

$$\Delta(m, n, p) := \langle \rho_0, \rho_1, \rho_2 \mid \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^m = (\rho_1\rho_2)^n = (\rho_0\rho_2)^p \rangle.$$

These groups are automorphism groups of tilings on the sphere, euclidean plane or the hyperbolic plane, depending on whether $\frac{1}{m} + \frac{1}{n} + \frac{1}{p}$ is greater, equal or less than 1, respectively.

The triangle groups $(m, n, 2)$, with $m, n \geq 3$, are Coxeter groups with string diagrams, giving polytopes of Schläfli type $\{m, n\}$. These triangle groups $(m, n, 2)$ give tilings (or tesselations) on the sphere, euclidean plane or the hyperbolic plane, depending on whether $\frac{1}{m} + \frac{1}{n}$ is greater, equal or less than $\frac{1}{2}$, respectively. When $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$, we get finite tilings on a sphere, usually called the platonic solids. On the other hand, if $\frac{1}{m}+\frac{1}{n}\leq\frac{1}{2}$ the tesselations are infinite. Hence, if $\frac{1}{m}+\frac{1}{n}<\frac{1}{2}$, we get tesselations of the hyperbolic plane. Lastly, if $\frac{1}{m}+\frac{1}{n}=\frac{1}{2}$, we get a tesselation of the euclidean plane.

In this last case, the only possible integers $m,n\geq 3$ that satisfy $\frac{1}{m}+\frac{1}{n}=\frac{1}{2}$ are $\{m,n\}\in\{\{3,6\},\{4,4\},\{6,3\}\}$, which represent the infinite tesselations of the euclidean plane by triangles, squares and hexagons, respectively. As previously said, the automorphism group of regular toroidal maps are quotients of triangle groups. Consider then a normal subgroup K, with finite index, of an infinite triangle group G of type $\{m,n\}\in\{\{3,6\},\{4,4\},\{6,3\}\}$. If its intersection with the maximal parabolic subgroups G_0, G_1 and G_2 is trivial, then the finite group G/K is the automorphism group of a tesselation of the torus, i.e. a toroidal map with type $\{m,n\}$. This subgroup K is generated by translations of the triangular, quadrangular and hexagonal tesselations of the euclidean plane, respectively. The conditions under which this factorization gives a regular toroidal map will be specified below.

3.1.1 Toroidal Maps of type $\{4,4\}$

Given a tesselation of the euclidean plane by squares, consider, for $s,t\geq 1$, a parallelogram with vertices $(0,0)$, (s,t), $(s-t,s+t)$ and $(-t,s)$, as shown in Figure 3.1. The resulting parallelogram can be seen as a toroidal map, designated as $\{4,4\}_{(s,t)}$, with $V=s^2+t^2$ vertices, $2V$ edges and V faces, that is the obtained by identifying opposite sides of the parallelogram. The product of two reflections ρ_i and ρ_j, with $i,j\in\{0,1,2\}$,

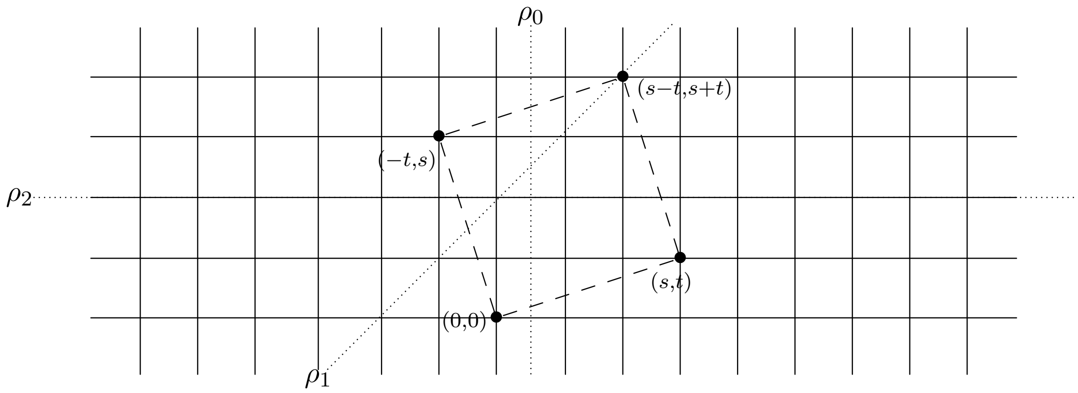


Figure 3.1: Toroidal map of type $\{4,4\}$.

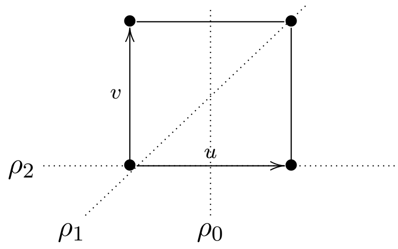
are still symmetries of the identified map $\{4,4\}_{(s,t)}$ (rotational symmetries), but ρ_0, ρ_1 and ρ_2 are reflexions of $\{4,4\}_{(s,t)}$ only if $st(s-t)=0$ [CM72], which is precisely when the map is regular, meaning that the automorphism group acts regularly on the set of flags.

Hence there are two families of toroidal regular maps, denoted by $\{4,4\}_{(s,0)}$ and $\{4,4\}_{(s,s)}$. Their automorphism groups are factorizations of the Coxeter group $[4,4]$ (same as the triangle group $(4,4,2)$) by

$$(\rho_0\rho_1\rho_2\rho_1)^s=id_{[4,4]} \text{ or } (\rho_0\rho_1\rho_2)^{2s}=id_{[4,4]},$$

respectively. The number of flags of $\{4,4\}_{(s,0)}$ is $8s^2$ while the number of flags of $\{4,4\}_{(s,s)}$ is $16s^2$. Moreover the automorphism group of $\{4,4\}_{(s,0)}$ is a quotient of the automorphism group of $\{4,4\}_{(s,s)}$ by $\langle(\rho_0\rho_1\rho_2\rho_1)^s\rangle$, while the automorphism group of $\{4,4\}_{(s,s)}$ is a quotient of the automorphism group of $\{4,4\}_{(2s,0)}$ by $\langle(\rho_0\rho_1\rho_2)^{2s}\rangle$.

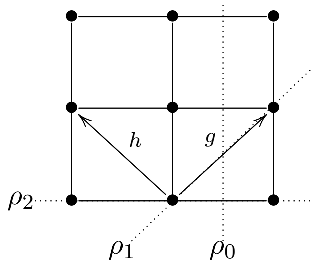
For the map $\{4,4\}_{(s,0)}$ consider the unitary translations $u = \rho_0\rho_1\rho_2\rho_1$ and $v = u^{\rho_1}$.



We have the following equalities

$$u^{\rho_0} = u^{-1},\ u^{\rho_2} = u,\ v^{\rho_0} = v \text{ and } v^{\rho_2} = v^{-1}. \tag{3.1}$$

In the case of the map $\{4,4\}_{(s,s)}$, consider as the unitary translations $g := uv = (\rho_0\rho_1\rho_2)^2$ and $h := u^{-1}v = g^{\rho_0}$.



We have the following equalities

$$g^{\rho_1} = g,\ g^{\rho_2} = h^{-1} \text{ and } h^{\rho_1} = h^{-1}. \tag{3.2}$$

3.1.2 Toroidal Maps of type $\{3,6\}$ and $\{6,3\}$

Consider the tesselation of the euclidean plane by triangles, whose automorphism group is the Coxeter group $[3,6]$, generated by three reflections ρ_0, ρ_1 and ρ_2, as shown in Figure 3.2 (the triangle group $(3,6,2)$). For $s,t \geq 1$, the vertices $(0,0)$, (s,t), $(-t,s+t)$ and $(s-t,s+2t)$ define a parallelogram, as shown in Figure 3.2, which gives a toroidal map, designated as $\{3,6\}_{(s,t)}$ having $V = s^2+st+t^2$ vertices, $3V$ edges and $2V$ faces, when opposite sides of the parallelogram are identified. As before, the involutions ρ_0, ρ_1 and ρ_2 are symmetries of $\{3,6\}_{(s,t)}$ only if $st(s-t)=0$ [CM72], which is the condition for regularity of $\{3,6\}_{(s,t)}$. Hence we distinguish two families of toroidal regular maps of type $\{3,6\}$: $\{3,6\}_{(s,0)}$ and $\{3,6\}_{(s,s)}$. The automorphism group of $\{3,6\}_{(s,0)}$ and $\{3,6\}_{(s,s)}$ are factorizations of the Coxeter group $[3,6]$ by

$$(\rho_0\rho_1\rho_2)^{2s} = id_{[3,6]} \text{ and } (\rho_0(\rho_1\rho_2)^2)^{2s} = id_{[3,6]},$$

respectively. The number of flags of $\{3,6\}_{(s,0)}$ is $12s^2$ while the number of flags of $\{3,6\}_{(s,s)}$ is $36s^2$. The automorphism group of $\{3,6\}_{(s,0)}$ is a quotient of the group

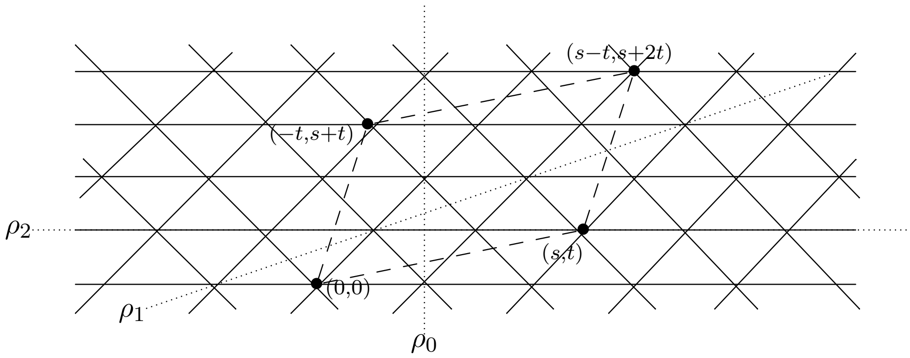
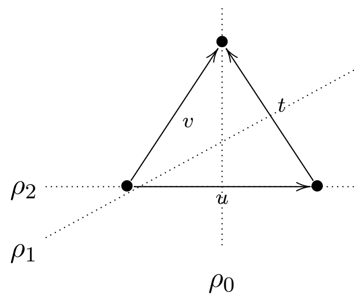


Figure 3.2: Toroidal map of type $\{3,6\}$.

$\{3,6\}_{(s,s)}$ by $\langle(\rho_0\rho_1\rho_2)^{2s}\rangle$, and the automorphism group of $\{3,6\}_{(s,s)}$ is a quotient of the group of the map $\{3,6\}_{(3s,0)}$ by $\langle(\rho_0(\rho_1\rho_2)^2)^{2s}\rangle$. For the map $\{3,6\}_{(s,0)}$ consider the unitary translations $u=\rho_0(\rho_1\rho_2)^2\rho_1$, $v=u^{\rho_1}=(\rho_0\rho_1\rho_2)^2$ and $t=u^{-1}v$.

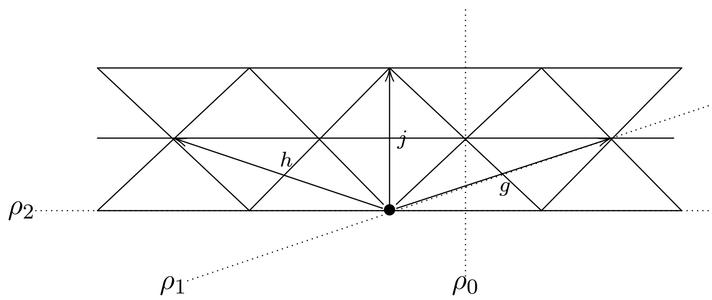


We have the following equalities

$$u^{\rho_0}=u^{-1},\ u^{\rho_2}=u,\ v^{\rho_0}=t \text{ and } v^{\rho_2}=t^{-1}. \tag{3.3}$$

In the case of the map $\{3,6\}_{(s,s)}$, consider as the unitary translations $g:=uv=(\rho_0(\rho_1\rho_2)^2)^2$, $h:=u^{-2}v=g^{\rho_0}$ and $j:=hg$. We have the following equalities

$$g^{\rho_1}=g,\ g^{\rho_2}=h^{-1} \text{ and } h^{\rho_1}=j^{-1}. \tag{3.4}$$



Consider lastly the tesselation of the euclidean plane by hexagons, whose automorphism is the Coxeter group $[6,3]$, generated by three reflections ρ_0, ρ_1 and ρ_2, as shown

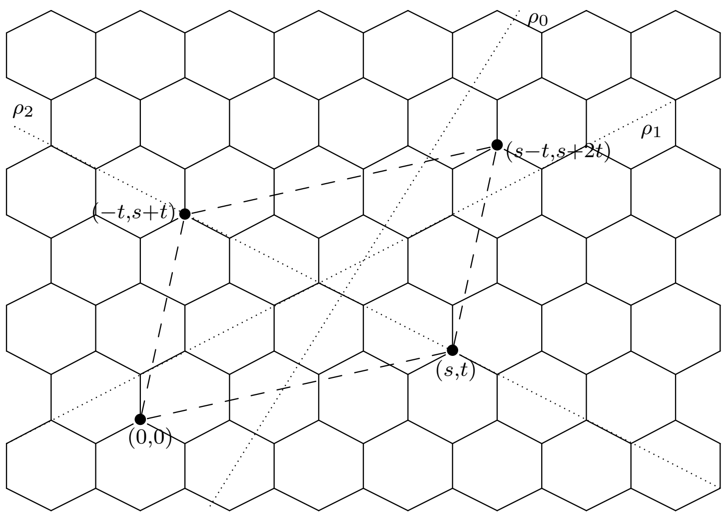


Figure 3.3: Toroidal map of type $\{6,3\}$

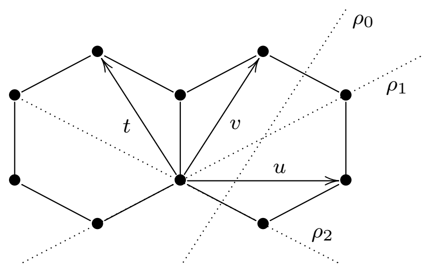
in Figure 3.3 (triangle group $(6,3,2)$). For $s,t \geq 1$, the parallelogram with vertices $(0,0)$, (s,t), $(-t,s+t)$ and $(s-t,s+2t)$, as shown in Figure 3.3, gives a toroidal map, denoted by $\{6,3\}_{(s,t)}$, with $F = s^2+st+t^2$ faces, $3F$ edges and $2F$ vertices, when opposite sides of the parallelogram are identified.

The map $\{6,3\}$ is said to be regular when the automorphism group acts regularly on the set of flags of the map, which is the case if and only if $st(s-t)=0$ [CM72]. Therefore, two families of toroidal regular maps of type $\{6,3\}$ arise: $\{6,3\}_{(s,0)}$ and $\{6,3\}_{(s,s)}$, which are factorizations of the infinite Coxeter group $[6,3]$ by

$$(\rho_0\rho_1\rho_2)^{2s} = id_{[6,3]} \text{ and } ((\rho_0\rho_1)^2\rho_2)^{2s} = id_{[6,3]},$$

respectively. The number of flags of $\{6,3\}_{(s,0)}$ is $12s^2$ while the number of flags of $\{6,3\}_{(s,s)}$ is $36s^2$. The automorphism group of $\{6,3\}_{(s,0)}$ is a quotient of the automorphism group of $\{6,3\}_{(s,s)}$ by $\langle(\rho_0\rho_1\rho_2)^{2s}\rangle$, and the automorphism group of $\{6,3\}_{(s,s)}$ is a quotient of the automorphism group of the map $\{6,3\}_{(3s,0)}$ by $\langle((\rho_0\rho_1)^2\rho_2)^{2s}\rangle$.

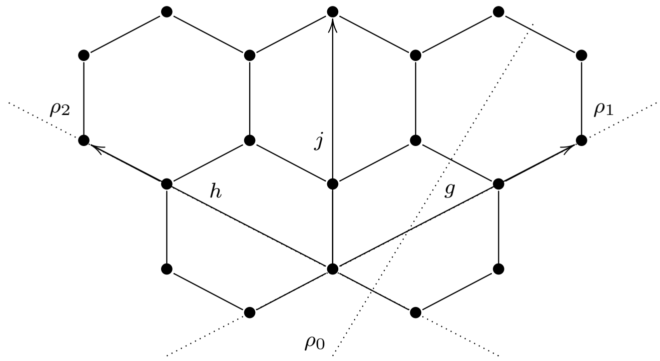
For the map $\{6,3\}_{(s,0)}$ consider the unitary translations $u = (\rho_0\rho_1\rho_2)^2$, $v = u^{\rho_1} = (\rho_1\rho_0)^2\rho_1\rho_2$ and $t = u^{-1}v$.



We have the following equalities

$$u^{\rho_0} = t,\ u^{\rho_2} = t^{-1},\ v^{\rho_0} = v \text{ and } v^{\rho_2} = v^{-1}. \tag{3.5}$$

In the case of the map $\{6,3\}_{(s,s)}$, consider as the unitary translations $g := uv = ((\rho_0\rho_1)^2\rho_2)^2$, $h := u^{-2}v = g^{\rho_0\rho_1\rho_0}$ and $j := gh$.



We have the following equalities

$$g^{\rho_1} = g,\ g^{\rho_0} = j \text{ and } h^{\rho_1} = j^{-1}. \tag{3.6}$$

Notice that the maps $\{3,6\}$ and $\{6,3\}$ are dual, for that reason, the degrees determined in Chapter 4 for the toroidal maps of type $\{3,6\}$, coincide withe the degres of the toroidal maps of type $\{6,3\}$ (since the group is isomorphic).

3.2 Regular Hypermaps

A hypermap can be seen as a generalization of a map, where edges can connect more than two vertices. It can be defined as a cellular embedding of a connected bipartite graph into a compact surface where the bipartition of vertices determines two types of vertices, in which one type will be the (hyper)vertices and the other the hyperedges. From a group-theoretical point of view, the full automorphism group of a hypermap is also a factorization of a triangle group (m,n,p), just like in the case of maps. When one of the parameters m, n or p is 2, then the hypermap is actually a map. Hence, we say a hypermap is *proper* if all parameters m, n, p are at least 3. Similarly to what we have seen before, a quotient of the triangle group (m,n,p) by a normal subgroup of finite index that intersects its maximal parabolic subgroups trivially gives the automorphism group of a hypermap.

We know that when $\frac{1}{m}+\frac{1}{n}+\frac{1}{p}=1$, the triangle group (m,n,p) gives a tesselation of the euclidean plane. For proper hypermaps, the only possible parameters that give an euclidean plane tiling is when $m=n=p=3$, hence the quotients of the triangle group $(3,3,3)$ give tilings on the torus by hypermaps.

3.2.1 Toroidal Hypermap of type $(3,3,3)$

The toroidal hypermap $(3,3,3)$ is obtained from a map of type $\{6,3\}$ by considering a bipartition on the set of its vertices (see Figure 3.4). The toroidal hypermap constructed from $\{6,3\}_{(s,t)}$ is denoted by $(3,3,3)_{(s,t)}$. The automorphism group G of the hypermap $(3,3,3)_{(s,t)}$ is a subgroup of index 2 of the automorphism group K of the map $\{6,3\}_{(s,t)}$,

$$G := \langle \tilde{\rho_0}, \rho_1, \rho_2\rangle, \text{ where } \tilde{\rho_0} := \rho_0\rho_1\rho_0 \text{ and } G \leq K := \langle \rho_0, \rho_1, \rho_2\rangle.$$

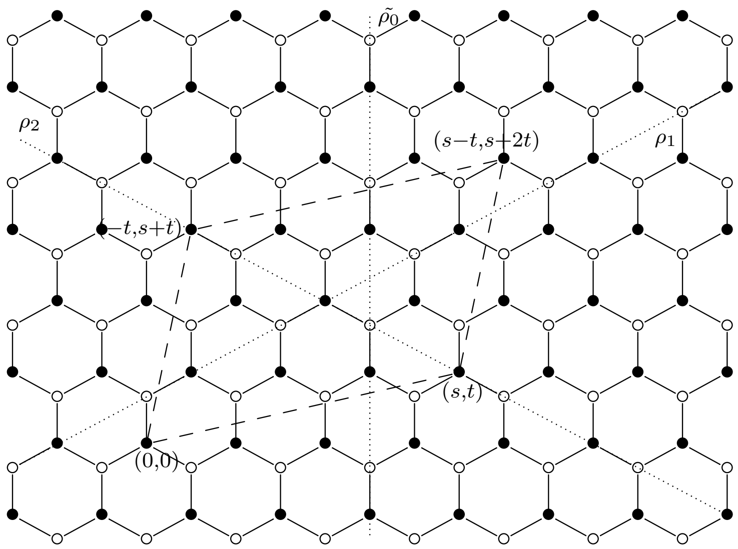


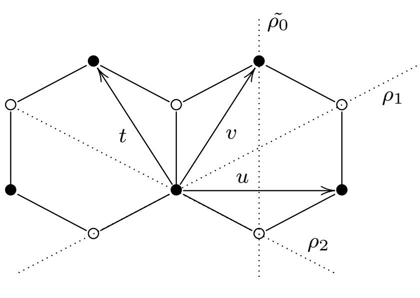
Figure 3.4: Toroidal map of type $(3,3,3)$

If the toroidal hypermap is regular, then G is the infinite Coxeter group of the triangle group $(3,3,3)$ factorized by either $(\tilde{\rho}_0\rho_1\rho_2\rho_1)^s$ or $(\tilde{\rho}_0\rho_1\rho_2)^{2s}$, depending on whether it is $(3,3,3)_{(s,0)}$ or $(3,3,3)_{(s,s)}$, respectively.

The automorphism group of the $(3,3,3)_{(s,0)}$ is a quotient of the group of $(3,3,3)_{(s,s)}$ by $\langle(\tilde{\rho}_0\rho_1\rho_2\rho_1)^s\rangle$, and the latter is a quotient of the automorphism group of $(3,3,3)_{(3s,0)}$ by $\langle(\tilde{\rho}_0\rho_1\rho_2)^{2s}\rangle$.

Let u, v and t denote the following translations of order s of the group of the hypermap $(3,3,3)_{(s,0)}$ (that are translations with the same order in the group of the map $\{6,3\}_{(s,0)}$).

$$u := \tilde{\rho}_0\rho_1\rho_2\rho_1,\ v := u^{\rho_1} = \rho_1\tilde{\rho}_0\rho_1\rho_2 \text{ and } t := u^{-1}v.$$

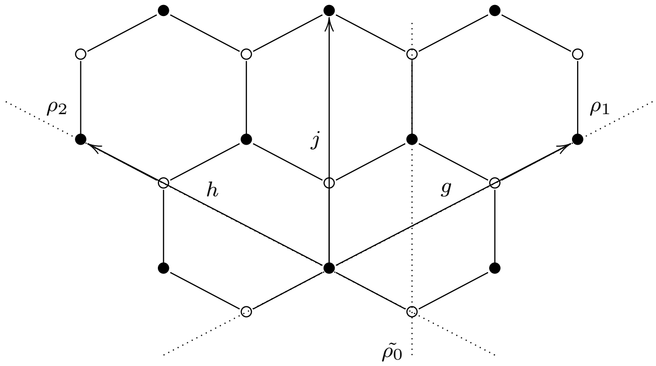


We have the equalities

$$u^{\tilde{\rho}_0} = u^{-1},\ u^{\rho_2} = t^{-1},\ v^{\rho_2} = v^{-1},\ v^{\tilde{\rho}_0} = t \ \text{ and } \ t^{\rho_1} = t^{-1}. \tag{3.7}$$

For the hypermap $(3,3,3)_{(s,s)}$, consider the translations $g := uv = (\tilde{\rho}_0\rho_1\rho_2)^2$, $h := g^{\tilde{\rho}_0}$

and $j := gh$.



In this case we have the following equalities

$$g^{\rho_1} = g, \; g^{\rho_2} = j^{-1} \text{ and } h^{\rho_1} = j^{-1}. \tag{3.8}$$

Chapter 4

Degrees of Toroidal Maps and Hypermaps

The classification of locally toroidal polytopes was one of the missions of Grünbaum [Grü78], motivating other researchers to follow him, leading to multiple publications on the area (see [MS02] for references). Recently, a generalization of the notion of an abstract regular polytope was introduced (see Chapter 6) and the classification of these structures called regular hypertopes has begun (see Chapters 7, 8 and 9, and [FLW15; CFHL18; FLPW21]). With this, a generalization of Grünbaum's classification problem to these new structures became of high interest for algebraists and geometers. In [FLW15] the usage of FTPRs of toroidal maps to build FTPRs of locally toroidal hypertopes was crucial, which motivated a research project that aims to determine the degrees of the automorphism groups of regular polytopes and hypertopes. In this chapter we will determine all possible degrees of FTPR of toroidal maps and hypermaps. This study will be continued in Chapter 5 where we study of locally toroidal polytopes of type $\{4,4,4\}$. This classification of the degrees of FTPRs give powerful tools to study and build new locally toroidal hypertopes (see Chapters 7 and 8).

The results presented in this chapter can be found in [FP20b; FP20a; FP21], giving the possible degrees of toroidal regular (hyper)maps $\{4,4\}$, $\{3,6\}$ and $(3,3,3)$.

4.1 Preliminary Results

In this section, some important results will be collected to be used repeatedly to prove this chapter's main results, for each of the toroidal (hyper)maps considered. Let G be the automorphism group of a toroidal regular map of type $\{4,4\}$ or $\{3,6\}$, or hypermap of type $(3,3,3)$, and suppose G has a faithful transitive permutation representation of degree n. Let u, v, g and h be the translations defined in Chapter 3 for each toroidal map, and $T := \langle u,v\rangle$, when G is the automorphism group of $\{4,4\}_{(s,0)}$, $\{3,6\}_{(s,0)}$ or $(3,3,3)_{(s,0)}$; and $T := \langle g,h\rangle$, when G is the automorphism group of $\{4,4\}_{(s,s)}$, $\{3,6\}_{(s,s)}$ or $(3,3,3)_{(s,s)}$.

Given a FTPR of degree n, the translation subgroup T can either be transitive or intransitive. Since T is a normal subgroup of G and it is a direct product of two cyclic groups of order s, the T-orbits form a block system (which might be trivial). Consider σ and τ the actions of the generators of T restricted to a block and let $K := \langle \sigma,\, \tau\rangle$.

Lemma 4.1.1. *If $B := |K : \langle\sigma\rangle|$ and $C := |K : \langle\tau\rangle|$ then the size of a T-orbit is $k = ds$ where $d = gcd(B, C)$.*

Proof. Consider that σ and τ are the actions of the generators of T on a block of size k. Then $K := \langle\sigma, \tau\rangle$, $A := o(\sigma)$, $B := |K : \langle\sigma\rangle|$ and $C := |K : \langle\tau\rangle|$. We have that K has order AB and acts regularly on the block, hence $k = AB$. As σ and τ commute, we have the following

$$K/\langle\sigma\rangle = \{\langle\sigma\rangle, \langle\sigma\rangle\tau, \langle\sigma\rangle\tau^2, \ldots, \langle\sigma\rangle\tau^{B-1}\} \text{ and}$$

$$K/\langle\tau\rangle = \{\langle\tau\rangle, \langle\tau\rangle\sigma, \langle\tau\rangle\sigma^2, \ldots, \langle\tau\rangle\sigma^{C-1}\}.$$

Thus B divides $o(\tau)$ and C divides $o(\sigma) = A$. Let $D := A/C$. As $k = AB = o(\tau)C$ we have $o(\tau) = DB$. Now

$$s = lcm(o(\sigma), o(\tau)) = lcm(CD,\, BD) = D\, lcm(C, B)$$

and

$$k = AB = DCB = D\, lcm(C, B)\, gcd(C, B) = s\, gcd(C, B).$$

To conclude the proof consider $d = gcd(C, B)$. □

Corollary 4.1.2. *If T is transitive, then $n = s^2$. In this case T is regular and $G \cong T \rtimes Stab_G(id)$.*

Proof. As T is transitive, $K = T$ and since T is a direct product of cyclic groups of order s, using Lemma 4.1.1, $B = C = s$, hence $k = s^2$. Since T is transitive, $n = s^2$. Moreover the action of T is regular. Hence, $G \cong T \rtimes Stab_G(id)$ [Cam99, Section 1.7]. □

Proposition 4.1.3. *If G is the automorphism group of $\{4, 4\}_{(s,s)}$, $\{3, 6\}_{(s,s)}$ and $(3, 3, 3)_{(s,s)}$, then T is intransitive.*

Proof. Suppose that G is the automorphism group of the map $\{4, 4\}_{(s,s)}$ and that T is transitive. Let $\alpha = \rho_0\rho_1\rho_2\rho_1$ (which the reader can identify as the unitary horizontal translation of $\{4, 4\}$). By the previous result we have that T is regular, and hence the n points can be seen as elements of the subgroup T, one of which is id_G, therefore $id_G\alpha = g^a h^b$. As α commutes with both g and h, $g^i h^j \alpha = g^{i+a} h^{j+b}$ and $g^i h^j \alpha^s = g^{i+sa} h^{j+sb} = g^i h^j$. Hence the order of α is at most s, a contradiction.

Analogously, for the map $\{3, 6\}_{(s,s)}$ and hypermap $(3, 3, 3)_{(s,s)}$, if T is transitive then the order of $\alpha = (\rho_0\rho_1\rho_2)^2$ and $\alpha = \tilde{\rho}_0\rho_1\rho_2\rho_1$, respectively, is at most s, giving a contradiction. □

This leads to the following lemma, that combines the previous two results.

Lemma 4.1.4. *If T is transitive, then G is the automorphism group of $\{4, 4\}_{(s,0)}$, $\{3, 6\}_{(s,0)}$ or $(3, 3, 3)_{(s,0)}$ and $n = s^2$.*

Proof. This result is a consequence of Corollary 4.1.2 and Proposition 4.1.3. □

Consider now only the cases where T is intransitive. Then from Corollary 4.1.2, $n \neq s^2$. To deal with this results, we have the following proposition.

Proposition 4.1.5. *If $n \neq s^2$ then G is embedded into $S_k \wr S_m$ with $n = km$ $(m, k > 1)$ and we have*

(i) $k = ab$ *where* $s = lcm(a, b)$, *and*

(ii) m *is a divisor of* $\frac{|G|}{s^2}$.

Proof. By Corollary 4.1.2, T is intransitive, and thus G is embedded into $S_k \wr S_m$, where k is the size of an orbit of T and $n = km$.

(i) This is a consequence of Lemma 4.1.1, since if $lcm(a, b) = s$, then $ab = ds$, where $d = gcd(a, b)$.

(ii) Consider the induced action of G on the set of m blocks and the induced homomorphim $f : G \to S_m$. The kernel of this homomorphism has size at least s^2, as it contains T. Hence, the size of $Im(f)$ is a divisor $\frac{|G|}{s^2}$.

□

From now on we use m for number of blocks (the T-orbits) and k size of a T-orbit.

The correspondence between FTPRs of G and its core-free subgroups was pointed out in Section 2.6. Notice that any subgroup of a core-free subgroups is also core-free. Furthermore, there is an upwards correspondence between core-free subgroups that leads the following the result.

Lemma 4.1.6. *Let H, G and K be groups such that $H < G < K$. If H is a core-free subgroup of G, then H is a core-free subgroup of K.*

Moreover, if G has a faithful transitive permutation representation of degree n and is a subgroup of index α of K, then K has a faithful transitive permutation representation of degree αn.

Proof. Let H be the core-free subgroup of G with index n, that is, $\bigcap_{g\in G} H^g = \{id\}$. As G is a subgroup of K, then $\bigcap_{g\in K} H^g = \{id\}$, meaning that H is also a core-free subgroup of K with index $|K : H| = \alpha n$, where $|K : G| = \alpha$. Therefore, there is a FTPR of K with degree αn. □

Since the automorphism groups of the (hyper)maps $\{4, 4\}_{(s,0)}$, $\{3, 6\}_{(s,0)}$ $(3, 3, 3)_{(s,0)}$ are quotients of the automorphism groups of index 2 of $\{4, 4\}_{(s,s)}$, and of index 3 of $\{3, 6\}_{(s,s)}$ and $(3, 3, 3)_{(s,s)}$, respectively, we have the following corollary.

Corollary 4.1.7. *Let n be a degree of G. Then:*

- *if G is the automorphism group of the map $\{4, 4\}_{(s,0)}$ (resp. $\{4, 4\}_{(s,s)}$) then $2n$ is a degree of $\{4, 4\}_{(s,s)}$ (resp. $\{4, 4\}_{(2s,0)}$);*
- *if G is the automorphism group of the map $\{3, 6\}_{(s,0)}$ (resp. $\{3, 6\}_{(s,s)}$) then $3n$ is a degree of $\{3, 6\}_{(s,s)}$ (resp. $\{3, 6\}_{(3s,0)}$); and*
- *if G is the automorphism group of the hypermap $(3, 3, 3)_{(s,0)}$ (resp. $(3, 3, 3)_{(s,s)}$) then $3n$ is a degree of $(3, 3, 3)_{(s,s)}$ (resp. $(3, 3, 3)_{(3s,0)}$).*

Proof. This result is a consequence of Lemma 4.1.6 □

Also, as seen in Section 3.2.1 of Chapter 3, we have that the automorphism group of a toroidal hypermap $(3,3,3)_{(s,t)}$ is a subgroup of index 2 of the automorphism group of the toroidal map $\{6,3\}_{(s,t)}$. Hence, we have this immediate result from the Lemma 4.1.6.

Corollary 4.1.8. *If n is a degree of $(3,3,3)_{(s,0)}$ (resp. $(3,3,3)_{(s,s)}$) then $2n$ is a degree of $\{6,3\}_{(s,0)}$ (resp. $\{6,3\}_{(s,s)}$).*

Proof. This result is a consequence of Lemma 4.1.6 □

We remind again the reader that the degrees of the maps $\{6,3\}_{(s,0)}$ and $\{6,3\}_{(s,s)}$ are the same as the ones of $\{3,6\}_{(s,0)}$ and $\{3,6\}_{(s,s)}$, respectively.

4.2 Degrees of regular maps of type $\{4,4\}$

In this section the degrees of the regular maps of type $\{4,4\}$ will be determined. Moreover, using some of the core-free subgroups determined, examples of FTPR graphs will be given for some of the degrees.

4.2.1 The possible degrees for the map $\{4,4\}_{(s,0)}$

Consider here that G is the automorphism group of the regular map $\{4,4\}_{(s,0)}$.

Proposition 4.2.1. *Let G be the automorphism group of $\{4,4\}_{(s,0)}$ ($s>2$). Then its dihedral subgroups $\langle\rho_i,\rho_j\rangle$ and respective subgroups are core-free, for $i,j\in\{0,1,2\}$.*

Proof. We need only prove that the dihedrals are core-free, since their subgroups are core-free as a consequence. Consider the dihedral subgroup $H=\langle\rho_0,\rho_1\rangle$. Since $s>2$, it can be seen that

$$H\cap H^{\rho_2}\cap H^{\rho_2\rho_1}=\langle\rho_0,\rho_1\rangle\cap\langle\rho_0,\rho_1^{\rho_2}\rangle\cap\langle\rho_0^{\rho_1},\rho_1^{\rho_2}\rangle=\langle id_G\rangle.$$

Hence, H is core-free.

Analogously, the same can be done for the other two dihedral subgroups. □

The dihedral groups of Proposition 4.2.1 and their subgroups are stabilizers of vertices, edges, faces, darts and flags of the toroidal map $\{4,4\}_{(s,0)}$. Considering these core-free subgroups, we can have faithful transitive permutation representations acting on the set of vertices, edges, faces, darts and flags, respectively. Therefore, the regular map $\{4,4\}_{(s,0)}$ ($s>2$) has the following degrees.

Corollary 4.2.2. *Let G be the automorphism group of $\{4,4\}_{(s,0)}$ ($s>2$). Then n is a degree of G if*

$$n\in\{s^2,2s^2,4s^2,8s^2\}.$$

Proof. Since the order o of the dihedrals $\langle\rho_i,\rho_j\rangle$ (for $i,j\in\{0,1,2\}$) and their subgroups is $o\in\{8,4,2,1\}$, and they are core-free based on Proposition 4.2.1, then G has degrees $\frac{|G|}{o}\in\{s^2,2s^2,4s^2,8s^2\}$. □

Consider now the exceptional cases where $s \in \{1,2\}$. For $s = 1$, the only core-free subgroups are either trivial or have order 2, and thus the only possible degrees for the map $\{4,4\}_{(1,0)}$ are 8 and 4. If $s = 2$, the subgroups $\langle \rho_0, \rho_2 \rangle$, $\langle \rho_0 \rangle$ and the trivial subgroup are core-free, having a faithful action on the set of edges, darts and flags. However, $\langle \rho_1, \rho_2 \rangle$ and $\langle \rho_0, \rho_1 \rangle$ have nontrivial core, that is $\langle \rho_2^{\rho_1} \rangle$ and $\langle \rho_0^{\rho_1} \rangle$, respectively. Thus the map $\{4,4\}_{(2,0)}$ is an example of a map whose actions on the vertices and faces are non-faithful. Therefore, the only possible degrees are 8, 16 and 32 for $\{4,4\}_{(2,0)}$.

In what follows the other possible degrees for the maps $\{4,4\}_{(s,0)}$ ($s > 2$) are determined.

Proposition 4.2.3. *Let G be the automorphism group of $\{4,4\}_{(s,0)}$ ($s > 2$). If a and b are nonnegative integers and $s = lcm(a,b)$ then*

(1) $H = \langle u^a, v^b \rangle$ is core-free and $|G:H| = 8ab$,

(2) $H = \langle u^a, v^b \rangle \rtimes \langle \rho_0 \rangle$ is core-free and $|G:H| = 4ab$,

(3) if $ab \neq s$ then $H = \langle u^a, v^b \rangle \rtimes \langle \rho_0, \rho_2 \rangle$ is core-free and $|G:H| = 2ab$, and

(4) $H = \langle u \rangle \rtimes \langle \rho_0, \rho_2 \rangle$ is core-free and $|G:H| = 2s$.

Proof. (1) Suppose that $x \in H \cap H^{\rho_1} = \langle u^a, v^b \rangle \cap \langle u^b, v^a \rangle$. Then, since u and v commute, we have that $x = (u^a)^i (v^b)^j = (u^b)^k (v^a)^l$. Hence, we have that

$$ai \equiv bk \bmod s$$
$$bj \equiv al \bmod s.$$

Since ai is a multiple of both a and b, it is also a multiple of s and $ai \equiv 0 \bmod s$. The same reasoning can be used for bj, leading to $bj \equiv 0 \bmod s$. Hence, $x = id_G$ and H is core-free. The order of H is $\frac{s^2}{ab}$ thus $|G:H| = 8ab$.

(2) Suppose that $x \in H \cap H^{\rho_1} = \langle u^a, v^b \rangle \rtimes \langle \rho_0 \rangle \cap \langle u^b, v^a \rangle \rtimes \langle \rho_0^{\rho_1} \rangle$. If $x \notin T$ then $\rho_0 \rho_0^{\rho_1} \in T$, a contradiction. Thus $x \in T$ and therefore as in (1) we conclude that $x = id_G$. The order of H is $\frac{2s^2}{ab}$ thus $|G:H| = 4ab$.

(3) Suppose that $x \in H \cap H^{\rho_1} = \langle u^a, v^b \rangle \rtimes \langle \rho_0, \rho_2 \rangle \cap \langle u^b, v^a \rangle \rtimes \langle \rho_0^{\rho_1}, \rho_2^{\rho_1} \rangle$. If $x \notin T$ then $\rho_0^i \rho_2^j (\rho_0^{\rho_1})^k (\rho_2^{\rho_1})^l \in K := \langle u^a, v^b, u^b, v^a \rangle$ for some $i,\ j,\ k,\ l \in \{0,1\}$. This is only possible for $(i,\ j,\ k,\ l) \in \{(0,1,1,0), (1,1,1,1), (1,0,0,1)\}$. Then we get either $v \in K$, $u \in K$ or $v^{-1}u \in K$ which is possible only if a and b are co-primes, and, since $s = lcm(a,b)$, we would have $s = ab$, a contradiction. Thus $x \in T$ and therefore, as in (1), we conclude that $x = id_G$. The order of H is $\frac{4s^2}{ab}$ thus $|G:H| = 2ab$.

(4) Following a similar argument as (3), we get that $\langle u, \rho_0, \rho_2 \rangle \cap \langle v, \rho_0^{\rho_1}, \rho_2^{\rho_1} \rangle = \langle \rho_2, \rho_2^{\rho_1} \rangle$ which is a subgroup of a dihedral subgroup, and hence is core-free.

□

Theorem 4.2.4. *Let $s > 2$. A faithful transitive permutation representation of the automorphism group of $\{4,4\}_{(s,0)}$ has degree n if and only if*

$$n \in \{s^2, 2ab, 4ab, 8ab\}$$

where a and b are positive integers with $s = lcm(a,b)$.

Proof. The theorem follows from Corollary 4.1.2, Proposition 4.1.5, Corollary 4.2.2 and Proposition 4.2.3. □

4.2.2 The possible degrees for the map $\{4,4\}_{(s,s)}$

The automorphism group of $\{4,4\}_{(1,0)}$ is isomorphic to a subgroup of $\{4,4\}_{(1,1)}$, and hence, by Corollary 4.1.7 we have that $\{4,4\}_{(1,1)}$ has FTPRs with degrees 8 and 16. Since it is not possible to have a faithful transitive permutation representation of the automorphism group of $\{4,4\}_{(1,1)}$ on only 4 points, the possible degrees for $\{4,4\}_{(1,1)}$ are 8 and 16. For the case of $s=2$, by Corollary 4.1.7, the automorphism group of $\{4,4\}_{(2,2)}$ has degrees 16, 32 and 64. Also, the subgroup $\langle \rho_0, \rho_1 \rangle$ has trivial core, adding the degree 8 to the list of possible degrees of $\{4,4\}_{(2,2)}$. The complete list of degrees of $\{4,4\}_{(2,2)}$ is then 8, 16, 32 and 64.

Consider now $s > 2$. By Corollary 4.1.7 and Theorem 4.2.4, there are FTPRs for $\{4,4\}_{(s,s)}$ for $n \in \{2s^2, 4ab, 8ab, 16ab\}$ with $s = lcm(a,b)$. We only need to prove that these are all the degrees for the map $\{4,4\}_{(s,s)}$. Moreover, from Proposition 4.1.3 we know that $m \neq 1$, meaning that we need only to check for other possibilities for $m = 2$.

Lemma 4.2.5. *If $m = 2$ then $n = 2s^2$.*

Proof. Let G be the automorphism group of $\{4,4\}_{(s,s)}$ acting faithfully on n points. Suppose that $T = \langle g, h \rangle$ has two orbits. Consider the group K, isomorphic to the automorphism group of $\{4,4\}_{(2s,0)}$, where $G \cong K/\langle (uv)^s \rangle$. We have that K acts faithfully on two copies of the set of n points. Let $H = \langle u, v \rangle < K$ be the translation group for $\{4,4\}_{(2s,0)}$, as in Chapter 3.1.1. We have that $|u| = |v| = 2s$. Moreover, if x is a point on one of the copies, $x(uv)^s$ is on the other copy. In addition T is a proper subgroup of H, thus H must be transitive on $2n$ points and therefore it acts regularly on $2n$ points. Hence, H has order $(2s)^2$, $2n = (2s)^2$, as wanted. ☐

Theorem 4.2.6. *Let $s > 2$. A faithful transitive permutation representation of the automorphism group of $\{4,4\}_{(s,s)}$ has degree n if and only if*

$$n \in \{2s^2, 4ab, 8ab, 16ab\}$$

where a and b are positive integers with $s = lcm(a,b)$.

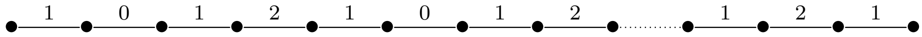
Proof. This result follows from Proposition 4.1.3, Lemma 4.1.5, Corollary 4.1.7, Theorem 4.2.4 and Lemma 4.2.5. ☐

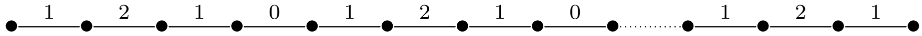
4.2.3 Examples of Faithful Transitive Permutation Representation Graphs

Any of the core-free subgroups presented in Propositions 4.2.1 and 4.2.3 (or conjugates) can be used to build FTPR graphs. Moreover, for groups of small order, we can actually compute all the possible core-free subgroups using GAP [GAP21]. Having a group G and a core-free subgroup H of G, we can use GAP to get the coset action of our group G on the set of cosets of H by permutations, allowing us to get examples of FTPR graphs. Moreover, by fixing the core-free subgroup and increasing the parameter s of the automorphism groups of the toroidal maps $\{4,4\}_{(s,0)}$ and $\{4,4\}_{(s,s)}$, we can check for patterns of the resulting FTPR graphs, making it possible to generalize for any s.

It is important to notice that not all core-free subgroups of G were given in this study, since our focus was to determine the possible degrees. From our computational research on GAP [GAP21], some different core-free subgroups from those given above will be presented in this section as stabilizers of vertices of the graphs.

Lemma 4.2.7. *The following graphs are faithful transitive permutation representation graphs of the automorphism group of* $\{4,4\}_{(s,0)}$ *with degree* $2s$ *(*$s \geq 3$*).*

s odd: 

s even: 

Moreover the stabilizer of a point is, up to a conjugacy, $\langle v\rangle \rtimes \langle \rho_0, \rho_2\rangle$*.*

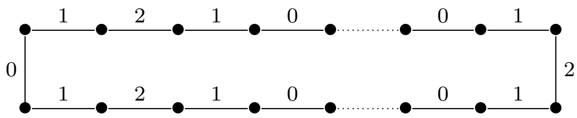
Proof. Let $G = \langle \rho_0, \rho_1, \rho_2\rangle$ be the group with one of the given permutation representation graphs. Let x be the vertex on the left. When s is odd, both x and all the $s-1$ vertices that are swapped by ρ_2 are fixed by $\rho_0\rho_1\rho_2\rho_1 = u$, while the other are cyclicly permuted. If s is even, u fixes all s vertices swapped by ρ_2 and cyclicly permutes the remaining ones. In both cases, $u^s = id_G$. In addition, by the graph it can also be seen that $\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^4 = (\rho_1\rho_2)^4 = (\rho_0\rho_1)^2 = id_G$. Hence, G is a quotient of the automorphism group of $\{4,4\}_{(s,0)}$ and $|G| \leq 8s^2$. Let us prove that $|G| = 8s^2$.

First consider the case s odd. The stabilizer $Stab_G(x)$ of x contains ρ_0, ρ_2 and u. Hence $Stab_G(x)$ contains $\langle \rho_0, \rho_2, u\rangle \cong \langle \rho_2\rangle \times D_s$, thus $|Stab_G(x)| \geq 4s$. Hence, by the orbit-stabilizer theorem, $|G| = |Stab_G(x)||Orb_G(x)| \geq 4s \times 2s = 8s^2$. Therefore, $|G| = 8s^2$.

Now let s be even. The stabilizer $Stab_G(x)$ of x contains ρ_0, ρ_2 and $v := \rho_1\rho_0\rho_1\rho_2 = u^{\rho_1}$. As before $|G| \geq 8s^2$, implying that $|G| = 8s^2$.

Then, the graph is a faithful transitive permutation representation of the automorphism group of $\{4,4\}_{(s,0)}$. □

Lemma 4.2.8. *The following graph is a faithful transitive permutation representation graph of the automorphism group of* $\{4,4\}_{(s,0)}$ *with degree* $4s$ *(*$s \geq 2$*).*

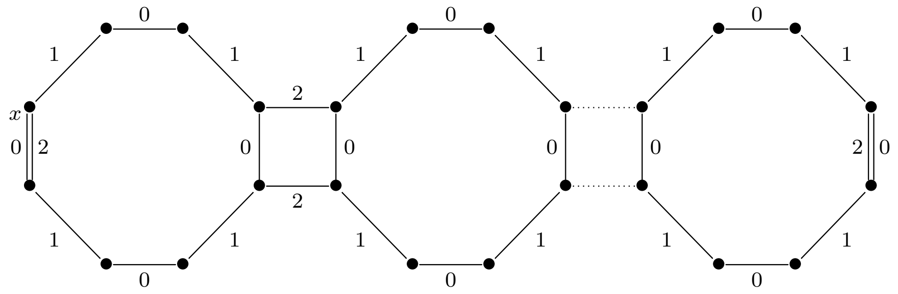


Moreover the stabilizer of a point is, up to a conjugacy, $\langle u\rangle \rtimes \langle \rho_0\rangle$*.*

Proof. Let $G = \langle \rho_0, \rho_1, \rho_2\rangle$ be the group with the given transitive permutation representation graph. It can be easily seen that $\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^4 = (\rho_1\rho_2)^4 = (\rho_0\rho_2)^2 = id_G$. Moreover, $u := \rho_0\rho_1\rho_2\rho_1$ fixes the $2s$ vertices that ρ_2 swaps and has two cycles of order s for the remaining vertices. Hence, $u^s = id_G$. With this, it can be concluded that G is a quotient of the automorphism group of $\{4,4\}_{(s,0)}$, particularly $|G| \leq 8s^2$. Now consider a vertex x not fixed by ρ_2. The stabilizer $Stab_G(x)$ of x contains ρ_0 and u. Thus $Stab_G(x)$ contains $\langle \rho_0, u\rangle \cong D_s$, a subgroup of order $2s$. Since, $|Stab_G(x)| \geq 2s$, the order $|G| \geq 2s \times 4s$, which proves that $|G| = 8s^2$. Consequently, the graph is a faithful transitive permutation representation graph of $\{4,4\}_{(s,0)}$.

□

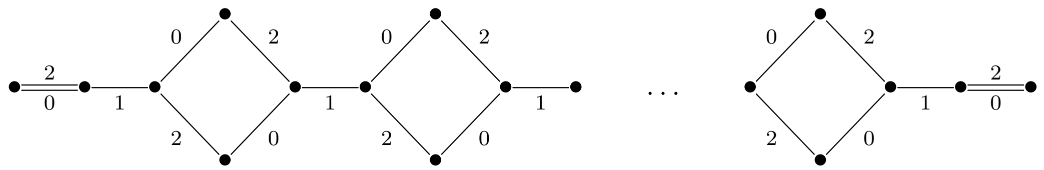
Lemma 4.2.9. *Let s be even. The following graph is a faithful transitive permutation* *representation graph of the automorphism group of* $\{4,4\}_{(s,0)}$ *with degree* $4s$ *(*$s \geq 2$*).*



Moreover the stabilizer of a point is, up to a conjugacy, $\langle \rho_0\rho_2, \rho_1\rho_2\rho_1 \rangle$*.*

Proof. Let $G = \langle \rho_0, \rho_1, \rho_2 \rangle$ be the group with the given transitive permutation representation graph. First, through the graph it can be seen that $(\rho_0\rho_1)^4 = (\rho_1\rho_2)^4 = (\rho_0\rho_2)^2 = (\rho_0\rho_1\rho_2\rho_1)^s = id_G$. Hence, G is a quotient of the automorphism group of $\{4,4\}_{(s,0)}$, particularly $|G| \leq 8s^2$. Consider the vertex x of the graph. The stabilizer $Stab_G(x)$ of x contains both $\rho_0\rho_2$ and $\rho_1\rho_2\rho_1$. Thus, $Stab_G(x)$ has $\langle \rho_0\rho_2, \rho_1\rho_2\rho_1 \rangle \cong D_s$ with the element $u\rho_2$, in which $(u\rho_2)^s = id_G$ since s is even. Thus $|Stab_G(x)| \geq 2s$ and $|G| = 8s^2$. Consequently the graph is a faithful transitive permutation representation graph of $\{4,4\}_{(s,0)}$. □

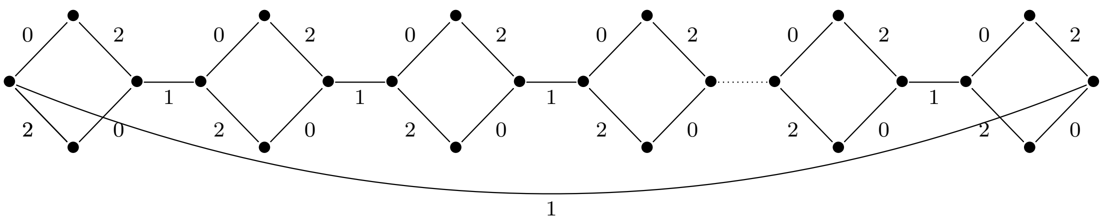
Lemma 4.2.10. *The following graph is a faithful transitive permutation representation graph of the automorphism group of* $\{4,4\}_{(s,s)}$ *with degree* $4s$ *(*$s \geq 2$*).*



Moreover the stabilizer of a point is, up to a conjugacy, $\langle \rho_0\rho_2, \rho_0\rho_1\rho_2 \rangle$*.*

Proof. Let $G = \langle \rho_0, \rho_1, \rho_2 \rangle$ be the group with the given transitive permutation representation graph ($s \geq 2$). First, it can be seen that $\rho_i^2 = id_G$, for $i \in \{0,1,2\}$, and also that $(\rho_0\rho_1)^4 = (\rho_1\rho_2)^4 = (\rho_0\rho_1)^2 = (\rho_0\rho_1\rho_2)^{2s} = id_G$. Hence, G is a subgroup of the automorphism group of $\{4,4\}_{(s,s)}$, with order $|G| \leq 16s^2$. Let x be the second vertex starting from the left of the graph. The stabilizer $Stab_G(x)$ has the elements $\rho_0\rho_2$ and $\rho_0\rho_1\rho_2$. Hence $D_{2s} \cong \langle \rho_0\rho_2, \rho_0\rho_1\rho_2 \rangle \leq Stab_G(x)$, and thus $|Stab_G(x)| \geq 4s$. Therefore, $|G| \geq 16s^2$, proving that the graph is a faithful transitive permutation representation of the automorphism group of $\{4,4\}_{(s,s)}$. □

Lemma 4.2.11. *The following graph is a faithful transitive permutation representation graph of the automorphism group of* $\{4,4\}_{(s,s)}$ *with degree* $8s$ *(*$s \geq 2$*).*



Moreover the stabilizer of a point is, up to a conjugacy, $\langle g\rangle \rtimes \langle \rho_0\rho_1\rho_0\rangle$.

Proof. Let $G = \langle \rho_0, \rho_1, \rho_2\rangle$ be the group with the given transitive permutation representation graph ($s \geq 2$). First, it can be seen that $\rho_i^2 = id_G$, for $i \in \{0,1,2\}$, and also that $(\rho_0\rho_1)^4 = (\rho_1\rho_2)^4 = (\rho_0\rho_1)^2 = (\rho_0\rho_1\rho_2)^{2s} = id_G$. Hence, G is a subgroup of the automorphism group of $\{4,4\}_{(s,s)}$, with order $|G| \leq 16s^2$. Consider a vertex x that is not fixed by ρ_1. The permutation $g := (\rho_0\rho_1\rho_2)^2$ and $\rho_1^{\rho_0}$ is in the stabilizer of x. Hence, $D_s \cong \langle g\rangle \rtimes \langle \rho_1^{\rho_0}\rangle \leq Stab_G(x)$, and thus $|Stab_G(x)| \geq 2s$. By the orbit-stabilizer theorem $|G| \geq 16s^2$, which proves that indeed G is the automorphism group of the toroidal regular map $\{4,4\}_{(s,s)}$ and the graph is a faithful transitive permutation representation. □

4.3 Degrees of regular maps of type $\{3,6\}$

In this section, the degrees of the regular maps of type $\{3,6\}$ will be determined. Moreover, examples of FTPR graphs will be given for some of these degrees.

4.3.1 The possible degrees for the map $\{3,6\}_{(s,0)}$

The smallest map of type $\{3,6\}$, the map $\{3,6\}_{(1,0)}$, has only one vertex, three edges, two faces and a automorphism group with order 12. It has the trivial core-free subgroup and core-free subgroups of order 2. Hence, this map has FTPRs of degree 12 (on the set of flags) and of degree 6 (on the set of darts) but not on the vertices, edges and faces. The map $\{3,6\}_{(2,0)}$ has FTPRs on the set of edges, faces, darts and flags but not on the set of vertices since the subgroup $\langle \rho_1, \rho_2\rangle$ has non-trivial core $\langle(\rho_1\rho_2)^3\rangle$.

Consider now maps of type $\{3,6\}_{(s,0)}$ with $s > 2$. Similarly to the case of the maps of type $\{4,4\}$, we have the following results.

Proposition 4.3.1. *Let G be the automorphism group of $\{3,6\}_{(s,0)}$ ($s > 2$). Then its dihedral subgroups $\langle \rho_i, \rho_j\rangle$ and respective subgroups are core-free, for $i, j \in \{0,1,2\}$.*

Proof. The proof is similiar to the one given in Proposition 4.2.1. □

Corollary 4.3.2. *Let G be the automorphism group of $\{3,6\}_{(s,0)}$ ($s > 2$). Then n is a degree of G if*

$$n \in \{s^2, 2s^2, 3s^2, 4s^2, 6s^2, 12s^2\}.$$

Proof. Since the order o of the dihedrals $\langle \rho_i, \rho_j\rangle$ (for $i, j \in \{0,1,2\}$) and their subgroups is $o \in \{12,6,4,3,2,1\}$, and they are core-free based on Proposition 4.3.1, then G has degrees $\frac{|G|}{o} \in \{s^2, 2s^2, 3s^2, 4s^2, 6s^2, 12s^2\}$. □

In what follows, the other possible degrees for the maps $\{3,6\}_{(s,0)}$ ($s > 2$) are determined.

Proposition 4.3.3. *Let G be the automorphism group of $\{3,6\}_{(s,0)}$ ($s \geq 2$).*

(1) $H = \langle u^a, v^b\rangle$ is core-free and $|G : H| = 12ab$, where $s = lcm(a,b)$.

(2) If d is a divisor of s then $H = \langle u^d\rangle \rtimes \langle \rho_0, \rho_2\rangle$ and $H' = \langle u^d\rangle \rtimes \langle \rho_0\rho_2\rangle$ are core-free. Moreover $|G : H| = 3ds$ and $|G : H'| = 6ds$.

Proof. The proof of (1) is similar the proof of Proposition 4.2.3(1). Let us now prove (2).

(2) Suppose that

$$x \in H \cap H^{\rho_1} = \langle u^d \rangle \rtimes \langle \rho_0, \rho_2 \rangle \cap \langle v^d \rangle \rtimes \langle \rho_0^{\rho_1}, \rho_2^{\rho_1} \rangle.$$

If $x \notin T$, then there is a non-trivial element

$$\rho_0^i \rho_2^j (\rho_0^{\rho_1})^k (\rho_2^{\rho_1})^l \in K := \langle u^d, v^d \rangle$$

for some $i, j, k, l \in \{0, 1\}$. This implies that $(i, j, k, l) = (1, 1, 1, 1)$ and that $t^{-1} \in K$, which is only possible if $d = 1$. If $x \in T$, then $x \in \langle u^d \rangle \cap \langle v^d \rangle = \{id_G\}$. Hence H is core free when $d \neq 1$.

Consider $d = 1$. In this case $u^{-1}\rho_0\rho_2 = v^{-1}\rho_0^{\rho_1}\rho_2^{\rho_1} = (\rho_1\rho_2)^3$ is the unique nontrivial element in $H \cap H^{\rho_1}$, hence $H \cap H^{\rho_1} = \langle (\rho_1\rho_2)^3 \rangle$ which is a subgroup of a dihedral subgroup of G. Thus also in this case H is core-free. As H' is a subgroup of H, then the latter is also core-free.

The order of H is $\frac{4s}{d}$ and the order of H' is $\frac{2s}{d}$ thus $|G : H| = 3ds$ and $|G : H'| = 6ds$. □

In order to determine the remaining degrees of $\{3, 6\}_{(s,0)}$, we will need the following result, for which we give a proof.

Proposition 4.3.4. *Let q be an odd number. The modular equation*

$$x^2 - x + 1 \equiv 0 \bmod q$$

has a solution if and only if all prime divisors p of q are such that $p \equiv 1 \bmod 6$.

Proof. Let 2^* be an inverse of 2 modulo q (which exists because q is odd). Then the equation $x^2 - x + 1 \equiv 0 \bmod q$, which is equivalent to $(x - 2^*)^2 \equiv (-3)(2^*)^2 \bmod q$, has a solution if and only if -3 is a quadratic residue modulo p for every prime divisor of q [Rib01]. In addition -3 is a quadratic residue modulo p for every prime divisor of q if and only if $p \equiv 1 \bmod 3$, or, as by assumption p is odd, $p \equiv 1 \bmod 6$. □

Proposition 4.3.5. *Let d be a divisor of s. Suppose that there exists α, coprime with s/d, such that $\alpha^2 - \alpha + 1 \equiv 0 \bmod (s/d)$. Then $\langle (v^{-\alpha}u)^d,\ \rho_0\rho_1 \rangle$ and $\langle (v^{-\alpha}u)^d,\ \rho_1\rho_2 \rangle$ are core-free subgroups of G with indexes $4ds$ and $2ds$ respectively.*

Proof. Let us consider $H := \langle (v^{-\alpha}u)^d,\ \rho_0\rho_1 \rangle$. We have

$$(v^{-\alpha}u)^{\rho_0\rho_1} = t^{\alpha}v^{-1} = v^{\alpha-1}u^{-\alpha} = v^{\alpha^2}u^{-\alpha} = (v^{-\alpha}u)^{-\alpha}.$$

Hence $H = \langle (v^{-\alpha}u)^d \rangle \rtimes \langle \rho_0\rho_1 \rangle$. Furthermore $|H| = 3s/d$, and therefore $|G : H| = 4ds$.

Now let us prove that H is core-free. Suppose that $H \cap H^{\rho_0}$ is nontrivial. Then there exist $l,\ l' \in \{0, \ldots, s/d - 1\}$ and $j,\ j' \in \{0, 1, 2\}$ such that $(v^{-\alpha}u)^{ld}(\rho_0\rho_1)^j = (t^{-\alpha}u^{-1})^{l'd}(\rho_0\rho_1)^{j'}$. Clearly this is only possible when $j = j'$, $ld = l'd$ and $(\alpha - 1)ld \equiv ld \bmod s$, or equivalently $\alpha^2 l \equiv l \bmod (s/d)$. As $\alpha^2 - \alpha + 1 \equiv 0 \bmod (s/d)$, α is a cubic root of -1 modulo (s/d), $l \equiv -\alpha l \bmod (s/d)$. Consequently, $\alpha^2 l - \alpha l + l \equiv 0 \bmod (s/d)$ can be rewritten as $l \equiv -2l \bmod (s/d)$. As, by Proposition 4.3.4, 3 does not divide (s/d), we obtain $l = 0$. Consequently $H \cap H^{\rho_0} \leq \langle \rho_0\rho_1 \rangle$, which is a subgroup of a dihedral subgroup. Hence H is core-free.

For $H := \langle (v^{-\alpha}u)^d,\ \rho_1\rho_2 \rangle$ the proof is analogous.

□

Corollary 4.3.6. *Let $s > 2$. There exists faithful transitive permutation representations of the automorphism group of the toroidal map $\{3,6\}_{(s,0)}$ when $n = s^2$ and the following:*

1. *$3ds$, $6ds$ or $12ds$ for any divisor d of s,*
2. *$2ds$ and $4ds$ if and only if d is a divisor of s and all prime divisors of s/d are equal to* $1 \bmod 6$.

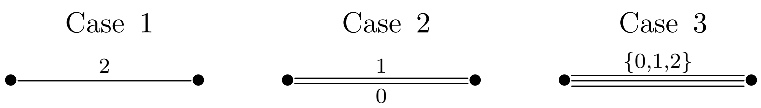
Proof. This a consequence from Corollary 4.1.2, Corollary 4.3.2, Proposition 4.3.3 and Proposition 4.3.5. □

For when $s = 2$, the only case of Corollary 4.3.6 that is not a degree for the map $\{3,6\}_{(2,0)}$ is $n = s^2$ ($n = 4$).

In what follows we prove that the degrees given in Corollary 4.3.6 are the only possible degrees for the automorphism group of the map $\{3,6\}_{(s,0)}$ with $s > 2$. By Lemma 4.1.5, consider now that G is embedded into $S_k \wr S_m$, where $n = km$ with $m \in \{2, 4\}$ being the number of orbits of $T = \langle u, v\rangle$ where u, v and t are as in Section 3.1.2, and k the size of those orbits. The actions of u, v and t on the block i are denoted by u_i, v_i and t_i, respectively. In addition let $K = \langle u_1, v_1\rangle$ be the action of T on the block 1.

Proposition 4.3.7. *If $m = 2$, then $k = sd$ where d is a divisor of s and all prime divisors p of s/d are such that $p \equiv 1 \bmod 6$.*

Proof. Let $m = 2$. The following graphs represent all the possible block actions determined in GAP [GAP21].



In any of the three cases $o(u_1) = o(u_2) = o(v_1) = o(v_2) = o(t_1) = o(t_2) = s$. Let $d = |K : \langle u_1\rangle| = |K : \langle v_1\rangle|$, then $k = ds$. Assume $d \neq s$ since the degree $2s^2$ is already known to exist.

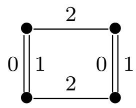
Case 1: Let $u_1^d = v_1^j$. Conjugating by ρ_0 we obtain $u_1^{-d} = t_1^j = u_1^{-j}v_1^j = u_1^{d-j}$, hence $j \equiv 2d \bmod s$. Now conjugating the equation $u_1^d = v_1^{2d}$ by ρ_2 we obtain $u_2^d = t_2^{-2d} = u_2^{2d}v_2^{-2d}$, thus $u_2^d = v_2^{2d}$, which gives $u^d = v^{2d}$, a contradiction.

Case 2: Let $u_1^d = v_1^j$. Conjugating by ρ_2 we obtain $u_1^d = t_1^{-j} = u_1^j v_1^{-j} = u_1^{j-d}$, hence $j \equiv 2d \bmod s$. Now conjugating the equation $u_1^d = v_1^{2d}$ by $\rho_1\rho_0$ we obtain $t_1^d = u_1^{-2d}$, thus $v_1^d = u_1^{-d} = v_1^{-2d}$. Hence $3d \equiv 0 \bmod s$ and therefore $u_1^d = v_1^{-d}$ and, conjugating by ρ_1, $v_2^d = u_2^{-d}$. Consequently $(u_1u_2)^d = (v_1v_2)^{-d}$, a contradiction.

Case 3: Let $u_1^d = v_1^j$. As $|K : \langle u_1\rangle| = d$, j must be a multiple of d, say $j = \alpha d$. Also, we have that s/d is the smallest integers such that $u_1^{ds/d} = id_G$. From the equality $u_1^d = v_1^{\alpha d}$ and as $o(u_1) = o(v_1)$, we have that α and s/d must be coprimes, i.e. $gcd(\alpha, s/d) = 1$. Conjugating the equation $u_1^d = v_1^{\alpha d}$ by $\rho_0\rho_1$ we obtain $v_1^{-d} = t_1^{-\alpha d} = u_1^{\alpha d}v_1^{-\alpha d}$. Thus $u_1^{\alpha d} = v_1^{\alpha d - d}$. Consequently $v_1^{\alpha^2 d} = v_1^{\alpha d-d}$, which implies $d(\alpha^2 - \alpha + 1) \equiv 0 \bmod s$, or equivalently $\alpha^2 - \alpha + 1 \equiv 0 \bmod (s/d)$. The rest follows from Proposition 4.3.4. □

Proposition 4.3.8. *If $m = 4$, then $k = ds$ where d is a divisor of s and all prime divisors p of s/d are such that $p \equiv 1 \bmod 6$.*

Proof. Using GAP [GAP21] it can be checked that there is only one possibility for the action of G given by the following graph.



Let Δ_i denote the block i, $i \in \{1, \dots, 4\}$, as follows.

$$\Delta_2 = \Delta_1\rho_0 = \Delta_1\rho_1,\ \Delta_3 = \Delta_1\rho_2 \text{ and } \Delta_4 = \Delta_3\rho_0 = \Delta_3\rho_1$$

We have $o(u_i) = s$ for $i \in \{1, 2, 3, 4\}$ and the same holds for $o(v_i)$ and $o(t_i)$.

Let $|K : \langle u_1\rangle| = |K : \langle v_1\rangle| = d$. Assume that $d \neq s$. Analogously to Proposition 4.3.7, we may write $u_1 = v_1^{\alpha d}$ with $gcd(\alpha, s/d) = 1$. Then, conjugating by $\rho_0\rho_1$, we get $v_1^{(\alpha-1)d} = v_1^{\alpha^2 d}$, which implies $\alpha^2 - \alpha + 1 \equiv 0 \bmod (s/d)$. The rest follows from Proposition 4.3.4. □

Theorem 4.3.9. *Let $s > 2$. The degrees of a faithful transitive permutation representation of a toroidal regular map of type $\{3,6\}_{(s,0)}$ has degree n if and only if $n = s^2$ or if n is one of the following.*

1. *$3ds$, $6ds$ or $12ds$ for any divisor d of s,*

2. *$2ds$ and $4ds$ if and only if d is a divisor of s and all prime divisors of s/d are equal to $1 \bmod 6$.*

Proof. This is a consequence of Corollary 4.1.2, Proposition 4.1.5, Corollary 4.3.6 and Propositions 4.3.7 and 4.3.8. □

4.3.2 The possible degrees for the map $\{3,6\}_{(s,s)}$

In this section we determine the degrees of $\{3,6\}_{(s,s)}$ using the degrees of $\{3,6\}_{(s,0)}$ and $\{3,6\}_{(3s,0)}$, given in Theorem 4.3.9. Let G be the automorphism group of $\{3,6\}_{(s,s)}$. Consider first the particular case $s = 1$. In this case $\langle(\rho_1\rho_2)^2\rangle$ is a normal subgroup of G, hence the subgroup $\langle\rho_1, \rho_2\rangle$ has non-trivial core, being the action on the set of vertices not faithful. On the other hand, the subgroup $\langle\rho_0, \rho_1\rangle$ is core-free, hence G has a faithful action on the set of faces. All the other possible degrees for $\{3,6\}_{(1,1)}$ are the degrees of $\{3,6\}_{(1,0)}$ multiplied by three. Then, the possible degree of $\{3,6\}_{(1,0)}$ are 36, 18 and 6.

Contrarily to what happens with the map $\{3,6\}_{(2,0)}$, the map $\{3,6\}_{(2,2)}$ has a faithful action on the set of vertices, with degree 12. The remaining degrees for $\{3,6\}_{(2,2)}$ are obtained multiplying each degree of the map $\{3,6\}_{(2,0)}$ by three, hence the set of degrees is 12, 24, 36, 72 and 144. In what follows we deal with the case $s > 2$.

Theorem 4.3.10. *Let $s > 2$. A faithful transitive permutation representation of a toroidal regular map of type $\{3,6\}_{(s,s)}$ has degree n if and only if $n = 3s^2$ or n is one of the following.*

1. *$9ds$, $18ds$ or $36ds$ for any divisor d of s,*

2. *$6ds$ and $12ds$ if and only if d is a divisor of s and all prime divisors of s/d are equal to $1 \bmod 6$.*

Proof. By Corollary 4.1.7 all the degrees given on Theorem 4.3.9 multiplied by 3 are degrees for the map $\{3,6\}_{(s,s)}$, which are those presented in the statement of the theorem. Let us prove that this list is complete.

We have that the degrees of $\{3,6\}_{(s,s)}$ multiplied by 3 are degrees of $\{3,6\}_{(3s,0)}$. Then, by Theorem 4.3.9, we can divide the degrees of $\{3,6\}_{(3s,0)}$ by 3. Hence, the set of degrees of $\{3,6\}_{(s,s)}$ must be contained in

$$\{3s^2,\, 3\delta s,\, 6\delta s, 12\delta s\}$$

with δ being any divisor of $3s$, or in

$$\{2\delta s,\, 4\delta s\}.$$

if and only if δ is a divisor of $3s$ and all prime divisors of $3s/\delta$ are equal to 1 mod 6.

If all prime factors of $3s/\delta$ are 1 mod 6, δ must be divisible by 3. Say $\delta = 3d$, thus $\{2\delta s,\, 4\delta s\} = \{6ds,\, 12ds\}$ where d is a divisor of s and all prime divisors of s/d are equal to 1 mod 6. These degrees are in (2).

Let us now prove that the degrees $3\delta s$, $6\delta s$ and $12\delta s$ correspond to the ones listed in (1). We need only to prove that δ is divisible by 3.

Consider now that G is the automorphism group of $\{3,6\}_{(3s,0)}$ and K is the action of T on block 1, where $T = \langle u, v\rangle$ is the translation group of order $(3s)^2$. The automorphism group of $\{3,6\}_{(s,s)}$ is a factorization of the automorphism group of $\{3,6\}_{(3s,0)}$ by $\langle (uv)^s\rangle$. A faithful transitive permutation representation of $\{3,6\}_{(3s,0)}$ on n points corresponds to a transitive permutation representation (not necessarily faithful) of $\{3,6\}_{(s,s)}$ on $n/3$ triples of points of the form

$$\{x, x(uv)^s, x(uv)^{2s}\}.$$

Note that these points are in the same T-orbit. Let $B := |K : \langle u_1\rangle|$ and $C := |K : \langle v_1\rangle|$, where u_1 and v_1 are the action of u and v on block 1. By Lemma 4.1.1, $\delta = gcd(B, C)$.

Suppose that B divides s. Then $(u_1v_1)^s = u_1^{is}$ for some integer $i \in \{0, 1, 2\}$. In this case the triples of points are as follows

$$\{x, xu_1^{is}, xu_1^{2is}\},$$

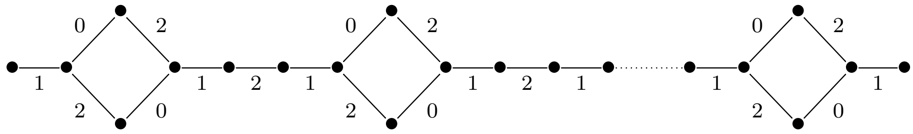
for x in block 1. But then both u^s and v^s fix these triples. By conjugation with ρ_0, ρ_1 and ρ_2 we get that the order of u and v in $\{3,6\}_{(s,s)}$ is at most s, a contradiction (i.e. it does not give a faithful transitive permutation representation). Thus B does not divide s. Particularly 3 must divide B. Analogously 3 divides C and therefore, $\delta = gcd(B, C)$ is divisible by 3, as wanted. $\square$

4.3.3 Examples of Faithful Transitive Permutation Representation Graphs

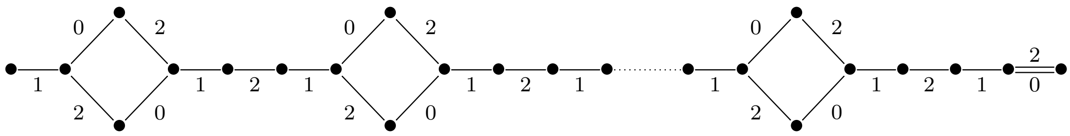
In this section, examples of FTPR graphs of the automorphism groups of $\{3,6\}_{(s,0)}$ and $\{3,6\}_{(s,s)}$ are given. Contrary to what happened in Section 4.2.3, here it will not be given proofs as they follow the same ideas as the ones presented in the referred section.

Lemma 4.3.11. *The following graphs are faithful transitive permutation representation graphs of the automorphism group of $\{3,6\}_{(s,0)}$ with degree $3s$ ($s \geq 3$).Moreover the stabilizer of a point is, up to a conjugacy, $\langle u \rangle \rtimes \langle \rho_0, \rho_2 \rangle$.*

s even

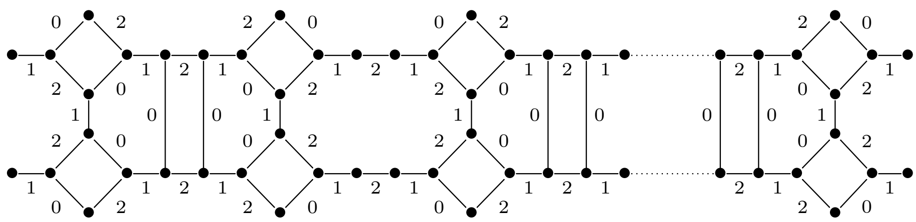


s odd

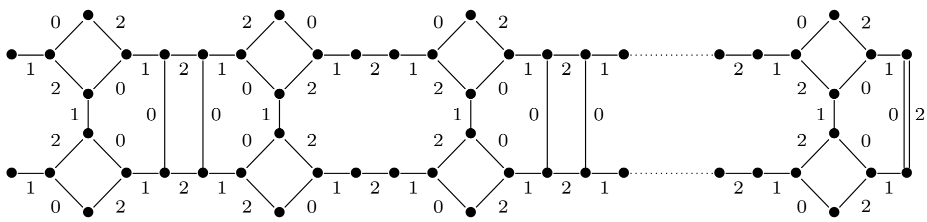


Lemma 4.3.12. *Let s be even. The following graphs are faithful transitive permutation representation graphs of the automorphism group of $\{3,6\}_{(s,0)}$ with degree $6s$ ($s \geq 4$). Moreover the stabilizer of a point is, up to a conjugacy, $\langle u^2 \rangle \rtimes \langle \rho_0, \rho_2 \rangle$.*

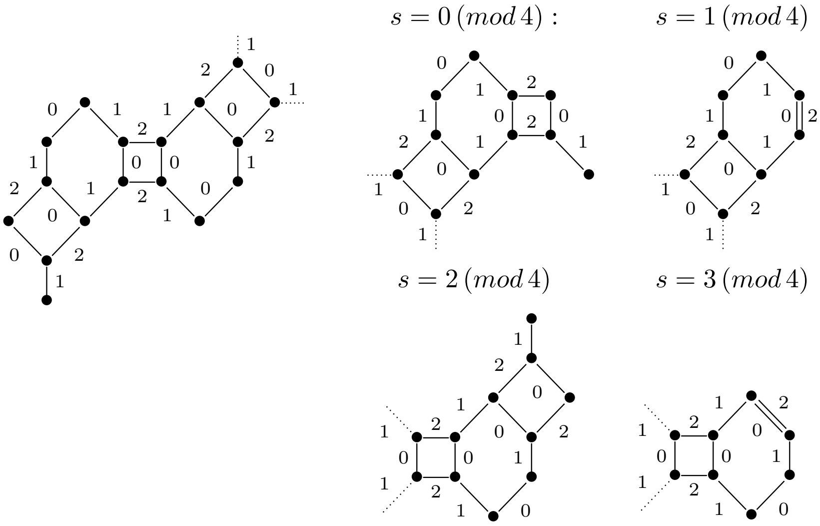
$s = 0\,(mod\,4)$



$s = 2\,(mod\,4)$

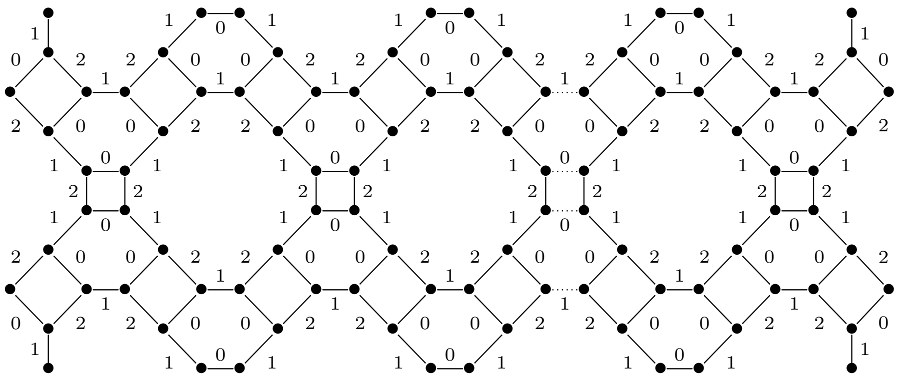


Lemma 4.3.13. *The following graphs are faithful transitive permutation representation graphs of the automorphism group of $\{3,6\}_{(s,s)}$ with degree $9s$ ($s \geq 3$). Moreover the stabilizer of a point is, up to a conjugacy, $\langle gh \rangle \rtimes \langle \rho_0, \rho_2 \rangle$.*

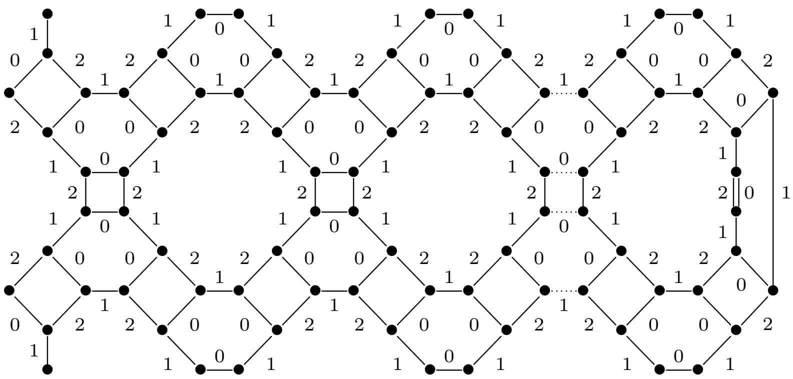


Lemma 4.3.14. *Let s be even. The following graphs are faithful transitive permutation representation graphs of the automorphism group of $\{3,6\}_{(s,s)}$ with degree $18s$ ($s \geq 4$). Moreover the stabilizer of a point is, up to a conjugacy, $\langle (gh)^2 \rangle \rtimes \langle \rho_0, \rho_2 \rangle$.*

$s = 0\,(mod\,4)$



$s = 2\,(mod\,4)$



4.4 Degrees of regular hypermaps of type $(3, 3, 3)$

In this section, the degrees of the regular hypermaps of type $(3, 3, 3)$ will be determined. Moreover an example of a FTPR graph is given.

4.4.1 The possible degrees for the map $(3, 3, 3)_{(s,0)}$

Consider $s \geq 2$, since if $s = 1$ the group would just be D_3. As in the groups of regular toroidal maps, the dihedrals and their subgroups are core-free on the automorphism group of the regular hypermaps $(3, 3, 3)$. Notice that here $\tilde{\rho}_0$ is defined as in Section 3.2.1.

Proposition 4.4.1. *Let G be the automorphism group of $(3, 3, 3)_{(s,0)}$ ($s > 2$). Then its dihedral subgroups $\langle \rho_i, \rho_j \rangle$ and respective subgroups are core-free, for $i, j \in \{\tilde{0}, 1, 2\}$.*

Proof. The proof is similiar to the one given in Proposition 4.2.1. □

Corollary 4.4.2. *Let G be the automorphism group of $(3, 3, 3)_{(s,0)}$ ($s > 2$). Then n is a degree of G if*

$$n \in \{s^2, 2s^2, 3s^2, 6s^2\}.$$

Proof. Since the order o of the dihedrals $\langle \rho_i, \rho_j \rangle$ (for $i, j \in \{\tilde{0}, 1, 2\}$) and their subgroups is $o \in \{6, 3, 2, 1\}$, and they are core-free based on Proposition 4.4.1, then G has degrees $\frac{|G|}{o} \in \{s^2, 2s^2, 3s^2, 6s^2\}$. □

In the following proposition, the remaining degrees are determinded.

Proposition 4.4.3. *Let G be the automorphism group of $(3, 3, 3)_{(s,0)}$ ($s \geq 2$). If d is a divisor of s and a and b are positive integers such that $s = lcm(a, b)$, then*

(1) $H = \langle u^a, v^b \rangle$ is core-free and $|G : H| = 6ab$;

(2) $H = \langle u^d \rangle \rtimes \langle \tilde{\rho}_0 \rangle$ is core-free and $|G : H| = 3ds$; and

(3) $H = \langle (v^{-\alpha}u)^d, \rho_1\rho_2 \rangle$ is core-free, for an α coprime with s/d such that $\alpha^2 - \alpha + 1 \equiv 0 \bmod (s/d)$, and $|G : H| = 2ds$.

Proof. For the case of (1) and (3), the proof is similiar to Proposition 4.2.3(1) and Proposition 4.3.5, respectively. Consider then case (2).

(2) Let $H := \langle u^d \rangle \rtimes \langle \tilde{\rho}_0 \rangle$. Suppose that $x \in H \cap H^{\rho_1} = \langle u^d \rangle \rtimes \langle \tilde{\rho}_0 \rangle \cap \langle v^d \rangle \rtimes \langle \tilde{\rho}_0{}^{\rho_1} \rangle$. If $x \notin T$ then $\tilde{\rho}_0\tilde{\rho}_0{}^{\rho_1} \in T$, a contradiction. Thus $x \in T$ and therefore $x \in \langle u^d \rangle \cap \langle v^d \rangle$, which implies that x is trivial. The order of H is $\frac{2s}{d}$ thus $|G : H| = 3ds$. □

By Corollary 4.1.2, T can be considered to be intransitive, and therefore, by Lemma 4.1.5, G is embedded into $S_k \wr S_m$, where $n = km$ with $m \in \{2, 3, 6\}$ (m being the number of orbits of $T = \langle u, v \rangle$). Moreover $k = ab$ with $s = lcm(a, b)$, or alternatively $k = ds$ with d divisor of s. Since there is a core-free subgroup of G with index $6ab$ and $3ds$, it is only needed to prove that the degrees given before for $m = 2$ (i.e. $n = 2ds$, where d is a divisor of s and all prime factors of s/d are equal to 1 mod 6) are the only possible degrees for the automorphism group of the map $(3, 3, 3)_{(s,0)}$ with $s \geq 2$.

Proposition 4.4.4. *If $m = 2$, then $k = ds$, where d is a divisor of s and all prime factors of s/d are equal to* 1 mod 6.

Proof. If $m = 2$ then T has two orbits of size $k = ds$, with d being a divisor of s, and G has a core-free subgroup H of index $2ds$. But then H is also a core-free subgroup of the automorphism group of the map $\{6,3\}_{(s,0)}$, of index $4ds$. Let $G/H = \{Hg_1, \ldots, Hg_n\}$ and let K be the automorphism group of $\{6,3\}_{(s,0)}$. Since $K/G = \{G, G\rho_0\}$, then $K/H = \{Hg_1, \ldots, Hg_n\} \cup \{Hg_1\rho_0, \ldots, Hg_n\rho_0\}$. We have that $\{Hg_1, \ldots, Hg_n\}$ and $\{Hg_1\rho_0, \ldots, Hg_n\rho_0\}$ can either be in the same T-orbits or in different T-orbits. Hence, the action of K on K/H gives a faithful transitive permutation representation for the map $\{6,3\}_{(s,0)}$ for which T can have either 2 or 4 orbits of size $k = ds$. But then by Propositions 4.3.7 and 4.3.8, $k = ds$ where d is a divisor of s and all prime factors of s/d are equal to 1 mod 6.. □

Theorem 4.4.5. *Let $s \geq 2$. A faithful transitive permutation representation of the automorphism group of $(3,3,3)_{(s,0)}$ has degree n if and only if $n \in \{s^2, 3ds, 6ds\}$ where d is a divisor of s or; $n = 2ds$ where d is a divisor of s and all prime factors of s/d are equal* 1 mod 6.

Proof. This is a consequence of Corollary 4.1.2, Lemmas 4.1.3 and 4.1.5, Corollary 4.4.2, and Propositions 4.4.3 and 4.4.4. □

4.4.2 The possible degrees for the map $(3,3,3)_{(s,s)}$

In this section we determine the degrees of $(3,3,3)_{(s,s)}$ using the degrees of $(3,3,3)_{(s,0)}$ and $(3,3,3)_{(3s,0)}$, given by Theorem 4.4.5. Let n be the degree of a faithful transitive permutation representation of $(3,3,3)_{(s,s)}$ and let $T = \langle u, v\rangle$ be the translation group of $(3,3,3)_{(3s,0)}$ of order $(3s)^2$.

Theorem 4.4.6. *Let $s \geq 2$. A faithful transitive permutation representation of the automorphism group of $(3,3,3)_{(s,s)}$ has degree n if and only if $n \in \{3s^2, 9ds, 18ds\}$ where d is a divisor of s or; $n = 6ds$ where d is a divisor of s and all prime factors of s/d are equal* 1 mod 6.

Proof. Let G be the automorphism group of $(3,3,3)_{(s,s)}$. From Theorem 4.4.5 and Corollary 4.1.7 there are faithful transitive permutation representations with the degrees given in the statement of this theorem. By Theorem 4.4.5, a degree of $(3,3,3)_{(3s,0)}$ is either equal to $(3s)^2$, $3\delta(3s)$ and $6\delta(3s)$, with δ being a divisor of $3s$, or to $2\delta(3s)$, with δ being a divisor of $3s$ and all prime factors of $3s/\delta$ equal 1 mod 6.

Dividing the possible degrees of $(3,3,3)_{(3s,0)}$ by 3, we get that

$$n \in \{3s^2, 2\delta s, 3\delta s, 6\delta s\}$$

with δ dividing $3s$.

The degree $n = 3s^2$ is in set given in the statement of the theorem. If $n = 2\delta s$ then, since δ is a divisor of $3s$ and all prime divisors of $3s/\delta$ must be equal 1 mod 6, $\delta = 3d$ for some divisor d of s. Hence this degree is already included in the set given in the statement of this theorem as $6ds$, with d a divisor of s and all prime factors of s/d are equal 1 mod 6. Let us prove that also on the remaining cases $\delta = 3d$, for some divisor d of s.

The hypermap $(3,3,3)_{(3s,0)}$ contains three copies of the hypermap $(3,3,3)_{(s,s)}$. To be more precise, the automorphism group of $(3,3,3)_{(s,s)}$ is the automorphism group of

$(3,3,3)_{(3s,0)}$ factorized by the translation $(uv)^s$ of order 3. Hence, the points x, $x(uv)^s$ and $x(uv)^{2s}$ of any faithful transitive permutation representation of $(3,3,3)_{(3s,0)}$ are identified under this factorization. Any faithful transitive permutation representation of an action of $(3,3,3)_{(3s,0)}$ on a set X gives a transitive permutation representation (not necessarily faithful), of degree $|X|/3$, of $(3,3,3)_{(s,s)}$ on triples of points of X of the form

$$\left\{x, x(uv)^s, x(uv)^{2s}\right\}$$

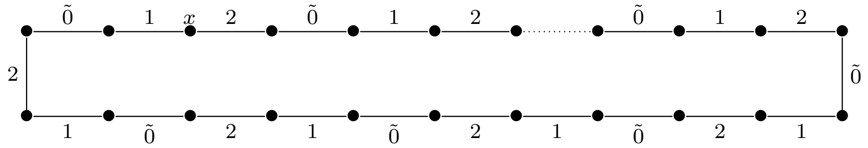
with $x \in X$. Note that these points are in the same T-orbit. Hence the number m of T-orbits is unchanged under this factorization.

To prove that the action on the triple of points is faithful only if $\delta = 3d$, for some divisor d, we can follow an identical proof as the one presented for Theorem 4.3.10. We note that Lemma 4.1.1, that establishes the size of a T-orbit, can be used here. $\square$

4.4.3 Example of Faithful Transitive Permutation Representation Graph

Lastly, we can construct FTPR graphs with any of the core-free subgroups H presented in Proposition 4.4.3 (or conjugates). Here we will only present one possible FTPR graph. Notice that the FTPR graph given on Proposition 4.4.7 is of minimal degree whenever s is not a prime number congruent with 1 mod 6.

Proposition 4.4.7. *Let $s \geq 2$. The following graph is a faithful transitive permutation representation graph of the automorphism group of $(3,3,3)_{(s,0)}$ with degree $3s$.*



Moreover, the stabilizer of a point is, up to conjugacy, $\langle u \rangle \rtimes \langle \rho_0 \rangle$.

Proof. Let $G = \langle \tilde{\rho_0}, \rho_1, \rho_2 \rangle$ be the group with the given permutation representation graph. It is clear from the graph that $\tilde{\rho_0}^2 = \rho_1^2 = \rho_2^2 = (\tilde{\rho_0}\rho_1)^3 = (\tilde{\rho_0}\rho_2)^3 = (\rho_1\rho_2)^3 = (\tilde{\rho_0}\rho_1\rho_2\rho_1)^s = 1$. Hence G must be a subgroup of the automorphism group of the regular hypermap $(3,3,3)_{(s,0)}$ and $|G| \leq 6s^2$.

Consider the vertex x of the permutation representation. Its stabilizer $Stab_G(x)$ contains the subgroup $\langle \tilde{\rho_0}, u \rangle$ of order $2s$. Then, $|Stab_G(x)| \geq 2s$ and, by the Orbit-Stabilizer theorem, $|G| \geq 6s^2$. Consequently, the graph is a faithful transitive permutation representation of the automorphism group of $(3,3,3)_{(s,0)}$. $\square$

Chapter 5

Degrees of Locally Toroidal 4-Polytopes of type $\{4, 4, 4\}$

The known finite locally toroidal regular polytopes can be found in [MS02], particularly, the classification of the finite universal locally toroidal regular 4-polytopes whose facets and vertex-figures are maps $\{4,4\}_{(s,0)}$ or $\{4,4\}_{(s,s)}$, also known as polytopes of Euclidean type $[4,4,4]$ [FLW20]. This classification is almost complete and listed in the following table, where all parameters, corresponding finite regular polytopes $\{\{4,4\}_{(t_1,t_2)}, \{4,4\}_{(s_1,s_2)}\}$, and the respective automorphism groups G are given. The classification of the universal finite regular polytopes $\{\{4,4\}_{(t,0)}, \{4,4\}_{(s,0)}\}$, for $s, t \geq 3$ and both odd, is still an open problem, being conjectured in [MS02] that those given in Table 5.1 are the only finite ones.

Table 5.1: The known finite universal regular polytopes $\{\{4,4\}_{(t_1,t_2)}, \{4,4\}_{(s_1,s_2)}\}$.

(t_1, t_2)	(s_1, s_2)	$\lvert G\rvert$	G
(2,0)	(s,s), $s \geq 2$	$64s^2$	$(C_2 \times C_2) \rtimes [4,4]_{(s,s)}$
(2,0)	$(2s,0)$, $s \geq 1$	$128s^2$	$(C_2 \times C_2) \rtimes [4,4]_{(2,0)}$, $s = 1$ $((C_2 \times C_2) \rtimes [4,4]_{(s,s)}) \times C_2$, $s \geq 2$
(3,0)	(3,0)	1440	$S_6 \times C_2$
(3,0)	(4,0)	36864	$C_2 \wr [4,4]_{(3,0)}$
(3,0)	(2,2)	2304	$(S_4 \times S_4) \rtimes (C_2 \times C_2)$
(2,2)	(2,2)	1024	$C_2^4 \rtimes [4,4]_{(2,2)}$
(2,2)	(3,3)	9216	$C_2^6 \rtimes [4,4]_{(3,3)}$
(3,0)	(5,0)	3916800	$Sp_4(4) \times C_2 \times C_2$

In this chapter we will determine all possible degrees of the infinite families of lines 1 and 2 of Table 5.1. The degrees of the remaining polytopes of Table 5.1 can be determined computationally. Indeed we were able to compute them in GAP [GAP21] (see Table 5.2). Using the same algorithm we found the degrees of the polytopes of lines 1 and 2 of Table 5.1 only up to $s = 79$ and $s = 47$, respectively. To determine the degrees in Table 5.2, we can use GAP [GAP21] to calculate all the subgroups, up to conjugacy, of the automorphism group of the polytopes and then proceed to determine which are core-free or not by checking the size of the core, taking records of the index of those which are core-free. For most cases we can use the group as a finitely presented group on GAP. However, for polytopes $\{\{4,4\}_{(3,0)},\{4,4\}_{(4,0)}\}$ and $\{\{4,4\}_{(3,0)},\{4,4\}_{(5,0)}\}$, using it as a finitely presented group ends up not being efficient. Using an isomorphic permutation group solves the problem with the automorphism group of these two polytopes.

Table 5.2: The degrees of the known finite universal regular polytopes $\{\{4,4\}_{(t_1,t_2)},\{4,4\}_{(s_1,s_2)}\}$.

(t_1,t_2)	(s_1,s_2)	Set of Possible Degrees	Minimal degree Core-free subgroups
(2,0)	(2,0)	$\{m \mid m \text{ a divisor of } 128 \wedge m \geq 32\}$	$\langle\rho_0,\rho_2\rangle$
(2,0)	(4,0)	$\{m \mid m \text{ a divisor of } 512 \wedge m \geq 32\}$	$\langle\rho_0,\rho_3,(\rho_1\rho_2)^2\rangle$
(3,0)	(3,0)	$\{m \mid m \text{ a divisor of } 1440 \wedge m \geq 60\wedge$ $\wedge m \neq 96\} \cup \{40,30,24,20,12\}$	$\langle\rho_0,\rho_2,\rho_1\rho_3\rangle$
(3,0)	(4,0)	$\{m \mid m \text{ a divisor of } 36864 \wedge m \geq 72\}$ $\cup\,\{18,36,48\}$	$\langle\rho_0,\rho_1,\rho_3^{\rho_2},\rho_3^{\rho_2\rho_1\rho_2}\rangle$
(3,0)	(2,2)	$\{m \mid m \text{ a divisor of } 2304 \wedge m \geq 12\}$	$\langle\rho_0,\rho_2,\rho_3,\rho_1\rho_2\rho_1\rangle$
(2,2)	(2,2)	$\{m \mid m \text{ a divisor of } 1024 \wedge m \geq 16\}$	$\langle\rho_0,\rho_1,\rho_2\rangle$
(2,2)	(3,3)	$\{m \mid m \text{ a divisor of } 9216 \wedge m \geq 24\}$	$\langle\rho_0,\rho_1\rho_0\rho_1,\rho_3,\rho_2\rho_3\rho_2,\rho_3^{\rho_2\rho_1\rho_2}\rangle$
(3,0)	(5,0)	$\{2^i\cdot 255,\ 2^i\cdot 1275,$ $2^i\cdot 3825,\ 2^i\cdot 425 \mid 2\leq i\leq 10\}$ $\cup\,\{2^i\cdot 765 \mid 3\leq i\leq 10\}$ $\cup\,\{2^i\cdot 15,\ 2^i\cdot 17 \mid 5\leq i\leq 6\}$ $\cup\,\{2^i\cdot 85 \mid i\in\{2,6,7,8\}\}$ $\cup\,\{2^i\cdot 225 \mid 8\leq i\leq 10\}$ $\cup\,\{2^i\cdot 153 \mid 7\leq i\leq 10\}$ $\cup\,\{2^i\cdot 51 \mid 7\leq i\leq 8\}$	$\langle(\rho_0\rho_1\rho_2)^2,(\rho_1\rho_2\rho_0)^2,[(\rho_1\rho_2)^2]^{\rho_3}\rangle$

5.1 The finite universal regular polytopes $\{\{4,4\}_{(t_1,t_2)},\{4,4\}_{(s_1,s_2)}\}$

Consider the regular toroidal maps $\{4,4\}_{(s,0)}$ and $\{4,4\}_{(s,s)}$ introduced in Chapter 3 and their respective unitary translations. The universal regular polytope $\{\{4,4\}_{(t_1,t_2)},\{4,4\}_{(s_1,s_2)}\}$, where $(t_1,t_2)\in\{(t,t),(t,0)\}$ and $(s_1,s_2)\in\{(s,s),(s,0)\}$ with $t,s\geq 2$, is the Coxeter group $[4,4,4]=\langle\rho_0,\ldots,\rho_3\rangle$, factored out by two relations of the following set: one with parameter t and the other with parameter s.

$$\{(\rho_0\rho_1\rho_2\rho_1)^t,\ (\rho_1\rho_2\rho_3\rho_2)^s,\ (\rho_0\rho_1\rho_2)^{2t},\ (\rho_1\rho_2\rho_3)^{2s}\}$$

The effect of this factorization is that the facets and vertex figures of the honeycomb $\{4,4,4\}$, which are planar infinite tilings $\{4,4\}$, collapse to a finite toroidal regular map, $\{4,4\}_{(t_1,t_2)}$ and $\{4,4\}_{(s_1,s_2)}$, respectively. That is, $G_3=\langle\rho_0,\rho_1,\rho_2\rangle$ and $G_0=\langle\rho_1,\rho_2,\rho_3\rangle$ are the automorphism groups of the toroidal maps $\{4,4\}_{(t_1,t_2)}$ and $\{4,4\}_{(s_1,s_2)}$, respectively. This construction gives always a regular polytope of type $\{4,4,4\}$, but the known finite ones are those given in Table 5.1.

5.2 Relations between the degrees of types $[4,4]$ and $[4,4,4]$

As mentioned in Section 2.6, there is a correspondence between core-free subgroups and faithful transitive actions. Also, as seen in Lemma 4.1.6, if H is a core-free subgroup of G, and G is a subgroup of K of index κ, then H is also core-free in K and $|K:H| = \kappa|G:H|$. Hence, if G has a faithful transitive permutation representation of degree n, then K has a faithful transitive permutation representation of degree κn. Consequently, we have the following result.

Corollary 5.2.1. *If n is a degree of the toroidal map $\{4,4\}_{(s,s)}$ (resp. $\{4,4\}_{(2s,0)}$), then $4n$ is a degree of the locally toroidal polytope $\{\{4,4\}_{(2,0)}, \{4,4\}_{(s,s)}\}$ (resp. $\{\{4,4\}_{(2,0)}, \{4,4\}_{(2s,0)}\}$).*

This guarantees that $\{\{4,4\}_{(2,0)}, \{4,4\}_{(s,s)}\}$ has FTPRs with degrees

$$8s^2,\ 16ab,\ 32ab \text{ and } 64ab,$$

with $s = lcm(a,b)$; while $\{\{4,4\}_{(2,0)}, \{4,4\}_{(2s,0)}\}$ has faithful transitive permutation representations with degrees

$$16s^2,\ 32ab,\ 64ab \text{ and } 128ab$$

with $s = lcm(a,b)$. We will prove that these lists are incomplete.

In what follows we give conditions under which there is a one-to-one correspondence between the degrees of $\{4,4\}_{(s,s)}$ and $\{\{4,4\}_{(2,0)}, \{4,4\}_{(s,s)}\}$. Before that, we prove the following result that can be used for any group having a central involution.

Proposition 5.2.2. *Let G be a transitive group of degree n containing a central involution α. Then G is embedded into $S_2 \wr S_{\frac{n}{2}}$, where the blocks are the $\langle\alpha\rangle$-orbits. If $\langle\alpha\rangle$ is the kernel of this embedding, then $\frac{n}{2}$ is the degree of a faithful transitive permutation representation of $G/\langle\alpha\rangle$.*

Proof. The orbits of α, of size two, form a block system for G. Consider the group homomorphism $f : G \to S_{\frac{n}{2}}$ induced by the action of G on these blocks. Therefore the isomorphism $G/\langle\alpha\rangle \cong f(G)$, determines a FTPR of $G/\langle\alpha\rangle$ on $\frac{n}{2}$ points. □

Proposition 5.2.3. *Let $s > 2$. Let $x \in \{1,\dots,n\}$ be a point of a faithful transitive permutation representation of $\{\{4,4\}_{(2,0)}, \{4,4\}_{(s,s)}\}$ whose group is $G = \langle\rho_0,\rho_1,\rho_2,\rho_3\rangle$. Let $G_0 = \langle\rho_1,\rho_2,\rho_3\rangle$ (the group of $\{4,4\}_{(s,s)}$). If ρ_0 is fixed-point free, then G_0 acts faithfully and transitively on the 4-sets $\{x, x\rho_0, x(\rho_0\rho_1)^2, x\rho_1\rho_0\rho_1\}$. In particular, G_0 has a faithful transitive permutation representation of degree $n/4$.*

Proof. Let $\delta := (\rho_0\rho_1)^2$. Let $f : G \to S_{\frac{n}{2}}$ be the embedding of G into $S_2 \wr S_{\frac{n}{2}}$ determined by the $\langle\delta\rangle$-orbits. Firstly let us prove that $Ker(f) = \langle\delta\rangle$.

Suppose that $\langle g, h\rangle \cap Ker(f)$ is nontrivial. As $Ker(f)$ is embedded into $C_2^{\frac{n}{2}}$, all the elements of the kernel are involutions. The only involutions of $\langle g, h\rangle$ are $g^{s/2}$, $h^{s/2}$ or $(gh)^{s/2}$ (which can only happen if s is even). As the $Ker(f)$ is normal in G, $(gh)^{s/2} \in Ker(f)$. As $(gh)^{s/2}$ is a central involution of G, the actions of $(gh)^{s/2}$ and δ coincide, hence $(gh)^{s/2} = \delta$, a contradiction. Then, $\langle g, h\rangle \cap Ker(f)$ is trivial. Consequently $f(\rho_1)$, $f(\rho_2)$ and $f(\rho_3)$ are involutions and the group generated by these involutions satisfies all the defining relations of $\{4,4\}_{(s,s)}$. This implies that $H=f(G_0)$ must be the group of a toroidal map $\{4,4\}_{(s,s)}$. As ρ_0 is fixed-point-free, it cannot be in $Ker(f)$ or its action would also coincide with the one of δ, a contradiction. As $\langle\rho_0,(\rho_0\rho_1)^2\rangle$ is a normal subgroup of G, its orbits have the same size, hence we may conclude that they are 4-sets of the form $\{x,\, x\rho_0,\, x(\rho_0\rho_1)^2,\, x\rho_1\rho_0\rho_1\}$ with $x\in\{1,\ldots,n\}$. In particular, $f(\rho_0)$ is nontrivial and commutes with $f(G_0)$, since $\rho_0\notin Ker(f)$ and $f((\rho_0\rho_1)^2)=id_{f(G)}$, which implies that $f(\rho_0)f(\rho_1)=f(\rho_1)f(\rho_0)$. This shows that $G/Ker(f)\cong C_2\times H$, which is precisely the string C-group with disconnected Coxeter diagram obtained factoring G by $\langle\delta\rangle$. In particular, $Ker(f)=\langle\delta\rangle$.

By Proposition 5.2.2 $G/\langle\delta\rangle$ has a FTPR of degree $\frac{n}{2}$. Furthermore $G/\langle\delta\rangle$ is isomorphic to $\langle\alpha\rangle\times H$, where $\alpha=f(\rho_0)$ is an involution (ρ_0 acting on 2-sets $\{x,x(\rho_0\rho_1)^2\}$).

Now $\langle\alpha\rangle\times H$ is embedded into $S_2\wr S_{\frac{n}{4}}$. Moreover we may use a similar argument to the one before to conclude that the kernel of this embedding is $\langle\alpha\rangle$.

Factoring $\langle\alpha\rangle\times H$ by $\langle\alpha\rangle$ gives a group isomorphic to G_0. Thus by Proposition 5.2.2 G_0 has a FTPR of degree $\frac{n}{4}$.

Moreover, the orbits of α are pairs of 2-sets $\{\{x,\, x(\rho_0\rho_1)^2\},\, \{x\rho_0,\, x\rho_0(\rho_0\rho_1)^2\}\}$ and the action of G_0 is faithful on the 4-sets $\{x,\, x(\rho_0\rho_1)^2,\, x\rho_0,\, x\rho_0(\rho_0\rho_1)^2\}$. □

The following corollary to Proposition 5.2.2 gives sufficient conditions that guarantee an one-to-one correspondence between the degrees of $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$ and $\{\{4,4\}_{(2,0)},\{4,4\}_{(2s,0)}\}$ for $s\geq 2$.

Corollary 5.2.4. *Let G be the group of $\{\{4,4\}_{(2,0)},\{4,4\}_{(2s,0)}\}$ ($s\geq 2$) acting transitively and faithfully on n points. Suppose that f is the embedding of G into $S_{\frac{n}{2}}$ determined by the connected components of $\delta:=(\rho_1\rho_2\rho_3)^{2s}$. If $Ker(f)=\langle\delta\rangle$ then $n=2n'$ where n' is a degree of a faithful transitive permutation representation of the group of $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$.*

Proof. Let $G=\langle\rho_0,\rho_1,\rho_2,\rho_3\rangle$ be the group of $\{\{4,4\}_{(2,0)},\{4,4\}_{(2s,0)}\}$. The translation $\delta:=(\rho_1\rho_2\rho_3)^{2s}$ is a central involution in G. Moreover $G/\langle\delta\rangle$ is the group of $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$. Thus by Proposition 5.2.2 we get the correspondence between the degrees of $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$ and $\{\{4,4\}_{(2,0)},\{4,4\}_{(2s,0)}\}$ stated. □

5.3 The degrees of $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$ and $\{\{4,4\}_{(2,0)},\{4,4\}_{(2s,0)}\}$

5.3.1 The possible degrees of $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$

Let n be the degree of a FTPR of the group $G=\langle\rho_0,\rho_1,\rho_2,\rho_3\rangle$ of the polytope $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$ with $s\geq 2$. Let us denote by G_0 the maximal parabolic subgroup of G generated by $\{\rho_1,\,\rho_2,\,\rho_3\}$, that is, the group of $\{4,4\}_{(s,s)}$. Consider the subgroup T of G_0 generated by $g:=(\rho_1\rho_2\rho_3)^2$ and $h:=g^{\rho_1}$, as defined in Section 3.1.1. We have the following relations.

$$h=h^{\rho_0},\ g=g^{\rho_0},\ g=g^{\rho_2},\ h^{-1}=h^{\rho_2}\text{ and }h^{-1}=g^{\rho_3}.$$

Proposition 5.3.1. *The group G has a faithful transitive permutation representation of degree $8s^2$, $16ab$, $32ab$ and $64ab$, where and a and b are positive integers such that $s=lcm(a,b)$.*

Proof. This follows from Theorem 4.2.6 and Corollary 5.2.1. □

The degrees given above are in correspondence with the indexes of core-free subgroups of $\{4, 4\}_{(s,s)}$. Moreover, the core-free subgroups of G_0 can be used to build FTPR graphs of G. Let us now give other core-free subgroups corresponding to degrees that are not listed in Proposition 5.3.1.

Proposition 5.3.2. *Let a and b be positive integers such that $s = lcm(a, b)$. The subgroups*

1. $\langle\rho_0\rangle \times \langle\rho_2, \rho_3\rangle$;
2. $\big(\langle\rho_0\rangle \times \langle g^{a/2}, h^b\rangle\big) \rtimes \langle\rho_2, \rho_1\rho_2\rho_1\rangle$ *if a is even and* $lcm(a/2, b) = s$ *and* $\big(\langle\rho_0\rangle \times \langle g^a, h^{b/2}\rangle\big) \rtimes \langle\rho_2, \rho_1\rho_2\rho_1\rangle$ *if b is even and* $lcm(a, b/2) = s$;
3. $\big(\langle\rho_0\rangle \times \langle g^a, h^b\rangle\big) \rtimes \langle\rho_2, \rho_1\rho_2\rho_1\rangle$

are core-free subgroups of G with indexes $4s^2$, $4ab$ and $8ab$, respectively.

Proof. (1) Let $H = \langle\rho_0\rangle\times\langle\rho_2, \rho_3\rangle \cong C_2\times D_8$. As $\langle\rho_0\rangle$ and $\langle\rho_0^{\rho_1}\rangle$ have a trivial intersection, we have

$$H \cap H^{\rho_1} = \langle\rho_2, \rho_3\rangle \cap \langle\rho_2^{\rho_1}, \rho_3\rangle = \langle\rho_3\rangle.$$

In addition,

$$H \cap H^{\rho_1\rho_2} = \langle\rho_2, \rho_3\rangle \cap \langle\rho_2^{\rho_1}, \rho_3^{\rho_1\rho_2}\rangle = \langle\rho_3^{\rho_2}\rangle,$$

hence $H \cap H^{\rho_1} \cap H^{\rho_1\rho_2}$ is trivial. Since $|H| = 16$, we have that $|G : H| = 4s^2$.

(2) Let $H = \big(\langle\rho_0\rangle \times \langle g^{a/2}, h^b\rangle\big) \rtimes \langle\rho_2, \rho_1\rho_2\rho_1\rangle$. As $lcm(a/2, b) = s$, $\langle g^{a/2}, h^b\rangle$ and $\langle h^{a/2}, g^b\rangle$ have trivial intersection. In addition the intersections $\langle\rho_2, \rho_2^{\rho_1}\rangle \cap \langle\rho_2^{\rho_3}, \rho_2^{\rho_3\rho_1}\rangle$ and $\langle\rho_0\rangle \cap \langle\rho_0^{\rho_1}\rangle$ are trivial. Therefore $H \cap H^{\rho_1} \cap H^{\rho_3}$ is trivial. Since $|H| = 8\frac{s^2}{\frac{a}{2}b}$, we have that $|G : H| = 4ab$.

The proofs for the other subgroup given in (2) and for the subgroup given in (3) follow similar arguments. □

Lemma 5.3.3. *The following two conditions are equivalent.*

1. *a and b are even numbers.*
2. *a is even and $lcm(a/2, b) = lcm(a, b)$, or b is even and $lcm(a, b/2) = lcm(a, b)$.*

Proof. Suppose that a and b are both even. Let α and β be the maximal integers such that 2^α divides a and 2^β divides b. Then if $\alpha \leq \beta$, then $lcm(a/2, b) = lcm(a, b)$, otherwise $lcm(a, b/2) = lcm(a, b)$.

To prove that (2) implies (1), observe that if a is even and b is odd, then $lcm(a/2, b) < lcm(a, b)$. □

In what follows, it will be proven that the degrees given in Proposition 5.3.2 are the only ones missing in the list of degrees of $\{\{4, 4\}_{(2,0)}, \{4, 4\}_{(s,s)}\}$ obtained by Proposition 5.3.1.

Similarly to Lemma 4.1.5 we have the following result.

Lemma 5.3.4. *If $n \neq s^2$, then G is embedded into $S_k \wr S_m$ with $n = km$ $(m,\ k > 1)$ and* *(i)* $k = ab$ *where* $s = lcm(a, b)$ *and,*

(ii) m *is a divisor of* $\frac{|G|}{s^2} = 64$.

Proof. As $T = \langle g, h\rangle$ is a normal subgroup of G, in the proof of Lemma 4.1.5 replace the group of a toroidal map by our G, the group of the locally toroidal polytope $\{\{4, 4\}_{(2,0)}, \{4, 4\}_{(s,s)}\}$. □

Let m be the number of T-orbits and k be the size of a T-orbit (thus $n = km$) of a FTPR of G. We will consider $m \in \{1, 2, 4\}$, as the existence of FTPRs of degrees $n = mab$ for $m \in \{8, 16, 32, 64\}$ (for any integers a and b with $lcm(a, b) = s$) is guaranteed by Propositions 5.3.1 and 5.3.2. Given a numbering on the T-orbits, let us denote by g_i and h_i the actions of g and h on block i (or T-orbit i), respectively. Let Δ_i denote the block i.

We will consider cases $m = \{1, 2, 4\}$, but before we proceed, let us prove a result that will be used later in both cases.

Proposition 5.3.5. *Let* K *be a transitive group containing the regular subgroup* $H = \langle\alpha, \beta \,|\, \alpha^a = \beta^b = [\alpha, \beta]\rangle$ *with* $a, b \geq 1$*. If* $\delta \in K$ *is an involution commuting with both* α *and* β*, then* $\delta \in H$ *and one of the following situations must occur:*

1. $\delta = \alpha^{a/2}$ *and* a *is even;*

2. $\delta = \beta^{b/2}$ *and* b *is even or;*

3. $\delta = \alpha^{a/2}\beta^{b/2}$ *and both* a *and* b *are even.*

Proof. If $\delta \in H$, then, as δ is an involution, there are at most the three possibilities for δ given in the statement of this proposition. Suppose that $\delta \notin H$. Then for some integers i and j, $\delta\alpha^i\beta^j$ is in the stabilizer of a point x [Cam99, page 9]. As H is regular, any point y can be written as xh with $h \in H$. But then, $y\delta\alpha^i\beta^j = xh\delta\alpha^i\beta^j = x\delta\alpha^i\beta^j h = xh = y$. This implies that $\delta\alpha^i\beta^j$ is trivial, a contradiction. □

In Proposition 5.3.5, when $a \neq 1$ and $b = 1$, the group H, is a cyclic group of order a. Then there is only one possibility for an involution commuting with the generator α of H, that is, $\alpha^{a/2}$.

Proposition 5.3.6. *There cannot be faithful transitive permutation representation with a single* T*-orbit.*

Proof. Suppose that $m = 1$, , that is, T is transitive. In this case T is regular, hence $n = s^2$. If ρ_0 has a fixed point then, as ρ_0 commutes with g and h, ρ_0 is trivial, a contradiction. Thus ρ_0 is fixed-point free. Hence, by Proposition 5.2.3, there exists a FTPR of the group of the toroidal map $\{4, 4\}_{(s,s)}$ on 4-sets with T being transitive. But T cannot be transitive, as proven in Proposition 4.1.3. □

Proposition 5.3.7. *If there are two* T*-orbits, then* $k \neq s$*.*

Proof. Suppose that $k = s$. If ρ_0 is fixed-point free, then by Proposition 5.2.3, $\{4, 4\}_{(s,s)}$ has a FTPR of degree $n = s/2$, contradicting Theorem 4.2.6. Hence, ρ_0 must have a fixed point. Consequently, $(\rho_0\rho_1)^2$ fixes the blocks and therefore, s is even. Moreover, as

ρ_0 commutes with both g and h, it fixes a block point-wise. In addition, ρ_1 must swap the two blocks, otherwise ρ_0 would be trivial.

As $k = s$, either the action of g within a block, say Δ_1, has order s or $gcd(|g_1|, |g_2|) = 1$. Let us consider the two cases separately.

Firstly assume g_1 and h_2 are cycles of order s. Since g and h commute,

$$g = g_1 h_2^{\alpha} \text{ and } h = g_1^{\beta} h_2$$

for some integers α and β. As $(\rho_0\rho_1)^2$ is a fixed-point free central involution, by Proposition 5.3.5, we have $(\rho_0\rho_1)^2 = g_1^{s/2} h_2^{s/2}$. Assume without loss of generality that ρ_0 fixes a point in Δ_1, so that $\rho_0^{\rho_1} = g_1^{s/2}$ and $\rho_0 = h_2^{s/2}$.

If α and β are even, one gets $g^{s/2} = g_1^{s/2}$ and $h^{s/2} = h_2^{s/2}$, hence $(\rho_0\rho_1)^2 = (gh)^{s/2}$, a contradiction.

If α is odd and β is even, one gets $g^{s/2} = g_1^{s/2} h_2^{s/2}$ and $h^{s/2} = h_2^{s/2}$, hence $(gh)^{s/2} = g_1^{s/2} = \rho_0^{\rho_1}$, a contradiction. Similarly if α is even and β is odd one gets the contradiction $(gh)^{s/2} = h_2^{s/2} = \rho_0$.

If α and β are both odd, one gets $g^{s/2} = h^{s/2}$, hence $(gh)^{s/2}$ is trivial, a contradiction.

Now consider the case $gcd(|g_1|, |g_2|) = 1$. Let $|g_1| = a$, $|g_2| = b$ with b odd. Then $ab = s$. In this case, $(\rho_0\rho_1)^2 = g_1^{a/2} h_2^{a/2}$. But also, as b is odd, $(gh)^{s/2} = (g_1h_2)^{a/2}$, a contradiction.

□

Proposition 5.3.8. *There cannot be faithful transitive permutation representation with two T-orbits.*

Proof. Let $m = 2$. By Lemma 4.1.1 and Proposition 5.3.7, $k > s$. Then both $\langle g \rangle$ and $\langle h \rangle$ act intransitively within a block. If ρ_0 is fixed-point free, then the group of $\{4, 4\}_{(s,s)}$ has a FTPR on $n/4$ points by Proposition 5.2.3, with T having either one or two orbits. We know T cannot have one orbit by Proposition 4.1.3 and if T has two orbits, then $n/4 = 2s^2$ by Lemma 4.2.5, meaning that the size of a T-orbit acting on n points is $k = (2s)^2$, a contradiction by Lemma 4.1.1. Thus ρ_0 must have a fixed point and thus must fix an entire block point-wise.

The permutation ρ_0 cannot have a trivial action in both blocks, hence $\Delta_1\rho_1 = \Delta_2$ and $(\rho_0\rho_1)^2$ fixes the blocks. In particular, s is even. In addition, neither ρ_2 nor ρ_3 can swap the blocks. Hence, since ρ_3 must fix a block, therefore $|g_i| = |g_i^{\rho_3}| = |h_i^{-1}| = s$. As in addition $|g_1| = |g_2|$ and $|h_1| = |h_2|$, we must have $|g_i| = |h_i|$ for $i = 1, 2$ (meaning that each cycle of the cyclic decomposition of g, and h, has order s).

Assume that ρ_0 acts trivially on block Δ_1. As $(\rho_0\rho_1)^2$ is a central involution, Proposition 5.3.5 determines the possibilities for $(\rho_0\rho_1)^2$. The action of $(\rho_0\rho_1)^2$ on block Δ_1 cannot be $(g_1h_1)^{s/2}$ otherwise $(\rho_0\rho_1)^2 = (gh)^{s/2}$. Thus either $(\rho_0\rho_1)^2 = (g_1h_2)^{s/2}$ or $(\rho_0\rho_1)^2 = (h_1g_2)^{s/2}$. Since $(\rho_0\rho_1)^2 = ((\rho_0\rho_1)^2)^{\rho_3}$, in any case one gets $g_i^{s/2} = h_i^{s/2}$ (for $i \in \{1, 2\}$). This gives $g^{s/2} = h^{s/2}$, a contradiction. □

With the cases $m \in \{1, 2\}$ out of the way, we will focus now on the case where we have four T-orbits. Before that, consider the following two results.

Proposition 5.3.9. *Let $w = w_1w_2 \dots w_l$ with $w_j \in \{\rho_i \,|\, i = 0, 1, 2, 3\}$ for $j \in \{1, \dots, l\}$ and such that*

$$|\{j \in \{1, \dots, l\} \,:\, w_j = \rho_1 \vee w_j = \rho_3\}|$$

is odd. If w acts non-trivially within a T-orbit, then $k = s^2$.

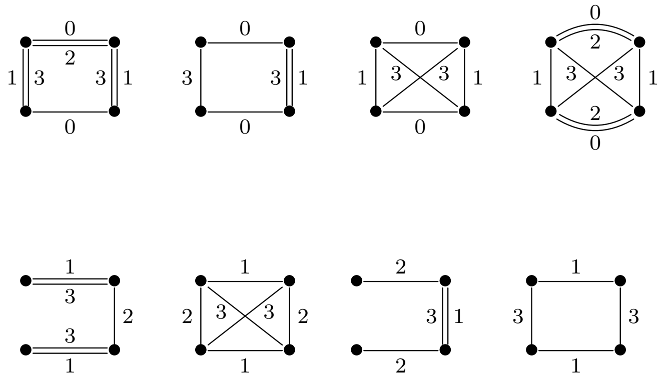
Proof. Suppose that w acts non-trivially on Δ_1 and let $K = \langle g_1, h_1 \rangle$. As $g_1^w = h_1^{\pm 1}$, we have $|g_1| = |h_1|$. Moreover, by conjugation, we get $|g_i| = |h_i|$ for $i \in \{1, \dots, 4\}$. Hence $|g_1| = |h_1| = s$. Let $B = |K : \langle g_1 \rangle| = |K : \langle h_1 \rangle|$. We have $k = |K| = Bs$. Let us prove that $B = s$. There exists an integer j such that $g_1^B = h_1^{Bj}$. Conjugating by w, this implies that $h_1^B = g_1^{Bj}$. Hence, $(g_1h_1)^B = (g_1h_1)^{Bj}$. As $|g_1h_1| = s$, $B \equiv Bj \bmod s$. Now the equality $g_1^B = h_1^{Bj}$ can be rewritten as $g_1^B = h_1^B$, or equivalently $(g_1h_1^{-1})^B$ is trivial. As $|g_1h_1^{-1}| = s$, we have $B = s$. □

Proposition 5.3.10. *The element $u := \rho_1\rho_2\rho_3\rho_2$ cannot fix all T-orbits.*

Proof. Suppose that u fixes Δ_i for some $i \in \{1, \dots, 4\}$. Then there exist a pair of integers r and t such that ug^rh^t fixes a point $x \in \Delta_i$. Hence u^s fixes x. Moreover, as u^s commutes with both g and h, it fixes every point in Δ_i.

Thus if u fixes every block, then u^s is trivial, a contradiction. □

Proposition 5.3.11. *If $m = 4$ and $k \neq s^2$ then the action of G on the blocks is described by one of the following graphs.*



Proof. The group G acting on the 4 blocks is a group satisfying the defining relations of G and such that

$$(\rho_0\rho_1\rho_2\rho_1)^2 \text{ and } (\rho_1\rho_2\rho_3)^2$$

are both trivial. Under these conditions, using GAP [GAP21], we found the 51 block actions given in Table 5.3 on page 50. By Propositions 5.3.9 and 5.3.10 this list can be reduced to only eight possibilities, those given in this proposition. □

Proposition 5.3.12. *If $m = 4$ and $k \neq s^2$, then $k = ab$, with a and b being even divisors of s such that $s = lcm(a, b)$.*

Proof. Let us deal with two cases separately: (1) ρ_0 is fixed-point free; (2) ρ_0 has a fixed-point.

(1) If ρ_0 is fixed-point free, then the group of $\{4,4\}_{(s,s)}$ has a FTPR on $n/4$ points by Proposition 5.2.3, with T having either one, two or four orbits. The first possibility, T having exactly one orbit, cannot happen by Proposition 4.1.3. This also excludes the second graph of the first row of Proposition 5.3.11 (note that x, $x\rho_0, x\rho_1\rho_0\rho_1$ and

$x(\rho_0\rho_1)^2$ belong to different blocks). If T has two orbits, then by Lemma 4.2.5, $n/4 = 2s^2$, meaning that the size of a T-orbit acting on n points is $k = 2s^2$, a contradiction. This excludes the remaining graphs of the first row of Proposition 5.3.11.

Finally, suppose that T has four orbits when acting on the quadruples. Then the size k of a T-orbit on the set of size n must be divisible by 4. Thus, by Lemma 4.1.5, $k/4 = a'b'$ with $lcm(a',b') = s$ and, by Lemma 5.3.4, $k = ab$ with $lcm(a,b) = s$. Hence we have $k = 4gcd(a',b')s = gcd(a,b)s$, and therefore $gcd(a,b)$ is even, as desired.

(2) Suppose now that ρ_0 has a fixed point. Hence the action on the blocks cannot be given by the first four graphs given in Proposition 5.3.11, where ρ_0 is fixed-point free. Whenever ρ_0 has a fixed point in a block, say Δ_i, then since it commutes with both g and h, it must act trivially on Δ_i.

Now consider the first three block actions described by the graphs given on the second row of Proposition 5.3.11. If ρ_0 is trivial on a block, then, as it commutes with ρ_2 and ρ_3, we get that ρ_0 acts as the identity, a contradiction. Thus the remaining possibility for the block action is described by the alternating $\{1,3\}$-square, the one on the right side of the second row of Proposition 5.3.11.

Let $\Delta_2 = \Delta_1\rho_1$, $\Delta_3 = \Delta_2\rho_3$ and $\Delta_4 = \Delta_3\rho_1 = \Delta_1\rho_3$. As $(\rho_0\rho_1)^2$ fixes the blocks, k must be even and, consequently s is even.

Let $K = \langle g_1, h_1\rangle$ be the action of T on the block Δ_1 and let $B := |K : \langle g_1\rangle|$ and $C := |K : \langle h_1\rangle|$. By Lemma 4.1.1 $k = gcd(C,B)s$ and, as seen in the proof of Lemma 4.1.1, there exists some D such that $|g_1| = DC$ and $|h_1| = DB$. One may consider $a = gcd(B,C)$ and $b = s$. Then it is sufficient to prove that both B and C are even numbers.

Let us first prove that both $|g_1|$ and $|h_1|$ are even. Note that $|g_1|$ and $|h_1|$ cannot both be odd, since $s = lcm(|g_1|,|h_1|)$. Hence, suppose $|g_1|$ is even and $|h_1|$ is odd. Since $|h_1|$ is odd, we must have $(\rho_0\rho_1)^2 = (g_1h_2h_3g_4)^{|g_1|/2}$. We have $s/2 \equiv 0 \mod |h_i|$ for $i \in \{1,4\}$ and $s/2 \equiv 0 \mod |g_i|$ for $i \in \{2,3\}$, hence $(h_1g_2g_3h_4)^{s/2}$ is trivial. In addition, note that $s/2 \equiv |g_1|/2 \mod |g_1|$. Consequently,

$$(gh)^{s/2} = (g_1h_2h_3g_4)^{s/2}(h_1g_2g_3h_4)^{s/2} = (g_1h_2h_3g_4)^{s/2} = (g_1h_2h_3g_4)^{|g_1|/2} = (\rho_0\rho_1)^2$$

a contradiction. The case where $|g_1|$ is odd and $|h_1|$ is even can be treated similarly. Then both $|g_1|$ and $|h_1|$ are even.

Suppose that $gcd(C,B)$ is odd. Assume that either C or B is odd, but not both. Then, since the orders of both g_1 and h_1 are even, D must be even. Suppose first that that B is even and C is odd. Let i and j be such that $h_1^B = g_1^{Ci}$ and $g_1^C = h_1^{Bj}$. As $|h_1^B| = |g_1^C| = D$ both i and j must be coprime with D. Hence i and j are odd numbers. Then $h_1^{\frac{Bs}{2}} = g_1^{\frac{Cis}{2}}$, implies that $\frac{Cis}{2} = 0 \mod s$, a contradiction, since C and i are odd. We get the same contradiction if we assume that B is odd and C is even. Thus B and C are both odd. Let α and β be such that $lcm(B,C) = \alpha B = \beta C$. Then, both α and β are odd. Thus, considering a i such that $h_1^B = g_1^{Ci}$, we get the equalities below, meaning that $(gh)^{s/2}$ is trivial:

$$\begin{aligned}(gh)^{s/2} &= (g_1h_2h_3g_4)^{s/2}(h_1g_2g_3h_4)^{s/2}\\ &= (g_1h_2h_3g_4)^{\frac{D}{2}lcm(B,C)}(h_1g_2g_3h_4)^{\frac{D}{2}lcm(B,C)}\\ &= (g_1h_2h_3g_4)^{\frac{D}{2}\beta C}(h_1g_2g_3h_4)^{\frac{D}{2}\alpha B}\\ &= (g_1h_2h_3g_4)^{\frac{DC}{2}(\beta+\alpha i)}\\ &= id,\end{aligned}$$

a contradiction. Hence $gcd(C,B)$ must be even. □

Theorem 5.3.13. *Let a and b be positive integers such that $s = lcm(a,b)$ and $s \geq 2$. Then the locally toroidal polytope $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$ has a faithful transitive permutation representation of degree n if and only if*

$$n \in \{4s^2, 8ab, 16ab, 32ab, 64ab\} \text{ or } n = 4ab \text{ if } a \text{ and } b \text{ are both even.}$$

Proof. This result follows from Propositions 5.3.1, 5.3.2, 5.3.8 and 5.3.12 and Lemma 5.3.3. □

5.3.2 The possible degrees of $\{\{4,4\}_{(2,0)},\{4,4\}_{(2s,0)}\}$

Let us now focus on the locally toroidal polytope $\{\{4,4\}_{(2,0)},\{4,4\}_{(2s,0)}\}$, for $s \geq 2$.

Theorem 5.3.14. *Let a and b be positive integers such that $s = lcm(a,b)$ and $s \geq 2$. Then the locally toroidal polytope $\{\{4,4\}_{(2,0)},\{4,4\}_{(2s,0)}\}$ has a faithful transitive permutation representation of degree n if and only if*

$$n \in \{8s^2, 16ab, 32ab, 64ab, 128ab\} \text{ or } n = 8ab \text{ if } a \text{ and } b \text{ are both even.}$$

Proof. Notice that the theorem holds when $s = 2$ (see Table 5.2). From now on assume that $s \geq 3$. Let G be the group of $\{\{4,4\}_{(2,0)},\{4,4\}_{(2s,0)}\}$ and let n be a degree of a FTPR of G. Consider the normal subgroup T of G generated by u^2 and v^2 where $u = \rho_1\rho_2\rho_3\rho_2$ and $v = u^{\rho_2}$. Let $\delta := (\rho_1\rho_2\rho_3)^{2s} = (uv)^s$ and $\beta := (\rho_0\rho_1)^2$. As δ is a central involution, it determines an embedding of G into $S_2 \wr S_{\frac{n}{2}}$ where the blocks are the connected components of δ. Let f denote the homomorphism $G \to S_{\frac{n}{2}}$ determined by this embedding. If $Ker(f) = \langle\delta\rangle$ then, by Theorem 5.3.13 and Corollary 5.2.4, n is one of the degrees listed in the statement of this theorem. In what follows we lead with the case $Ker(f) \neq \langle\delta\rangle$.

Firstly consider the case s odd. In this case T has no involutions. Hence, as all the elements of $Ker(f)$ are involutions, the intersection of T with $Ker(f)$ is trivial, therefore $f(G_0)$ must be is the group of the map $\{4,4\}_{(s,s)}$. If $\rho_0 \in Ker(f)$ then $\beta \in Ker(f)$. But as β is a central involution in G, we get $\beta = \delta$, a contradiction. Hence $|f(\rho_0)| = |f((\rho_0\rho_1)^2)| = 2$. This shows that $f(G)$ is the group of $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$, or equivalently, $Ker(f) = \langle\delta\rangle$, a contradiction.

Let us now lead with the case s even. In this case $\delta \in T$. Suppose that $Ker(f) \cap T$ is not $\langle\delta\rangle$. Then $\langle u^s, v^s\rangle \leq Ker(f)$. Indeed $\langle u^s, v^s\rangle$ is the maximal subgroup of T contained in $Ker(f)$. Consequently, $f(G_0)$ is, in this case, the group of the map $\{4,4\}_{(s,0)}$. Now suppose that $\rho_0 \in Ker(f)$ then $\beta := (\rho_0\rho_1)^2 \in Ker(f)$, as before one gets a contradiction. Hence $Ker(f) = \langle u^s, v^s\rangle$ and $G/Ker(f)$ is isomorphic to the group of $\{\{4,4\}_{(2,0)}, \{4,4\}_{(2s',0)}\}$ where $s' := s/2$.

We may assume by induction the degrees of $\{\{4,4\}_{(2,0)}, \{4,4\}_{(2s',0)}\}$ are precisely those of the following list where $lcm(a',b') = s'$.

$$8s'^2,\ 16a'b',\ 32a'b',\ 64a'b',\ 128a'b' \text{ or } n = 8a'b' \text{ if } a' \text{ and } b' \text{ are both even.}$$

Assume without loss of generality that $lcm(2a',b') = s$. Then the degrees of G are contained in the following list.

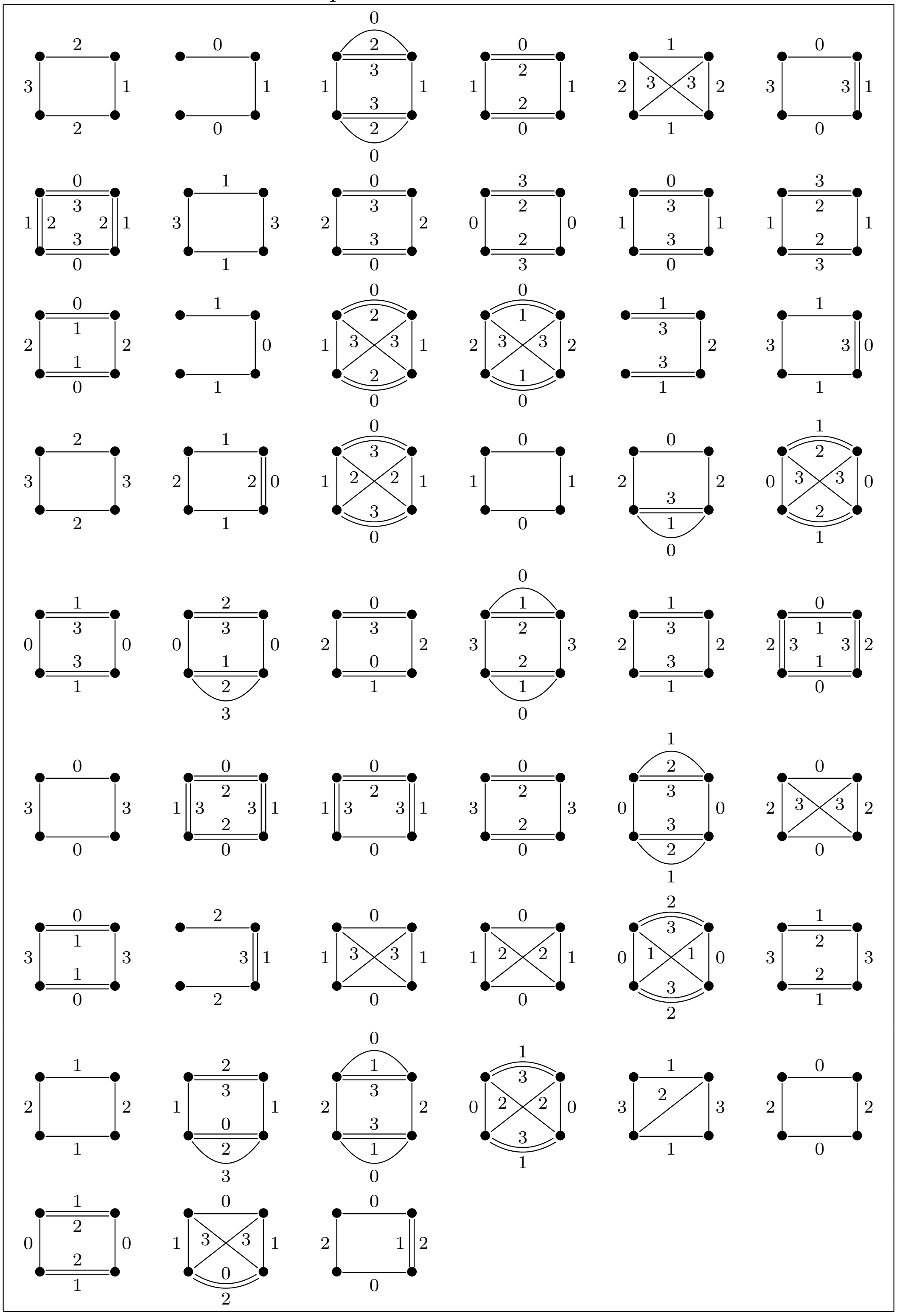
$$4s^2, 16(2a')b', 32(2a')b', 64(2a')b', 128(2a')b' \text{ or } n = 8(2a')b' \text{ if } a' \text{ and } b' \text{ are both even.}$$

All these degrees correspond to the ones given in the statement of this theorem with one exception, $n = 4s^2$. Let us now rule out this possibility.

Suppose that $n = 4s^2$ then, the number m of T-orbits is at most 4. We remind that $T = \langle u^2, v^2\rangle$, $\delta = u^s v^s$, $\beta = (\rho_0\rho_1)^2$ and, since s is even, $Ker(f) = \langle u^s, v^s\rangle$. As δ is a fixed-point-free involution (swapping $n/2$ pairs of points), and u^2 and v^2 have the same cyclic decomposition, u^s swaps exactly $\frac{n}{4}$ pairs of points while v^s swaps the remaining $\frac{n}{4}$ pairs of points. As the orbits of T have the same size and T acts regularly on each orbit, there exist exactly two possible sizes, say a and b, of a cycle of the cyclic decomposition of u^2 (and for v^2). Moreover, as u^s has fixed points, a and b must be distinct. Let us see that this implies that $m = 2$. Firstly, as the case where $a = b = s$ cannot happen, T cannot be transitive, thus $m \neq 1$. Secondly, again as $a = b = s$ cannot happen, $n/4 \neq s^2$, hence $m \neq 4$. Let Δ_1 and Δ_2 be the T-orbits. As $a \neq b$, $\Delta_1\rho_2 = \Delta_2$. Furthermore, ρ_2 is the unique permutation of the generating set of G, permuting the blocks. Indeed, as ρ_0 commutes with u and v, $u^{\rho_1} = u^{-1}$, $u^{\rho_3} = u$, $v^{\rho_1} = v$ and $v^{\rho_3} = v^{-1}$, the other involutions, ρ_0, ρ_1 and ρ_3, cannot swap the blocks as this would force the cyclic decomposition of u^2 and v^2 to be the same on the blocks, and $a = b = s$. Let u_1, v_1, u_2 and v_2 denote the actions of u and v on Δ_1 and Δ_2, respectively. Let a and b be the orders of u_1^2 and u_2^2, respectively. Then $u^s = u_1^a$, $v^s = v_2^a$. The orbits of $\langle\beta,\delta\rangle$ have the same size, equal to 4. Thus, with no other possibilities, either $\beta = v_1^b u_2^b$ or $\beta = \delta v_1^b u_2^b$. In addition, as ρ_0 fixes the blocks and commutes with u^2 and v^2, we get $\rho_0 \in \langle\beta,\delta\rangle$, a contradiction.

□

Table 5.3: The possible actions on the blocks when $m = 4$



Chapter 6

Regular Hypertopes

In the previous chapters, we have dealt with regular maps, hypermaps and locally toroidal polytopes. In particular, the introduction of hypermaps and Coxeter groups without string diagrams (such as the triangle groups) as a generalization of maps, gives way to new geometrical structures which are not polytopes, since they do not possess a string Coxeter diagram, but still have symmetry. Inspired by this generalization, the authors of [FLW16] generalized the concept of abstract polytopes to similar structures whose automorphism groups are C-groups without a string Coxeter diagram and named them a hypertopes.

In this chapter, we will give an introduction to incidence systems and incidence geometries, which are necessary to define a (regular) hypertope. This chapter will have mainly results presented in [FLW16], [FLW20] and [BC13].

6.1 Incidence Systems, Geometries and Hypertopes

An *incidence system* $\Gamma := (X, *, t, I)$ is a 4-tuple where

- X is the set of *elements* of Γ;
- I is the set of *types* of Γ;
- $t : X \to I$ is a *type function* that associates to each element $x \in X$ a type $t(x) \in I$;
- $*$ is an *incidence relation* on X that is reflexive, symmetric and such that, for all $x, y \in X$, if $x * y$ and $t(x) = t(y)$ then $x = y$.

With this incidence relation, we can build an *incidence graph* of Γ, where the elements of X are the vertices and two vertices $x, y \in X$ are connected if $x * y$. The *rank* of Γ is the number of types of Γ, i.e. the cardinality of I.

A *flag*[1] is a subset of X such that all elements are pairwise incident. This can be seen in the incidence graph as a clique. The *type of a flag* F is the set of types of the flag, i.e. $t(F) := \{t(x) : x \in F\}$. A *chamber*[2] is a flag of type I. We say that Γ is a *geometry* or *incidence geometry* if all flags of Γ are contained in a chamber.

[1] Note that the definition of a flag in the context of incidence systems is not the same as in abstract polytope.

[2] Chambers in incidence systems correspond to flags in abstract polytopes (seen as posets).

An element $x \in X$ is said to be *incident to a flag* F, denoted as $x * F$, if x is incident with every element of F. The *residue* of a flag F in Γ is the incidence geometry $\Gamma_F := (X_F, *_F, t_F, I_F)$ where

- $X_F := \{x \in X \ : \ x * F, x \notin F\}$;
- $I_F := I \backslash t(F)$;
- t_F and $*_F$ are the restrictions of t and $*$ to X_F and I_F.

We say Γ is *connected* if its incidence graph is connected. Moreover, we say that Γ is *residually connected* if all residues of rank at least two of Γ, including Γ, have a connected incidence graph. Furthermore, we say Γ is *chamber-connected* if, for every two chambers C and $\tilde{C}$, there exists a sequence of chambers $C =: C_0, C_1, \ldots, C_k := \tilde{C}$ such that consecutive chambers in the sequence differ in exactly one element of a certain type $i \in I$, i.e. $|C_i \cap C_{i+1}| = |I| - 1$. Γ is said to be *strongly chamber-connected* if all its residues of rank at least two, including Γ, are chamber-connected.

An incidence system Γ is said to be *thin* (respectively *firm*) if all rank one residues of Γ contain exactly two (resp. at least two) elements. The thinness condition is similar to the diamond condition of abstract polytopes. Moreover, if an incidence system Γ is thin, then for every chamber C of Γ there is one and only one chamber that differs on the element of type i, called the i-adjacent chamber and denoted as C^i. Consider now the following result.

Proposition 6.1.1. *[FLW16, Proposition 2.1] Let Γ be a firm incidence geometry. Then Γ is residually connected if and only if Γ is strongly chamber-connected.*

We call *pre-hypertope* to a thin incidence geometry and we define a *hypertope* to be a strongly chamber-connected (or residually connected) pre-hypertope.

The incidence graph of a hypertope is then an n-coloured graph (where n is the number of types of the hypertope, i.e. the rank) such that

1. The maximal cliques (which are the chambers) all have size n and have an element of each type;
2. All $(n-1)$-clique can be augmented to form an n-clique in exactly two ways (a consequence of the thinness condition); and
3. The graph is connected and for all k-cliques, where $1 \leq k \leq n-2$, the subgraph of elements incident to all elements of that clique is still connected (the residually connected condition).

In order to fully understand the concept of a hypertope, take as an example the cube. Let us build an incidence system from the cube. Let the set of elements X be the union of the set of vertices, edges and faces of the cube and the set of types I is $\{0, 1, 2\}$. The type function t is defined as follows: $t(x)$ is 0, 1 or 2 if x is a vertex, an edge or a face, respectively. The incidence graph is as in Figure 6.2, where $\bullet$, $\blacktriangle$ and $\blacklozenge$ denote elements of type 0, 1 and 2, respectively.

Chambers correspond to maximal cliques of this graph, with type $\{0, 1, 2\}$, i.e. a set with vertice-edge-face ($\bullet - \blacktriangle - \blacklozenge$). This incidence system is connected since the graph is connected, and it is a geometry since any flag is a subset of a chamber. Also,

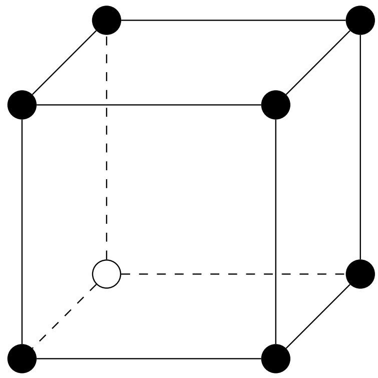


Figure 6.1: A cube.

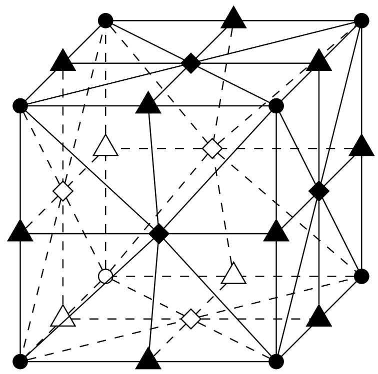


Figure 6.2: Incidence graph of a cube.

residually connectedness is easy to see since the residue of any flag of rank one is an incidence geometry of rank 2 with an incidence graph that is still connected. Finally, we can see that this incidence geometry is thin since, picking any flag of rank 2, its rank one residues only have two elements (i.e. a flag of rank 2 can be augmented to be a chamber in exactly two different ways). Hence, the cube is a hypertope. Notice that in this case this incidence relation is a partial order, and for this reason this incidence geometry is a polytope. In general, a hypertope is an abstract polytope when the incidence relation is a partial order. Hypertopes that not abstract polytopes are called *proper*.

6.2 Regular Hypertopes

Let Γ be an incidence geometry $(X, *, t, I)$. An *automorphism* of Γ is a mapping $\Gamma \to \Gamma$ such that

- α is a bijection on X;
- for each $x, y \in X$, $x * y$ if and only if $\alpha(x) * \alpha(y)$;
- for each $x, y \in X$, $t(x) = t(y)$ if and only if $t(\alpha(x)) = t(\alpha(y))$.

We say an automorphism α is *type preserving* if, for any $x \in X$, $t(x) = t(\alpha(x))$. The set of all automorphisms of Γ is denoted as $Aut(\Gamma)$ and the set of all type-preserving automorphisms is denoted as $Aut_I(\Gamma)$. Both sets are groups and it is obvious that $Aut_I(\Gamma) \leq Aut(\Gamma)$. A non-type-preserving automorphism $\sigma \in Aut(\Gamma) \backslash Aut_I(\Gamma)$ is called a *correlation*. This generalizes the concept of duality in abstract polytopes.

An incidence geometry is said to be *flag-transitive* if the type-preserving automorphism group $Aut_I(\Gamma)$ is transitive on all flags of a given type J, for each type $J \subseteq I$. Equivalently, Γ is *chamber-transitive* if $Aut_I(\Gamma)$ is transitive on all chambers of Γ and we have the following result.

Proposition 6.2.1. *[FLW16, Proposition 2.2] Let Γ be an incidence geometry. Γ is chamber-transitive if and only if Γ is flag-transitive.*

An incidence geometry Γ is *regular* if $Aut_I(\Gamma)$ acts regularly on the chambers, i.e. if the action is semi-regular and transitive. Hence, we say a *regular hypertope* is a flag-transitive (or equivalently chamber-transitive) hypertope. We need not impose the semi-regularity since thinness already implies it: an element $g \in Aut_I(\Gamma)$ that fixes a chamber C will fix all the i-adjacent chambers C^i, for each $i \in I$, which by residually connectedness, implies that g must fix all chambers, and hence $g = id$.

Let C be a (base) chamber of a regular hypertope Γ with type set $I := \{0, \ldots, n-1\}$. We will denote as ρ_i, for each $i \in I$, the automorphism from $Aut_I(\Gamma)$ that maps the chamber C to its i-adjacent chamber C^i. Due to thinness, it is easy to see that each ρ_i is an involution. Then we have that $S := \{\rho_0, \ldots, \rho_{n-1}\}$ is a generating set of $Aut_I(\Gamma)$ and the following result.

Theorem 6.2.2. *[FLW16, Theorem 4.1] Let $I := \{0, \ldots, n-1\}$ and let Γ be a regular hypertope of rank n. Then $Aut_I(\Gamma)$ is a C-group of rank n.*

Notice that regular hypertopes in which its type-preserving automorphisms is a string C-group are precisely abstract regular polytopes. If the Coxeter diagram of a regular hypertope is connected we say it is an *irreducible* regular hypertope; otherwise, we say it is *reducible*.

6.3 Coset Geometries

As in the case of abstract regular polytopes, regular hypertopes can be built from C-groups. However, contrary to string C-groups, not all C-groups give a regular hypertope.

Proposition 6.3.1. *[Tit56] Let n be a positive integer and $I := \{0, \ldots, n-1\}$. Let G be a group together with a family of subgroups $(G_i)_{i \in I}$, X the set consisting of all cosets $G_i g$ with $g \in G$ and $i \in I$, and $t : X \to I$ defined by $t(G_i g) = i$. Define an incidence relation $*$ on $X \times X$ by:*

$$G_i g_1 * G_j g_2 \text{ iff } G_i g_1 \cap G_j g_2 \neq \emptyset.$$

Then the 4-tuple $\Gamma := (X, *, t, I)$ is an incidence system having a chamber. Moreover, the group G acts by right multiplication as an automorphism group on Γ as a group of type preserving automorphisms. Finally, the group G is transitive on the flags of rank less than 3.

An incidence system built using the previous proposition is denoted as $\Gamma(G;(G_i)_{i\in I})$, where the subgroups G_i, for $i \in I$, are designated as the *maximal parabolic subgroups.* If $\Gamma(G;(G_i)_{i\in I})$ is a geometry, it is designated a *coset geometry.* If G is a C-group, there is no guarantee that the incidence system $\Gamma(G;(G_i)_{i\in I})$ is a regular hypertope, or even if it is a geometry (examples are given in [FLW16] and [FLW20]). Nonetheless, the following result gives a sufficient condition for obtaining regular hypertopes.

Theorem 6.3.2. *[FLW16, Theorem 4.6] Let $G = \langle \rho_0, \dots, \rho_{n-1}\rangle$ be a C-group of rank n and let $\Gamma := \Gamma(G;(G_i)_{i\in I})$ with $G_i := \langle \rho_j \mid j \in I\backslash\{i\}\rangle$ for all $i \in I := \{0, \dots, n-1\}$. If G is flag-transitive on Γ, then Γ is a regular hypertope.*

Hence, it is sufficient to prove that a coset geometry constructed from a C-group is flag-transitive to guarantee that we have a regular hypertope. In order to prove the flag-transitivity of a C-group on its coset geometry, we have that following theorem by Buekenhout and Cohen [BC13].

Theorem 6.3.3. *[BC13, Theorem 1.8.10 (iii)] Let $\Gamma(G,(G_i)_{i\in I})$ be the coset incidence system of G over $(G_i)_{i\in I}$. Then Γ is flag-transitive if and only if for each subset J of I of size three, the group G is transitive on the set of flags of type J, and for each $i \in I$ the subgroup G_i is flag-transitive on $\Gamma(G_i,(G_{i,j})_{j\in I\backslash\{i\}})$. If one (whence both) these properties hold, then Γ is a geometry.*

To prove that G is transitive on the set of flags of type $J \subseteq I$ ($|J| = 3$), we need to prove one of the three equivalent conditions given in the following Lemma (see Proposition 1.4.1 in [Tit74])

Lemma 6.3.4. *[Tit74] Let G_i, G_j, G_k be three subgroups of a group G. The following conditions are equivalent.*

1. $G_iG_j \cap G_iG_k = G_i(G_j \cap G_k)$.
2. $(G_i \cap G_j)(G_i \cap G_k) = G_i \cap (G_jG_k)$.
3. *If the three cosets G_ix, G_jy and G_kz have pairwise non-empty intersections, then $G_ix \cap G_jy \cap G_kz \neq \emptyset$.*

Whenever we have a rank 3 coset incidence system of a C-group, the following condition is sufficient to prove we have a regular hypertope.

Proposition 6.3.5. *[FLW16, Proposition 4.3] If $G := \langle \rho_0, \rho_1, \rho_2\rangle$ is a C-group of rank 3, and $\Gamma(G;(G_i)_{i\in\{0,1,2\}})$ with $G_i = \langle \rho_j \,|\, j \neq i\rangle$ is thin, then Γ is a regular hypertope.*

A special case is when we are dealing with string C-groups, where in this case, as observed previously, we have an improper regular hypertope.

Theorem 6.3.6. *[Sch83; Asc83] Let $(G, \{\rho_0, \dots, \rho_{n-1}\}$ be a string C-group of rank n and let $\Gamma := \Gamma(G,(G_i)_{i\in I})$ with $G_i := \langle \rho_j \mid j \in I\backslash\{i\}\rangle$ for all $i \in I := \{0, \dots, n-1\}$. Then Γ is thin, residually connected and regular, i.e. a regular hypertope. Moreover, Γ has a string diagram.*

To classify regular hypertopes, the authors of [FLW20] give an important result about quotients of the automorphism group of regular hypertopes, which generalizes a known result of abstract regular polytopes [MS02, Theorem 2E17], called the *quotient criterion.*

Theorem 6.3.7. *[FLW20, Theorem 3.1] Let $G := \langle \rho_0, \dots, \rho_{n-1} \rangle$ be a group generated by involutions and $H := \langle \delta_0, \dots, \delta_{n-1} \rangle$ be a C-group. If the mapping $\sigma : G \to H$ with $\sigma(\rho_i) = \delta_i$ for each $i = 0, \dots, n-1$ is a homomorphism which is one-to-one on G_i for each $i = 0, \dots, n-1$, then G is also a C-group.*

Consider the incidence system $\Gamma := (X, *, t, I)$ and a normal subgroup $N \leq Aut_I(\Gamma)$. The *quotient of* Γ *with respect to* N is an incidence system $\Gamma/N := (\overline{X}, *_N, t_N, I)$ where

- $\overline{X}$ is the set $\{F \cdot N \ : F \in X\}$ of orbits of N in X;
- for $F_1,\ F_2 \in X$, $(F_1 \cdot N) *_N (F_2 \cdot N)$ if and only if there exist a face F in $F_1 \cdot N$ and a face G in $F_2 \cdot N$ such that $F * G$; and
- $t_N(F \cdot N) = t(F)$.

Theorem 6.3.8. *[FLW20, Theorem 3.2] Let $\mathcal{U}$ be a regular hypertope and $U := \langle \rho_0, \dots, \rho_{n-1} \rangle$ be its type-preserving automorphism group. Let $N \triangleleft U$ be such that $N \cap U_i$ is trivial for all $i \in \{0, \dots, n-1\}$. Let $H_i = \langle \rho_j N \ : \ j \in \{0, \dots, n-1\} \setminus \{i\} \rangle$ for all $i \in \{0, \dots, n-1\}$. If $\Gamma(U/N; \{H_0, \dots, H_{n-1}\})$ is a flag-transitive coset geometry, then it is a regular hypertope and it is isomorphic to $\mathcal{U}/N$.*

6.4 (Locally) Spherical and Toroidal Hypertopes

Every Coxeter group is the type-preserving automorphism group of a regular hypertope, which more recently was named the *universal hypertope associated with the Coxeter group* [FLW20]. As explained in Section 2.4, C-groups are quotients of Coxeter groups and the type-preserving automorphism groups of regular hypertopes are C-groups. Hence, the universal hypertope associated with the Coxeter group W is then called the *universal cover of the hypertope* Γ and the Coxeter diagram of Γ is the diagram of its universal cover. We say that a regular hypertope Γ is of *type* W if its type-preserving automorphism group is a quotient of a Coxeter group W.

A regular hypertope is of *spherical type* if its type-preserving automorphism group is a smooth quotient of a finite irreducible Coxeter group or its diagram is a union of diagrams of finite irreducible Coxeter groups. A *locally spherical regular hypertope* is a regular hypertope whose maximal parabolic residues are of spherical type. A *projective hypertope* is a quotient of a (locally) spherical regular hypertope by a central symmetry. These hypertopes are listed in Table 6.1[FLW20, Table 1].

We say a locally spherical regular hypertope is of *euclidean type* if its Coxeter diagram correspondes to an affine irreducible Coxeter group of Euclidean type, listed in Table 6.2 [FLW20, Table 2]. A *regular toroidal hypertope* is a smooth quotient of a regular universal hypertope of euclidean type [FLW20]. As examples of regular toroidal hypertopes, we have the regular (hyper)maps, described in Chapter 3, or the tesselation of the euclidean space by n-cubes (with diagram $\tilde{C}_{n-1}$) described in [MS02, Section 6D] Moreover, Ens characterized all the proper rank 4 regular hypertopes of toroidal type [Ens18], i.e. with diagram $\tilde{A}_3$ and $\tilde{B}_3$. In section 9.2 we give an infinite family of regular hypertopes with diagrams $\tilde{B}_{n-1}$.

A locally spherical regular hypertope is of *hyperbolic type* if its type-preserving automorphism group of its universal cover is an irreducible compact hyperbolic Coxeter group [FLW20, Table 2]. These compact hyperbolic Coxeter groups exist only in ranks

Table 6.1: Finite irreducible Coxeter groups and respective locally spherical hypertopes of spherical type [FLW20, Table 1].

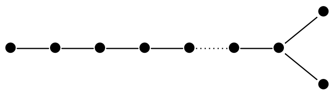
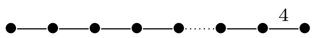
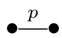
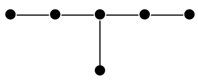
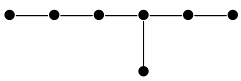
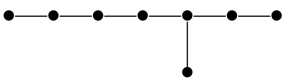
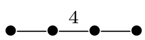
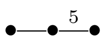
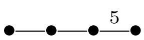
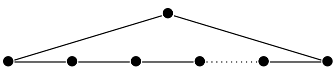
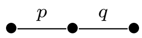
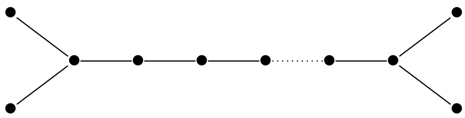
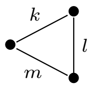
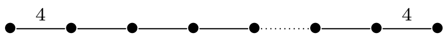
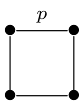
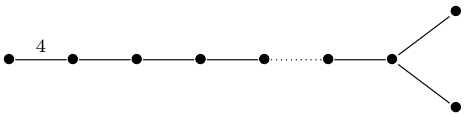
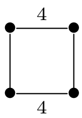
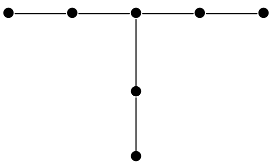
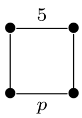
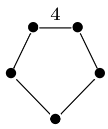
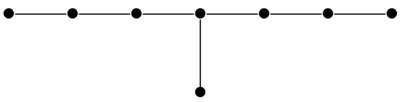
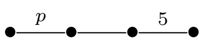
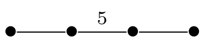
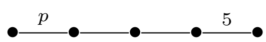
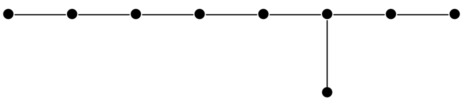
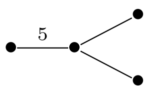
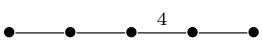
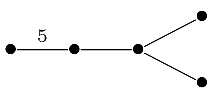
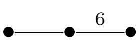
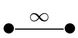
Diagram		Group	Order	Universal hypertope	Projective hypertope
A_n $(n \geq 1)$		$[3^{n-1}]$	$(n+1)!$	$\{3^{n-1}\}$	-
D_n $(n \geq 4)$		$[3^{n-3,1,1}]$	$2^{n-1} \cdot n!$	$\{3^{n-3}, {3 \atop 3}\}$	-
B_n $(n \geq 3)$		$[3^{n-2}, 4]$	$2^n \cdot n!$	$\{3^{n-2}, 4\}$	$\{3^{n-2}, 4\}_n$
I_2^p $(p \geq 3)$		$[p]$	$2p$	$\{p\}$	-
E_6		$[3^{2,2,1}]$	$12 \cdot 6!$	$\{2_{2,1}\}$	-
E_7		$[3^{3,2,1}]$	$8 \cdot 9!$	$\{3_{2,1}\}$	$\{3_{2,1}\}_9$
E_8		$[3^{4,2,1}]$	$192 \cdot 10!$	$\{4_{2,1}\}$	$\{4_{2,1}\}_{15}$
F_4		$[3, 4, 3]$	1152	$\{3, 4, 3\}$	$\{3, 4, 3\}_{56}$
H_3		$[3, 5]$	120	$\{3, 5\}$	$\{3, 5\}_5$
H_4		$[3, 3, 5]$	14400	$\{3, 3, 5\}$	$\{3, 3, 5\}_{15}$

Table 6.2: Affine irreducible and compact hyperbolic Coxeter groups and respective locally spherical hypertopes of euclidean and hyperbolic types [FLW20, Table 2]
.

Euclidean	Hyperbolic
$\tilde{A}_{n-1}$ $(n \geq 3)$ 	$[p,\, q]$  $3 \leq p, q < \infty,\ \frac{1}{p} + \frac{1}{q} < \frac{1}{2}$
$\tilde{D}_{n-1}$ $(n \geq 5)$ 	$[(k,\, l,\, m)]$ $3 \leq k,\, l,\, m < \infty$ $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1$ 
$\tilde{C}_{n-1}$ $(n \geq 3)$ 	$[(3,\, 3,\, 3,\, p)]$ $p = 4,\, 5$ 
$\tilde{B}_{n-1}$ $(n \geq 4)$ 	$[(3,\, 4,\, 3,\, 4)]$ 
$\tilde{E}_6$ 	$[(3,\, 5,\, 3,\, p)]$ $p = 4,\, 5$  $[(3,\, 3,\, 3,\, 3,\, 4)]$ 
$\tilde{E}_7$ 	$[p,\, 3,\, 5]$ $p = 4,\, 5$  $[3,\, 5,\, 3]$  $[p,\, 3,\, 3,\, 5]$ $p = 3,\, 4,\, 5$ 
$\tilde{E}_8$ 	$[5,\, {}^{3}_{3}]$ 
$\tilde{F}_4$ 	$[5,\, 3,\, {}^{3}_{3}]$ 
$\tilde{G}_2$ 	
$\tilde{A}_1$ 	

3, 4 and 5, and not many finite examples of proper hypertopes of this type were known for rank 4 and 5, since their groups are quite big. Some examples of small size were obtained computationally in [FLW20, Table 3] and more examples and infinite families of proper regular hypertopes of hyperbolic type were characterized [MW20; MW21]. In Sections 9.3 and 9.3 the works of Weiss and Montero in [MW20] will be extended to infinite families of regular hypertopes of type $\{{3 \atop 3}, 5\}$ and $\{{3 \atop 3}, 3, 5\}$. Also, in Section 9.2 families of improper regular hypertopes of hyperbolic type $\{2p, 2p\}$ (with $p \geq 3$) will be given through the halving operation.

A regular n-polytope or a regular 4-hypertope is said to be *locally toroidal* if its maximal residues are either of spherical type or euclidean type, with at least one of them being of euclidean type [MS02; FLPW21]. A generalization of this concept for hypertopes of rank greater than 4 is yet to be established. When the defining relations of the type-preserving automorphism group of a locally toroidal hypertope are those corresponding to the Coxeter diagram with the only additional relations determining the toroidal residues, we say it is a *universal locally toroidal hypertope.* As examples of locally toroidal regular hypertopes, we have the polytopes of type $\{4, 4, 4\}$ of Chapter 5 or the proper hypertopes described in [CFHL18; FLW15]. Moreover, in Chapters 7 and 8 and Section 9.2 infinite families of locally toroidal regular hypertopes will be given.

6.5 Halving Operation of non-degenerate polytopes

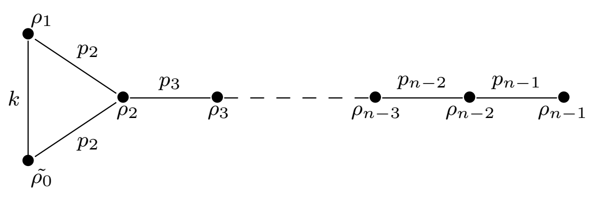
Recently, in [MW20], the halving operation was revisited and, furthermore, the conditions under which it gives a regular hypertope, starting from a regular polytope, were established. In what follows we recall important results that can be found in [MW20].

Let $n \geq 3$ and let $\mathcal{P}$ be a regular non-degenerate n-polytope of type $\{p_1, \ldots, p_{n-1}\}$ with automorphism group $G(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$.

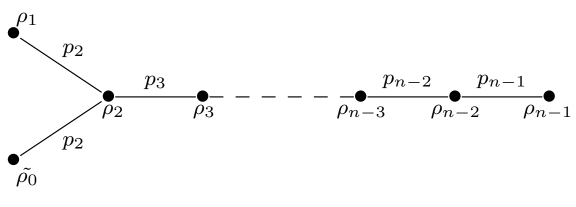
The *halving operation* is the map

$$\eta : \langle \rho_0, \rho_1, \rho_2, \ldots, \rho_{n-1} \rangle \to \langle \rho_0\rho_1\rho_0, \rho_1, \rho_2, \ldots, \rho_{n-1} \rangle = \langle \tilde{\rho_0}, \rho_1, \rho_2, \ldots, \rho_{n-1} \rangle.$$

The *halving group* of $\mathcal{P}$, denoted by $H(\mathcal{P})$, is the image of the halving operation on $G(\mathcal{P})$. If $n \geq 3$ and $\mathcal{P}$ is non-degenerate, then $H(\mathcal{P}) = \langle \tilde{\rho_0}, \rho_1, \ldots, \rho_{n-1} \rangle$ is a C-group [MW20, Theorem 3.1] with the following Coxeter diagram



where $k = p_1$ if p_1 is odd, otherwise $k = \frac{p_1}{2}$. In this paper, we will focus on polytopes of type $\{4, p_2, \ldots, p_{n-1}\}$, hence the Coxeter diagram will be as follows.



Let $\mathcal{H}(\mathcal{P})$ denote the coset indicence system $\Gamma(H(\mathcal{P}),(H_i)_{i\in\{0,\ldots,n-1\}})$ associated with the halving group of the non-generate regular polytope $\mathcal{P}$, where H_i are the maximal parabolic subgroups of $H(\mathcal{P})$. Then, $\mathcal{H}(\mathcal{P})$ is flag-transitive [MW20, Proposition 3.2]. Hence, using Theorem 6.3.2, we have the following corollary.

Corollary 6.5.1. *[MW20, Corollary 3.2] Let $\mathcal{P}$ be a non-degenerate regular n-polytope and $I=\{0,\ldots,n-1\}$. Let $H(\mathcal{P})$ be the halving group of $\mathcal{P}$. Then the incidence system $\mathcal{H}(\mathcal{P})=\Gamma(H(\mathcal{P}),(H_i)_{i\in I})$ is a regular hypertope such that $Aut_I(\mathcal{H}(\mathcal{P}))=H(\mathcal{P})$.*

If $\mathcal{P}$ is a polytope of type $\{4,p_2,p_3\ldots,p_{n-1}\}$, then we denote the type of $\mathcal{H}(\mathcal{P})$ as $\left\{{}^{p_2}_{p_2},p_3,\ldots,p_{n-1}\right\}$, and its universal automorphism group as $\left[{}^{p_2}_{p_2},p_3,\ldots,p_{n-1}\right]$. Moreover, as stated in [MW20], $H(\mathcal{P})$ has index 2 on $G(\mathcal{P})$ if the set of vertices of $\mathcal{P}$ is bipartite, which is only possible if p_1 is even.

Chapter 7

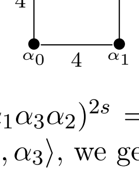
Two families of locally toroidal hypertopes

Locally toroidal hypertopes were one of the first type of hypertopes to be extensively studied in [FLW15]. In this paper, the authors used Tits's coset geometries to computationally obtain lists of finite universal locally toroidal regular hypertopes with toroidal residues of type $\{3, 6\}$. This was done using Coxeter groups with the intended diagram which were then parameterized using the parameters of their toroidal residues. Contrary to what happens in abstract regular polytopes, this parametrization can lead to hypertopes with toroidal residues smaller than what would be expected by the factorization. That is, the hypertopes had the expected diagrams but the residues did not correspond to the given parameters (this will be explained in detail in Section 7.3). Hence, this raises the importance of confirming the correctness of the residues, since it may mislead when the parameters and the residues differ.

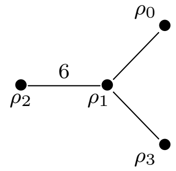
In this chapter, we will revisit two families of hypertopes that were introduced in [FLW15], prove that their automorphism groups are C-groups, and confirm the correctness of the toroidal residues. Also, faithful transitive permutation representations will prove their usefulness when proving the C-group condition in Section 7.1. The results presented in this chapter are the product of a visit of Dimitri Leemans and Asia Ivić Weiss to the University of Aveiro and can be found in [FLPW21].

7.1 A family of locally toroidal hypertopes arising from $\{4, 3, 4\}_{(s,s,0)}$

Consider the cubic toroids $\{4, 3, 4\}_{(s,s,0)}$ with automorphism group $[4, 3, 4] := \langle \tau_0, \tau_1, \tau_2, \tau_3 \rangle$ factorized by the extra relation $(\tau_0\tau_1\tau_2\tau_3\tau_2)^{2s}$. Using a Petrie operation, such that $\alpha_0 := \tau_0$, $\alpha_1 := \tau_1$, $\alpha_2 := \tau_2$ and $\alpha_3 := \tau_1\tau_3$, we get a group with diagram



and satisfying the relations $(\alpha_0\alpha_1\alpha_2\alpha_1\alpha_3\alpha_2)^{2s} = id$ and $(\alpha_1\alpha_2\alpha_3)^4 = id$. Now taking its index 2 subgroup $G := \langle \alpha_1^{\alpha_0}, \alpha_1, \alpha_2, \alpha_3 \rangle$, we get an infinite family of universal locally toroidal regular hypertopes with the following Coxeter diagram where $\rho_0 : \alpha_1$, $\rho_1 := \alpha_2$, $\rho_2 := \alpha_3$ and $\rho_3 := \alpha_1^{\alpha_0}$.



As this diagram can be obtained from the diagram of the tetrahedron by adding an edge with label 6, this is called a *hexagonal extension of the diagram of a tetrahedron.* The family of groups found is an incidence system with toroidal residues $\{3,6\}_{(2,0)}$ (corresponding to G_0) and $\{3,6\}_{(s,0)}$ (corresponding to G_3, for $s \geq 3$). However, the result of theese operations, the Petrie operation and the second one, which corresponds with doubling the fundamental region, do not guarantee that G is a C-group nor that its coset incidence system is flag-transitive. Hence, our goal is to prove the following theorem.

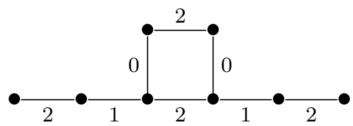
Theorem 7.1.1. *Let G be the group with the following presentation*

$$G := \langle \rho_0, \rho_1, \rho_2, \rho_3 \mid \rho_0^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = (\rho_0\rho_1)^3 = (\rho_0\rho_2)^2 = (\rho_0\rho_3)^2 = (\rho_1\rho_2)^6 = (\rho_1\rho_3)^3 = (\rho_2\rho_3)^2 = (\rho_0(\rho_1\rho_2)^2\rho_1)^s = (\rho_3(\rho_1\rho_2)^2\rho_1)^2\rangle.$$

Then $\Gamma(G; (G_i)_{\{0,1,2,3\}})$ is an universal locally toroidal regular hypertope with toroidal residues $\{3,6\}_{(2,0)}$ and $\{3,6\}_{(s,0)}$.

Proof. This follows from Propositions 7.1.2 and 7.1.3. □

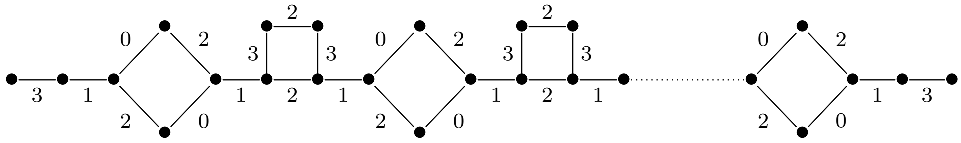
Consider the FTPR graphs of degree $3s$ given in Lemma 4.3.11 for s even and odd in Section 4.3.3, and the following FTPR graph of $\{3,6\}_{(2,0)}$ acting on the set of faces.



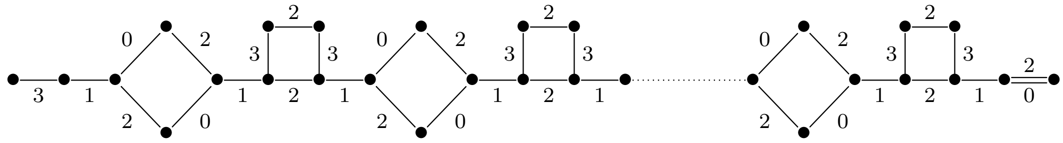
Combining these graphs we get a transitive permutation representation graph that encompasses the nature of both toroidal residues of G which are the automophism groups of $\{3,6\}_{(2,0)}$ and $\{3,6\}_{(s,0)}$. We now prove that this graph gives a FTPR of G and, with this, we will be able to prove that G is a C-group.

Proposition 7.1.2. *Let $s \geq 3$. The following graphs are faithful transitive permutation representation graphs, of degree $4s$, of the group G given in Theorem 7.1.1. Moreover, G is a C-group.*

s even



s odd



Proof. Let H be the group presented by the transitive permutation representation on either graphs. First of all, it can be easily that $H \leq G$ since $\rho_0^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = (\rho_0\rho_1)^3 = (\rho_0\rho_2)^2 = (\rho_0\rho_3)^2 = (\rho_1\rho_2)^6 = (\rho_1\rho_3)^3 = (\rho_2\rho_3)^2 = (\rho_0(\rho_1\rho_2)^2\rho_1)^s = (\rho_3(\rho_1\rho_2)^2\rho_1)^2 = id_H$. Hence, $|H| \leq 48s^3$ Consider in both graphs the vertex on the left. The stabilizer of this vertex is the group generated by ρ_0, ρ_1 and ρ_2, of size $12s^2$. Since H is transitive on $4s$ vertices, then $|H| \geq 48s^3$. Hence, the graph is a FTPR of G. In order to prove that G is a C-group, let us use the FTPR graphs and the intersection property in Equation 2.1. Let $\mathcal{G}$ be either of the graphs presented and let $\mathcal{G}_J$ be the subgraph of $\mathcal{G}$ with the same vertices of $\mathcal{G}$ but only with edges with label in $J \subset I = \{0,1,2,3\}$. It is easy to see that all subgraphs of $\mathcal{G}_{I\setminus\{i\}}$ are faithful (intransitive) permutation representations of C-groups and, hence, the maximal parabolic subgroups G_i are also C-groups. Hence, by Proposition 2.3.3, we need only to prove the intersection property on the maximal parabolic subgroups. Graph wise, it can be easily seen that $\mathcal{G}_{I\setminus\{i\}} \cap \mathcal{G}_{I\setminus\{j\}} = \mathcal{G}_{I\setminus\{i,j\}}$, with $i, j \in I$. Hence, it is clear that the intersection condition verifies, and so G is a C-group. □

The proposition above describes one FTPR graph with $4s$ vertices for the C-group G. It is most likely the minimal degree of a FTPR of that group, having in mind the results obtained in Section 4.

By Theorem 6.3.2, to complete the proof of Theorem 7.1.1 we only need to prove flag-transitivity.

Proposition 7.1.3. *If G is the group of Theorem 7.1.1 and $G_i := \langle \rho_j \,|\, j \in I \setminus \{i\}\rangle$ with $i \in I := \{0,1,2,3\}$, then $\Gamma(G; (G_i)_{i\in I})$ is flag-transitive.*

Proof. To prove that G is flag-transitive on Γ, by Lemma 6.3.4 it is sufficient to prove the following equality for every possible subset $\{i, j, k\}$ of I with three distinct elements.

$$(G_i \cap G_j)(G_i \cap G_k) = G_i \cap (G_jG_k),$$

since all maximal parabolic subgroups G_i are flag-transitive in $\Gamma(G_i, (G_j)_{j\in I\setminus\{i\}})$ (they are automophism groups of abstract regular polytopes). In any of these cases, the inclusion $(G_i \cap G_j)(G_i \cap G_k) \subseteq G_i \cap (G_jG_k)$ is trivial. Let us prove the other inclusion for each case separately.

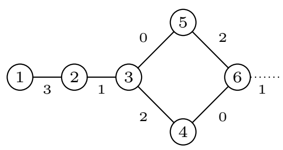
Case $\{i, j, k\} = \{0, 1, 2\}$: Since $G_1 = \{id_G, \rho_0, \rho_2, \rho_3, \rho_0\rho_2, \rho_0\rho_3, \rho_2\rho_3, \rho_0\rho_2\rho_3\}$ and $G_1G_2 = \bigcup_{g\in G_1} gG_2$, it follows that $G_1G_2 = G_2 \cup \rho_2G_2$ and thus $G_0 \cap G_1G_2 = G_{0,2} \cup (G_0 \cap \rho_2G_2)$. As $\rho_2 \in G_0$, we get $G_0 \cap \rho_2G_2 = \rho_2(G_0 \cap G_2) = \rho_2G_{0,2} \subseteq G_{0,1}G_{0,2}$. It is now clear that $G_0 \cap G_1G_2 = G_{0,1}G_{0,2}$, as wanted.

The cases where $\{i, j, k\}$ are equal to $\{0, 1, 3\}$ and $\{2, 1, 3\}$ follow a similar proof.

Case $\{i, j, k\} = \{0, 2, 3\}$: Now we have $G_2G_3 = G_3 \cup \rho_3G_3 \cup \rho_1\rho_3G_3 \cup \rho_0\rho_1\rho_3G_3$ and therefore

$$G_0 \cap G_2G_3 = G_{0,3} \cup (G_0 \cap \rho_3G_3) \cup (G_0 \cap \rho_1\rho_3G_3) \cup (G_0 \cap \rho_0\rho_1\rho_3G_3).$$

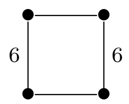
Since $G_0 \cap \rho_3G_3 \subseteq \rho_3G_{0,3}$ and $G_0 \cap \rho_1\rho_3G_3 \subseteq \rho_1\rho_3G_{0,3}$, we get $G_0 \cap G_2G_3 \subseteq G_{0,3} \cup \rho_3G_{0,3} \cup \rho_1\rho_3G_{0,3} \cup (G_0 \cap \rho_0\rho_1\rho_3G_3)$. The only difficulty now is to prove that $G_0 \cap \rho_0\rho_1\rho_3G_3 \subseteq G_{0,2}G_{0,3}$. Let $\alpha \in G_0 \cap \rho_0\rho_1\rho_3G_3$ and consider the following numbering of the left side vertices of one of the FTPR graphs given in Proposition 7.1.2.



For every $\alpha \in \rho_0\rho_1\rho_3G_3$, α acts on the FTPR graph by sending vertex 5 to vertex 1. On the other hand 1 and 5 are in different orbits of G_0 and hence $G_0 \cap \rho_0\rho_1\rho_3G_3 = \emptyset$. With this we conclude that $G_0 \cap G_2G_3 \subseteq G_{0,2}G_{0,3}$ and therefore Γ is flag-transitive. $\square$

7.2 A family of locally toroidal hypertopes arising from $\{3, 3, 4, 3\}_{(s,0,0,0)}$

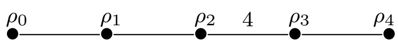
In [FLW15], the authors suggested the existence of an infinite family of universal finite locally toroidal regular hypertopes whose Coxeter diagram is a 4-circuit of type $(3, 6, 3, 6)$, as pictured below, with four toroidal rank 3 residues being $\{3, 6\}_{(s,s)}$, $\{3, 6\}_{(1,1)}$, $\{3, 6\}_{(2,0)}$, and $\{3, 6\}_{(2,0)}$.



However, the authors did not prove that those were exactly the toroidal residues of this potential regular hypertope, nor that it was actually a regular hypertope. The authors did observe that this family could be obtained from toroids of type $\{3, 3, 4, 3\}_{(s,0,0,0)}$ (for more information on this family, we refer to [MS02, Section 6E]). In the following proposition, we use this observation given in [FLW15] to derive a ggi whose maximal parabolic subgroups are automorphism groups of the toroidal maps specified above. Moreover, we will prove that the group obtained satisfies the intersection property, hence is a C-group, and that it is flag-transitive on its coset incidence system.

Proposition 7.2.1. *Let $s \geq 2$ and $H = \langle \rho_0, \rho_1, \rho_2, \rho_3, \rho_4 \rangle$ be the automorphism group of the regular toroid $\{3, 3, 4, 3\}_{(s,0,0,0)}$. The group $G = \langle \tau_0, \tau_1, \tau_2, \tau_3 \rangle$ with $\tau_0 := \rho_1$, $\tau_1 := \rho_2$, $\tau_2 := \rho_1\rho_3$, $\tau_3 := \rho_0\rho_4$ is a ggi such that G_0, G_1, G_2 and G_3 are the automorphism groups of the toroidal maps $\{3, 6\}_{(s,s)}$, $\{3, 6\}_{(1,1)}$, $\{3, 6\}_{(2,0)}$, and $\{3, 6\}_{(2,0)}$ respectively. Moreover, the groups G and H are isomorphic.*

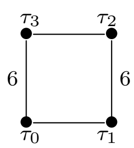
Proof. Let H be the automorphism group of the rank 5 toroid $\{3, 3, 4, 3\}_{(s,0,0,0)}$. The group H is obtained as the quotient of the affine Coxeter group $[3, 3, 4, 3]$ having the Coxeter diagram



by adding the relation $(\rho_0\rho_1(\rho_2\rho_3\rho_4)^3)^{2s} = id_H$. Consider the group $G := \langle \tau_0, \tau_1, \tau_2, \tau_3 \rangle$ generated by the involutions

$$\tau_0 := \rho_1, \ \tau_1 := \rho_2, \ \tau_2 := \rho_1\rho_3, \ \tau_3 := \rho_0\rho_4.$$

As it was shown in [FLW15], we can write $\rho_0 = (\tau_0\tau_3)^2\tau_0$ and $\rho_4 = (\tau_0\tau_3)^3$, meaning that $G \cong H$. We have that G has the following Coxeter diagram (as a ggi), as wanted.



We now prove that the toroidal residues G_i are as stated in the proposition. From the diagram we see that every G_i determines a regular toroidal map of type $\{3,6\}$, and hence either a map $\{3,6\}_{(k,0)}$ with automorphism group of order $12k^2$ or a map $\{3,6\}_{(k,k)}$ with automorphism group of order $36k^2$. We now determine each group G_i for $i \in \{0,\ 1,\ 2,\ 3\}$.

The group $G_1 = \langle \tau_2, \tau_3, \tau_0 \rangle = \langle \rho_1\rho_3, \rho_0\rho_4, \rho_1 \rangle$. As $\rho_4 = (\rho_1\rho_0\rho_4)^3$, we have that $G_1 = \langle \rho_1, \rho_3, \rho_0, \rho_4 \rangle$, i.e. $G_1 \cong H_2 \cong D_3 \times D_3$. Thus G_1 must be the automorphism group of the toroidal map $\{3,6\}_{(1,1)}$.

Now consider $G_2 = \langle \tau_1, \tau_0, \tau_3 \rangle = \langle \rho_2, \rho_1, \rho_0\rho_4 \rangle$. As $\rho_4 = (\rho_1\rho_0\rho_4)^3$, we conclude that $G_2 \cong H_3 \cong C_2 \times S_4$. Hence G_2 has order 48 and is the automorphism group of $\{3,6\}_{(2,0)}$.

Similarly, as $G_3 = \langle \tau_0, \tau_1, \tau_2 \rangle = \langle \rho_1, \rho_2, \rho_1\rho_3 \rangle$, $G_3 \cong H_{0,4} \cong C_2 \times S_4$. Hence G_3 has order 48 and is the automorphism group of $\{3,6\}_{(2,0)}$.

Let us now consider the group G_0. We have $G_0 = \langle \tau_3, \tau_2, \tau_1 \rangle = \langle \rho_0\rho_4, \rho_1\rho_3, \rho_2 \rangle$. To prove that G_0 is the automorphism group of $\{3,6\}_{(s,s)}$ we will need to prove that $(\tau_3(\tau_2\tau_1)^2)^2$ has order s and that $\tau_3(\tau_2\tau_1)^2\tau_2$ has order $3s$. Indeed, as the automorphism group of $\{3,6\}_{(s,0)}$ is a quotient of the automorphism group of $\{3,6\}_{(s,s)}$, we have to guarantee that the group is not smaller than expected.

We may take the vertex-set of the toroid $\{3,3,4,3\}$ to be $\mathbb{Z}^4 \cup (\mathbb{Z}^4 + (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}))$, i.e. the set of points of the euclidian 4-space whose cartesian coordinates are all integers or all halves of odd integers [MS02, Section 6E]. As a space-form, the toroid $\{3,3,4,3\}_{(s,0,0,0)}$ is the euclidian 4-space factorized by $\Lambda_{(s,0,0,0)} = s\Lambda_{(1,0,0,0)}$ whose basis is $\{e_1, e_2, e_3, \frac{1}{2}(e_1 + e_2 + e_3 + e_4)\}$ (where e_i, $i \in \{1, \ldots, 4\}$ are the vectors of the canonical basis).

Let us now prove that $v := \tau_3(\tau_2\tau_1)^2\tau_2 = \rho_0\rho_4(\rho_1\rho_3\rho_2)^2\rho_1\rho_3$ is a translation of order $3s$. Consider the involution $\sigma := \rho_1\rho_2\rho_3\rho_2\rho_1$ and the hyperplane reflexions R_i (for $i \in \{0, \ldots, 4\}$) and S as defined on page 171 of [MS02, Section 6E].

$$
\begin{aligned}
&(x_1,x_2,x_3,x_4)R_0 = (1-x_1,x_2,x_3,x_4)\\
&(x_1,x_2,x_3,x_4)R_1 = (x_1,x_2,x_3,x_4) - \tfrac{1}{2}(x_1-x_2-x_3-x_4)(1,-1,-1,-1)\\
&(x_1,x_2,x_3,x_4)R_2 = (x_1,x_2,x_3,-x_4)\\
&(x_1,x_2,x_3,x_4)R_3 = (x_1,x_2,x_4,x_3)\\
&(x_1,x_2,x_3,x_4)R_4 = (x_1,x_3,x_2,x_4)\\
&(x_1,x_2,x_3,x_4)S = (x_2,x_1,x_3,x_4)
\end{aligned}
$$

Then $v = \rho_0\rho_4\rho_3\rho_2\sigma\rho_3$ and its action on the euclidean 4-space is given by $V = R_0R_4R_3R_2SR_3$. We find that $xV = (x_3, 1-x_1, -x_2, x_4)$ and $(x_1,x_2,x_3,x_4)V^3 = (-1,1,-1,0)+(x_1,x_2,x_3,x_4)$. Consequently, V is a translation of order $3s$. Therefore G_0 is the automorphism group of either $\{3,6\}_{(s,s)}$ or of $\{3,6\}_{(3s,0)}$.

Let us now prove that $(\tau_3(\tau_2\tau_1)^2)^2 = (\rho_0\rho_4(\rho_1\rho_3\rho_2)^2)^2$ has order s, or equivalently that $U = (R_0R_4(R_1R_3R_2)^2)^2$ is a translation of order s. Let $P = R_0R_4(R_1R_3R_2)^2$. Then $U = P^2$. The action of P on the euclidean 4-space is described by the following equality.

$$
(x_1,x_2,x_3,x_4)P = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) + \frac{1}{2}\begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
$$

Let M be the 4×4 matrix above. As $M^2 = 4I_4$ (where I_4 denotes the 4 by 4 identity matrix) and $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})M = (-2,0,0,0)$, we get that $U(= P^2)$ is a translation defined by $(x_1,x_2,x_3,x_4)U = (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) + (x_1,x_2,x_3,x_4)$ that has order s. With this we have shown that G_0 is the automorphism group of $\{3,6\}_{(s,s)}$. □

Proposition 7.2.2. *The group $G = \langle \tau_0, \tau_1, \tau_2, \tau_3 \rangle$ of Proposition 7.2.1 is a C-group.*

Proof. Let H be the automorphism group of the rank 5 toroid $\{3, 3, 4, 3\}_{(s,0,0,0)}$ and let G be as in the previous proof. We have shown in Proposition 7.2.1 that $G_1 = H_2$, $G_2 = H_3$ and $G_3 = H_{0,4}$. Since H is a C-group, $G_i \cap G_j = G_{i,j}$ whenever $\{i, j\} \subset \{1, 2, 3\}$. Since G_0 is a C-group (the automorphism group of a toroidal map $\{3, 6\}_{(s,s)}$), by Proposition 2.3.3 we only need to prove that $G_0 \cap G_k = G_{0,k}$ for every $k \in \{1, 2, 3\}$.

Recall that $G_0 = \langle \tau_1, \tau_2, \tau_3 \rangle = \langle \rho_2, \rho_1\rho_3, \rho_0\rho_4 \rangle$. For each k, the fact that $G_0 \cap G_k \supseteq G_{0,k}$ is trivial. To prove the other inclusion we consider each case in turn.

Case $k = 1$: We know that $G_1 = \langle \rho_1\rho_3, \rho_0\rho_4, \rho_1 \rangle = \langle \rho_3, \rho_0, \rho_4, \rho_1 \rangle = H_2 \cong D_3 \times D_3$ and that $G_{0,1} \cong D_3$. With this, we have the inclusion chain $D_3 \cong G_{0,1} \subseteq G_0 \cap G_1 \subseteq G_1 \cong D_3 \times D_3$. Suppose that $G_0 \cap G_1 \ncong D_3$. Then, from the subgroup lattice of $D_3 \times D_3$, $G_0 \cap G_1$ can either be $C_3 \times D_3$, $C_2 \times D_3$ or $(C_3 \times C_3) \rtimes C_2$. For the first two cases, there would exist $\alpha \in G_0 \cap G_1$ centralizing $G_{0,1}$. But there is no such element in $[3, 6]_{(1,1)}$. Consider then $G_0 \cap G_1 \cong (C_3 \times C_3) \rtimes C_2$. It can be checked computationally that $\langle \rho_0\rho_1, \rho_3\rho_4 \rangle \leq G_0 \cap G_1$. Moreover, that would mean that $\rho_0\rho_1 \in G_0$, which in turn results in $\rho_0\rho_1\rho_2\rho_1\rho_0\rho_2\rho_0\rho_1\rho_2 = \rho_0 \in G_0$, which implies $G_0 = G$, a contradiction. As a consequence, $G_{0,1} = G_0 \cap G_1$.

Case $k = 2$: We have $G_{0,2} \cong \langle \rho_2, \rho_0\rho_4 \rangle \cong C_2 \times C_2$ and $G_2 \cong H_3 \cong C_2 \times S_4$, and suppose that $G_0 \cap G_2 \neq G_{0,2}$. Since we have the inclusion $G_{0,2} \leq G_0 \cap G_2 \leq G_2$, we can check computationally the subgroup chain of $G_2 \cong C_2 \times S_4$ and determine the proper subgroups of G_2 that contain $\langle \rho_2, \rho_0\rho_4 \rangle = G_{0,2}$ but are distinct from it. These are $\langle \rho_0, \rho_2, \rho_4 \rangle \cong C_2 \times C_2 \times C_2$ and $\langle \rho_0, \rho_1\rho_2\rho_0\rho_1, \rho_2, \rho_4 \rangle \cong C_2 \times D_4$. In either cases, this implies that both ρ_0 and ρ_4 are in G_0, which in turn result in $\rho_1\rho_3\rho_4\rho_3\rho_1\rho_4\rho_1\rho_3\rho_4 = \rho_1 \in G_0$, a contradiction as in case $k = 1$. Consequently $G_{0,2} = G_0 \cap G_2$.

Case $k = 3$: Lastly, we have $G_{0,3} \cong \langle \rho_2, \rho_1\rho_3 \rangle \cong D_6$ and $G_3 = \langle \rho_1, \rho_2, \rho_1\rho_3 \rangle = H_{0,4} \cong C_2 \times S_4$. There is no other subgroup in the subgroup chain between D_6 and $C_2 \times S_4$. Hence $G_0 \cap G_3 = G_{0,3}$.

As a result, we have that G is a C-group. □

Theorem 7.2.3. *Let $G = \langle \tau_0, \tau_1, \tau_2, \tau_3 \rangle$ be as defined in Proposition 7.2.1. The incidence system given by $\Gamma(G; (G_i)_{i \in I})$ with $G_i := \langle \tau_j | j \in I \setminus \{i\} \rangle$ for all $i \in I := \{0, 1, 2, 3\}$ is a finite universal locally toroidal regular hypertope whose Coxeter diagram is a 4-circuit of type $(3, 6, 3, 6)$ having the rank 3 residues $\{3, 6\}_{(s,s)}$, $\{3, 6\}_{(1,1)}$, $\{3, 6\}_{(2,0)}$, and $\{3, 6\}_{(2,0)}$ (with type-preserving automorphism groups G_0, G_1, G_2 and G_3, respectively).*

Proof. From Theorem 6.3.2, Propositions 7.2.1 and 7.2.2 we only need to prove that G is flag-transitive in Γ. For that, by Lemma 6.3.4, we need to prove one of the two equivalent equalities $G_{i,j}G_{i,k} = G_i \cap (G_jG_k)$ or $G_iG_{j,k} = G_iG_j \cap G_iG_k$ for $\{i, j, k\} \subset \{0, 1, 2, 3\}$, since all G_i are flag-transitive in $\Gamma(G_i, (G_j)_{j \in I \setminus \{i\}})$. In both equalities, the direct inclusion is trivial. We now prove the other inclusion for each case separately.

As in Proposition 7.2.1, let $H := \langle \rho_0, \rho_1, \rho_2, \rho_3, \rho_4 \rangle$ be the group of the regular polytope $\{3, 3, 4, 3\}_{(s,0,0,0)}$.

Case $\{i, j, k\} = \{1, 2, 3\}$: Since H is a string C-group of a regular polytope and since $G_1 = H_2$, $G_2 = H_3$, and $G_3 = H_{0,4}$, the equivalent equalities are satisfied.

Case $\{i, j, k\} = \{0, 1, 2\}$: In this case $G_1G_2 = G_2 \cup \rho_3G_2 \cup \rho_4\rho_3G_2$. Hence, $G_0 \cap (G_1G_2) = G_{0,2} \cup (G_0 \cap \rho_3G_2) \cup (G_0 \cap \rho_4\rho_3G_2)$. It is easy to see that $G_0 \cap \rho_3G_2 = \rho_1\rho_3G_0 \cap \rho_1\rho_3G_2 = \rho_1\rho_3G_{0,2}$ and that $G_0 \cap \rho_4\rho_3G_2 = G_0 \cap \rho_4\rho_0\rho_1\rho_3G_2 = \rho_4\rho_0\rho_1\rho_3(G_{0,2})$.

Since both $\rho_3\rho_1$ and $\rho_3\rho_1\rho_4\rho_0 \in G_{0,1}$, we conclude that $G_0 \cap (G_1G_2) \subseteq G_{0,1}G_{0,2}$, as wanted.

Case $\{i,j,k\} = \{0,1,3\}$: We have $G_0G_1 = G_0 \cup G_0\rho_0 \cup G_0\rho_1 \cup G_0\rho_0\rho_1 \cup G_0\rho_0\rho_3 \cup G_0\rho_0\rho_1\rho_0$ and $G_0G_3 = G_0 \cup G_0\rho_1 \cup G_0\rho_1\rho_2 \cup G_0\rho_1\rho_2\rho_3$. We claim that the cardinality of the following set C of cosets is 8 (meaning that all cosets are pairwise distinct).

$$C := \{G_0,\ G_0\rho_0,\ G_0\rho_0\rho_1,\ G_0\rho_1\rho_0, G_0\rho_0\rho_1\rho_0,\ G_0\rho_1,\ G_0\rho_1\rho_2,\ G_0\rho_1\rho_2\rho_3\}.$$

As $G_0 \cap \langle\rho_0,\rho_1\rangle \subseteq G_0 \cap G_1 \cap \langle\rho_0,\rho_1\rangle = G_{0,1} \cap \langle\rho_0,\rho_1\rangle$, and the right-hand intersection is trivial, the first six cosets must be distinct. To prove that the remaining cosets are also distinct some more calculations are needed. Let us prove that $G_0\rho_0 \neq G_0\rho_1\rho_2\rho_3$. Assume that $\rho_0\rho_3\rho_2\rho_1 \in G_0$. Since $\rho_1\rho_3 \in G_0$ and ρ_3 commutes with both ρ_0 and ρ_1, we have that $\rho_3\rho_1(\rho_0\rho_3\rho_2\rho_1) = \rho_1\rho_0\rho_2\rho_1 \in G_0$. As $\rho_1\rho_0\rho_2\rho_1$ also belongs to G_2 it must belong to $G_0 \cap G_2 = \langle\rho_2,\rho_0\rho_4\rangle = \{id_G,\rho_2,\rho_0\rho_4,\rho_2\rho_4\rho_0\}$. We can easily see that any case leads to a contradiction: if $\rho_1\rho_0\rho_2\rho_1 = id_G$, then $\rho_0\rho_2 = id_G$; if $\rho_1\rho_0\rho_2\rho_1 = \rho_2$, then $\rho_1\rho_0\rho_2 = \rho_2\rho_1$, but $\rho_1\rho_0\rho_2$ has order 4 and $\rho_2\rho_1$ has order 3; if $\rho_1\rho_0\rho_2\rho_1 = \rho_0\rho_4$, then $\rho_4 = \rho_0\rho_1\rho_0\rho_2\rho_1$; at last if $\rho_1\rho_0\rho_2\rho_1 = \rho_2\rho_0\rho_4$, then $\rho_4 = \rho_0\rho_2\rho_1\rho_0\rho_2\rho_1$.

Similar calculations made us conclude that $|C| = 8$. From this we get $G_0G_1 \cap G_0G_3 = G_0 \cup G_0\rho_1 \subseteq G_0G_{1,3}$, as desired.

Case $\{i,j,k\} = \{0,2,3\}$: In this case $G_0G_2 = G_0 \cup G_0\rho_0 \cup G_0\rho_1 \cup G_0\rho_0\rho_1 \cup G_0\rho_1\rho_0 \cup G_0\rho_1\rho_2 \cup G_0\rho_0\rho_1\rho_0 \cup G_0\rho_0\rho_1\rho_2 \cup G_0\rho_1\rho_0\rho_2 \cup G_0\rho_0\rho_1\rho_0\rho_2 \cup G_0\rho_1\rho_0\rho_2\rho_1 \cup G_0\rho_0\rho_1\rho_0\rho_2\rho_1$ and $G_0G_3 = G_0 \cup G_0\rho_1 \cup G_0\rho_1\rho_2 \cup G_0\rho_1\rho_2\rho_3$. We claim that all the cosets of these partition decompositions of G_0G_2 and G_0G_3 are distinct when the representatives of the coset are different. We omit the calculations as the arguments are similar to those used in the above cases and can easily be checked. Then, $G_0G_2 \cap G_0G_3 = G_0 \cup G_0\rho_1 \cup G_0\rho_1\rho_2$ and consequently $G_0G_2 \cap G_0G_3 \subseteq G_0(G_2 \cap G_3)$, as desired.

With this we conclude that Γ is flag-transitive and therefore a regular hypertope. $\square$

7.3 Locally Toroidal Hypertopes with unexpected residues

In the previous section, we proved the existence of a family of locally toroidal hypertopes which was suggested in the first line of Table 8 of [FLW15]. In this same table a list of universal locally toroidal hypertopes with a 4-circuit diagram of type $(3,6,3,6)$ was given, results which were obtained using Magma [BCP97]. Tits algorithm was used in that research to construct regular (and also chiral) incidence geometries from groups with the defining relations for the groups with the 4-circuit diagram of type $(3,6,3,6)$ together with four additional relations which determine the toroidal maps $\{3,6\}_{\mathbf{s}}$, $\{3,6\}_{\mathbf{t}}$, $\{3,6\}_{\mathbf{u}}$, and $\{3,6\}_{\mathbf{v}}$.

In all cases, from lines 2 to 12, this construction leads to a hypertope. However, the authors assumed that the toroidal rank 3 residues for each hypertope in the table would be determined by the values of the eight parameters $\mathbf{s} = (a,b)$, $\mathbf{t} = (c,d)$, $\mathbf{u} = (e,f)$, $\mathbf{v} = (g,h)$, which was not the case. Specifically, in lines 2 and 3 this fails as some of the toroidal residues are in fact smaller than what would be expected by the defining relations of that toroidal residue.

The type-preserving automorphism group of the hypertope of line 2 of Table 8 of [FLW15] is the group $G := G_{(\mathbf{s},\mathbf{t},\mathbf{u},\mathbf{v})}$ with $(\mathbf{s},\mathbf{t},\mathbf{u},\mathbf{v}) = ((2,0),\ (2,0),\ (1,1),\ (3,0))$ as having the following presentation (in [FLW15] the authors considered the rotational subgroup instead).

$$\begin{aligned} G &= \langle \rho_0, \rho_1, \rho_2, \rho_3 | \rho_0^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = (\rho_0\rho_1)^3 = (\rho_0\rho_2)^6 = (\rho_0\rho_3)^2 = \\ &= (\rho_1\rho_2)^2 = (\rho_1\rho_3)^6 = (\rho_2\rho_3)^3 = (\rho_2(\rho_3\rho_1)^2\rho_3)^3 = (\rho_3(\rho_2\rho_0)^2\rho_2)^2 = \\ &= (\rho_0(\rho_1\rho_3)^2)^2 = (\rho_1(\rho_0\rho_2)^2\rho_0)^2 = id_G \rangle \end{aligned} \tag{7.1}$$

The maximal parabolic subgroups G_1, G_2, and G_3 are the type-preserving automorphism groups of $\{3,6\}_{(2,0)}$, $\{3,6\}_{(1,1)}$, and $\{3,6\}_{(2,0)}$ respectively. However, after a closer inspection, G_0 turns out to be the automorphism group of a toroidal regular map $\{3,6\}_{(1,1)}$, which is covered by the automorphism group of $\{3,6\}_{(3,0)}$ (the toroidal residue that was expected). We note that changing the presentation of G given above, replacing the relator $(\rho_2(\rho_3\rho_1)^2\rho_3)^3$ by $(\rho_2(\rho_3\rho_1)^2))^2$ (the relator of the toroidal regular map $\{3,6\}_{(1,1)}$), we get $(\rho_1\rho_3)^2 = id_G$, and the resulting hypertope turns out the universal rank 4 locally toroidal regular polytope $\{\{3,6\}_{(2,0)}, \{6,3\}_{(2,0)}\}$ of type $\{3,6,3\}$ with 240 chambers (see [Wei84] and also Table 11E1 of [MS02]) and not the hypertope determined by the group presentation 7.1.

The same happens to the hypertope in line 3 of Table 8 of [FLW15] that was constructed from the rotational subgroup of the C-group $G := G_{(\mathbf{s},\mathbf{t},\mathbf{u},\mathbf{v})}$ with $(\mathbf{s},\mathbf{t},\mathbf{u},\mathbf{v}) = ((2,0), (2,0), (1,1), (6,0))$ with the following presentation.

$$\begin{aligned} G &= \langle \rho_0, \rho_1, \rho_2, \rho_3 | \rho_0^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = (\rho_0\rho_1)^3 = (\rho_0\rho_2)^6 = (\rho_0\rho_3)^2 = \\ &= (\rho_1\rho_2)^2 = (\rho_1\rho_3)^6 = (\rho_2\rho_3)^3 = (\rho_2(\rho_3\rho_1)^2\rho_3)^6 = (\rho_3(\rho_2\rho_0)^2\rho_2)^2 = \\ &= (\rho_0(\rho_1\rho_3)^2)^2 = (\rho_1(\rho_0\rho_2)^2\rho_0)^2 = id_G \rangle \end{aligned} \tag{7.2}$$

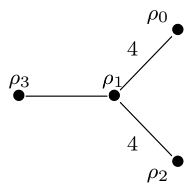
In this case, G_0 is the automorphism group of the toroidal map $\{3,6\}_{(2,2)}$ and not of $\{3,6\}_{(6,0)}$, which covers it. It can easily be checked that replacing the relator $(\rho_2(\rho_3\rho_1)^2\rho_3)^6$ by $(\rho_2(\rho_3\rho_1)^2)^4$ yields the group of the same locally toroidal regular polytope $\{3,6,3\}$ as above.

Another important aspect to be noted is that the ordering of the residues (arising from the Coxeter diagram) matters. In the previous section, we have a family of regular hypertopes with toroidal residues $\{3,6\}_{(s,s)}$, $\{3,6\}_{(1,1)}$, $\{3,6\}_{(2,0)}$, and $\{3,6\}_{(2,0)}$ for the maximal parabolic subgroups G_0, G_1, G_2 and G_3, respectively, but a different order of these residues on the 4-circuit $(3,6,3,6)$ may lead to different hypertopes or even may not lead to hypertopes at all.

Chapter 8

Locally toroidal hypertopes from FTPR graphs

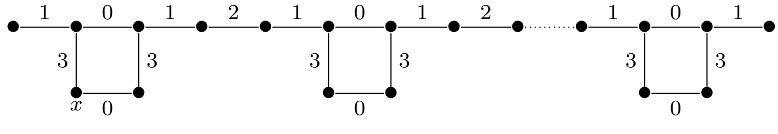
In this chapter we give two new infinite families of regular 4-hypertopes having the following diagram that we will denote by $\{3, {}^{4}_{4}\}$. We also say that these are hypertopes of type $\{3, {}^{4}_{4}\}$.



We construct these regular hypertopes from a C-group using what is known as Tits' algorithm, which was described in Proposition 6.3.1. The C-groups in this chapter were found by combining FTPR graphs of toroidal maps $\{4,4\}_{(s,0)}$ found in Chapter 4 (Propositions 4.2.7 and 4.2.9), with FTPR graphs of the cube and of the hemi-cube. Here, it will be evident how the use of FTPR graphs, together with Tits algorithm, is an efficient method to discover new families regular hypertopes.

8.1 Star 4-hypertopes having the map $\{4,4\}_{(s,0)}$, the hemi-cube and the cube as rank 3 residues

Let s be an even number and $s \geq 4$. Consider the following FTPR graph, having $3s$ vertices, which was obtained combining FTPR graphs of the cube, the hemi-cube and of a FTPR graph of degree $2s$ of $\{4,4\}_{(s,0)}$ given in Proposition 4.2.7 with labels 0 and 2 interchanged (which is also, by duality, a FTPR graph of a map of $\{4,4\}_{(s,0)}$).

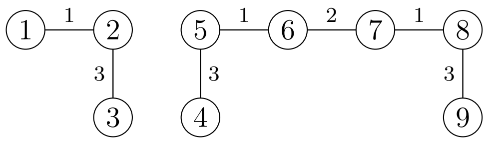


Let $G = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$ be the group with the above FTPR graph. Consider the incidence system $(G, (G_i)_{i \in \{0,1,2,3\}})$ where $G_i = \langle \rho_j \mid j \neq i \rangle$. It can be easily seen that the maximal parabolic subgroups of G are $G_0 \cong [4,3]$, $G_1 \cong C_2 \times C_2 \times C_2$, $G_2 \cong [4,3]_3$ and $G_3 \cong [4,4]_{(s,0)}$.

Proposition 8.1.1. *G is a C-group.*

Proof. For $i \in \{0,1,2,3\}$, G_i is known to be a C-group, thus, by Proposition 2.3.3, we need only to prove that $G_i \cap G_j = G_{i,j}$, for distinct $i,j \in \{0,1,2,3\}$. As $G_{i,j} \leq G_i \cap G_j$ we only need to prove the other inclusion. First $G_1 \cap G_j \cong C_2 \times C_2 \cong G_{1,j}$ for $j \in \{0,2,3\}$. Now since $G_{0,2} = \langle \rho_1, \rho_3 \rangle \cong S_3$ is a maximal subgroup of G_2 and $G_{2,3} = \langle \rho_0, \rho_1 \rangle \cong D_4$ is a maximal subgroup of G_2, we have, $G_0 \cap G_2 = G_{0,2}$ and $G_2 \cap G_3 = G_{2,3}$.

Consider the subgroups G_0 and G_3. The subgroup G_0 acts faithfully on nine points, as follows.



The subgroup $G_0 \cap G_3$ must be a subgroup of the stabilizer of vertex 4 in G_0, which is $\langle \rho_1, \rho_2 \rangle$. Hence, $G_0 \cap G_3 = G_{0,3}$.

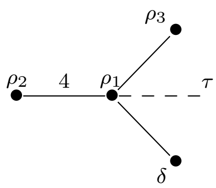
□

Proposition 8.1.2. *The group G has the following presentation.*

$$\langle \rho_0, \rho_1, \rho_2, \rho_3 \mid \rho_0^2,\ \rho_1^2,\ \rho_2^2,\ \rho_3^2,\ (\rho_0\rho_1)^4,\ (\rho_0\rho_2)^2,\ (\rho_0\rho_3)^2,$$
$$(\rho_1\rho_2)^4,\ (\rho_1\rho_3)^3,\ (\rho_2\rho_3)^2,\ (\rho_0\rho_1\rho_2\rho_1)^s,\ (\rho_0\rho_1\rho_3)^3 \rangle$$

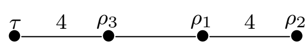
Moreover, the order of G is $24s^3$.

Proof. Let H be the group with the presentation given in this proposition. We easily get from the FTPR graph that $G \leq H$. Let $\delta := \rho_0\rho_3$. The group generated by the set $\{\delta,\ \rho_1,\ \rho_2,\ \rho_3\}$, which is isomorphic to H, is a quotient of an infinite Coxeter group having the following diagram.



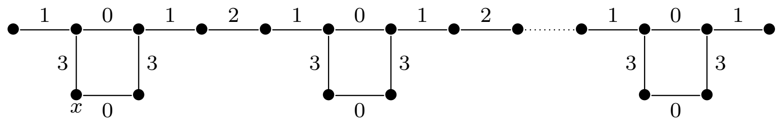
Since $(\rho_2\rho_1\delta)^3 = \rho_2\rho_2^{\rho_1}\rho_2^{\rho_1\delta}$ and the involutions ρ_2, $\rho_2^{\rho_1}$ and $\rho_2^{\rho_1\delta}$ commute with each other, the order of $\rho_2\rho_1\delta$ is 6. Thus $\langle \delta, \rho_1, \rho_2 \rangle \cong [4,3]$. In addition $(\delta\rho_3\rho_1\rho_2\rho_1)^s = id$. Thus H admits a duality τ as suggested by the diagram above.

Consider the group $H^* := \langle \tau, \rho_3, \rho_1, \rho_2 \rangle$, where τ is an involution commuting with both ρ_1 and ρ_2, and such that $\rho_3^\tau = \delta$. It turns out that H^* is a factorization of a Coxeter group with linear diagram



by the relation $(\tau\rho_3\tau\rho_3\rho_1\rho_2\rho_1)^s = id$. Let $u := \tau\rho_3\rho_1\rho_2\rho_1\rho_3$ and $v := u^{\rho_3}$. As s is even, $(\tau\rho_3\tau\rho_3\rho_1\rho_2\rho_1)^s = (\tau v)^s = (\tau v \tau v)^{s/2} = v^s = id$. Hence, H^* is the automorphism group of the toroid $\{4,3,4\}_{(s,0,0)}$, of order $48s^3$. Since H is a index 2 subgroup of H^*, we have that $|H| = 24s^3$.

Now consider the vertex x of the FTPR graph of G.



The stabilizer of x, $Stab_G(x)$, is a subgroup of $\langle \rho_1, \rho_2, \rho_0\rho_1\rho_0, \rho_3\rho_1\rho_2\rho_1\rho_3\rangle$. We have that $K := \langle \rho_1, \rho_2, \rho_0\rho_1\rho_0\rangle$ is an index 2 subgroup of the automorphism group of the map $\{4,4\}_{(s,0)}$. Hence, $|K| = 4s^2$. By Proposition 8.1.1, $\rho_3\rho_1\rho_2\rho_1\rho_3 \notin K$, hence $|Stab_G(x)| \geq 2|K| = 8s^2$. Therefore, by the Orbit-Stabilizer Theorem, we have that $|G| \geq 8s^2 \cdot 3s = 24s^3$. Consequently $G \cong H$.

□

Proposition 8.1.3. *For* $(i,j,k) \in \{(0,1,2),(0,1,3),(2,1,3),(0,2,3)\}$,

$$G_i \cap G_j G_k = G_{i,j} G_{i,k}.$$

Proof. For $j = 1$ we have $G_1 G_k = G_k \cup \rho_k G_k$ for $k \in \{0,2,3\}$. Hence for $(i,k) \in \{(0,2),(0,3),(2,3)\}$ we have $G_i \cap G_1 G_k = G_i \cap (G_k \cup \rho_k G_k) = (G_i \cap G_k) \cup (G_i \cap \rho_k G_k) = G_{i,k} \cup \rho_k G_{i,k} \subseteq G_{i,1} G_{i,k}$.

Now let $(i,j,k) = (0,2,3)$. We have that $G_2 G_3 = G_3 \cup \rho_3 G_3 \cup \rho_1\rho_3 G_3$. Then, $G_0 \cap G_2 G_3 = G_{0,3} \cup \rho_3 G_{0,3} \cup \rho_1\rho_3 G_{0,3} \subseteq G_{0,2} G_{0,3}$.

□

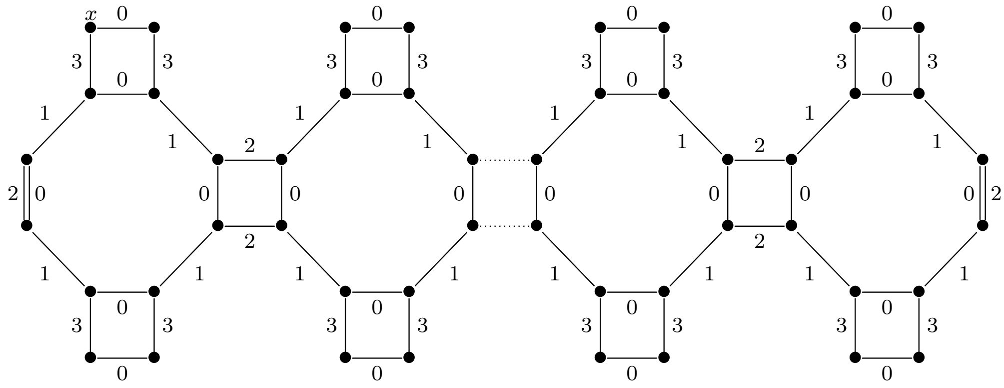
Theorem 8.1.4. *Let s be even and $s \geq 4$. The group G with the following presentation is the automorphism group of a regular hypertope of type $\{3, {4 \atop 4}\}$ whose residues of rank 3 are either cubes, hemi-cubes or toroidal maps $\{4,4\}_{(s,0)}$.*

$$\langle \rho_0, \rho_1, \rho_2, \rho_3 \,|\, \rho_0^2,\ \rho_1^2,\ \rho_2^2,\ \rho_3^2,\ (\rho_0\rho_1)^4,\ (\rho_0\rho_2)^2,\ (\rho_0\rho_3)^2,$$
$$(\rho_1\rho_2)^4,\ (\rho_1\rho_3)^3,\ (\rho_2\rho_3)^2,\ (\rho_0\rho_1\rho_2\rho_1)^s,\ (\rho_0\rho_1\rho_3)^3\rangle$$

Proof. This is a consequence of Theorem 6.3.2, Lemma 6.3.4 and Propositions 8.1.1, 8.1.2, 8.1.3. □

8.2 Star 4-hypertopes having the map $\{4,4\}_{(s,0)}$ and cubes as rank 3 residues

Let s be even and $s \geq 4$. Let G be the group described by the following FTPR graph, having $6s$ vertices, that was obtained combining FTPR graphs of the cube and the FTPR graph of the map $\{4,4\}_{(s,0)}$ in Proposition 4.2.9.

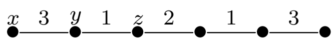


From the graph, it can be easily seen that the maximal parabolic subgroups of G are $G_0 := \langle \rho_1, \rho_2, \rho_3 \rangle \cong [4,3]$, $G_1 := \langle \rho_0, \rho_2, \rho_3 \rangle \cong C_2 \times C_2 \times C_2$, $G_2 := \langle \rho_0, \rho_1, \rho_3 \rangle \cong [4,3]$ and $G_3 := \langle \rho_0, \rho_1, \rho_2 \rangle \cong [4,4]_{(s,0)}$.

Proposition 8.2.1. *The group G is a C-group.*

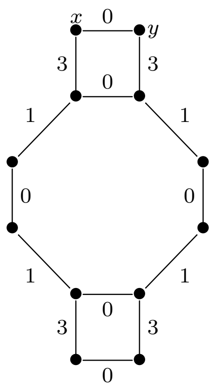
Proof. Since G_i is a C-group for all i, by Proposition 2.3.3, we need only to prove that $G_i \cap G_j = G_{i,j}$ for $i, j \in \{0,1,2,3\}$ and $i \neq j$. As $G_1 \cong C_2 \times C_2 \times C_2$ and $G_{1,k} \cong C_2 \times C_2$ for all $k \in \{0,2,3\}$, then it is trivial to see that $G_{1,k} = G_1 \cap G_k$.

The group G_0 acts faithfully on 6 points with the following FTPR graph.



Consider the points x, y, z in the graph. The group $G_0 \cap G_2$ is in the stabilizer of the set $\{x, y, z\}$ that is isomorphic to S_3. Hence $G_0 \cap G_2 = G_{0,2}$. The group $G_0 \cap G_3$ is in the stabilizer of x that is isomorphic to D_4. Hence $G_0 \cap G_3 = G_{0,3}$.

Now consider the faithful action of G_2 on 10 points given by the following FTPR graph.



The group $G_2 \cap G_3$ is in the stabilizer of $\{x, y\}$ that is isomorphic to D_4, hence $G_2 \cap G_3 \cong G_{2,3}$. □

Proposition 8.2.2. *Let $\lambda := \rho_0\rho_1\rho_3\rho_0\rho_1\rho_2\rho_1\rho_3$. The group G has the following presentation with s even and $s \geq 4$.*

$$\langle \rho_0, \rho_1, \rho_2, \rho_3 \mid \rho_0^2,\ \rho_1^2,\ \rho_2^2,\ \rho_3^2,\ (\rho_0\rho_1)^4,\ (\rho_0\rho_2)^2,\ (\rho_0\rho_3)^2,\ (\rho_1\rho_2)^4,\ (\rho_1\rho_3)^3,\ (\rho_2\rho_3)^2,\ (\rho_0\rho_1\rho_2\rho_1)^s,\ \lambda^2 \rangle$$

Moreover, the order of G is $48s^3$.

Proof. Consider the point x on the FTPR graph of G. The stabilizer of x, $Stab_G(x)$, contains $\langle \rho_0\rho_1\rho_0, \rho_1, \rho_2, \lambda\rho_3\rho_1\rho_0\rho_1\rho_3\rangle$. The group $K := \langle \rho_0\rho_1\rho_0, \rho_1, \rho_2\rangle$ is a index 2 subgroup of the automorphism group of the map $\{4,4\}_{(s,0)}$. Hence, $|K| = 4s^2$. By Proposition 8.2.1 K is a proper subgroup of $Stab_G(x)$, then $|Stab_G(x)| \geq 8s^2$. As the FTPR graph of G has $6s$ points, by the Orbit-Stabilizer Theorem, $|G| \geq 8s^2 \cdot 6s = 48s^3$.

Let H be the group with the presentation given in this proposition. We have $G \leq H$. The element $(\rho_0\rho_1\rho_3)^3$ is a central involution in H. Indeed $\rho_2(\rho_0\rho_1\rho_3)^3\rho_2 = \rho_0\rho_1\rho_3\lambda\rho_3\rho_1\rho_3\rho_2 = (\rho_0\rho_1\rho_3)^3$. Factorizing H by $\langle(\rho_0\rho_1\rho_3)^3\rangle$ we get the automorphism group of the regular hypertope given in Theorem 8.1.4. Thus $|H| = 48s^2$. Consequently $G \cong H$.

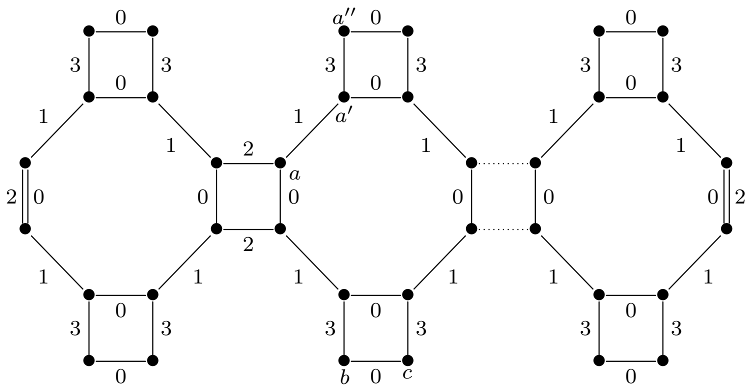
□

Proposition 8.2.3. *For $(i,j,k) \in \{(0,1,2),(0,1,3),(2,1,3),(0,2,3)\}$,*

$$G_i \cap G_jG_k = G_{i,j}G_{i,k}.$$

Proof. For $j = 1$ the proof is similar to case $j = 1$ of Proposition 8.1.3. Let $(i,j,k) = (0,2,3)$. We have that $G_2G_3 = G_3 \cup \rho_3G_3 \cup \rho_1\rho_3G_3 \cup \rho_0\rho_1\rho_3G_3 \cup \rho_1\rho_0\rho_1\rho_3G_3 \cup \rho_3\rho_1\rho_0\rho_1\rho_3G_3$. Then, $G_0 \cap G_2G_3 = G_{0,3} \cup \rho_3G_{0,3} \cup \rho_1\rho_3G_{0,3} \cup (G_0 \cap \rho_0\rho_1\rho_3G_3) \cup (G_0 \cap \rho_1\rho_0\rho_1\rho_3G_3) \cup (G_0 \cap \rho_3\rho_1\rho_0\rho_1\rho_3G_3)$.

Consider the FTPR graph of G and the vertices a, b, c.



For all $\alpha \in \rho_0\rho_1\rho_3G_3$, $a\alpha \in \{b,c\}$. But neither b nor c are in the G_0-orbit of a. Hence, $G_0 \cap \rho_0\rho_1\rho_3G_3 = \emptyset$. Similarly, replacing a by a'(resp. a'') we get $G_0 \cap \rho_1\rho_0\rho_1\rho_3G_3 = \emptyset$ (resp. $G_0 \cap \rho_3\rho_1\rho_0\rho_1\rho_3G_3 = \emptyset$). Consequently, $G_0 \cap G_2G_3 = G_{0,3} \cup \rho_3G_{0,3} \cup \rho_1\rho_3G_{0,3} \subseteq G_{0,2}G_{0,3}$.

□

Theorem 8.2.4. *Let $\lambda := \rho_0\rho_1\rho_3\rho_0\rho_1\rho_2\rho_1\rho_3$, s be even and $s \geq 4$. The group G with the following presentation is the automorphism group of a regular hypertope of type $\{3, {}^4_4\}$ whose residues of rank 3 are either cubes or toroidal maps $\{4,4\}_{(s,0)}$.*

$$\langle \rho_0,\rho_1,\rho_2,\rho_3 \mid \rho_0^2,\ \rho_1^2,\ \rho_2^2,\ \rho_3^2,\ (\rho_0\rho_1)^4,\ (\rho_0\rho_2)^2,\ (\rho_0\rho_3)^2,$$
$$(\rho_1\rho_2)^4,\ (\rho_1\rho_3)^3,\ (\rho_2\rho_3)^2,\ (\rho_0\rho_1\rho_2\rho_1)^s,\ \lambda^2\rangle$$

Proof. This is a consequence of Theorem 6.3.2, Lemma 6.3.4 and Propositions 8.2.1, 8.2.2, 8.2.3. □

8.3 Quotients of regular hypertopes that give rise to regular hypertopes

In Theorem 6.3.8, we have given sufficient conditions under which a quotient of the type-preserting automorphism group G of a regular n-hypertope by a normal subgroup N give another regular n-hypertope. Among these conditions $N \cap G_i = \{id_G\}$, for all $i \in \{0, \dots, n-1\}$, is required. This theorem together with Theorem 6.3.7 allows to prove that G/N is a C-group. In the proof of Proposition 8.2.2, we use the fact that factoring the automorphism group given in Section 8.2 by $\langle(\rho_0\rho_1\rho_3)^3\rangle$, results in the automorphism group given in Section 8.1. Although $\langle(\rho_0\rho_1\rho_3)^3\rangle$ is a normal subgroup of one of its maximal parabolic subgroups, failing the condition above, the resulting group is still a C-group. This led us to believe that we can have a more general result, without imposing that condition. Here, we will prove a weaker version, where we consider a normal subgroup of a maximal parabolic subgroup.

Theorem 8.3.1. *Let G be the automorphism group of a regular hypertope and N be a normal subgroup of G_i for some $i \in \{0, \dots, r-1\}$, whose normal closure $\overline{N}$, in G, has trivial intersection with G_j for $j \in \{0, \dots, r-1\} \setminus \{i\}$. If G_i/N is a C-group then the maximal parabolic subgroups of $\tilde{G} := G/\overline{N}$ are C-groups.*

Proof. Assume, without loss of generality, that $i = 0$. We have that $N \lhd G_0$ and G_0/N is a C-group. Let $\tilde{G} := G/\overline{N}$ such that $\tilde{G} = \langle \rho_0\overline{N}, \dots, \rho_{n-1}\overline{N}\rangle$ and its maximal parabolic subgroups $\tilde{G}_j = \langle \rho_k\overline{N} \mid k \in \{0, \dots, n-1\} \setminus \{j\}\rangle$. When $j \neq 0$ we have that $G_j \cap \overline{N}$ is trivial, meaning $\tilde{G}_j \cong G_j$ and hence $\tilde{G}_j$ is a C-group.

Based on the result of Theorem 6.3.7, consider the mapping $\sigma : \tilde{G}_0 \to G_0/N$ such that $\sigma(x\overline{N}) = xN$. This map is a homomorphism, as shown below.

$$\begin{aligned}
\sigma(x\overline{N}y\overline{N}) &= (\text{since } \overline{N} \lhd G)\\
&= \sigma(xy\overline{N}\,\overline{N})\\
&= \sigma(xy\overline{N})\\
&= xyN\\
&= xyNN = (\text{since } y \in G_0 \text{ and } N \lhd G_0)\\
&= xNyN = \sigma(x\overline{N})\sigma(y\overline{N})
\end{aligned}$$

Moreover, it is easy to see that it is one-to-one on the maximal parabolic subgroups $\tilde{G}_{0,j}$ (for each $j \in \{1, \dots, n-1\}$). Hence, $\tilde{G}_0$ is a C-group. □

Note that the conditions of Theorem 8.3.1 are not sufficient to guarantee that $\tilde{G}$ is a C-group. Nevertheless, we know from Proposition 2.3.3 that if the maximal parabolic subgroups are C-groups, we need only to prove that $\tilde{G}_i \cap \tilde{G}_j = \tilde{G}_{i,j}$, for all $0 \leq i, j \leq n-1$, in order to prove that $\tilde{G}$ is a C-group.

8.3.1 Star 4-hypertopes having the map $\{4,4\}_{(s,s)}$, the hemi-cube and the cube as rank 3 residues

Proposition 8.3.2. *Let $s \geq 2$. Consider the automorphism group G of the regular hypertope of Section 8.1, with toroidal residue $\{4,4\}_{(2s,0)}$ and let $N := \langle(\rho_0\rho_1\rho_2)^{2s}\rangle$.*

Then $\tilde{G} := G/\overline{N}$ *has the following presentation*

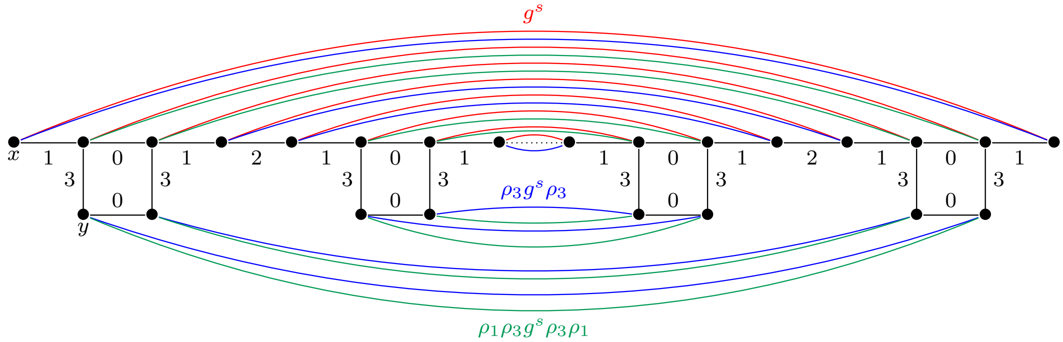
$$\langle \rho_0, \rho_1, \rho_2, \rho_3 \mid \rho_0^2,\ \rho_1^2,\ \rho_2^2,\ \rho_3^2,\ (\rho_0\rho_1)^4,\ (\rho_0\rho_2)^2,\ (\rho_0\rho_3)^2,\ (\rho_1\rho_2)^4,\ (\rho_1\rho_3)^3,\ (\rho_2\rho_3)^2,\ (\rho_0\rho_1\rho_2)^{2s},\ (\rho_0\rho_1\rho_3)^3\rangle$$

with toroidal residue $\{4,4\}_{(s,s)}$*. Moreover,* $\tilde{G}$ *has order* $48s^3$*.*

Proof. Consider $u = \rho_0\rho_1\rho_2\rho_1$ and $g = (\rho_0\rho_1\rho_2)^2$. First, let us prove that

$$\overline{N} = \{id_G, g^s, \rho_3 g^s \rho_3, \rho_1\rho_3 g^s \rho_3\rho_1\}.$$

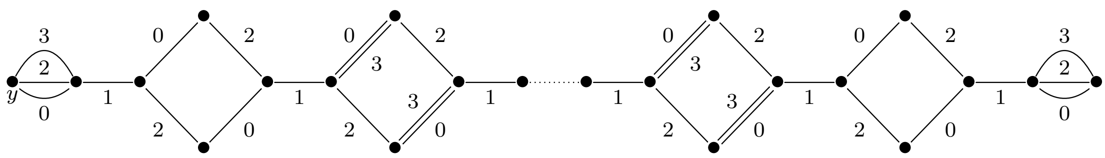
Consider the FTPR graph of G, that was introduced in Section 8.1, and consider the action of the involutions g^s, $(g^s)^{\rho_3}$ and $(g^s)^{\rho_3\rho_1}$.



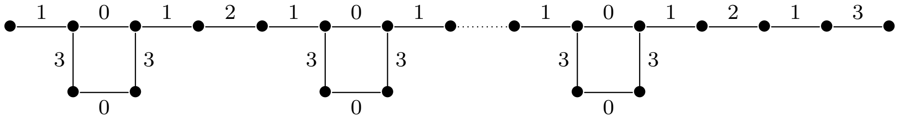
Suppose that there is an element of $\overline{N}$ that is not in the four elements stated above. We can check that both $(g^s)^{\rho_3\rho_1\rho_0}$ and $(g^s)^{\rho_3\rho_1\rho_2}$ are equal to $(g^s)^{\rho_3\rho_1}$. Hence, these four elements are the only ones of $\overline{N}$. Moreover, it can be seen that $(g^s)^{\rho_3\rho_1} = g^s(g^s)^{\rho_3}$, making $\overline{N} \cong C_2 \times C_2$. With this, we have that $\tilde{G} = G/\overline{N}$ has order $48s^3$.

From Theorem 8.3.1, we already have that all maximal parabolic subgroups are C-groups. Additionally, we know that all $G_i \cong \tilde{G}_i$, for $i \in \{0,1,2\}$, proving all the relations of the presentation above, except for the ones of the toroidal residue $\{4,4\}$. We have that $\tilde{G}_3$ can be isomorphic to the automorphism groups of either the map $\{4,4\}_{(s,s)}$ and $\{4,4\}_{(s,0)}$. Let us prove it *cannot* be the latter and, to do so, suppose that $u^s\overline{N} = \overline{N}$. Then we have that $u^s \in \overline{N}$. Since $u^s \notin N$, this implies that u^s is $\rho_3 g^s \rho_3$ or one of its conjugates. Consider the vertex labelled x. The action of u^s fixes this vertex, impling that $u^s = (g^s)^{\rho_3\rho_1}$. However, by considering the vertex labelled y, the action of u^s either fixes the vertex or swaps it to its ρ_0-adjacent (depending on whether s is even or odd, respectively), a contradiction. Hence, $u^s \notin \overline{N}$, proving that the toroidal residue of $\tilde{G}$ is $\{4,4\}_{(s,s)}$. □

Using Theorem 8.3.1, it is already known that all maximal parabolic subgroups of $\tilde{G}$ are C-groups. Then to prove that $\tilde{G}$ is the automorphism group of a regular hypertope with the above presentation, we need only to prove the intersection property between these maximal parabolic subgroups and flag-transitivity. To do so, we can use the following transitive permutation representation graphs, for s even



and for s odd.



Conjecture 8.3.3. $\tilde{G}$ is the automorphism group of a regular hypertope of type $\{3, \begin{smallmatrix}4\\4\end{smallmatrix}\}$ whose residues are $\{4,3\}$, $\{4,3\}_3$ and the toroidal map $\{4,4\}_{(s,s)}$.

Chapter 9

Families of hyperbolic hypertopes

As previously said in Section 6.4, compact hyperbolic Coxeter groups exist only in ranks 3, 4 and 5, and until very recently there were not many examples of finite proper regular hypertopes of hyperbolic type for rank 4 and 5. Computational examples were of small size [FLW20, Table 3]. However the works of Weiss and Montero expanded the numbers of examples and families of proper regular hypertopes of hyperbolic type [MW20; MW21]. Some of these examples were obtained using the halving operation, an operation well characterized for abstract regular polytopes (see [MS02, Section 7B]). Weiss and Montero in [MW20] expanded this operation to obtain regular hypertopes from non-degenerate abstract regular polytopes, as presented in Section 6.5.

As described in Chapter 2, we will consider as a *non-degenerate* abstract regular polytope those which their partial order induces a lattice. Some authors consider a polytope degenerate whenever its Schläfli type has at least a 2 [MS02; Cun17]. However, polytopes such as the toroidal map $\{3,6\}_{(1,1)}$ have all their vertices incident with each and every one of their facets (i.e. it is a flat polytope), having also a slight degeneracy that cannot be allowed when applying the halving operation. Hence, by restricting non-degenerate polytopes to those whose partial order is a lattice guarantees that no type of degeneracy can happen. Moreover, the lattice condition will be useful when building new polytopes from centrally symmetric ones, as described in Lemma 9.1.3 of Section 9.1.

In this chapter, we will start from centrally symmetric regular non-degenerate polytopes of spherical type and, by applying the halving operation, we obtain families of regular hypertopes, some of which of hyperbolic type. As expressing in Section 6.5, if the set of vertices of $\mathcal{P}$ is bipartite, then $H(\mathcal{P})$ has index 2 on $G(\mathcal{P})$. In Sections 9.2 and 9.3 the halving operation will be used on polytopes of type $\{4, p_2, \ldots, p_{n-1}\}$ factorized by relations with even number of ρ_0 and ρ_1. Hence, in all cases that we will deal with in this chapter, $|H(\mathcal{P})| = |G(\mathcal{P})|/2$.

In Table 9.1, a list of all centrally symmetric regular non-degenerate polytopes of spherical type (i.e. finite irreducible Coxeter groups with linear diagram) is given, whose automorphism group is $\langle \tau_0, \ldots, \tau_{n-1} \rangle$ and with central involution α.

The only centrally symmetric polygons have an even number of vertices and their proper central involution is a 180 degrees rotation. For rank 3 and 4, the spherical regular polytopes that are centrally symmetric can be easily computed. For rank $n \geq 5$, the only spherical regular polytopes are the n-simplex and the n-cube $\{4, 3^{n-2}\}$ (and its dual). Since the group of the n-simplex is centerless, only the n-cube $\{4, 3^{n-2}\}$ (and its dual) are centrally symmetric, with $\alpha = (\tau_0\tau_1 \ldots \tau_{n-1})^n$ [HL08]. Moreover, all the

Table 9.1: The centrally symmetric non-degenerate regular polytopes of spherical type

Rank	Schläfli type	Number of vertices	α	$\lvert\mathcal{P}\rvert$
2	$\{2p\}$, for $2 \leq p < \infty$	$2p$	$(\tau_0\tau_1)^p$	$4p$
3	$\{3,4\}$	6	$(\tau_0\tau_1\tau_2)^3$	48
	$\{4,3\}$	8		
	$\{3,5\}$	12	$(\tau_0\tau_1\tau_2)^5$	120
	$\{5,3\}$	20		
4	$\{3,3,4\}$	8	$(\tau_0\tau_1\tau_2\tau_3)^4$	384
	$\{4,3,3\}$	16		
	$\{3,4,3\}$	24	$(\tau_0\tau_1\tau_2\tau_3)^6$	1152
	$\{3,3,5\}$	120	$(\tau_0\tau_1\tau_2\tau_3)^{15}$	14400
	$\{5,3,3\}$	600		
$n \geq 5$	$\{3^{n-2},4\}$	$2n$	$(\tau_0\tau_1\ldots\tau_{n-1})^n$	$2^n n!$
	$\{4,3^{n-2}\}$	2^n		

polytopes of Table 9.1 are convex polytopes, meaning that their poset form a face-lattice.

9.1 The $2^{\mathcal{P},\mathcal{G}(s)}$ polytopes

Consider a Coxeter group W generated by k involutions $\sigma_0, \ldots, \sigma_{k-1}$ with Coxeter diagram $\mathcal{G}$, and $\tau_0, \ldots, \tau_{n-1}$ to be involutory automorphisms of W, permuting its generators. Then W can be extended to a semidirect product $G = W \rtimes \Lambda$, where Λ is the group of involutory automophisms of W which permute its generators. In the case W is a C-group represented by a Coxeter diagram $\mathcal{G}$, the automorphisms τ_i can be seen as symmetries of $\mathcal{G}$.

Definition 9.1.1. [MS02] Let $\mathcal{G}$ be the Coxeter diagram of a C-group and let $\mathcal{P}$ be a regular n-polytope with automorphism group $G(\mathcal{P}) = \langle \tau_0, \ldots, \tau_{n-1} \rangle$. We say $\mathcal{G}$ is $\mathcal{P}$-*admissible* if:

- The Coxeter diagram $\mathcal{G}$ has more than one node;
- $G(\mathcal{P})$ acts transitively on the set of nodes of $\mathcal{G}$, $V(\mathcal{G})$;
- The subgroup $\langle \tau_1, \ldots, \tau_{n-1} \rangle$ of $G(\mathcal{P})$ fixes at least one node of $\mathcal{G}$, which we will designate as F_0;
- The action of $G(\mathcal{P})$ on the diagram $\mathcal{G}$, with respect to F_0, respects the intersection property, i.e., for $I \subseteq \{0, \ldots, n-1\}$ and denoting $V(\mathcal{G}, I)$ as the set of nodes of $\mathcal{G}$ that the subgroup $\langle \tau_i | i \in I \rangle$ maps the node F_0 to, then

$$V(\mathcal{G}, I) \cap V(\mathcal{G}, J) = V(\mathcal{G}, I \cap J) \text{ if } I, J \subseteq \{0, \ldots, n-1\}$$

Let us consider the case $V(\mathcal{G}) = V(\mathcal{P})$ and let F_0 be a vertex of $\mathcal{P}$. Then $\mathcal{G}$ is $\mathcal{P}$-admissible [MS02]. The number of possible choices for the proper branches of the diagram $\mathcal{G}$ depends on the number of diagonal classes of $\mathcal{P}$. When $\mathcal{P}$ is centrally symmetric, there is an involution α permuting pairs of antipodal vertices of $\mathcal{P}$, forming one diagonal class of $\mathcal{P}$. When this diagonal class is the only one represented in the Coxeter diagram $\mathcal{G}$ by proper branches all with the same label s, then the diagram $\mathcal{G} =: \mathcal{G}(s)$ is a matching and the corresponding Coxeter group is a direct product of dihedral groups of degree s, D_s, which is finite if and only if $\mathcal{P}$ is finite.

Consider a Coxeter group W with diagram $\mathcal{G}(s)$,

$$W := W(\mathcal{G}(s)) = \langle \sigma_F \mid F \in V(\mathcal{G}(s)) \rangle, \tag{9.1}$$

and a centrally symmetric regular polytope $\mathcal{P}$, with $G(\mathcal{P}) = \langle \tau_0, \ldots, \tau_{n-1} \rangle$ and where $V(\mathcal{G}(s)) = V(\mathcal{P})$. Let F_0 be a vertex of $\mathcal{P}$. Then the $(n+1)$-polytope $2^{\mathcal{P},\mathcal{G}(s)}$ is defined by the group

$$G(2^{\mathcal{P},\mathcal{G}(s)}) := W \rtimes G(\mathcal{P}) = \langle \rho_0, \ldots, \rho_n \rangle$$

where

$$\rho_i := \begin{cases} \sigma_{F_0}, & \text{for } i = 0, \\ \tau_{i-1}, & \text{for } i = 1, \ldots, n. \end{cases} \tag{9.2}$$

For each $F \in V(\mathcal{G})$, there exists an element $\tau \in G(\mathcal{P})$ and an involution σ_F of W such that

$$\sigma_F = \sigma_{F_0\tau} = \tau^{-1}\sigma_{F_0}\tau = \tau^{-1}\rho_0\tau. \tag{9.3}$$

When $\mathcal{P}$ is centrally symmetric, with central involution α, $\mathcal{G}(s)$ is a matching, as described before, with label s, making $(\sigma_F\sigma_{F\alpha})^s = id$. When $s \geq 3$, all generators σ_F of W commute with each other, except with $\sigma_{F\alpha}$. Then we have $W \cong D_s^{|V(\mathcal{G})|/2}$. Also, for each $F \in V(\mathcal{G})$ and $\tau \in G(\mathcal{P})$, such that $F_0\tau = F$, we have that

$$\sigma_F\sigma_{F\alpha} = \tau^{-1}\rho_0\tau\alpha^{-1}\tau^{-1}\rho_0\tau\alpha = \tau^{-1}\rho_0\alpha\rho_0\alpha\tau. \tag{9.4}$$

In particular, for $s = 2$, the diagram $\mathcal{G}(2)$ only has improper branches and $2^{\mathcal{P},\mathcal{G}(2)}$ is the Danzer polytope $2^{\mathcal{P}}$ [Dan84].

The following theorem gives some properties of $2^{\mathcal{P},\mathcal{G}(s)}$ which will be of great importance for our results.

Theorem 9.1.2. *[MS02, Theorem 8C5] Let $n \geq 1$, and let $\mathcal{P}$ be a centrally symmetric regular n-polytope of type $\{p_1, \ldots, p_{n-1}\}$ with $p_1 \geq 3$. Then the regular $(n+1)$-polytope $2^{\mathcal{P},\mathcal{G}(s)}$ has the following properties.*

1. *$2^{\mathcal{P},\mathcal{G}(s)}$ is of type $\{4, p_1, \ldots, p_{n-1}\}$;*
2. *$G(2^{\mathcal{P},\mathcal{G}(s)}) = D_s^q \rtimes G(\mathcal{P})$, with $q := |V(\mathcal{P})|/2$, where the action of $G(\mathcal{P})$ on D_s^q $(= W)$ is induced by the action on $\mathcal{G}(s)$. In particular, $2^{\mathcal{P},\mathcal{G}(s)}$ is finite if and only if $\mathcal{P}$ is finite, in which case*

$$\left|G(2^{\mathcal{P},\mathcal{G}(s)})\right| = |D_s|^q \cdot |G(\mathcal{P})| = (2s)^{|V(\mathcal{P})|/2}|G(\mathcal{P})|;$$

3. *If s is even and $\mathcal{P}$ has only finitely many vertices, then $2^{\mathcal{P},\mathcal{G}(s)}$ is also centrally symmetric.*

Notice that if $\mathcal{P}$ is the Coxeter group $[p_1, \ldots, p_{n-1}]$ factorized by a set of relations R, then $2^{\mathcal{P},\mathcal{G}(s)}$ is a Coxeter group $[4, p_1, \ldots, p_{n-1}]$ factorized by the relations in R and the extra relations

$$(\rho_0\alpha\rho_0\alpha)^s = id,$$

where α is the central involution of $\mathcal{P}$, and

$$(\rho_0\tau^{-1}\rho_0\tau)^2 = id$$

for all $\tau \in G(\mathcal{P})$ such that $\tau \neq \alpha$ and $\{F_0, F_0\tau\}$ give distinct diagonal classes of $\mathcal{P}$.

When $\mathcal{P}$ is non-degenerate, this construction gives a non-degenerate polytope, as expressed the next lemma.

Lemma 9.1.3. *[MS02, pp. 264] Let $\mathcal{P}$ be a centrally symmetric regular n-polytope of type $\{p_1, \dots, p_{n-1}\}$ with $p_1 \geq 3$. If the poset of $\mathcal{P}$ is a lattice, then the poset of $2^{\mathcal{P},\mathcal{G}(s)}$ is a lattice.*

9.2 Polytopes $2^{\mathcal{P},\mathcal{G}(s)}$ and Hypertopes $\mathcal{H}(2^{\mathcal{P},\mathcal{G}(s)})$ when $\mathcal{P}$ is a $2p$-gon, n-cube or n-orthoplex

In Table 9.1 we were introduced to the centrally symmetric polytopes $\mathcal{P}$ that will be considered in this chapter. There are three infinite families of polytopes given in that table: the $2p$-gons with type $\{2p\}$, the n-cube with type $\{4, 3^{n-2}\}$ and the n-orthoplex with type $\{3^{n-2}, 4\}$. In the following sections, we will construct extensions of these polytopes and then apply the halving operation as defined in Section 6.5 to obtain families of hypertopes. Moreover, families of proper regular toroidal hypertopes $\left\{\begin{smallmatrix}3\\3\end{smallmatrix}, 3^{n-3}, 4\right\}_{(2s,0^{n-1})}$ will be given for an arbitrary rank and s, extending the results of [Ens18] and [MW20], where the duals of the hypertopes $\left\{\begin{smallmatrix}3\\3\end{smallmatrix}, 4\right\}_{(2s,0,0)}$ and $\left\{\begin{smallmatrix}3\\3\end{smallmatrix}, 3^{n-3}, 4\right\}_{(4,0^{n-1})}$ are presented, respectively.

- **The polytope $2^{\{2p\},\mathcal{G}(s)}$ and hypertope $\mathcal{H}(2^{\{2p\},\mathcal{G}(s)})$**

Consider the following polytopes, defined as below.

Definition 9.2.1. [MS02, Section 7B] Let $2 \leq j \leq k := \lfloor \frac{1}{2}q \rfloor$. Then, we define the polytope $\mathcal{P} := \{p, q \mid h_2, \dots, h_k\}$ such that its automorphism group $G(\mathcal{P})$ has the following presentation

$$\begin{aligned} G(\mathcal{P}) := \; & \langle\, \rho_0, \rho_1, \rho_2 \mid \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^p = (\rho_1\rho_2)^q = (\rho_0\rho_2)^2 = id, \\ & \{(\rho_0\rho_1(\rho_2\rho_1)^{j-1})^{h_j} = id, \text{ for } 2 \leq j \leq k\} \,\rangle \end{aligned}$$

Let $\mathcal{P}$ be the polygons with even number of vertices, i.e. of type $\{2p\}$, for $p \geq 2$. From Corollary 8C7 of [MS02], we have the following result.

Corollary 9.2.2. *[MS02, Corollary 8C7] Let $2 \leq p < \infty$ and $2 \leq s < \infty$. Then $2^{\{2p\},\mathcal{G}(s)} = \{4, 2p \mid 4^{p-2}, 2s\}$, with group $D_s^p \rtimes D_{2p}$, of order $(2s)^p \cdot 4p$. If $p = 2$, this is the torus map $\{4,4\}_{(2s,0)}$, with group $(D_s \times D_s) \rtimes D_4$ of order $32s^2$.*

We write 4^{p-2} to mean a row of 4's of size $(p-2)$. Using this result and Definition 9.2.1, we have the following proposition.

Proposition 9.2.3. *Let $2 \leq p < \infty$ and $2 \leq s < \infty$. Then the group $G(2^{\{2p\},\mathcal{G}(s)}) = [4, 2p \mid 4^{p-2}, 2s]$ has the following presentation*

$$G(2^{\{2p\},\mathcal{G}(s)}) := \langle\, \rho_0, \rho_1, \rho_2 \mid \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^4 = (\rho_1\rho_2)^{2p} = (\rho_0\rho_2)^2 = id,$$

$$\{(\rho_0\rho_1(\rho_2\rho_1)^{j-1})^4 = id, \textit{ for } 2 \leq j \leq p-1\}, (\rho_0\rho_1(\rho_2\rho_1)^{p-1})^{2s} = id \,\rangle.$$

Proof. The proof follows from Corollary 9.2.2 and Definition 9.2.1. □

From the polytopes of the previous proposition, we derive a family of polytopes using the halving operation.

Proposition 9.2.4. *Let $2 \leq p < \infty$, $2 \leq s < \infty$. The incidence system*

$$\mathcal{H}(2^{\{2p\},\mathcal{G}(s)}) = \Gamma(H(2^{\{2p\},\mathcal{G}(s)}), (H_i)_{i\in\{0,1,2\}}),$$

where $H(2^{\{2p\},\mathcal{G}(s)}) := \langle \rho_0\rho_1\rho_0, \rho_1, \rho_2\rangle = \langle \tilde{\rho}_0, \rho_1, \rho_2\rangle$, is a regular polytope of type $\{2p, 2p\}$, where its automorphism group, of size $(2s)^p \cdot 2p$, is the quotient of the Coxeter group $[2p, 2p]$ by the relations $((\tilde{\rho}_0\rho_2)^{j-1}\tilde{\rho}_0\rho_1(\rho_2\rho_1)^{j-1})^2 = id$, for $2 \leq j \leq p-1$, and $((\tilde{\rho}_0\rho_2)^{p-1}\tilde{\rho}_0\rho_1(\rho_2\rho_1)^{p-1})^s = id$.

Proof. Let $2 \leq p < \infty$, $2 \leq s < \infty$. The fact that the incidence system of the halving group $H(2^{\{2p\},\mathcal{G}(s)})$ is a regular hypertope follows from the fact that $2^{\{2p\},\mathcal{G}(s)}$ is non-degenerate (a lattice, by Lemma 9.1.3) and from Corollary 6.5.1. Moreover, in Section 7B [MS02], it is given that the halving operation on regular polytope of type $\{4, k\}$ results in a regular polytope of type $\{k, k\}$. Let us write the relations that are not of the infinite Coxeter group $[2p, 2p]$.

Firstly,

$$\begin{aligned}
id &= (\rho_0\rho_1(\rho_2\rho_1)^{j-1})^4\\
&= (\rho_0\rho_1(\rho_2\rho_1)^{j-1}\rho_0\rho_1(\rho_2\rho_1)^{j-1})^2\\
&= (\rho_0\rho_1(\rho_0\rho_2\rho_0\rho_1)^{j-1}\rho_0\rho_1(\rho_2\rho_1)^{j-1})^2\\
&= ((\rho_0\rho_1\rho_0\rho_2)^{j-1}\rho_0\rho_1\rho_0\rho_1(\rho_2\rho_1)^{j-1})^2\\
&= ((\tilde{\rho}_0\rho_2)^{j-1}\tilde{\rho}_0\rho_1(\rho_2\rho_1)^{j-1})^2.
\end{aligned}$$

For the relation $(\rho_0\rho_1(\rho_2\rho_1)^{p-1})^{2s} = id$, similar arguments give

$$id = (\rho_0\rho_1(\rho_2\rho_1)^{p-1})^{2s} = ((\tilde{\rho}_0\rho_2)^{p-1}\tilde{\rho}_0\rho_1(\rho_2\rho_1)^{p-1})^s.$$

□

Notice that, if $p = 2$, the regular hypertope obtained from the halving of $\{4, 4\}_{(2s,0)}$ is the regular map $\{4, 4\}_{(s,s)}$.

• The polytope $2^{\{3^{n-2},4\},\mathcal{G}(s)}$ and hypertope $\mathcal{H}(2^{\{3^{n-2},4\},\mathcal{G}(s)})$

Let $\mathcal{P}$ be the n-orthoplex, with $n \geq 3$. From Corollary 8C6 of [MS02], we have the following result.

Corollary 9.2.5. *[MS02, Corollary 8C6] Let $n \geq 3$ and $2 \leq s < \infty$. The polytope $2^{\{3^{n-2},4\},\mathcal{G}(s)}$ is the cubical regular $(n+1)$-toroid $\{4, 3^{n-2}, 4\}_{(2s,0^{n-1})}$, with group $D^n_s \rtimes [3^{n-2}, 4]$ of order $(4s)^n n!$.*

The defining relations of the regular polytope $\{4, 3^{n-2}, 4\}_{(2s,0^{n-1})}$ are those given by its Schläfli type and the extra relation [MS02, Section 6D]

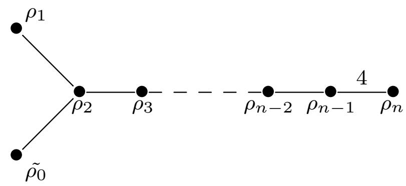
$$(\rho_0\rho_1\rho_2 \dots \rho_{n-1}\rho_n\rho_{n-1} \dots \rho_2\rho_1)^{2s} = id.$$

Now, using the halving operation, we get a family of proper regular toroidal hypertopes, as shown in the following result.

Proposition 9.2.6. *Let $n \geq 3$, $2 \leq s < \infty$. The incidence system*

$$\mathcal{H}(2^{\{3^{n-2},4\},\mathcal{G}(s)}) = \Gamma(H(2^{\{3^{n-2},4\},\mathcal{G}(s)}), (H_i)_{i\in\{0,\dots,n\}}),$$

where $H(2^{\{3^{n-2},4\},\mathcal{G}(s)}) := \langle \rho_0\rho_1\rho_0, \rho_1, \dots, \rho_n\rangle = \langle \tilde{\rho_0}, \rho_1, \dots, \rho_n\rangle$, is a regular hypertope whose automorphism group, of size $(4s)^{n-1}(2s)n!$, is the quotient of the Coxeter group with diagram



factorized by

$$(\tilde{\rho_0}\rho_2\rho_3 \dots \rho_{n-1}\rho_n\rho_{n-1} \dots \rho_3\rho_2\rho_1)^{2s} = id.$$

Proof. The incidence system of the halving group $H(2^{\{3^{n-2},4\},\mathcal{G}(s)})$ is a regular hypertope since the poset of the polytope $\{3^{n-2}, 4\}$ is a lattice, making $2^{\{3^{n-2},4\},\mathcal{G}(s)}$ non-degenerate (by Lemma 9.1.3), which is under the conditions of Corollary 6.5.1. The relations of the Coxeter diagram above follow naturally from the definition of the halving operation.

Consider now the extra relation

$$(\rho_0\rho_1\rho_2 \dots \rho_{n-1}\rho_n\rho_{n-1} \dots \rho_2\rho_1)^{2s} = id$$

of $\{4, 3^{n-2}, 4\}_{(2s,0^{n-1})}$. Then,

$$\begin{aligned}
id &= (\rho_0\rho_1\rho_2 \dots \rho_{n-1}\rho_n\rho_{n-1} \dots \rho_2\rho_1)^{2s} \\
&= (\rho_0\rho_1\rho_2 \dots \rho_{n-1}\rho_n\rho_{n-1} \dots \rho_2\rho_1\rho_0\rho_1\rho_2 \dots \rho_{n-1}\rho_n\rho_{n-1} \dots \rho_2\rho_1)^{s} \\
&= (\rho_0\rho_1\rho_2 \dots \rho_{n-1}\rho_n\rho_{n-1} \dots \rho_2\rho_0\rho_1\rho_0\rho_1\rho_0\rho_2 \dots \rho_{n-1}\rho_n\rho_{n-1} \dots \rho_2\rho_1)^{s} \\
&= (\rho_0\rho_1\rho_0\rho_2 \dots \rho_{n-1}\rho_n\rho_{n-1} \dots \rho_2\rho_1\rho_0\rho_1\rho_0\rho_2 \dots \rho_{n-1}\rho_n\rho_{n-1} \dots \rho_2\rho_1)^{s} \\
&= (\tilde{\rho_0}\rho_2 \dots \rho_{n-1}\rho_n\rho_{n-1} \dots \rho_2\rho_1\tilde{\rho_0}\rho_2 \dots \rho_{n-1}\rho_n\rho_{n-1} \dots \rho_2\rho_1)^{s} \\
&= (\tilde{\rho_0}\rho_2 \dots \rho_{n-1}\rho_n\rho_{n-1} \dots \rho_2\rho_1)^{2s}
\end{aligned}$$

□

Following the notation of Ens [Ens18] and Weiss and Montero [MW20], we denote these regular toroidal hypertopes by $\left\{{}^{3}_{3}, 3^{n-3}, 4\right\}_{(2s,0^{n-1})}$.

• The polytope $2^{\{4,3^{n-2}\},\mathcal{G}(s)}$ and hypertope $\mathcal{H}(2^{\{4,3^{n-2}\},\mathcal{G}(s)})$

Consider the n-cube with automorphism group $[4, 3^{n-2}] = \langle \tau_0, \dots, \tau_{n-1}\rangle$, with $n \geq 3$.

Let $(v_1, v_2, \dots, v_n)$ be the coordinates of a vertex of the n-cube in an Euclidean space and let $v_i \in \{\pm 1\}$. In addition, let

$$\begin{gathered}
(v_1, v_2, \dots, v_n)\tau_0 := (-v_1, v_2, \dots, v_n) \\
(v_1, \dots, v_{j-1}, v_j, v_{j+1}, v_{j+2}, \dots, v_n)\tau_j := (v_1, \dots, v_{j-1}, v_{j+1}, v_j, v_{j+2}, \dots, v_n),
\end{gathered}$$

for $j \in \{1, \dots, n-1\}$. Let $F_0 := (1^n)$ be the vertex having all coordinates equal to 1, and let $\beta := \tau_0\tau_1\tau_2 \dots \tau_{n-2}\tau_{n-1}$. Then,

$$F_0\beta = (1^{n-1}, -1)$$

where 1^{n-1} means we have a row of $+1$'s of size $(n-1)$. Moreover, it is easily seen that

$$F_0\beta^i = (1^{n-i}, -1^i)$$

Particularly,

$$F_0\beta^n = (-1^n).$$

Thus β^n is clearly the central involution of the n-cube.

Lemma 9.2.7. *The n-cube has exactly n diagonal classes which can be represented by $\{F_0, F_0\beta^i\}$, for $1 \leq i \leq n$, where F_0 is a vertex, $\beta = \tau_0\tau_1\tau_2\ldots\tau_{n-2}\tau_{n-1}$, and $F_0\beta^i$ is the vertex of the action of β^i on the vertex F_0.*

Proof. Consider the construction of the vertices of the cube as above and let $F_0 := (1^n)$. Let $1 \leq i, j \leq n$ and consider the vertices $F_0\beta^i = (1^{n-i}, -1^i)$ and $F_0\beta^j = (1^{n-j}, -1^j)$. Suppose that the diagonals $\{F_0, F_0\beta^i\}$ and $\{F_0, F_0\beta^j\}$ are in the same diagonal class. Then, they share the same square length as their diagonal class representative

$$||F_0 - F_0\beta^i||^2 = ||F_0 - F_0\beta^j||^2.$$

Hence,

$$\begin{gathered} ||F_0 - F_0\beta^i||^2 = ||F_0 - F_0\beta^j|| \Leftrightarrow \\ \Leftrightarrow ||(0^{n-i}, 2^i)||^2 = ||(0^{n-j}, 2^j)||^2 \Rightarrow \\ \Leftrightarrow i = j. \end{gathered}$$

Since there are n distinct diagonal classes of the n-cube [MS02, Section 5B] and we can represent n distinct diagonal classes as above, we have proven the statement of the lemma. □

With the above lemma, we are able to give the relations of the group of automorphisms of $2^{\{4,3^{n-2}\},\mathcal{G}(s)}$.

Corollary 9.2.8. *Let $n \geq 3$ and $2 \leq s < \infty$. Then $2^{\{4,3^{n-2}\},\mathcal{G}(s)}$ is a $(n+1)$-polytope with type $\{4, 4, 3^{n-2}\}$ and automorphism group $D_s^{2^{n-1}} \rtimes [4, 3^{n-2}]$ of order $(2s)^{2^{n-1}}2^n n!$ with the relations given by its Coxeter diagram and the following extra relations*

$$(\rho_0\beta^{-i}\rho_0\beta^i)^2 = id \text{ for } 2 \leq i \leq n-1$$

$$(\rho_0\beta^n\rho_0\beta^n)^s = id,$$

where $\beta = \rho_1\rho_2\ldots\rho_n$. Moreover, its toroidal residue is the map $\{4,4\}_{(4,0)}$.

Proof. The polytopes above are obtained by Theorem 9.1.2 and, when $i = n$, β^n is the central involution of the polytope $\{4, 3^{n-2}\}$, meaning that

$$(\rho_0\beta^n\rho_0\beta^n)^s = id.$$

The remaining extra relations of the statement of this corollary come from diagonal classes of improper branches of the diagram $\mathcal{G}(s)$. Particularly, when $i = 1$, we have

$$id = (\rho_0\beta^{-1}\rho_0\beta)^2 = (\rho_0\rho_{n-1}\ldots\rho_2\rho_1\rho_0\rho_1\rho_2\ldots\rho_n)^2$$

which implies that

$$id = (\rho_0\rho_1\rho_0\rho_1)^2 = (\rho_0\rho_1)^4,$$

a relation given by the type of the polytope. Hence, $(\rho_0\beta^n\rho_0\beta^n)^s = id$ and $(\rho_0\beta^{-i}\rho_0\beta^i)^2 = id$, for $2 \leq i \leq n-1$, are the only extra relations needed to define the automorphism group of $2^{\{4,3^{n-2}\},\mathcal{G}(s)}$.

To prove that the toroidal residue is the map $\{4,4\}_{(4,0)}$, observe that from $id = (\rho_0\beta^{-2}\rho_0\beta^2)^2$ we have

$$id = (\rho_0\rho_1\rho_{n-1}\dots\rho_2\rho_1\rho_0\rho_1\rho_2\dots\rho_n\rho_1)^2$$

which implies

$$id = (\rho_0\rho_1\rho_2\rho_1\rho_0\rho_1\rho_2\rho_1)^2 = (\rho_0\rho_1\rho_2\rho_1)^4,$$

showing that the toroidal residue is the map $\{4,4\}_{(4,0)}$. □

If $n = 3$, the resulting abstract polytopes are quotients of the locally toroidal polytope of type $\{4,4,3\}$, satisfying the following relations

$$(\rho_0\rho_1\rho_2\rho_1)^4 = (\rho_0(\rho_1\rho_2\rho_3)^3)^{2s} = id,$$

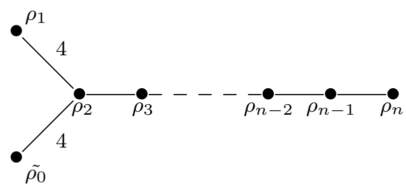
which do not give an universal locally toroidal polytope. Therefore, these polytopes do not appear in [MS02, Section 10].

Let us construct the regular hypertopes corresponding to this family. As before, we will use the halving operation.

Proposition 9.2.9. *Let $n \geq 3$ and $2 \leq s < \infty$. The incidence system*

$$\mathcal{H}(2^{\{4,3^{n-2}\},\mathcal{G}(s)}) = \Gamma(H(2^{\{4,3^{n-2}\},\mathcal{G}(s)}), (H_i)_{i\in\{0,\dots,n\}}),$$

where $H(2^{\{4,3^{n-2}\},\mathcal{G}(s)}) := \langle\rho_0\rho_1\rho_0, \rho_1, \dots, \rho_n\rangle = \langle\tilde{\rho}_0, \rho_1, \dots, \rho_n\rangle$, is a regular hypertope and its automorphism group, of size $(2s)^{2^{n-1}}2^{n-1}n!$, has the relations given by its Coxeter diagram



and the extra relations $(\tilde{\beta}^{-i}\beta^i)^2 = id$, for $2 \leq i \leq n-1$, and $(\tilde{\beta}^n\beta^n)^s = id$, where $\beta = \rho_1\rho_2\dots\rho_n$ and $\tilde{\beta} = \tilde{\rho}_0\rho_2\dots\rho_n$. Moreover the toroidal residue is the map $\{4,4\}_{(2,2)}$.

Proof. Let $n \geq 3$ and $2 \leq s < \infty$. The incidence system of the halving group $H(2^{\{4,3^{n-2}\},\mathcal{G}(s)})$ is a regular hypertope by Corollary 6.5.1 since the poset of the polytope $\{4,3^{n-2}\}$ is a lattice, making $2^{\{4,3^{n-2}\},\mathcal{G}(s)}$ non-degenerate, by Lemma 9.1.3.

Then, if we denote $\beta = \rho_1\rho_2\dots\rho_n$ and $\tilde{\beta} = \tilde{\rho}_0\rho_2\dots\rho_n$, we have

$$\begin{aligned} id &= (\rho_0\beta^{-i}\rho_0\beta^i)^k \\ &= (\rho_0(\rho_{n-1}\dots\rho_2\rho_1)^i\rho_0(\rho_1\rho_2\dots\rho_n)^i)^k \\ &= ((\rho_{n-1}\dots\rho_2\rho_0\rho_1\rho_0)^i(\rho_1\rho_2\dots\rho_n)^i)^k \\ &= ((\rho_{n-1}\dots\rho_2\tilde{\rho}_0)^i(\rho_1\rho_2\dots\rho_n)^i)^k \\ &= (\tilde{\beta}^{-i}\beta^i)^k, \end{aligned}$$

where $k = 2$ if $2 \leq i \leq n-1$, and $k = s$ if $i = n$. Moreover, we have that $\tilde{\beta}^n = \tilde{\beta}^{-n}$, meaning that

$$(\tilde{\beta}^{-n}\beta^n)^s = (\tilde{\beta}^n\beta^n)^s.$$

Let us prove that the toroidal residue is the map $\{4,4\}_{(2,2)}$. Consider the translations $u := \tilde{\rho_0}\rho_2\rho_1\rho_2$ and $g := (\tilde{\rho_0}\rho_2\rho_1)^2$ of the toroidal map residue $\{4,4\}$ of the above hypertope. Then, we have that

$$u = \tilde{\rho_0}\rho_2\rho_1\rho_2 = \rho_0\rho_1\rho_0\rho_2\rho_1\rho_2 = \rho_0\rho_1\rho_2\rho_0\rho_1\rho_2 = (\rho_0\rho_1\rho_2)^2,$$

which is a translation of order 4 of the toroidal residue $\{4,4\}_{(4,0)}$ of $2^{\{4,3^{n-2}\},\mathcal{G}(s)}$. Furthermore, we have that

$$g = (\tilde{\rho_0}\rho_2\rho_1)^2 = (\rho_0\rho_1\rho_0\rho_2\rho_1)^2 = \rho_0\rho_1\rho_2\rho_0\rho_1\rho_0\rho_1\rho_0\rho_2\rho_1 = \rho_0\rho_1\rho_2\rho_1\rho_0\rho_1\rho_2\rho_1,$$

which is a conjugate of $\rho_0\beta^{-2}\rho_0\beta^2$, meaning $o(g) = 2$. Since $o(u) = 4$ and $o(g) = 2$, then the toroidal residue of our regular hypertope is the map $\{4,4\}_{(2,2)}$.

□

Particularly, when $n = 3$, the regular hypertope of type $\left\{ {}^{4}_{4}, 3 \right\}$ given by Proposition 9.2.9 is locally toroidal, with toroidal residue $\{4,4\}_{(2,2)}$, and satisfies the relation $((\tilde{\rho_0}\rho_2\rho_3)^3(\rho_1\rho_2\rho_3)^3)^s = id$.

9.3 Polytopes $2^{\mathcal{P},\mathcal{G}(s)}$ and Hypertopes $\mathcal{H}(2^{\mathcal{P},\mathcal{G}(s)})$ when $\mathcal{P}$ has rank 3 or 4

In this section we consider that $\mathcal{P}$ is one of the remaining regular polytopes of Table 9.1: the icosahedron, the dodecahedron, the 24-cell, the 600-cell and the 120-cell. In what follows, similarly to the previous section, we construct extensions of these polytopes and then we apply the halving operation to obtain regular hypertopes. In [MW20], two locally spherical regular hypertopes of hyperbolic type are given: $\left\{ {}^{3}_{3}, 5 \right\}$, with automorphism group of order $60 \cdot 2^{12}$, and $\left\{ {}^{3}_{3}, 3, 5 \right\}$, with automophism group of order 7200×2^{120}. These two hypertopes will correspond to our hypertopes $\mathcal{H}(2^{\{3,5\},\mathcal{G}(2)})$ and $\mathcal{H}(2^{\{3,3,5\},\mathcal{G}(2)})$, respectively. Here, we will give an infinite family of these hypertopes. In addition, we will give a family of hypertopes of type $\left\{ {}^{3}_{3}, 4, 3 \right\}$ with toroidal residue $\left\{ {}^{3}_{3}, 4 \right\}_{(4,0,0)}$. Most proofs of the following results will be omitted as they follow the same ideas present in the proofs of Corollary 9.2.8 and Proposition 9.2.9.

• The polytope $2^{\{3,5\},\mathcal{G}(s)}$ and hypertope $\mathcal{H}(2^{\{3,5\},\mathcal{G}(s)})$

Let $\mathcal{P}$ be the icosahedron with automorphism group $G(\mathcal{P}) := \langle \tau_0, \tau_1, \tau_2 \rangle$. The icosahedron has three distinct diagonal classes, which can be determined computationally with GAP[GAP21]: $\{F_0, F_0\beta\}$, $\{F_0, F_0\beta^3\}$ and $\{F_0, F_0\beta^5\}$, where $\beta := \tau_0\tau_1\tau_2$. The vertices of the latter diagonal class are antipodal. In fact, the diagonals $\{F_0, F_0\beta\}$ and $\{F_0, F_0\beta^2\}$ are in the same diagonal class, since the double G_0-cosets coincide, that is,

$$\begin{aligned} G_0\beta^2G_0 = G_0(\tau_0\tau_1\tau_2)^2G_0 = G_0\tau_0\tau_1\tau_2\tau_0G_0 = \\ = G_0\tau_0\tau_1\tau_0\tau_2G_0 = G_0\tau_1\tau_0\tau_1\tau_2G_0 = G_0\beta G_0. \end{aligned}$$

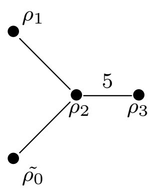
The same can be proven for the diagonals $\{F_0, F_0\beta^3\}$ and $\{F_0, F_0\beta^4\}$. With this, we can provide the polytope $2^{\{3,5\},\mathcal{G}(s)}$ and the hypertope $\mathcal{H}(2^{\{3,5\},\mathcal{G}(s)})$.

Corollary 9.3.1. *Let $2 \leq s < \infty$. Then $2^{\{3,5\},\mathcal{G}(s)}$ is a 4-polytope of type $\{4,3,5\}$ with automophism group $D_s^6 \rtimes [3,5]$ of order $120 \cdot (2s)^6$. Moreover, the group $G(2^{\{3,5\},\mathcal{G}(s)}) := \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$ is the quotient of the Coxeter group $[4,3,5]$ by the relations $(\rho_0\beta^{-3}\rho_0\beta^3)^2 = id$ and $(\rho_0\beta^5\rho_0\beta^5)^s = id$, where $\beta = \rho_1\rho_2\rho_3$.*

Proposition 9.3.2. *Let $2 \leq s < \infty$. The incidence system*

$$\mathcal{H}(2^{\{3,5\},\mathcal{G}(s)}) = \Gamma(H(2^{\{3,5\},\mathcal{G}(s)}), (H_i)_{i \in \{0,\ldots,3\}}),$$

where $H(2^{\{3,5\},\mathcal{G}(s)}) := \langle \rho_0\rho_1\rho_0, \rho_1, \rho_2, \rho_3 \rangle = \langle \tilde{\rho_0}, \rho_1, \rho_2, \rho_3 \rangle$, is a regular hypertope and its automorphism group, of size $60 \cdot (2s)^6$, is the quotient of the Coxeter group with diagram



factorized by $(\tilde{\beta}^{-3}\beta^3)^2 = id$ and $(\tilde{\beta}^5\beta^5)^s = id$, where $\beta := \rho_1\rho_2\rho_3$ and $\tilde{\beta} := \tilde{\rho_0}\rho_2\rho_3$.

• The polytope $2^{\{5,3\},\mathcal{G}(s)}$ and hypertope $\mathcal{H}(2^{\{5,3\},\mathcal{G}(s)})$

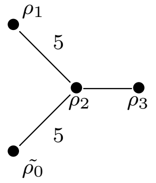
Let $\mathcal{P}$ be the dual of the icosahedron, the dodecahedron. Using GAP[GAP21] and the double coset action described in equation 2.3, we can determine the diagonal classes of the dodecahedron: $\{F_0, F_0\beta^i\}$, for $1 \leq i \leq 5$, where $\beta = \tau_0\tau_1\tau_2$ is an element of the group $[5,3] = \langle \tau_0, \tau_1, \tau_2 \rangle$. As before, we give the polytope $2^{\{5,3\},\mathcal{G}(s)}$ and the hypertope $\mathcal{H}(2^{\{5,3\},\mathcal{G}(s)})$.

Corollary 9.3.3. *Let $2 \leq s < \infty$. Then $2^{\{5,3\},\mathcal{G}(s)}$ is a family of 4-polytopes with type $\{4,5,3\}$ and automorphism group $D_s^{10} \rtimes [5,3]$ of order $120 \cdot (2s)^{10}$. Moreover, the group $G(2^{\{5,3\},\mathcal{G}(s)}) := \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$ is the quotient of the Coxeter group $[4,5,3]$ by the relations $(\rho_0\beta^{-i}\rho_0\beta^i)^2 = id$, for $2 \leq i \leq 4$, and $(\rho_0\beta^5\rho_0\beta^5)^s = id$, where $\beta = \rho_1\rho_2\rho_3$.*

Proposition 9.3.4. *Let $2 \leq s < \infty$. The incidence system*

$$\mathcal{H}(2^{\{5,3\},\mathcal{G}(s)}) = \Gamma(H(2^{\{5,3\},\mathcal{G}(s)}), (H_i)_{i \in \{0,\ldots,3\}}),$$

where $H(2^{\{5,3\},\mathcal{G}(s)}) := \langle \rho_0\rho_1\rho_0, \rho_1, \rho_2, \rho_3 \rangle = \langle \tilde{\rho_0}, \rho_1, \rho_2, \rho_3 \rangle$, is a regular hypertope and its automorphism group, of size $60 \cdot (2s)^{10}$, has the relations given by its Coxeter diagram



and the extra relations $(\tilde{\beta}^{-i}\beta^i)^2 = id$, for $2 \leq i \leq 4$, and $(\tilde{\beta}^5\beta^5)^s = id$, where $\beta := \rho_1\rho_2\rho_3$ and $\tilde{\beta} := \tilde{\rho_0}\rho_2\rho_3$.

• The polytope $2^{\{3,4,3\},\mathcal{G}(s)}$ and hypertope $\mathcal{H}(2^{\{3,4,3\},\mathcal{G}(s)})$

Let $\mathcal{P}$ be the self-dual polytope of type $\{3,4,3\}$. Using GAP[GAP21] we have that the 24-cell has 4 distinct diagonal classes: $\{F_0, F_0\beta^i\}$, for $i \in \{1,3,4,6\}$, where $\beta = \tau_0\tau_1\tau_2\tau_3$ is an element of the group $[3,4,3] = \langle \tau_0, \tau_1, \tau_2, \tau_3 \rangle$. We will determine the polytope $2^{\{3,4,3\},\mathcal{G}(s)}$ and regular hypertope $\mathcal{H}(2^{\{3,4,3\},\mathcal{G}(s)})$.

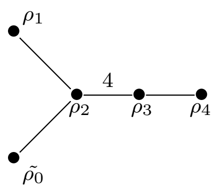
Corollary 9.3.5. *Let $2 \leq s < \infty$. Then $2^{\{3,4,3\},\mathcal{G}(s)}$ is a 5-polytope of type $\{4,3,4,3\}$ and automorphism group $D_s^{12} \rtimes [3,4,3]$ of order $1152 \cdot (2s)^{12}$. Moreover, the group $G(2^{\{3,4,3\},\mathcal{G}(s)}) := \langle \rho_0, \rho_1, \rho_2, \rho_3, \rho_4 \rangle$ is the quotient of the locally toroidal Coxeter group $[4,3,4,3]$ factorized by the relations $(\rho_0\beta^{-i}\rho_0\beta^i)^2 = id$, for $i \in \{3,4\}$, and $(\rho_0\beta^6\rho_0\beta^6)^s = id$, where $\beta = \rho_1\rho_2\rho_3\rho_4$ and its toroidal residue is the cubic toroid $\{4,3,4\}_{(4,0,0)}$.*

Proof. The proof follows the same idea as in Corollary 9.2.8. To prove that the toroidal residue is the cubic toroid $\{4,3,4\}_{(4,0,0)}$, observe that the relation $(\rho_0\beta^{-3}\rho_0\beta^3)^2 = id$ implies that $(\rho_0\rho_1\rho_2\rho_3\rho_2\rho_1)^4 = id$. □

Proposition 9.3.6. *Let $2 \leq s < \infty$. The incidence system*

$$\mathcal{H}(2^{\{3,4,3\},\mathcal{G}(s)}) = \Gamma(H(2^{\{3,4,3\},\mathcal{G}(s)}), (H_i)_{i\in\{0,\dots,4\}}),$$

where $H(2^{\{3,4,3\},\mathcal{G}(s)}) := \langle \rho_0\rho_1\rho_0, \rho_1, \rho_2, \rho_3, \rho_4 \rangle = \langle \tilde{\rho}_0, \rho_1, \rho_2, \rho_3, \rho_4 \rangle$, is a regular hypertope and its automorphism group, of size $576 \cdot (2s)^{12}$, has the relations given by its Coxeter diagram



and the extra relations $(\tilde{\beta}^{-i}\beta^i)^2 = id$, for $i \in \{3,4\}$, and $(\tilde{\beta}^6\beta^6)^s = id$, where $\beta := \rho_1\rho_2\rho_3\rho_4$ and $\tilde{\beta} := \tilde{\rho}_0\rho_2\rho_3\rho_4$. Moreover, its toroidal residue is $\left\{ {}^{3}_{3}, 4 \right\}_{(4,0,0)}$.

Proof. The proof follows the same idea as in Proposition 9.2.9. Moreover, as seen in the proof of Proposition 9.2.6, we can rewrite the relation $(\rho_0\rho_1\rho_2\rho_3\rho_2\rho_1)^4 = id$, given in Corollary 9.3.5, as $(\tilde{\rho}_0\rho_2\rho_3\rho_2\rho_1)^4 = id$, which is the factorizing relation of the regular hypertope $\left\{ {}^{3}_{3}, 4 \right\}_{(4,0,0)}$. □

• The polytope $2^{\{3,3,5\},\mathcal{G}(s)}$ and hypertope $\mathcal{H}(2^{\{3,3,5\},\mathcal{G}(s)})$

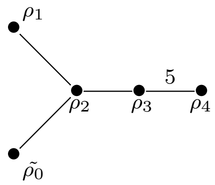
Let $\mathcal{P}$ be the 600-cell. The 600-cell has 8 distinct diagonal classes, which can be obtained with GAP[GAP21] and can be represented by $\{F_0, F_0\beta^i\}$, for $i \in \{1,4,6,7,9,10,12,15\}$, where $\beta = \tau_0\tau_1\tau_2\tau_3$ is an element of the group $[3,3,5] = \langle \tau_0, \tau_1, \tau_2, \tau_3 \rangle$. Let us determine the polytope $2^{\{3,3,5\},\mathcal{G}(s)}$ and the hypertope $\mathcal{H}(2^{\{3,3,5\},\mathcal{G}(s)})$.

Corollary 9.3.7. *Let $2 \leq s < \infty$. Then $2^{\{3,3,5\},\mathcal{G}(s)}$ is a 5-polytope of type $\{4,3,3,5\}$ and automophism group $D_s^{60} \rtimes [3,3,5]$ of order $14400 \cdot (2s)^{60}$. Moreover, the group $G(2^{\{3,3,5\},\mathcal{G}(s)}) := \langle \rho_0, \rho_1, \rho_2, \rho_3, \rho_4 \rangle$ is the quotient of the Coxeter group $[4,3,3,5]$ by the relations $(\rho_0\beta^{-i}\rho_0\beta^i)^2 = id$, for $i \in \{4,6,7,9,10,12\}$, and $(\rho_0\beta^{15}\rho_0\beta^{15})^s = id$, where $\beta = \rho_1\rho_2\rho_3\rho_4$.*

Proposition 9.3.8. *Let $2 \leq s < \infty$. The incidence system*

$$\mathcal{H}(2^{\{3,3,5\},\mathcal{G}(s)}) = \Gamma(H(2^{\{3,3,5\},\mathcal{G}(s)}), (H_i)_{i\in\{0,\ldots,4\}}),$$

where $H(2^{\{3,3,5\},\mathcal{G}(s)}) := \langle \rho_0\rho_1\rho_0, \rho_1, \rho_2, \rho_3, \rho_4 \rangle = \langle \tilde{\rho}_0, \rho_1, \rho_2, \rho_3, \rho_4 \rangle$*, is a regular hypertope and its automorphism group, of size* $7200 \cdot (2s)^{60}$*, has the relations given by its Coxeter diagram*



and the extra relations $(\tilde{\beta}^{-i}\beta^i)^2 = id$*, for* $i \in \{4, 6, 7, 9, 10, 12\}$*, and* $(\tilde{\beta}^{15}\beta^{15})^s = id$*, where* $\beta := \rho_1\rho_2\rho_3\rho_4$ *and* $\tilde{\beta} := \tilde{\rho}_0\rho_2\rho_3\rho_4$.

• **The polytope** $2^{\{5,3,3\},\mathcal{G}(s)}$ **and hypertope** $\mathcal{H}(2^{\{5,3,3\},\mathcal{G}(s)})$

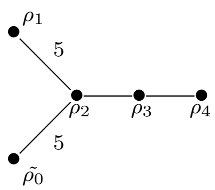
Lastly, let $\mathcal{P}$ be the 120-cell polytope. As previously, we can determine with the help of GAP[GAP21] that the 120-cell has 15 distinct diagonal classes, using the double coset action described in equation 2.3. These diagonal classes can be represented by $\{F_0, F_0\beta^i\}$, for $1 \leq i \leq 15$, where $\beta = \tau_0\tau_1\tau_2\tau_3$ is an element of the group $[5,3,3] = \langle \tau_0, \tau_1, \tau_2, \tau_3 \rangle$. Let us determine the polytope $2^{\{5,3,3\},\mathcal{G}(s)}$ and hypertope $\mathcal{H}(2^{\{5,3,3\},\mathcal{G}(s)})$.

Corollary 9.3.9. *Let $2 \leq s < \infty$. Then $2^{\{5,3,3\},\mathcal{G}(s)}$ is a 5-polytope of type $\{4,5,3,3\}$ and automophism group $D_s^{300} \rtimes [5,3,3]$ of order $14400 \cdot (2s)^{300}$. Moreover, the group $G(2^{\{3,3,5\},\mathcal{G}(s)}) := \langle \rho_0, \rho_1, \rho_2, \rho_3, \rho_4 \rangle$ is the quotient of the Coxeter group $[4,5,3,3]$ by the relations $(\rho_0\beta^{-i}\rho_0\beta^i)^2 = id$, for $2 \leq i \leq 14$, and $(\rho_0\beta^{15}\rho_0\beta^{15})^s = id$, where $\beta = \rho_1\rho_2\rho_3\rho_4$.*

Proposition 9.3.10. *Let $2 \leq s < \infty$. The incidence system*

$$\mathcal{H}(2^{\{5,3,3\},\mathcal{G}(s)}) = \Gamma(H(2^{\{5,3,3\},\mathcal{G}(s)}), (H_i)_{i\in\{0,\ldots,4\}}),$$

where $H(2^{\{5,3,3\},\mathcal{G}(s)}) := \langle \rho_0\rho_1\rho_0, \rho_1, \rho_2, \rho_3, \rho_4 \rangle = \langle \tilde{\rho}_0, \rho_1, \rho_2, \rho_3, \rho_4 \rangle$*, is a regular hypertope and its automorphism group, of size* $7200 \cdot (2s)^{300}$*, has the relations given by its Coxeter diagram*



and the extra relations $(\tilde{\beta}^{-i}\beta^i)^2 = id$*, for* $2 \leq i \leq 14$*, and* $(\tilde{\beta}^{15}\beta^{15})^s = id$*, where* $\beta := \rho_1\rho_2\rho_3\rho_4$ *and* $\tilde{\beta} := \tilde{\rho}_0\rho_2\rho_3\rho_4$.

9.4 Expanding the results further

In this chapter we have given a list of infinite families of regular hypertopes of arbitrary rank, constructed from finite centrally symmetric non-degenerate spherical polytopes $\mathcal{P}$. We notice that Theorem 9.1.2 establishes the following:

- If s is even and $\mathcal{P}$ has only finitely many vertices, then $2^{\mathcal{P},\mathcal{G}(s)}$ is also centrally symmetric.

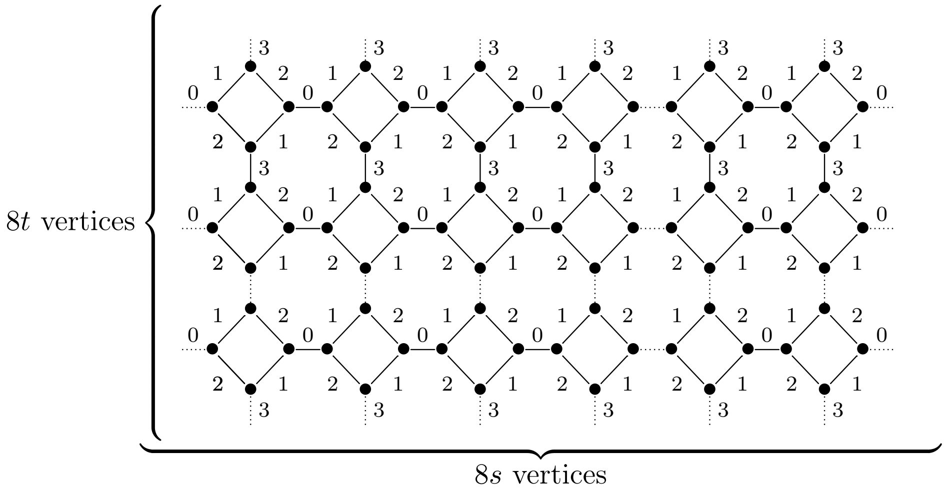
This implies that if s is even, all the polytopes determined in Sections 9.2 and 9.3 are centrally symmetric, being eligible to be used in Theorem 9.1.2, giving other families of polytopes of type $\{4, 4, p_1, \ldots, p_{n-1}\}$. Moreover, since we know that these polytopes $2^{\mathcal{P},\mathcal{G}(s)}$ are also non-degenerate, these new families would also be eligible for the halving operation, giving families of hypertopes.

Here the focus was on spherical polytopes but the same idea can be applied to centrally symmetric toroidal polytopes. Indeed, from Corollary 9.2.2 and the point above, it follows that the toroidal maps $\{4, 4\}_{(2s,0)}$ are centrally symmetric and non-degenerate (for $s \geq 4$ and even), and therefore extendable by the same processed that was used in this chapter to create polytopes $2^{\{4,4\}_{(2s,0)},G(k)}$. Moreover, the toroidal map $\{4, 4\}_{(2s,0)}$ is just a case of the toroidal $(n+1)$-cubic tesselation $\{4, 3^{n-2}, 4\}_{(2s,0^{n-1})}$ for $n = 2$. We can repeat the process to this more general case and extend further this cubic tesselation to $2^{\{4,3^{n-2},4\}_{(2s,0^{n-1})},G(k)}$, for $s \geq 4$ and even. To these new polytopes the halving operation can be applied, giving new hypertope families.

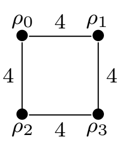
Chapter 10

Conclusion and Future Research

As stated in the introduction, faithful transitive permutation representations are a powerful tool to characterize groups, particularly automorphism groups of abstract regular polytopes and regular hypertopes, which are C-groups. Moreover, the FTPR graphs are extremely useful to detect patterns of the action of these groups. These patterns can be exploited by fusing strategically FTPR graphs of the maximal parabolic subgroups in order to obtain a new permutation representation graph, which will potentially be FTPR graphs of other C-groups. Although the results presented in this thesis are very well-behaved, these constructions do not always give a FTPR graph of a C-group, since the intersection property might be lost. Consider the following permutation representation graph.



This permutation representation graph can be obtained by combining copies of the FTPR given in Proposition 4.2.11, being a permutation representation of a ggi G with diagram



and maximal parabolic subgroups $G_0 \cong [4,4]_{(t,t)}$, $G_3 \cong [4,4]_{(s,s)}$ and $G_1 \cong G_2 \cong [4,4]_{(l,0)}$, where $l = 2lcm(s,t)$. It can be proven computationally that for $(s,t) \in$

$\{(2,2),(2,3),(2,4),(2,5),(3,4),(3,5)\}$ the group having this permutation representation graph is not a C-group. Hence, one must always be cautious when using this method of constructing FTPR graphs in order to obtain new families of hypertopes. In addition, we remind the reader of the difference between the parameters which we factor the Coxeter group of a hypertope and the respective resulting maximal parabolic subgroups, which in Section 7.3 we saw that it can differ.

Having in mind the construction of new families of regular hypertopes, particularly locally toroidal hypertopes, we started by determining the degrees of FTPR of toroidal regular maps and hypermaps. The results obtained for the regular toroidal maps $\{4,4\}$ ended up being crucial for the extension of this result to locally toroidal polytopes of type $\{4,4,4\}$. In [MS02], we may find the full classification of the finite locally toroidal regular polytopes of type $\{6,3,p\}$, for $p \in \{3,4,5,6\}$, and partial results for type $\{3,6,3\}$. Following the same line of research, we can classify in the future all the degrees of FTPR of these locally toroidal regular polytopes. Furthermore, another line of research would be to study all the possible degrees of FTPRs of the regular $(n+1)$-cubic toroids $\{4,3^{n-2},4\}_{\mathbf{s}}$, for $\mathbf{s} \in \{(s,0^{n-1}),(s,s,0^{n-2}),(s^n)\}$. The case where $n=2$ coincides with the toroidal maps $\{4,4\}_{(s,0)}$ and $\{4,4\}_{(s,s)}$. For any $n \geq 3$, one could think it would be easy to have a general result for the set of degrees of the FTPRs of these cubic toroids. However, as the group of the $(n-1)$-simplex is a subgroup of the $(n+1)$-cubic toroids, this depends on the degrees of S_n, which is not yet known. This seems to be a hard question.

This thesis started during a period where the theory of regular hypertopes was giving the first steps and my commitment was to give a contribution to the classification of these new structures. In this thesis some families of regular hypertopes of euclidean, hyperbolic and locally toroidal type were given, and a family of locally toroidal regular hypertopes of type $\{3,{}^{4}_{4}\}$ was let to future work. Related to this, other problem came into the scene: to find sufficient conditions to obtain a C-group using a "weak" factorization. If we consider all abstract regular polytopes which are non-degenerate (i.e. its poset is a lattice) of rank greater than 2 with at most 2000 flags, listed in [Har06], we have that this factorization is sufficient to get a C-group except for some abstract regular polytopes of type $\{4,6\}$, $\{4,8\}$, $\{4,10\}$, $\{4,12\}$ and $\{6,6\}$. By studying why this exceptions fail, we might understand better how to define the sufficient conditions that would allow this factorization to be a C-group.

In Chapter 9, many examples of families of regular hypertopes were obtained through the halving of the non-degenerate abstract regular polytopes $2^{\mathcal{P},\mathcal{G}(s)}$. Moreover, if s is even, the centrally symmetric abstract regular polytopes $2^{\mathcal{P},\mathcal{G}(s)}$ obtained of type $\{4,p_1,\ldots,p_{n-1}\}$ can be used to build new abstract regular polytopes $\{4,4,p_1,\ldots,p_{n-1}\}$. This would let us use again the halving operation to obtain families of regular hypertopes of type $\{{}^{4}_{4},p_1,\ldots,p_{n-1}\}$. The challenge into this approach is figuring out the diagonal classes of the centrally symmetric abstract regular polytopes $2^{\mathcal{P},\mathcal{G}(s)}$ derived for s even. In addiction, the application of the halving operation on the dual of the regular hypertopes of type $\{{}^{3}_{3},3,\ldots,3,4\}$ ($\tilde{B}_{n-1}$ Coxeter diagram) would allow us to obtain families of hypertopes of type $\{{}^{3}_{3},3,\ldots,3,{}^{3}_{3}\}$ ($\tilde{D}_{n-1}$ Coxeter diagram). However, the halving group here might not be a C-group nor flag-transitive, since it is only established for non-degenerate abstract regular polytopes. Furthermore, Chapter 9 started from centrally symmetric regular polytopes $\mathcal{P}$ of spherical type to get to the non-degenerate abstract regular polytopes $2^{\mathcal{P},\mathcal{G}(s)}$. These results could be expanded to the centrally symmetric regular polytopes of euclidean type, such as the toroidal regular maps of type $\{3,6\}_{\mathbf{s}}$, the cubic toroids of type $\{4, 3^{n-2}, 4\}$ and the honeycombs of type $\{3, 3, 4, 3\}$ (and their duals).

With the results of this thesis, we are able to give new ways to build and classify new regular hypertopes, either by building FTPRs of regular hypertopes based on the knowledge of FTPRs of the maximal parabolic subgroups, or by the halving operation on non-degenerate abstract regular polytopes. The focus of this thesis was mainly on the FTPRs' method, which led to the classification of all the degrees of FTPRs of the toroidal regular (hyper)maps $\{4, 4\}$, $\{3, 6\}$ and $(3, 3, 3)$. We believe the methodologies described here, combined with computational insight and experimentation, will have a heavy impact on the research of new regular hypertopes.

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