

Hyers-Ulam stability of a certain Fredholm integral equation *

Alberto SIMÕES^{1,2}, Ponmana SELVAN³

¹*CMA-UBI, Center of Mathematics and Applications*

University of Beira-Interior, Covilhã, 6201-001, Portugal

²*CIDMA, Center for Research and Development in Mathematics and Applications*

University of Aveiro, Aveiro, 3810-193, Portugal

³*Department of Mathematics, Sri Sai Ram Institute of Technology,
Tamil Nadu, Chennai, India*

Abstract

In this paper, by using Fixed point Theorem we establish the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of certain homogeneous Fredholm Integral equation of the second kind

$$\phi(x) = \lambda \int_0^1 (1+x+t) \phi(t) dt$$

and the non-homogeneous equation

$$\phi(x) = x + \lambda \int_0^1 (1+x+t) \phi(t) dt$$

for all $x \in [0, 1]$ and $0 < \lambda < \frac{2}{5}$.

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1 Introduction

The Ulam stability problem for various functional equation was initiated by S.M. Ulam [31] in 1940. Then, in the next year, D.H. Hyers [16] was solved the Ulam problem for Cauchy additive functional equation on Banach spaces. After that Aoki [3], Bourgin [6] and Rassias [25] have generalized the Hyers result. These days the Hyers-Ulam stability for different functional equations was proved by many mathematicians (see [4, 11, 26, 5]). A generalization Ulam problem was recently proposed by replacing functional equations with differential equations. In 1998, Alsina et al., [1] was proved the Hyers-Ulam stability of differential equation of first order of the form $y'(t) = y(t)$. This result was generalized

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by Takahasi [30] for Banach space valued differential equation $y'(t) = \lambda y(t)$. Then several researchers have studied the Hyers-Ulam stability of differential equations in various directions, for example (see [7, 10, 18, 17, 19, 21, 20, 23, 22, 24, 29, 32]).

Now a days, the Hyers-Ulam stability of integral equations has been given attention. In 2015, L. Hua et al., [15] studied the Hyers-Ulam stability of some kinds of Fredholm integral equations. Also, in 2015, Z. Gu and J. Huang [14] are investigated the Hyers-Ulam stability of the Fredholm integral equation

$$\phi(x) = f(x) + \lambda \int_a^b K(x, s) \phi(s) ds$$

by fixed point Theorem. Recently, only few authors are investigating the Hyers-Ulam stability of the various integral equations (see [2, 8, 9, 12, 13, 27, 28]). Motivated by the above ideas, our foremost aim is to study the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the certain Fredholm Integral equations of second kind

$$\phi(x) = \lambda \int_0^1 (1 + x + t) \phi(t) dt \quad (1)$$

and

$$\phi(x) = x + \lambda \int_0^1 (1 + x + t) \phi(t) dt \quad (2)$$

for all $x \in [0, 1]$ and $0 < \lambda < \frac{2}{5}$ in the sense of Z. Gu and J. Huang [14].

2 Preliminaries

The following Theorems and Definitions are very useful to prove our main results.

Theorem 1. (*Fixed Point Theorem*) Let (X, ρ) be a complete metric space. Assume that $T : X \rightarrow X$ is a strictly contractive operator with $\rho(Tx, Ty) \leq \theta \rho(x, y)$ where $0 < \theta < 1$. Then

- (i) there exists an unique fixed point x^* of T ;
- (ii) the sequence $\{T^n x\}_{n \in \mathbb{N}}$ converges to the fixed point x^* of T .

Theorem 2. (*Hölder's Inequality*) Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $x \in L^p(E)$ and $y \in L^q(E)$. Then $xy \in L(E)$ and

$$\int_E |x(t)y(t)| dt \leq \left(\int_E |x^p(t)| dt \right)^{\frac{1}{p}} \left(\int_E |y^q(t)| dt \right)^{\frac{1}{q}}.$$

Now, we give the definition of Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the Fredholm integral equations (1) and (2).

Definition 3. We say that the Fredholm integral equations (1) has the Hyers-Ulam stability, if there exists a real constant S which satisfies the following conditions: For every $\epsilon > 0$, and for each solution $\phi : [0, 1] \rightarrow \mathbb{R}$ satisfying the inequation

$$\left| \phi(x) - \lambda \int_0^1 (1 + x + t) \phi(t) dt \right| \leq \epsilon,$$

then there is some $\psi : [0, 1] \rightarrow \mathbb{R}$ satisfying the integral equation (1) such that

$$|\phi(x) - \psi(x)| \leq S\epsilon, \quad \forall x \in [0, 1].$$

Definition 4. We say that the Fredholm integral equations (2) has the Hyers-Ulam stability, if there exists a real constant S which satisfies the following conditions: For every $\epsilon > 0$, and for each solution $\phi : [0, 1] \rightarrow \mathbb{R}$ satisfying the inequality

$$\left| \phi(x) - x - \lambda \int_0^1 (1 + x + t) \phi(t) dt \right| \leq \epsilon,$$

then there exists a solution $\psi : [0, 1] \rightarrow \mathbb{R}$ satisfies the integral equation (2) such that

$$|\phi(x) - \psi(x)| \leq S\epsilon, \quad \forall x \in [0, 1].$$

Definition 5. The Fredholm integral equations (1) is said to have the Hyers-Ulam-Rassias stability, if there exists a real constant S which fulfill the following: For every $\theta \in C(\mathbb{R}_+, \mathbb{R}_+)$, and for each solution $\phi : [0, 1] \rightarrow \mathbb{R}$ satisfying the inequality

$$\left| \phi(x) - \lambda \int_0^1 (1 + x + t) \phi(t) dt \right| \leq \theta(x),$$

then there is a solution $\psi : [0, 1] \rightarrow \mathbb{R}$ satisfying the integral equation (1) such that

$$|\phi(x) - \psi(x)| \leq S\theta(x), \quad \forall x \in [0, 1].$$

Definition 6. We say that the Fredholm integral equations (2) has the Hyers-Ulam-Rassias stability, if there exists a real constant S which fulfill the following properties: For every $\theta \in C(\mathbb{R}_+, \mathbb{R}_+)$, and for each solution $\phi : [0, 1] \rightarrow \mathbb{R}$ satisfying the inequation

$$\left| \phi(x) - x - \lambda \int_0^1 (1 + x + t) \phi(t) dt \right| \leq \theta(x),$$

then there exists some $\psi : [0, 1] \rightarrow \mathbb{R}$ satisfying the integral equation (2) such that

$$|\phi(x) - \psi(x)| \leq S\theta(x), \quad \forall x \in [0, 1].$$

3 Main Results

In this section, we are going to prove the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability of the homogeneous and non-homogeneous Fredholm integral equations of second kind (1) and (2) with $\lambda < \frac{2}{5}$. First, we investigate the two stabilities of the homogeneous Fredholm integral equation of second kind (1).

Theorem 7. Consider H a fixed real number such that $H \geq \frac{5}{2}$ and $\lambda H < 1$. Let $\phi : [0, 1] \rightarrow \mathbb{R}$ a continuous function and the kernel $K : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ defined by $K(x, t) = 1 + x + t$. If ϕ is such that

$$\left| \phi(x) - \lambda \int_0^1 (1 + x + t) \phi(t) dt \right| \leq \epsilon, \tag{3}$$

where $\epsilon \geq 0$ then there exists a solution $\psi : [0, 1] \rightarrow \mathbb{R}$ of the Fredholm integral equation (1) and a real constant S such that $|\phi(x) - \psi(x)| \leq S\epsilon$ for all $x \in [0, 1]$.

Proof. Firstly, we define an operator T by,

$$(T\phi)(x) = \lambda \int_0^1 (1 + x + t) \phi(t) dt, \quad \phi \in L^2([0, 1]). \tag{4}$$

We have for each $x \in [0, 1]$,

$$\left| \int_0^1 (1+x+t) dt \right| \leq H \quad \text{and} \quad \left| \left(\int_0^1 \int_0^1 (1+x+t)^2 dt dx \right)^{\frac{1}{2}} \right| \leq H,$$

for any $H \geq \frac{5}{2}$.

Now, we define a metric ρ as follows,

$$\rho(\varphi_1, \varphi_2) = \left\{ \left(\int_0^1 \left| \frac{\varphi_1(x) - \varphi_2(x)}{\lambda H} \right|^2 dx \right)^{\frac{1}{2}} : \varphi_1, \varphi_2 \in L^2([0, 1]), \lambda H < 1 \right\}.$$

By using the Hölder's inequality, we obtain that

$$\begin{aligned} \int_0^1 \left| \int_0^1 (1+x+t) \phi(t) dt \right|^2 dx &\leq \int_0^1 \left(\int_0^1 (1+x+t)^2 dt \int_0^1 \phi^2(t) dt \right) dx \\ &\leq \int_0^1 \phi^2(t) dt \int_0^1 \int_0^1 (1+x+t)^2 dt dx < \infty. \end{aligned}$$

This implies that $T\phi \in L^2([0, 1])$ and T is a self-mapping of $L^2([0, 1])$. Thus, the solution of the equation (4) is the fixed point of T . So,

$$\begin{aligned} \rho(T\varphi_1, T\varphi_2) &= \left(\int_0^1 \left| \frac{(T\varphi_1)(x) - (T\varphi_2)(x)}{\lambda H} \right|^2 dx \right)^{\frac{1}{2}} \\ &= \frac{1}{H} \left(\int_0^1 \left| \int_0^1 (1+x+t) (\varphi_1(t) - \varphi_2(t)) dt \right|^2 dx \right)^{\frac{1}{2}} \\ &\leq \frac{1}{H} \left(\int_0^1 \int_0^1 (1+x+t)^2 dt dx \right)^{\frac{1}{2}} \left(\int_0^1 |\varphi_1(t) - \varphi_2(t)|^2 dt \right)^{\frac{1}{2}} \\ &\leq \left(\int_0^1 |\varphi_1(t) - \varphi_2(t)|^2 dt \right)^{\frac{1}{2}} \\ &= \lambda H \left(\int_0^1 \left| \frac{\varphi_1(t) - \varphi_2(t)}{\lambda H} \right|^2 dt \right)^{\frac{1}{2}} \\ &= \lambda H \rho(\varphi_1, \varphi_2). \end{aligned}$$

Since $\lambda H < 1$, T is a strictly contractive operator. Then by Theorem 1 the equation (4) has a unique solution $\phi^* \in L^2([0, 1])$, where $\phi^* = \lim_{r \rightarrow \infty} \phi_r$ for

$$\phi_r(x) = \lambda \int_0^1 (1+x+t) \phi_{r-1}(t) dt$$

and $\phi_0 \in L^2([0, 1])$ is an arbitrary function.

Let $\psi \in L^2([0, 1])$ be a solution of inequality (3) and

$$\psi(x) - \lambda \int_0^1 (1+x+t) \psi(t) dt =: h(x). \quad (5)$$

Obviously, we have $|h(x)| \leq \epsilon$ for all $x \in [0, 1]$. Then we can conclude that the solution of equation

$$\psi(x) = h(x) + \lambda \int_0^1 (1+x+t) \psi(t) dt$$

is $\psi^* \in L^2([0, 1])$, where $\psi^* = \lim_{r \rightarrow \infty} \psi_r$ for

$$\psi_r(x) = h(x) + \lambda \int_0^1 (1+x+t) \psi_{r-1}(t) dt$$

and $\psi_0 \in L^2([0, 1])$ is an arbitrary function.

For $\phi_0(x) = \psi_0(x) = 0$, we get,

$$\begin{aligned} |\phi_1(x) - \psi_1(x)| &= |h(x)| \leq \epsilon, \\ |\phi_2(x) - \psi_2(x)| &= \left| h(x) + \lambda \int_0^1 (1+x+t)(\psi_1(t) - \phi_1(t)) dt \right| \leq \epsilon \left(1 + \lambda \int_0^1 |1+x+t| dt \right) \\ |\phi_3(x) - \psi_3(x)| &= \left| h(x) + \lambda \int_0^1 (1+x+t_2)(\psi_2(t_2) - \phi_2(t_2)) dt_2 \right| \\ &\leq \epsilon + \epsilon \lambda \int_0^1 |1+x+t_2| \left(1 + \lambda \int_0^1 |1+t_2+t_1| dt_1 \right) dt_2 \\ &\leq \epsilon \left(1 + \lambda \int_0^1 |1+x+t_2| dt_2 + \lambda^2 \int_0^1 |1+x+t_2| \int_0^1 |1+t_2+t_1| dt_1 dt_2 \right) \\ &\dots\dots\dots \end{aligned}$$

$$\begin{aligned} |\phi_r(x) - \psi_r(x)| &= \left| h(x) + \lambda \int_0^1 (1+x+t) (\psi_{r-1}(x) - \phi_{r-1}(x)) dt \right| \\ &\leq \epsilon \left(1 + \lambda \int_0^1 |1+x+t_{r-1}| dt_{r-1} \right. \\ &\quad + \lambda^2 \int_0^1 |1+x+t_{r-1}| \int_0^1 |1+t_{r-1}+t_{r-2}| dt_{r-2} dt_{r-1} + \dots \\ &\quad \dots + \lambda^{r-1} \int_0^1 |1+x+t_{r-1}| \int_0^1 |1+t_{r-1}+t_{r-2}| \int_0^1 |1+t_{r-2}+t_{r-3}| \dots \\ &\quad \dots \left. \int_0^1 |1+t_2+t_1| dt_1 \dots dt_{r-3} dt_{r-2} dt_{r-1} \right) \\ &\leq \epsilon (1 + \lambda H + (\lambda H)^2 + \dots + (\lambda H)^{r-1}) = \epsilon \left(\frac{1 - (\lambda H)^r}{1 - \lambda H} \right), \end{aligned}$$

as $r \rightarrow \infty$, we obtain

$$|\phi^*(x) - \psi^*(x)| \leq \frac{1}{1 - \lambda H} \epsilon.$$

Let us choose $S = \frac{1}{1 - \lambda H}$, hence $|\phi^*(x) - \psi^*(x)| \leq S\epsilon$, and $0 < \lambda H < 1$, where S is the Hyers-Ulam stability constant for (1). Hence, by the virtue of Definition 3 the Fredholm integral equation (1) has the Hyers-Ulam stability. \square

The following theorem shows the Hyers-Ulam-Rassias stability of the homogeneous Fredholm integral equation of second kind (1).

Theorem 8. Consider H a fixed real number such that $H \geq \frac{5}{2}$ and $\lambda H < 1$. Let $\phi : [0, 1] \rightarrow \mathbb{R}$ a continuous function and the kernel $K : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ defined by $K(x, t) = 1 + x + t$ such that

$$\int_0^1 |1+x+t|\theta(t)dt \leq \theta(x) \int_0^1 |1+x+t|dt,$$

for all $x \in [0, 1]$, where $\theta \in C(\mathbb{R}_+, \mathbb{R}_+)$. If ϕ is such that

$$\left| \phi(x) - \lambda \int_0^1 (1+x+t) \phi(t) dt \right| \leq \theta(x), \quad (6)$$

then there exists a solution $\psi : [0, 1] \rightarrow \mathbb{R}$ of the Fredholm integral equation (1) and a real constant S such that $|\phi(x) - \psi(x)| \leq S \theta(x)$ for all $x \in [0, 1]$.

Proof. By a similar procedure to the previous we define a strictly contractive operator T as in (4) since $\lambda H < 1$. By (5) we have $|h(x)| \leq \theta(x)$ for all $x \in [0, 1]$. As in the previous proof, for $\phi_0(x) = \psi_0(x) = 0$, we get,

$$\begin{aligned} |\phi_1(x) - \psi_1(x)| &= |h(x)| \leq \theta(x), \\ |\phi_2(x) - \psi_2(x)| &= \left| h(x) + \lambda \int_0^1 (1+x+t)(\psi_1(t) - \phi_1(t)) dt \right| \leq \theta(x) \left(1 + \lambda \int_0^1 |1+x+t| dt \right) \\ |\phi_3(x) - \psi_3(x)| &= \left| h(x) + \lambda \int_0^1 (1+x+t_2)(\psi_2(t_2) - \phi_2(t_2)) dt_2 \right| \\ &\leq \theta(x) + \theta(x) \lambda \int_0^1 |1+x+t_2| \left(1 + \lambda \int_0^1 |1+t_2+t_1| dt_1 \right) dt_2 \\ &\leq \theta(x) \left(1 + \lambda \int_0^1 |1+x+t_2| dt_2 + \lambda^2 \int_0^1 |1+x+t_2| \int_0^1 |1+t_2+t_1| dt_1 dt_2 \right) \\ &\dots\dots\dots \end{aligned}$$

$$\begin{aligned} |\phi_r(x) - \psi_r(x)| &= \left| h(x) + \lambda \int_0^1 (1+x+t) (\psi_{r-1}(x) - \phi_{r-1}(x)) dt \right| \\ &\leq \theta(x) \left(1 + \lambda \int_0^1 |1+x+t_{r-1}| dt_{r-1} \right. \\ &\quad + \lambda^2 \int_0^1 |1+x+t_{r-1}| \int_0^1 |1+t_{r-1}+t_{r-2}| dt_{r-2} dt_{r-1} + \dots \\ &\quad \dots + \lambda^{r-1} \int_0^1 |1+x+t_{r-1}| \int_0^1 |1+t_{r-1}+t_{r-2}| \int_0^1 |1+t_{r-2}+t_{r-3}| \dots \\ &\quad \dots \int_0^1 |1+t_2+t_1| dt_1 \dots dt_{r-3} dt_{r-2} dt_{r-1} \left. \right) \\ &\leq \theta(x) (1 + \lambda H + (\lambda H)^2 + \dots + (\lambda H)^{r-1}) = \theta(x) \left(\frac{1 - (\lambda H)^r}{1 - \lambda H} \right), \end{aligned}$$

as $r \rightarrow \infty$, we obtain

$$|\phi^*(x) - \psi^*(x)| \leq \frac{1}{1 - \lambda H} \theta(x)$$

for all $x \in [0, 1]$. Let us choose $S = \frac{1}{1 - \lambda H}$, hence $|\phi^*(x) - \psi^*(x)| \leq S \theta(x)$, and $0 < \lambda H < 1$. Hence, by the virtue of Definition 5 the Fredholm integral equation (1) has the Hyers-Ulam-Rassias stability. \square

Now, we are going to establish the Hyers-Ulam stability of the non-homogeneous Fredholm integral equation of second kind (2).

Theorem 9. Consider H a fixed real number such that $H \geq \frac{5}{2}$ and $\lambda H < 1$. Let $\phi : [0, 1] \rightarrow \mathbb{R}$ a continuous function and the kernel $K : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ defined by $K(x, t) = 1 + x + t$. If ϕ is such that

$$\left| \phi(x) - x - \lambda \int_0^1 (1 + x + t) \phi(t) dt \right| \leq \epsilon, \quad (7)$$

where $\epsilon \geq 0$ then there exists a solution $\psi : [0, 1] \rightarrow \mathbb{R}$ of the non-homogeneous Fredholm integral equation (2) and a real constant S such that $|\phi(x) - \psi(x)| \leq S\epsilon$ for all $x \in [0, 1]$.

Proof. Let us define an operator T as

$$(T\phi)(x) = x + \lambda \int_0^1 (1 + x + t) \phi(t) dt, \quad \phi \in L^2([0, 1]). \quad (8)$$

We have $T\phi \in L^2([0, 1])$ and T a self-mapping of $L^2([0, 1])$. The solution of the equation (8) is the fixed point of the strictly contractive operator T since $\lambda H < 1$. By Theorem 1 the equation (8) has a unique solution $\phi^* \in L^2([0, 1])$, where $\phi^* = \lim_{r \rightarrow \infty} \phi_r$ for

$$\phi_r(x) = x + \lambda \int_0^1 (1 + x + t) \phi_{r-1}(t) dt$$

and $\phi_0 \in L^2([0, 1])$ is an arbitrary function.

Let $\psi \in L^2([0, 1])$ be a solution of inequality (4) and

$$\psi(x) - x - \lambda \int_0^1 (1 + x + t) \psi(t) dt =: h(x).$$

We have $|h(x)| \leq \epsilon$ for all $x \in [0, 1]$. Then we can conclude that the solution of equation

$$\psi(x) = h(x) + x + \lambda \int_0^1 (1 + x + t) \psi(t) dt$$

is $\psi^* \in L^2([0, 1])$, where $\psi^* = \lim_{r \rightarrow \infty} \psi_r$ for

$$\psi_r(x) = h(x) + x + \lambda \int_0^1 (1 + x + t) \psi_{r-1}(t) dt$$

and $\psi_0 \in L^2([0, 1])$ is an arbitrary function.

For $\phi_0(x) = \psi_0(x) = 0$, we get,

$$\begin{aligned} |\phi_r(x) - \psi_r(x)| &= \left| h(x) + \lambda \int_0^1 (1 + x + t) (\psi_{r-1}(x) - \phi_{r-1}(x)) dt \right| \\ &\leq \epsilon \left(1 + \lambda \int_0^1 |1 + x + t_{r-1}| dt_{r-1} \right. \\ &\quad + \lambda^2 \int_0^1 |1 + x + t_{r-1}| \int_0^1 |1 + t_{r-1} + t_{r-2}| dt_{r-2} dt_{r-1} + \dots \\ &\quad \dots + \lambda^{r-1} \int_0^1 |1 + x + t_{r-1}| \int_0^1 |1 + t_{r-1} + t_{r-2}| \int_0^1 |1 + t_{r-2} + t_{r-3}| \dots \\ &\quad \dots \int_0^1 |1 + t_2 + t_1| dt_1 \dots dt_{r-3} dt_{r-2} dt_{r-1} \left. \right) \\ &\leq \epsilon (1 + \lambda H + (\lambda H)^2 + \dots + (\lambda H)^{r-1}) = \epsilon \left(\frac{1 - (\lambda H)^r}{1 - \lambda H} \right), \end{aligned}$$

as $r \rightarrow \infty$, we obtain

$$|\phi^*(x) - \psi^*(x)| \leq \frac{1}{1 - \lambda H} \epsilon.$$

Let us choose $S = \frac{1}{1 - \lambda H}$, hence $|\phi^*(x) - \psi^*(x)| \leq S\epsilon$, and $0 < \lambda H < 1$, where S is the Hyers-Ulam stability constant for (2). Hence, by the virtue of Definition 4 the non-homogeneous Fredholm integral equation (2) has the Hyers-Ulam stability. \square

Finally, the following corollary proves the Hyers-Ulam-Rassias stability of the non-homogeneous Fredholm integral equation of second kind (2).

Corollary 10. *Consider H a fixed real number such that $H \geq \frac{5}{2}$ and $\lambda H < 1$. Let $\phi : [0, 1] \rightarrow \mathbb{R}$ a continuous function and the kernel $K : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ defined by $K(x, t) = 1 + x + t$ such that*

$$\int_0^1 |1 + x + t| \theta(t) dt \leq \theta(x) \int_0^1 |1 + x + t| dt,$$

for all $x \in [0, 1]$, where $\theta \in C(\mathbb{R}_+, \mathbb{R}_+)$. If ϕ is such that

$$\left| \phi(x) - x - \lambda \int_0^1 (1 + x + t) \phi(t) dt \right| \leq \theta(x), \quad (9)$$

then there exists a solution $\psi : [0, 1] \rightarrow \mathbb{R}$ of the non-homogeneous Fredholm integral equation (2) and a real constant S such that $|\phi(x) - \psi(x)| \leq S \theta(x)$ for all $x \in [0, 1]$.

4 Examples

In order to illustrate our results we will present some examples.

Let us consider the non-homogeneous Fredholm integral equation of second kind (2) defined by

$$\phi(x) = x + \lambda \int_0^1 (1 + x + t) \phi(t) dt$$

for all $x \in [0, 1]$ and $\lambda = \frac{1}{5}$. Let $H = \frac{13}{5}$ and the perturbation of the solution $\varphi(x) = \frac{587}{500}x + \frac{28}{100}$.

We realize that all conditions of Theorem 9 are satisfied. In fact $\lambda H = \frac{13}{25} < 1$ and φ is a continuous function such that

$$\left| \phi(x) - x - \lambda \int_0^1 (1 + x + t) \phi(t) dt \right| = \left| \frac{3}{5000}x + \frac{1}{3000} \right| \leq \frac{7}{7500} := \epsilon.$$

By the exact solution $\psi(x) = \frac{210}{179}x + \frac{50}{179}$, we realize that

$$|\phi(x) - \psi(x)| = \left| \frac{73}{89500}x + \frac{3}{4475} \right| \leq \frac{1}{1 - \lambda H} \epsilon = \frac{7}{3600}. \quad (10)$$

To illustrate the inequality (10), we have the Figure 1

Let us consider the same non-homogeneous Fredholm integral equation of second kind (2) but now with $\lambda = \frac{1}{100}$. Let $H = 3$ and the perturbation of the solution $\varphi(x) = \frac{10052}{10000}x + \frac{851}{100000}$. We have $\lambda H = \frac{3}{100} < 1$ and φ a continuous function such that

$$\left| \phi(x) - x - \lambda \int_0^1 (1 + x + t) \phi(t) dt \right| = \left| \frac{5334}{60000000}x + \frac{341}{60000000} \right| \leq \frac{227}{2400000} := \epsilon.$$

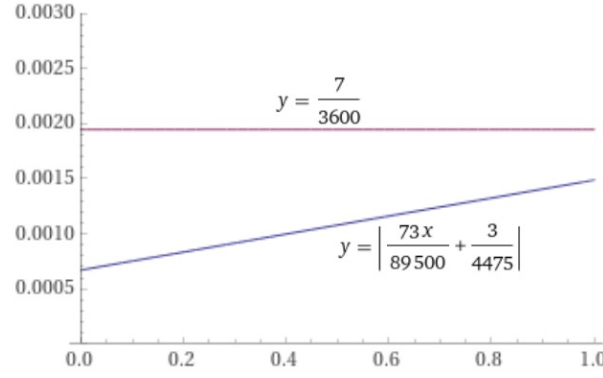


Figure 1:

By the exact solution $\psi(x) = \frac{118200}{117599}x + \frac{1000}{117599}$, we realize that

$$|\phi(x) - \psi(x)| = \left| \frac{26287}{293997500}x + \frac{76749}{11759900000} \right| \leq \frac{1}{1 - \lambda H} \epsilon = \frac{227}{2328000}. \quad (11)$$

If we consider $H = 30$, we get a worse result but still acceptable. We get,

$$|\phi(x) - \psi(x)| = \left| \frac{26287}{293997500}x + \frac{76749}{11759900000} \right| \leq \frac{1}{1 - \lambda H} \epsilon = \frac{227}{1680000}. \quad (12)$$

So we have that the non-homogeneous Fredholm integral equation of second kind (2) has the Hyers-Ulam stability.

To illustrate the inequalities (11) and (12), we have the Figure 2.

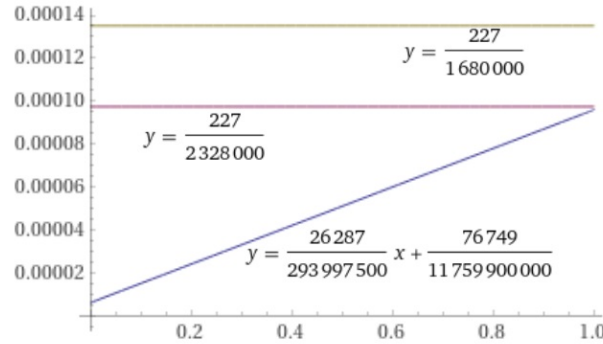


Figure 2:

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