A PRETORSION THEORY FOR THE CATEGORY OF ALL CATEGORIES

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Résumé. Une théorie de prétorsion pour la catégorie de toutes les catégories est présentée. Les prénoyaux et préconoyaux associés sont calculés pour chaque foncteur.

Abstract. A pretorsion theory for the category of all categories is presented. The associated prekernels and precokernels are calculated for every functor.

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1. Introduction

In [3] it was shown that, in the category $Preord$ of preordered sets, there is a natural notion of pretorsion theory, in which the partially ordered sets are the torsion-free objects and the sets endowed with an equivalence relation are the torsion objects. The notion of pretorsion theory given in [3] generalizes the notion of torsion theories for pointed categories given in [4].

In this paper we give what can be seen as an extension of the pretorsion theory for preordered sets to categories. A torsion-free object is now a category whose image by the well known reflection $Cat \rightarrow Preord$ (cf. [6]) is a partially ordered set. A torsion object is in turn the category whose image by the same functor is an equivalence relation. They are called below respectively the antisymmetric and the symmetric categories (see the beginning of

Section 3).

In this way, the trivial objects which were sets for $Preord$ are now the collections of monoids for the category of all categories Cat.

To assert that in fact this is a pretorsion theory, it was necessary to characterize prekernels and precokernels (corresponding to kernels and cokernels in torsion theories). The crucial result in this paper being the construction of precokernels in Cat (see Proposition 4.2).

2. Pretorsion theory for a general category

Consider two full replete[1] subcategories $\mathcal{T}$ and $\mathcal{F}$ of $\mathcal{C}$. They are called respectively the *torsion* and the *torsion-free subcategories*.

Set $\mathcal{Z} = \mathcal{T} \bigcap \mathcal{F}$, the full subcategory of $\mathcal{C}$ determined by the objects which are both in $\mathcal{T}$ and in $\mathcal{F}$. These objects are called *trivial*.

Let $Triv_{\mathcal{Z}}(X, Y)$ be the set of all morphisms $X \to Y$ in $\mathcal{C}$ that factor through an object of $\mathcal{Z}$. Such morphisms will be called $\mathcal{Z}$-*trivial* (or simply *trivial*, if there is no doubt about the subcategories considered). Notice that these trivial morphisms form an *ideal of morphisms* in the sense of [1].

The following definitions, proposition and example are to be considered in the context of the data just presented.

Definition 2.1. *A morphism $k : X \to A$ is a $\mathcal{Z}$-prekernel (or simply, prekernel) of a morphism $f : A \to A'$ if the following two conditions are satisfied:*

  1. *the composite $f \circ k$ is a trivial morphism;*

  2. *whenever $\lambda : Y \to A$ is a morphism in $\mathcal{C}$ and $f \circ \lambda$ is trivial, then there exists a unique morphism $\lambda' : Y \to X$ in $\mathcal{C}$ such that $\lambda = k \circ \lambda'$.*

Dually, one obtains the notion of ($\mathcal{Z}$-)precokernel.

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[1] I.e., whose objects are closed under isomorphisms.

Proposition 2.2. *Suppose that $\mathcal{Z}$-prekernels and $\mathcal{Z}$-precokernels exist in $\mathcal{C}$. Then, given any morphism f in $\mathcal{C}$,*

$$preker(precoker(preker f)) \cong preker f$$

$$and$$

$$precoker(preker(precoker f)) \cong precoker f,$$

where $preker f$ stands for the $\mathcal{Z}$-prekernel of f, and $precoker f$ stands for the $\mathcal{Z}$-precokernel of f.

Proof. This result is a consequence of the obvious Galois connections associated to each object in $\mathcal{C}$ (cf. [5, §VIII.1], for the classic and similar case of kernel and cokernel). $\square$

Definition 2.3. *It is said that*

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

is a short $\mathcal{Z}$-preexact sequence (or simply, short preexact sequence) in $\mathcal{C}$ if f is a ($\mathcal{Z}$-)prekernel of g and g is a ($\mathcal{Z}$-)precokernel of f.

Remark 2.4. The Proposition 2.2 gives canonical ways of constructing short $\mathcal{Z}$-preexact sequences, in a category with $\mathcal{Z}$-prekernels and $\mathcal{Z}$-precokernels (cf. [5, §VIII.1], for the classic and similar case of kernel and cokernel).

Definition 2.5. *The pair $(\mathcal{T}, \mathcal{F})$ is a pretorsion theory provided the following two conditions are satisfied:*

1. *$Hom_{\mathcal{C}}(T, F) = Triv_{\mathcal{Z}}(T, F)$, for every object $T \in \mathcal{T}$ and every object $F \in \mathcal{F}$;*

2. *for any object B of $\mathcal{C}$, there is a short $\mathcal{Z}$-preexact sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

   with $A \in \mathcal{T}$ and $C \in \mathcal{F}$.

Remark 2.6. The short $\mathcal{Z}$-preexact sequence, given in condition (2) in Definition 2.5 just above, is uniquely determined up to isomorphism (cf. Proposition 3.1 in [2]).

Example 2.7. The pair $(Equiv, Ord)$ is a pretorsion theory for the category $Preord$, whose objects are the reflexive and transitive relations, and the morphisms are the monotone maps. $Equiv$ denotes the subcategory of equivalence relations, and Ord the subcategory of partial orders, so that $Equiv \bigcap Ord = Set$ is the category of sets. Check [3] for details.

3. Symmetric and antisymmetric categories

Cat is the category whose objects are the small categories, and whose morphisms are the functors.

$CatEquiv$ will denote the full subcategory of Cat determined by the *symmetric categories* $\mathbb{T}$, meaning that for any $T, T' \in \mathbb{T}$, if $Hom_{\mathbb{T}}(T, T') \neq \emptyset$ then $Hom_{\mathbb{T}}(T', T) \neq \emptyset$.

$CatOrd$ will denote the full subcategory of Cat determined by the *antisymmetric categories* $\mathbb{F}$: for any $F, F' \in \mathbb{F}$, if $Hom_{\mathbb{F}}(F, F') \neq \emptyset$ and $Hom_{\mathbb{F}}(F', F) \neq \emptyset$, then $F = F'$.

Therefore, $CatMon = CatEquiv \bigcap CatOrd$ is the full (and replete) subcategory of Cat whose objects are classes of monoids.

The trivial functors in $Triv_{CatMon}(\mathbb{A}, \mathbb{B})$ (see the beginning of section 2), are going to be characterized in the following Lemma 3.1.

Lemma 3.1. *The functor* $F : \mathbb{A} \to \mathbb{B}$ *is trivial if and only if, for every* $A, A' \in \mathbb{A}$, *if* $Hom_{\mathbb{A}}(A, A') \neq \emptyset$ *then* $F(A) = F(A')$.

Proof. If $F : \mathbb{A} \to \mathbb{B}$ is trivial, by definition F factors through some $\mathbb{C} \in CatMon$:

$$\mathbb{A} \xrightarrow{\ \ G\ \ } \mathbb{C} \xrightarrow{\ \ H\ \ } \mathbb{B}.$$

Then, as $\mathbb{C}$ does not have morphisms between distinct objects, it follows trivially that if there exists a morphism $f : A \to A'$ then $G(A) = G(A')$ and $F(A) = H(G(A)) = H(G(A')) = F(A')$.

Conversely, supposing that

$$\forall_{A,A' \in \mathbb{A}} Hom_{\mathbb{A}}(A, A') \neq \emptyset \Rightarrow F(A) = F(A'),$$

it is obvious that
$$F = H \circ G : \mathbb{A} \xrightarrow{\quad G \quad} \mathbb{C} \xrightarrow{\quad H \quad} \mathbb{B},$$

with $obj(\mathbb{C}) = obj(\mathbb{B})$ ($\mathbb{C}$ and $\mathbb{B}$ have the same objects) and

$$Hom_{\mathbb{C}}(B, B') = \begin{cases} Hom_{\mathbb{B}}(B, B') \ if \ B = B' \\ \\ \emptyset \quad otherwise, \end{cases}$$

H being the inclusion functor, and G the restriction of the functor F to the codomain $\mathbb{C}$. Finally, notice that $\mathbb{C} \in CatMon$. $\square$

The following Proposition 3.2 asserts condition (1) in Definition 2.5. Notice that, in order to show that $(CatEquiv, CatOrd)$ is a pretorsion theory for Cat, it only remains to check condition (2) in Definition 2.5.

Proposition 3.2.

$$Hom_{Cat}(\mathbb{T}, \mathbb{F}) = Triv_{CatMon}(\mathbb{T}, \mathbb{F}),$$

for every $\mathbb{T} \in CatEquiv$ and every $\mathbb{F} \in CatOrd$.

Proof. The proof follows immediately from the respective definitions of symmetrical and antisymmetrical categories $\mathbb{T}$ and $\mathbb{F}$, and from Lemma 3.1.
$\square$

4. CatMon-Prekernels and CatMon-Precokernels

Proposition 4.1. *Let $F : \mathbb{A} \to \mathbb{A}'$ be any functor in Cat.*
Then, the functor $K : \mathbb{X} \to \mathbb{A}$ is a prekernel of F, where
$obj(\mathbb{X}) = obj(\mathbb{A})$ *($\mathbb{X}$ and $\mathbb{A}$ have the same objects),*

$$Hom_{\mathbb{X}}(A, A') = \begin{cases} Hom_{\mathbb{A}}(A, A') \ if \ F(A) = F(A') \\ \\ \emptyset \quad otherwise \ (F(A) \neq F(A')) \end{cases}$$

for every $A, A' \in \mathbb{A}$,
and K is the inclusion functor.

Proof. First, one has to establish that $\mathbb{X}$ is a category:

- $1_A \in Hom_{\mathbb{A}}(A, A) = Hom_{\mathbb{X}}(A, A)$, hence the identity of each object in $\mathbb{A}$ is also in $\mathbb{X}$;

- consider in $\mathbb{X}$ the composable morphisms $A \xrightarrow{\quad f \quad} A' \xrightarrow{\quad g \quad} A''$, then, necessarily, by the definition of $\mathbb{X}$,

$$F(A) = F(A') = F(A'') \Rightarrow F(A) = F(A'')$$

$$\Rightarrow g \circ f \in Hom_{\mathbb{A}}(A, A'') = Hom_{\mathbb{X}}(A, A''),$$

  hence the composition of any two morphisms of $\mathbb{X}$ is also in $\mathbb{X}$.

Secondly, one has to show that $F \circ K$ is a trivial functor (cf. Lemma 3.1): consider in $\mathbb{X}$ a morphism $f : A \to A'$ with $A \neq A'$; one wants to show that $F \circ K(A) = F \circ K(A')$; being K the inclusion functor, $F \circ K(f) = F(f) : F(A) \to F(A')$, and $F(A) = F(A')$ by the construction of $\mathbb{X}$, that is $F \circ K(A) = F \circ K(A')$.

Thirdly and finally, one has to check the universal property given in Definition 2.1: suppose $\Lambda : \mathbb{Y} \to \mathbb{A}$ is a functor such that $F \circ \Lambda$ is trivial; since K is the inclusion functor, one has to show that $\Lambda(\mathbb{Y}) \subseteq K(\mathbb{X})$; suppose by "reductio ad absurdum" that there is in $\mathbb{Y}$ a morphism $g : Y \to Y'$ such that $\Lambda(g) : \Lambda(Y) \to \Lambda(Y')$ is not in $K(\mathbb{X})$; then, by the construction of $\mathbb{X}$, $F(\Lambda(Y)) \neq F(\Lambda(Y'))$, which contradicts the assumption that $F \circ \Lambda$ is trivial (cf. Lemma 3.1).

$\square$

In the following Proposition 4.2, a construction of a precokernel of any functor is given.

Proposition 4.2. *Let $F : \mathbb{A} \to \mathbb{A}'$ be a functor in Cat.*

Consider the well-known adjunction $(G, U, \eta) : Graph \to Cat$, where G is the left-adjoint of the forgetful functor U from Cat into the category of graphs, and $\eta : 1_{Graph} \to UG$ is the unit of the adjunction (see [5, II.7]).

Let ζ_F be the equivalence relation on the set of objects of $\mathbb{A}'$, $obj(\mathbb{A}')$, generated by $\{(F(A_1), F(A_2)) | Hom_{\mathbb{A}}(A_1, A_2) \neq \emptyset; A_1, A_2 \in \mathbb{A}\}$.

Consider the graph morphism $(1_{mor(\mathbb{A}')}, \pi_{\zeta_F}) : U\mathbb{A}' \to \mathbb{P}$ from the underlying graph of $\mathbb{A}'$ into the graph $\mathbb{P} = (mor(\mathbb{A}'), obj(\mathbb{A}')/\zeta_F)$, where $1_{mor(\mathbb{A}')}$ is the identity on the arrows and π_{ζ_F} is the canonical projection of the set of nodes $obj(\mathbb{A}')$ into its equivalence classes.

Consider the unit morphism of $\mathbb{P}$, $\eta_{\mathbb{P}} : \mathbb{P} \to UG\mathbb{P}$, in the adjunction $G \dashv U$.

Consider finally the canonical functor $\Pi : G\mathbb{P} \to G\mathbb{P}/\equiv$, where $\equiv$ stands for the least congruence (in the sense of [5, II.8]) on $G\mathbb{P}$ which makes the graph morphism

$$U\pi = U\Pi \circ \eta_{\mathbb{P}} \circ (1_{mor(\mathbb{A}')}, \pi_{\zeta_F}) : U\mathbb{A}' \to U(G\mathbb{P}/\equiv)$$

a functor from $\mathbb{A}'$ into $G\mathbb{P}/\equiv$.

Then, $\pi : \mathbb{A}' \to G\mathbb{P}/\equiv$ is a precokernel of $F : \mathbb{A} \to \mathbb{A}'$.

Proof. Let $F' : \mathbb{A}' \to \mathbb{B}$ be a functor such that the composite $F' \circ F$ is trivial. One has to show that there is one and only one functor $H' : G\mathbb{P}/\equiv\to \mathbb{B}$ such that $H' \circ \pi = F'$.

Since $F' \circ F$ is trivial, there is one and only one morphism of graphs φ such that $\varphi \circ (1_{mor(\mathbb{A}')}, \pi_{\zeta_F}) = UF' : U\mathbb{A}' \to U\mathbb{B}$. In fact, define $\varphi(g : [A'_1] \to [A'_2]) = F'g : F'(A'_1) \to F'(A'_2)$. It is well defined since, for instance, if $A'_0 \in [A'_1]$ then there exists a sequence of morphisms ("zigzag") $A_0 \to \cdots \leftarrow A_n$ such that $F(A_0) = A'_0$ and $F(A_n) = A'_1$, which implies that $F'(A'_0) = F'(A'_1)$ because $F' \circ F$ is trivial (cf. Lemma 3.1).

There is also only one functor $H : G(\mathbb{P}) \to \mathbb{B}$ such that $\varphi = UH \circ \eta_{\mathbb{P}} : \mathbb{P} \to U\mathbb{B}$, being $\eta_{\mathbb{P}}$ the unit morphism of the adjunction $U \vdash G : Graph \to Cat$ (cf. Theorem 1 in [5, II.7]).

There is also a unique functor H' from the quotient category $G\mathbb{P}/\equiv$ into $\mathbb{B}$ such that $H' \circ \Pi = H$; in order to prove so, one has to show that (cf. [5, II.8]) H identifies $\eta_{\mathbb{P}} \circ (1_{mor(\mathbb{A}')}, \pi_{\zeta_F})(1_{A'}) =< 1_{A'} >$ and $1_{[A']}$, for every $A' \in \mathbb{A}'$, and that H identifies $\eta_{\mathbb{P}}((1_{mor(\mathbb{A}')}, \pi_{\zeta_F})(g)) \circ \eta_{\mathbb{P}}((1_{mor(\mathbb{A}')}, \pi_{\zeta_F})(f)) = \eta_{\mathbb{P}}(g) \circ \eta_{\mathbb{P}}(f) =< f, g >$ with $\eta_{\mathbb{P}}((1_{mor(\mathbb{A}')}, \pi_{\zeta_F})(g \circ f)) = \eta_{\mathbb{P}}(g \circ f) =< g \circ f >$, for every pair $(g : A'_2 \to A'_3, f : A'_1 \to A'_2)$ of composable morphisms in $\mathbb{A}'$; this is obvious since $UH \circ \eta_{\mathbb{P}} \circ (1_{mor(\mathbb{A}')}, \pi_{\zeta_F}) = UF'$ and F' is a functor.

It was proved just above that there is a functor H' such that $H' \circ \pi = F'$. It remains to check that such functor is the unique which satisfies $H' \circ \pi = F'$.

Suppose that S' is a functor such that $S' \circ \pi = F'$, then there is a functor S such that $S = S' \circ \Pi$, and then there is a graph morphism σ such that $US \circ \eta_{\mathbb{P}} = \sigma$, with $\sigma \circ (1_{mor(\mathbb{A}')}, \pi_{\zeta_F}) = UF'$, which implies that $\sigma = \varphi$ as defined above, and so $H' = S'$ going backwards.

$\square$

5. Short CatMon-preexact sequences

Let $\mathbb{A}'$ be any category, and let $\mathbb{A}$ be a category with the same objects, $obj(\mathbb{A}) = obj(\mathbb{A}')$, and such that, for any objects $A, B \in \mathbb{A}'$, if $Hom_{\mathbb{A}}(A, B) \neq \emptyset$ then $Hom_{\mathbb{A}'}(A, B) \neq \emptyset$ and $Hom_{\mathbb{A}'}(B, A) \neq \emptyset$ and $Hom_{\mathbb{A}}(A, B) = Hom_{\mathbb{A}'}(A, B)$ and $Hom_{\mathbb{A}}(B, A) \neq \emptyset$.

Let $F : \mathbb{A} \to \mathbb{A}'$ be the inclusion functor of $\mathbb{A}$ in $\mathbb{A}'$, and $\pi : \mathbb{A}' \to G\mathbb{P}/\equiv$ the precokernel of F constructed as in Proposition 4.2.

It is an immediate consequence of the characterization of $CatMon$-prekernel in Proposition 4.1 that F is the prekernel of π, so that we have constructed a short preexact sequence

$$\mathbb{A} \xrightarrow{\quad F \quad} \mathbb{A}' \xrightarrow{\quad \pi \quad} G\mathbb{P}/\equiv$$

for each $\mathbb{A}' \in Cat$, with $\mathbb{A} \in CatEquiv$.

Suppose that, one requires the category $\mathbb{A}$ just defined to satisfy in addition: for every $A, B \in \mathbb{A}'$, if $Hom_{\mathbb{A}'}(A, B) \neq \emptyset$ and $Hom_{\mathbb{A}'}(B, A) \neq \emptyset$ then $Hom_{\mathbb{A}}(A, B) \neq \emptyset$. Then, since $obj(G\mathbb{P}/\equiv) = obj(\mathbb{A}')/\zeta_F$ and by the nature of $\eta_{\mathbb{P}}$ and Π (cf. Proposition 4.2 and [5, II.7,8]), it is clear that $G\mathbb{P}/\equiv \in CatOrd$.

It was proved that, for every category $\mathbb{A}' \in Cat$, there is a short $CatMon$-preexact sequence

$$\mathbb{A} \to \mathbb{A}' \to \mathbb{A}''$$

with $\mathbb{A} \in CatEquiv$ and $\mathbb{A}'' \in CatOrd$.

Hence, the following Theorem can be stated.

Theorem 5.1. *The pair* $(CatEquiv, CatOrd)$ *is a pretorsion theory for the category of all categories* Cat.

Acknowledgement

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