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Generalized Grassmann algebras and applications to stochastic processes

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In this paper, we present the groundwork for an Itô/Malliavin stochastic calculus and Hida's white noise analysis in the context of a supersymmetry with $\mathbb{Z}_3$-graded algebras. To this end, we establish a ternary Fock space and the corresponding strong algebra of stochastic distributions and present its application in the study of stochastic processes in this context.

KEYWORDS

Fock space, hypersymmetry, stochastic process, ternary Grassmann algebra

MSC CLASSIFICATION

30G35; 60H45; 60G22; 15A75

1 | INTRODUCTION

Classic theories like Bose–Einstein or Fermi–Dirac statistics are based on SU(2)-symmetries and $\mathbb{Z}_2$-graded algebras. But theories like quantum chromodynamics where quarks are considered as fermions require a setting with $\mathbb{Z}_3$- (or $\mathbb{Z}_6$-) graded algebras for a convenient generalizations of Pauli's exclusion principle and the establishment of the corresponding statistics.[1] Such algebras and the corresponding Dirac operators have been studied in the recent past. But these theories raise an additional question about the necessary extension of the corresponding supersymmetry. While standard supersymmetry combines bosonic fields with fermionic fields to a $\mathbb{Z}_2$-graded Lie super algebra, a supersymmetric extension involving quarks requires a $\mathbb{Z}_3$-graded algebra of Grassmannian type which leads to a kind of hypersymmetry,[2] as $\mathbb{Z}_n$-graded versions of supersymmetry are called.

One important aspect of supersymmetry lies in its combination with stochastic dynamics, also called topological supersymmetry initiated by the seminal works of Parisi and Sourlas in 1979 (see, e.g., Parisi and Sourlas[3] or Junker[4]), in particular the more recent supersymmetric theory of stochastic dynamics.[5,6] But for the establishment of such a theory in the context of $\mathbb{Z}_3$- (or $\mathbb{Z}_6$-) graded algebras, that is, the development of topological hypersymmetry, a major problem arises. One needs the counterpart of the classic Itô/Malliavin stochastic calculus and Hida's white noise analysis in this context. This is the problem we are going to study in this paper. While differentiation and integration has been studied in superspace using Clifford algebras,[7,8] no such study has been made in the context of $\mathbb{Z}_3$-graded algebras to the knowledge of the authors.

To this end, let us recall the notion of classic (finite or infinite) Grassmann algebra. $\mathcal{G}_d$ is the unital algebra over the complex numbers generated by $\mathbf{e}_0 = 1$ and a finite set of elements $\mathbf{e}_i$, $i \in \{1, 2, \dots, d\}$, which do not belong to and are linearly independent over $\mathbb{C}$. Moreover, they satisfy

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 0, \tag{1.1}$$

where $i, j \in \{1, 2, \dots, d\}$, and in particular

$$\mathbf{e}_i^2 = 0. \tag{1.2}$$

An element of $\mathcal{G}_d$ is often referred to as a supernumber. The number of generators can be taken to be infinite. To do so, closures of the algebra with respect to a norm are considered—see, for example, previous studies.[9–12]

It is well known that the classic Grassmann algebra is $\mathbb{Z}_2$-graded. To introduce a $\mathbb{Z}_3$-grading, we present a generalization of the classic Grassmann algebra to a ternary Grassmanian setting. We consider once again the real generator $\mathbf{e}_0 = 1$ and basis elements $\mathbf{e}_1, \dots, \mathbf{e}_d$. Instead of (1.1), we assume the generators satisfy

$$\mathbf{e}_i\mathbf{e}_j = \omega\mathbf{e}_j\mathbf{e}_i \tag{1.3}$$

for every $i < j$, where $i, j \in \{1, 2, \dots, d\}$ and where $\omega \neq 1$ is a (fixed) nonzero complex number. As will be clear from the next section, these algebras have a natural $\mathbb{Z}_3$-grading.

Using these algebras, we are going to create the necessary algebraic and analytic tools for the establishment of the counterpart of Hida's white noise space theory, including the construction of a topological algebra associated with the above mentioned algebra based on a decreasing family of Hilbert spaces which allows us to obtain the link between this algebra and the Fock space. We will finish by showing the application to stochastic processes.

2 | FINITE TERNARY GRASSMANN ALGEBRAS

Let us start by introducing ternary Grassmann algebras. Let $\mathfrak{G} = \{\mathbf{e}_j, j \in \mathbb{N}\}$ be a countable set of linearly independent vectors over $\mathbb{C}$. We denote by V_d the complex linear space generated by the first d of such vectors $\mathbf{e}_1, \dots, \mathbf{e}_d$.

Definition 2.1. We define the *ternary Grassmann algebra* $\mathcal{G}_{3,d}$ associated to V_d as the free (nontrivial) algebra over $\mathbb{C}$ containing a copy of Span$\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ and of $\mathbb{C}$, and satisfying to the following relations:

(i) there exists a complex $\omega \neq 0$ such that

$$\mathbf{e}_i\mathbf{e}_j = \omega\mathbf{e}_j\mathbf{e}_i, \qquad \text{for all} \quad i < j; \tag{2.1}$$

(ii) the basis elements satisfy

$$\mathcal{T}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k) = 0, \qquad \text{for all} \quad i, j, k = 1, \dots, d, \tag{2.2}$$

where $\mathcal{T}$ denotes the ternary form (based on the anti-commutator):

$$\begin{aligned}\mathcal{T}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k) =& \mathbf{e}_i\{\mathbf{e}_j, \mathbf{e}_k\} + \mathbf{e}_j\{\mathbf{e}_k, \mathbf{e}_i\} + \mathbf{e}_k\{\mathbf{e}_i, \mathbf{e}_j\} \\ =& \mathbf{e}_i\mathbf{e}_j\mathbf{e}_k + \mathbf{e}_i\mathbf{e}_k\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i\mathbf{e}_k + \mathbf{e}_j\mathbf{e}_k\mathbf{e}_i + \mathbf{e}_k\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_k\mathbf{e}_j\mathbf{e}_i, \ 1 \le i \le j \le k \le d.\end{aligned} \tag{2.3}$$

Remark 2.2. By "nontrivial," we mean that the product of any two arbitrary basic elements of the algebra is either a new element of the algebra or it is zero. We will say that $\mathcal{G}_{3,d}$ is generated by $\mathfrak{G}_d = \{\mathbf{e}_1, \dots, \mathbf{e}_d\} \subset \mathfrak{G}$. Such algebras have already been studied in the literature. They are often referred to as *ternary Grassmann algebras*.[13,14]

An immediate consequence of the definition is the following lemma.

Lemma 2.3. *Under the conditions of Definition 2.1, it holds*

(i) $\mathbf{e}_i^3 = 0$ *for all* $i = 1, \ldots, d$;
(ii) ω *is a third root of the unit.*

Proof. Proposition (i) is immediate. For the second proposition, we have for $1 \le i < j \le d$

$$\begin{aligned} 0 = \mathcal{T}(\mathbf{e}_i, \mathbf{e}_i, \mathbf{e}_j) &= \mathbf{e}_i\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_i\mathbf{e}_j\mathbf{e}_i + \mathbf{e}_i\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_i\mathbf{e}_j\mathbf{e}_i + \mathbf{e}_j\mathbf{e}_i\mathbf{e}_i + \mathbf{e}_j\mathbf{e}_i\mathbf{e}_i \\ &= 2(\mathbf{e}_i^2\mathbf{e}_j + \mathbf{e}_i\mathbf{e}_j\mathbf{e}_i + \mathbf{e}_j\mathbf{e}_i^2) = 2(1+\omega+\omega^2)\mathbf{e}_i^2\mathbf{e}_j. \end{aligned}$$

Under the assumption of nontriviality (i.e., $\mathbf{e}_j\mathbf{e}_i^2 \neq 0$), we obtain $\omega^2 + \omega + 1 = 0$, and therefore, ω is a root of third order the unity. □

Remark 2.4. In what follows, we assume $\omega = e^{i2\pi/3}$.

Due to Lemma 2.3, one observes that the relevant powers of the vector basis elements are given by $\mathbf{e}_j^m$, $m = 0, 1, 2$, since $\mathbf{e}_j^m = 0$ for all $m \ge 3$. Furthermore, we identify $\mathbf{e}_j^0 = 1$, the identity of the field $\mathbb{C}$. Hence, we have

$$\mathbf{e}_j^m = \begin{cases} 1, & m = 0 \\ \mathbf{e}_j, & m = 1 \\ \mathbf{e}_j^2, & m = 2 \\ 0, & m \ge 3 \end{cases} \tag{2.4}$$

for all $j = 1, \ldots, d$. In consequence, a basis for the finite ternary Grassmann algebra $\mathcal{G}_{3,d}$ is expressible in terms of appropriated ordered d-tuples of powers less than 3, that is,

$$\mathbf{e}^{\boldsymbol{\nu}} := \mathbf{e}_1^{\nu_1} \ldots \mathbf{e}_d^{\nu_d}, \qquad \boldsymbol{\nu} = (\nu_1, \ldots, \nu_d) \in \{0,1,2\}^d. \tag{2.5}$$

Under the previous convention, we have $\mathbf{e}^{\boldsymbol{0}} = 1$ with $\boldsymbol{0} = (0, \ldots, 0)$. We shall denote by $\mathcal{I}_d$ the set of all such d-tuples, that is,

$$\mathcal{I}_d = \{0,1,2\}^d. \tag{2.6}$$

Notice that we have $\mathbf{e}^{\boldsymbol{\nu}} = \mathbf{e}^{\boldsymbol{\mu}}$ if and only if $\nu_j = \mu_j$ for all $j = 1, \ldots, d$.
Since there are 3^d, such basis elements every $z \in \mathcal{G}_{3,d}$ can be written as

$$\mathbf{z} = \sum_{|\boldsymbol{\nu}|=0}^{2d} z_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\nu}}, \qquad z_{\boldsymbol{\nu}} \in \mathbb{C},\ |\boldsymbol{\nu}| = \nu_1 + \ldots + \nu_d. \tag{2.7}$$

We observe that $\mathcal{G}_{3,d}$ is a $\mathbb{Z}_3$-graded algebra, that is, under multiplication, the grades add up modulus 3. This leads to the following blade decomposition of the generalized ternary Grassmann algebra:

$$\mathcal{G}_{3,d} = [\mathcal{G}_{3,d}]_0 \oplus [\mathcal{G}_{3,d}]_1 \oplus \ldots \oplus [\mathcal{G}_{3,d}]_{2d}, \tag{2.8}$$

where each k-blade is

$$[\mathcal{G}_{3,d}]_k = \left\{ \mathbf{z} \in \mathcal{G}_{3,d} : \mathbf{z} = \sum_{|\boldsymbol{\nu}|=k} z_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\nu}} \right\}, k = 0, \ldots, 2d. \tag{2.9}$$

In what follows, we denote by $[\mathbf{z}]_k := \sum_{|\boldsymbol{\nu}|=k} z_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\nu}}$ the projection of $\mathbf{z}$ into the blade $[\mathcal{G}_{3,d}]_k$, $k = 0, \ldots, 2d$.

Example 2.5. For dimension $d = 2$, we have

$$\mathbf{z} = \underbrace{z_{00}}_{\in[\mathcal{G}_{3,2}]_0} + \underbrace{z_{10}\mathbf{e}_1 + z_{01}\mathbf{e}_2}_{\in[\mathcal{G}_{3,2}]_1} + \underbrace{z_{20}\mathbf{e}_1^2 + z_{11}\mathbf{e}_1\mathbf{e}_2 + z_{02}\mathbf{e}_2^2}_{\in[\mathcal{G}_{3,2}]_2} + \underbrace{z_{21}\mathbf{e}_1^2\mathbf{e}_2 + z_{12}\mathbf{e}_1\mathbf{e}_2^2}_{\in[\mathcal{G}_{3,2}]_3} + \underbrace{z_{22}\mathbf{e}_1^2\mathbf{e}_2^2}_{\in[\mathcal{G}_{3,2}]_4},$$

where

$$[\mathbf{z}]_0 = z_{00},\ [\mathbf{z}]_1 = z_{10}\mathbf{e}_1 + z_{01}\mathbf{e}_2,\ \ [\mathbf{z}]_2 = z_{20}\mathbf{e}_1^2 + z_{11}\mathbf{e}_1\mathbf{e}_2 + z_{02}\mathbf{e}_2^2,$$

$$[\mathbf{z}]_3 = z_{21}\mathbf{e}_1^2\mathbf{e}_2 + z_{12}\mathbf{e}_1\mathbf{e}_2^2,\ \ [\mathbf{z}]_4 = z_{22}\mathbf{e}_1^2\mathbf{e}_2^2.$$

The generalized finite ternary Grassmann algebra $\mathcal{G}_{3,d}$ decomposes itself into the sum of three spaces

$$\mathcal{G}_{3,d} = \mathcal{G}_{3,d}^0 \oplus \mathcal{G}_{3,d}^1 \oplus \mathcal{G}_{3,d}^2, \tag{2.10}$$

where each $\mathcal{G}_{3,d}^k$

$$\mathcal{G}_{3,d}^k = \text{span}\ \ \{\mathbf{e}^{\boldsymbol{\nu}} : |\boldsymbol{\nu}| = k(\text{mod } 3)\},\ k = 0, 1, 2.$$

We observe that these spaces obey the following multiplication rules:

$$\mathcal{G}_{3,d}^0 \cdot \mathcal{G}_{3,d}^0 \subset \mathcal{G}_{3,d}^0, \qquad \mathcal{G}_{3,d}^1 \cdot \mathcal{G}_{3,d}^2 \subset \mathcal{G}_{3,d}^0, \qquad \mathcal{G}_{3,d}^2 \cdot \mathcal{G}_{3,d}^1 \subset \mathcal{G}_{3,d}^0.$$

Hence, only $\mathcal{G}_{3,d}^0$ is a subalgebra of $\mathcal{G}_{3,d}$ while the spaces $\mathcal{G}_{3,d}^1, \mathcal{G}_{3,d}^2$ *do not form an algebra.*

2.1 | Properties

We now present results on products in $\mathcal{G}_{3,d}$. Due to (2.1), we have

$$\mathbf{e}_i\mathbf{e}_j = \omega\mathbf{e}_j\mathbf{e}_i \iff \mathbf{e}_j\mathbf{e}_i = \omega^2\mathbf{e}_i\mathbf{e}_j, \tag{2.11}$$

for all $i<j$.

Hence, for each component, we have

$$\mathbf{e}_j^{\nu_j}\mathbf{e}_j^{\mu_j} = \begin{cases} \mathbf{e}_j^{\nu_j+\mu_j}, & \text{if } 0 \le \nu_j + \mu_j \le 2 \\ 0, & \text{otherwise.} \end{cases},\ j = 1, \dots, d. \tag{2.12}$$

We now observe that for all $\boldsymbol{\nu}, \boldsymbol{\mu} \in \mathcal{I}_d$, we have

$$\mathbf{e}^{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\mu}} = 0, \tag{2.13}$$

whenever $\nu_j + \mu_j \ge 3$ for some j. However, if $0 \le \nu_j + \mu_j < 3$ for all j, then by (2.11), we obtain

$$\mathbf{e}^{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\mu}} = \sigma(\boldsymbol{\nu}, \boldsymbol{\mu})\mathbf{e}^{\boldsymbol{\nu}+\boldsymbol{\mu}}, \tag{2.14}$$

where $\sigma(\boldsymbol{\nu}, \boldsymbol{\mu}) = \omega^{2\sum_{s=1}^{d-1}\sum_{j=s+1}^{d}\nu_j\mu_s}$ corresponds to $\omega = \exp\left(\frac{2\pi i}{3}\right)$ to the power of the number of permutations of the basis elements. This leads to the following multiplication rule:

$$\mathbf{e}^{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\mu}} = \sigma(\boldsymbol{\nu}, \boldsymbol{\mu})\mathbf{e}^{\boldsymbol{\nu}+\boldsymbol{\mu}}, \qquad \sigma(\boldsymbol{\nu}, \boldsymbol{\mu}) := \begin{cases} 0, & \text{if} \boldsymbol{\nu} + \boldsymbol{\mu} \notin \mathcal{I}_d, \\ \omega^{2\sum_{s=1}^{d-1}\sum_{j=s+1}^{d}\nu_j\mu_s}, & \text{otherwise.} \end{cases} \tag{2.15}$$

Lemma 2.6. *For every* $\mathbf{z} \in [\mathcal{G}_{3,d}]_k, \mathbf{w} \in [\mathcal{G}_{3,d}]_s$, $(0 \le k, s \le 2d)$, *we have that* $\mathbf{zw} = 0$ *if* $k+s > 2d$ *and*

$$\mathbf{zw} \in \{0\} \oplus [\mathcal{G}_{3,d}]_{k+s},\ \text{if}\ 0 \le k + s \le 2d. \tag{2.16}$$

Proof. Hence, we obtain

$$\begin{aligned} \mathbf{zw} &= \left(\sum_{|\boldsymbol{\nu}|=k} z_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\nu}}\right)\left(\sum_{|\boldsymbol{\mu}|=s} w_{\boldsymbol{\mu}}\mathbf{e}^{\boldsymbol{\mu}}\right) = \sum_{\substack{|\boldsymbol{\nu}|=k,|\boldsymbol{\mu}|=s \\ \boldsymbol{\nu}+\boldsymbol{\mu}\in\mathcal{I}_d}} z_{\boldsymbol{\nu}}w_{\boldsymbol{\mu}}\sigma(\boldsymbol{\nu}, \boldsymbol{\mu})\mathbf{e}^{\boldsymbol{\nu}+\boldsymbol{\mu}} \\ &= \sum_{\substack{|\boldsymbol{\nu}|=k,|\boldsymbol{\mu}|=s \\ \boldsymbol{\nu}+\boldsymbol{\mu}\in\mathcal{I}_d}} \sigma(\boldsymbol{\nu}, \boldsymbol{\mu}) z_{\nu_1,\dots,\nu_d} w_{\mu_1,\dots,\mu_d} \mathbf{e}_1^{\nu_1+\mu_1} \dots \mathbf{e}_d^{\nu_d+\mu_d}. \end{aligned}$$

The result follows trivially from (2.15). □

Lemma 2.7. *For every vector* $\mathbf{z} = z_1\mathbf{e}_1 + \ldots + z_d\mathbf{e}_d \in [\mathcal{G}_{3,d}]_1$, *it holds*

$$\mathbf{z}^3 = 0. \tag{2.17}$$

Proof. By direct computation, we get

$$\mathbf{z}^3 = \sum_{i,j,k} z_i z_j z_k \mathcal{T}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k) = 0.$$

since ω is the third root of the unit and $\mathbf{e}_i^3 = 0$. □

We now present some decomposition results for the finite ternary Grassmann algebra. The results will be expressed in terms of the basis element $\mathbf{e}_d$ but are easily extendable to any basis element $\mathbf{e}_j$ with appropriate modifications.

Lemma 2.8. *Every element* $\mathbf{z} \in \mathcal{G}_{3,d}$ *admits the following decomposition:*

$$\mathbf{z} = A + B\mathbf{e}_d + C\mathbf{e}_d^2, \ A, B, C \in \mathcal{G}_{3,d-1}. \tag{2.18}$$

The result is straightforward, and its proof will be omitted.

Lemma 2.9. *For every* $A \in \mathcal{G}_{3,d-1}$, *there exists* $A' \in \mathcal{G}_{3,d-1}$ *such that*

$$A\mathbf{e}_d = \mathbf{e}_d A'. \tag{2.19}$$

Proof. Recall that $\mathbf{e}_j\mathbf{e}_d = \omega\mathbf{e}_d\mathbf{e}_j$, $j = 1, \ldots, d-1$. Hence, we have for $A = \sum_{\boldsymbol{\nu}\in\mathcal{I}_{d-1}} a_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\nu}} \in \mathcal{G}_{3,d-1}$

$$\begin{aligned}
A\mathbf{e}_d &= \left(\sum_{\boldsymbol{\nu}\in\mathcal{I}_{d-1}} a_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\nu}}\right)\mathbf{e}_d = \sum_{\boldsymbol{\nu}\in\mathcal{I}_{d-1}} a_{\boldsymbol{\nu}}\mathbf{e}_1^{\nu_1} \ldots \mathbf{e}_{d-2}^{\nu_{d-2}}(\mathbf{e}_{d-1}^{\nu_{d-1}}\mathbf{e}_d) \\
&= \sum_{\boldsymbol{\nu}\in\mathcal{I}_{d-1}} a_{\boldsymbol{\nu}}\mathbf{e}_1^{\nu_1} \ldots \mathbf{e}_{d-2}^{\nu_{d-2}}\left(\omega^{\nu_{d-1}}\mathbf{e}_d\mathbf{e}_{d-1}^{\nu_{d-1}}\right) = \ldots = \sum_{\boldsymbol{\nu}\in\mathcal{I}_{d-1}} a_{\boldsymbol{\nu}}\omega^{\sum_{j=1}^{d}\nu_j}\mathbf{e}_d\mathbf{e}_1^{\nu_1} \ldots \mathbf{e}_{d-2}^{\nu_{d-2}}\mathbf{e}_{d-1}^{\nu_{d-1}} \\
&= \mathbf{e}_d\left(\sum_{\boldsymbol{\nu}\in\mathcal{I}_{d-1}} a_{\boldsymbol{\nu}}\omega^{\sum_{j=1}^{d}\nu_j}\mathbf{e}_1^{\nu_1} \ldots \mathbf{e}_{d-2}^{\nu_{d-2}}\mathbf{e}_{d-1}^{\nu_{d-1}}\right) := \mathbf{e}_d A'.
\end{aligned}$$

□

Corollary 2.10. *If we have* $A = 0$ *in decomposition (2.18), then* $\mathbf{z}^3 = 0$.

This is an obvious consequence of the two previous lemmas, as $\mathbf{z} = (B + C\mathbf{e}_d)\mathbf{e}_d$.

Lemma 2.11. *For every* $1 \le n \le d$, *there exists an element* $I_n = \mathbf{e}_{d-n+1}^2 \ldots \mathbf{e}_d^2$ *such that it satisfies:*

(i) I_n *is nilpotent, that is* $I_n^2 = 0$.

(ii) I_n *acts as a projector of* $\mathcal{G}_{3,d}$ *onto the subalgebra* $\mathcal{G}_{3,d-n}$, *that is to say, there exists a projector* $P_n : \mathcal{G}_{3,d} \to \mathcal{G}_{3,d-n}$ *given by*

$$\mathbf{z} = \sum_{\boldsymbol{\mu}\in\mathcal{I}_d} z_{\boldsymbol{\mu}}\mathbf{e}^{\boldsymbol{\mu}} \in \mathcal{G}_{3,d} \mapsto P_n(\mathbf{z}) := \sum_{\boldsymbol{\nu}\in\mathcal{I}_{d-n}} z_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\nu}} \in \mathcal{G}_{3,d-n},$$

where $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_{d-n})$.

(iii) In particular, P_d *acts as a projector of* $\mathcal{G}_{3,d}$ *onto* $\mathbb{C}$, *given by* $\mathbf{z} = \sum_{\boldsymbol{\mu}\in\mathcal{I}_d} z_{\boldsymbol{\mu}}\mathbf{e}^{\boldsymbol{\mu}} \in \mathcal{G}_{3,d} \mapsto P_d(\mathbf{z}) := [\mathbf{z}]_0 = z_{\mathbf{0}} := z_{(0,\ldots,0)}$.

Proof. The fact that I_n is nilpotent is straightforward. For the projection part, we observe that by (2.12) the product of an arbitrary $\mathbf{z}$ by I_n kills off all terms containing $\mathbf{e}_{d-n+1}, \dots, \mathbf{e}_d$, that is, given

$$\mathbf{z} = \sum_{\boldsymbol{\mu}\in\mathcal{I}_d} z_{\boldsymbol{\mu}}\mathbf{e}^{\boldsymbol{\mu}} = \sum_{\boldsymbol{\mu}\in\mathcal{I}_{d-n}} z_{\boldsymbol{\mu}}\mathbf{e}^{\boldsymbol{\mu}} + \sum_{\boldsymbol{\mu}\in\mathcal{I}_d\setminus\mathcal{I}_{d-n}} z_{\boldsymbol{\mu}}\mathbf{e}^{\boldsymbol{\mu}},$$

we have

$$\mathbf{z}I_n = \left(\sum_{\boldsymbol{\mu}\in\mathcal{I}_d} z_{\boldsymbol{\mu}}\mathbf{e}^{\boldsymbol{\mu}}\right)I_n = \sum_{\boldsymbol{\mu}\in\mathcal{I}_{d-n}} z_{\boldsymbol{\mu}}\mathbf{e}^{\boldsymbol{\mu}}I_n, \qquad \boldsymbol{\mu} = (\mu_1, \dots, \mu_{d-n}).$$

Hence, we identify the projection of $\mathbf{z}$ into $\mathcal{G}_{3,d-n}$ with $P_n(\mathbf{z}) := \sum_{\boldsymbol{\mu}\in\mathcal{I}_{d-n}} z_{\boldsymbol{\mu}}\mathbf{e}^{\boldsymbol{\mu}}$. The third proposition is now immediate. □

Remark 2.12. Based on the last proposition, the nonscalar part of an element $\mathbf{z} \in \mathcal{G}_{3,d}$ is obtained by

$$(1 - P_d)\mathbf{z} = \sum_{\substack{\boldsymbol{\mu}\in\mathcal{I}_d \\ \boldsymbol{\mu}\neq\mathbf{0}}} z_{\boldsymbol{\mu}}\mathbf{e}^{\boldsymbol{\mu}}.$$

Hence,

$$\mathbf{z} = P_d\mathbf{z} + (1 - P_d)\mathbf{z}.$$

Henceforward, we shall use the notations $z_{\mathbf{0}} := P_d\mathbf{z}$ for its scalar part (also, *body* of $\mathbf{z}$) and $\mathbf{z}_r := (1 - P_d)\mathbf{z} = \sum_{\substack{\boldsymbol{\mu}\in\mathcal{I}_d \\ \boldsymbol{\mu}\neq\mathbf{0}}} z_{\boldsymbol{\mu}}\mathbf{e}^{\boldsymbol{\mu}}$ for its remainder (also, *soul* of $\mathbf{z}$).

Lemma 2.13. *An arbitrary element $\mathbf{z} \in \mathcal{G}_{3,d}$ is invertible if and only if its scalar part $[\mathbf{z}]_0 = z_{\mathbf{0}}$ is nonzero.*

Proof. We begin our proof by showing that if the scalar part of an element $\mathbf{z} \in \mathcal{G}_{3,d}$ is zero, then this element cannot be invertible. Indeed, if $[\mathbf{z}]_0 = 0$, then $\mathbf{z} = \sum_{\substack{\nu\in\mathcal{I}_d \\ \nu\neq\mathbf{0}}} z_\nu\mathbf{e}^\nu$, and by Lemma 2.6, we get $[\mathbf{zw}]_0 = 0$ for all $\mathbf{w} \in \mathcal{G}_{3,d}$. Hence, $\mathbf{z}$ is not invertible.

Now, we consider $[\mathbf{z}]_0 \neq 0$. First, we observe that there exists $m \in \mathbb{N}$ such that $\mathbf{z}_r^m = 0$. Take $m(\mathbf{z}) := \min\{m \in \mathbb{N} : \mathbf{z}_r^m = 0\}$. Whenever $z_{\mathbf{0}} \neq 0$, we have

$$\mathbf{z}^n = z_{\mathbf{0}}^n\left(1 + \frac{1}{z_{\mathbf{0}}}\mathbf{z}_r\right)^n, \ n \in \mathbb{N}.$$

Hence, for $n = m(\mathbf{z}_r) - 1$, we obtain

$$\left(1 + \frac{1}{z_{\mathbf{0}}}\mathbf{z}_r\right)\left(1 - \frac{1}{z_{\mathbf{0}}}\mathbf{z}_r + \frac{1}{z_{\mathbf{0}}^2}\mathbf{z}_r^2 + \dots + (-1)^{m(\mathbf{z})-1}\frac{1}{z_{\mathbf{0}}^{m(\mathbf{z})-1}}\mathbf{z}_r{}^{m(\mathbf{z})-1}\right) = 1,$$

so that

$$\mathbf{z}^{-1} = \frac{1}{z_{\mathbf{0}}} - \frac{1}{z_{\mathbf{0}}^2}\mathbf{z}_r + \frac{1}{z_{\mathbf{0}}^3}\mathbf{z}_r^2 + \dots + (-1)^{m(\mathbf{z})-1}\frac{1}{z_{\mathbf{0}}^{m(\mathbf{z})}}\mathbf{z}_r{}^{m(\mathbf{z})-1}$$

is the right inverse of $\mathbf{z}$. The same construction holds for the left inverse which proves the unicity of $\mathbf{z}^{-1}$. □

2.2 | A conjugation in $\mathcal{G}_{3,d}$

We present a morphism acting on the finite generalized ternary algebra $\mathcal{G}_{3,d}$.

Definition 2.14 (Pseudoconjugation)**.** The *pseudoconjugation* in the ternary Grassmann algebra $\mathcal{G}_{3,d}$ is defined as the morphism $\bar{\cdot} : \mathcal{G}_{3,d} \to \mathcal{G}_{3,d}$, with $\mathbf{z} \mapsto \overline{\mathbf{z}} = \sum_{\boldsymbol{\nu}} \bar{z}_{\boldsymbol{\nu}}\overline{\mathbf{e}^{\boldsymbol{\nu}}}$, where $\bar{z}_{\boldsymbol{\nu}}$ denotes the standard complex conjugation, while its action on the basis elements $\mathbf{e}^{\boldsymbol{\nu}}$, $\boldsymbol{\nu} \in \mathcal{I}_d$, is given by

$$\overline{1} = 1, \ \overline{\mathbf{e}}_j = \mathbf{e}_j^2, \ j = 1, \dots, d, \tag{2.20}$$

and satisfying to

$$\overline{\mathbf{ab}+\mathbf{c}} = \overline{\mathbf{b}}\,\overline{\mathbf{a}} + \overline{\mathbf{c}},\ \textit{for all}\ \mathbf{a},\mathbf{b},\mathbf{c} \in \mathcal{G}_{3,d}. \tag{2.21}$$

Remark 2.15. As a consequence, we get $\overline{\mathbf{e}_j^2} = \overline{\mathbf{e}}_j\overline{\mathbf{e}}_j = \mathbf{e}_j^2\mathbf{e}_j^2 = 0$. Hence, $\overline{\mathbf{e}^{\boldsymbol{\nu}}} = 0$ if and only if $\boldsymbol{\nu} \notin \{0,1\}^d$. Furthermore, this morphism is not onto and it is not an involution since $(\bar{\bar{\mathbf{e}}}_j) = 0$.

Lemma 2.16. *For all* $\mathbf{z} \in \mathcal{G}_{3,d}$*, it holds:*

(i) $(\bar{\bar{\mathbf{z}}}) = z_{\mathbf{0}}$;
(ii) $[\mathbf{z}\overline{\mathbf{z}}]_0 = [\overline{\mathbf{z}}\mathbf{z}]_0 = |z_{\mathbf{0}}|^2$,

where we recall, $z_{\mathbf{0}}$ *denotes the scalar part of* $\mathbf{z}$.

Proof. The first proposition is obvious since the action of the pseudoconjugation on the basis elements is given by

$$\overline{\mathbf{e}^{\boldsymbol{\nu}}} = \mathbf{e}_d^{2\nu_d} \ldots \mathbf{e}_2^{2\nu_2}\mathbf{e}_1^{2\nu_1}, \qquad \text{whenever} \quad \boldsymbol{\nu} \in \{0,1\}^d,$$

and zero otherwise. Furthermore, as $\mathbf{e}_j^2\mathbf{e}_i^2 = \omega^2\mathbf{e}_i^2\mathbf{e}_j^2$, $i<j$, we obtain

$$\overline{\mathbf{e}^{\boldsymbol{\nu}}} = \sigma(\boldsymbol{\nu},\boldsymbol{\nu})\mathbf{e}^{2\boldsymbol{\nu}} := \omega^{2\left(\sum_{j=1}^{d-1}\sum_{s=j+1}^{d}\nu_j\nu_s\right)}\mathbf{e}^{2\boldsymbol{\nu}},\ \boldsymbol{\nu} \in \{0,1\}^d.$$

Hence,

$$(\bar{\bar{\mathbf{z}}}) = \overline{\sum_{\boldsymbol{\nu}\in\{0,1\}^d} \bar{z}_{\boldsymbol{\nu}}\mathbf{e}_d^{2\nu_d} \ldots \mathbf{e}_2^{2\nu_2}\mathbf{e}_1^{2\nu_1}} = \sum_{\boldsymbol{\nu}\in\{0,1\}^d} z_{\boldsymbol{\nu}}\overline{\mathbf{e}_1^{2\nu_1}}\,\overline{\mathbf{e}_2^{2\nu_2}} \ldots \overline{\mathbf{e}_d^{2\nu_d}} = \sum_{\boldsymbol{\nu}\in\{0,1\}^d} z_{\boldsymbol{\nu}}\mathbf{e}_1^{4\nu_1}\mathbf{e}_2^{4\nu_2} \ldots \mathbf{e}_d^{4\nu_d} = z_{\mathbf{0}}.$$

For the second proposition, we have

$$\begin{aligned}[\mathbf{z}\overline{\mathbf{z}}]_0 &= \left[\left(\sum_{\boldsymbol{\nu}\in\mathcal{I}_d} z_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\nu}}\right)\left(\overline{\sum_{\boldsymbol{\mu}\in\mathcal{I}_d} z_{\boldsymbol{\mu}}\mathbf{e}^{\boldsymbol{\mu}}}\right)\right]_0 \\ &= \left[\sum_{\substack{\boldsymbol{\nu}\in\mathcal{I}_d\\ \boldsymbol{\mu}\in\{0,1\}^d}} z_{\boldsymbol{\nu}}\bar{z}_{\boldsymbol{\mu}}\mathbf{e}_1^{\nu_1}\mathbf{e}_2^{\nu_2} \ldots \mathbf{e}_d^{\nu_d}\mathbf{e}_d^{2\mu_d} \ldots \mathbf{e}_2^{2\mu_2}\mathbf{e}_1^{2\mu_1}\right]_0 = z_{\mathbf{0}}\bar{z}_{\mathbf{0}} = |z_{\mathbf{0}}|^2.\end{aligned}$$

The same holds for $[\overline{\mathbf{z}}\mathbf{z}]_0$, which completes our proof. □

3 | COMPLETIONS OF GRASSMANN ALGEBRAS

We now consider the ternary Grassmann algebra generated by taking the formal limit $d \to \infty$ of $\mathcal{G}_{3,d}$. We denote the corresponding algebra by $\mathcal{G}_3$, associated to the countable set $\mathfrak{G} = \{\mathbf{e}_j, j \in \mathbb{N}\}$. Similar to the case of the infinite dimensional Grassmann algebra Λ_∞, the resulting ternary Grassmann algebra $\mathcal{G}_3$ is an associative but not commutative algebra over $\mathbb{C}$.

We denote its elements $\mathbf{z} = \sum_{\boldsymbol{\nu}\in\mathcal{I}} z_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\nu}} \in \mathcal{G}_3$ as *ternary supernumbers* where $\mathcal{I} = \{0,1,2\}^{\mathbb{N}}$ denotes the set of indexes, and we endow the ternary Grassmann algebra $\mathcal{G}_3$ with a p-norm. Remark that since the pseudoconjugation is not an isomorphism, it does not induce a norm. Hence, we will use the ℓ^p-norm where $\mathbf{z} = \sum_{\boldsymbol{\nu}\in\mathcal{I}} z_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\nu}}$ is to be identified with $(z_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{I}} \in \ell^p(\mathbb{C})$.

The conjugation and product in $\mathcal{G}_3$ (an infinite dimensional algebra) are well defined and provide that $\boldsymbol{\nu}$ satisfy $\#\boldsymbol{\nu} < \infty$, where $\#\boldsymbol{\nu}$ denotes the number of nonzero entries in the sequence $\boldsymbol{\nu}$ (equal to the number of $\mathbf{e}_j'$s present in the basis element $\mathbf{e}^{\boldsymbol{\nu}}$). For example, for $\boldsymbol{\nu} = (1,0,2,1,0,0,\ldots)$, we get $\#\boldsymbol{\nu} = 3$ corresponding to $\mathbf{e}^{\boldsymbol{\nu}} = \mathbf{e}_1\mathbf{e}_3^2\mathbf{e}_4$.

Hence, we have for the conjugation

$$\mathbf{w} = \sum_{\boldsymbol{\mu} \in \mathcal{I}} w_{\boldsymbol{\mu}} \mathbf{e}^{\boldsymbol{\mu}} \mapsto \overline{\mathbf{w}} = \sum_{\boldsymbol{\mu} \in \{0,1\}^{\mathbb{N}}} \sigma(\boldsymbol{\mu}, \boldsymbol{\mu}) \overline{w}_{\boldsymbol{\mu}} \mathbf{e}^{2\boldsymbol{\mu}}, \tag{3.1}$$

and for the product between $\mathbf{z} = \sum_{\boldsymbol{\nu} \in \mathcal{I}} z_{\boldsymbol{\nu}} \mathbf{e}^{\boldsymbol{\nu}}, \mathbf{w} = \sum_{\boldsymbol{\mu} \in \mathcal{I}} w_{\boldsymbol{\mu}} \mathbf{e}^{\boldsymbol{\mu}} \in \mathcal{G}_3$, we get

$$\mathbf{zw} = \sum_{\boldsymbol{\nu}, \boldsymbol{\mu} \in \mathcal{I}} \sigma(\boldsymbol{\nu}, \boldsymbol{\mu}) z_{\boldsymbol{\nu}} w_{\boldsymbol{\mu}} \mathbf{e}^{\boldsymbol{\nu}+\boldsymbol{\mu}}, \tag{3.2}$$

under the restriction $\#\boldsymbol{\nu}, \#\boldsymbol{\mu}$ are finite, and where $\sigma(\boldsymbol{\nu}, \boldsymbol{\mu})$ is defined as in (2.15). We stress that only under particular conditions we have $\sigma(\boldsymbol{\nu}, \boldsymbol{\mu}) = \overline{\sigma(\boldsymbol{\mu}, \boldsymbol{\nu})}$.

Henceforth, we assume all sequences to have a finite number of nonzero entries.

Definition 3.1. Let $p \in \mathbb{N}$. We define the p-norm of $\mathbf{z} \in \mathcal{G}_3$ is defined as

$$\|\mathbf{z}\|_p = \left(\sum_{\boldsymbol{\nu} \in \mathcal{I}} |z_{\boldsymbol{\nu}}|^p \right)^{1/p}, \tag{3.3}$$

where $|\cdot|$ is the usual modulus of a complex number.

We remark that Definition 3.1 holds also for any real $p \geq 1$.

Restricted to the finite-dimensional subalgebra $\mathcal{G}_{3,d}$, the p-norm satisfy the following properties:

Theorem 3.2. *For all* $\mathbf{z}, \mathbf{w} \in \mathcal{G}_{3,d}$ *it holds*

(i)

$$\|\mathbf{zw}\|_1 \leq \|\mathbf{z}\|_1 \|\mathbf{w}\|_1, \tag{3.4}$$

(ii) and for $p = 2, 3, \ldots$

$$\|\mathbf{zw}\|_p^p \leq \|\mathbf{z}\|_1^p \|\mathbf{w}\|_{2^{p-1}} \prod_{k=1}^{p-1} \|\mathbf{w}\|_{2^k}, \ \|\mathbf{zw}\|_p^p \leq \|\mathbf{w}\|_1^p \|\mathbf{z}\|_{2^{p-1}} \prod_{k=1}^{p-1} \|\mathbf{z}\|_{2^k}. \tag{3.5}$$

The proof of (3.4) is straightforward. Moreover, the proof of (3.5) follows in the same manner as the one presented in Alpay et al,[11] where the Cauchy–Schwarz inequality is used repeated times and having in mind that $|\omega| = 1$.

In order to study analytic properties of stochastic processes taking values in this algebra, one needs to consider its completion with respect to the ℓ^2-norm. In the next section, we study the completion of $\mathcal{G}_3$ with respect to the p-norm, which we denote by $\overline{\mathcal{G}}_3^{(p)}$. This closure is widely studied in the literature in the classical case of Grassmann algebras (see, e.g., previous studies[9,10,12]). Also, remark that by (3.4), we have that $\overline{\mathcal{G}}_3^{(1)}$ has a Banach algebra structure. The study of completion of the p-norm has the purpose of establish a ternary Fock space based on the ℓ^2-inner product between two supernumbers $\mathbf{z}, \mathbf{w} \in \mathcal{G}_3$ given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \sum_{\boldsymbol{\nu} \in \mathcal{I}} z_{\boldsymbol{\nu}} \overline{w}_{\boldsymbol{\nu}}. \tag{3.6}$$

3.1 | Topological algebra associated with $\mathcal{G}_3$

In order to establish an analysis and stochastic process theory in the framework of ternary Grassmann algebras, we need to establish an equivalent to the classic Gel'fand triple $(\mathcal{S}, \mathbf{L}^2(\mathbb{R}, dx), \mathcal{S}')$ where $\mathcal{S}$ is the space of test functions and $\mathcal{S}'$ denotes its dual (see, for instance, Schwartz[15]). The commutative setting was first introduced by Kondratiev and adapted to the framework of Hida's white noise space theory and commutative Fock space (see Holden et al[16] and Kuo[17]). Moreover, the commutative case is associated with bosons and applied to study solutions of stochastic differential equations and to model stochastic processes and their derivatives. The construction of the noncommutative counterpart of this theory in the classic setting (see, e.g., Alpay et al.[18,19]) was motivated by fermionic formulation. In our case of topological hypersymmetry, we need to establish the corresponding noncommutative counterpart in terms of our algebra and to construct a Gel'fand triple in this ternary Grassmann setting.

Gel'fand triples allow to define other products (on itself not necessarily laws of composition) different from the usual inner product in the Hilbert space. One such example is the Wick product in the white noise space which is not a law of composition. By embedding the white noise space into an analogous space of stochastic distributions, the Wick product becomes a law of composition by strict inclusion. Another important reason for such a construction of the space of stochastic distributions is the fact that such spaces are necessary for a future study of their derivatives (see Alpay et al.[18]).

Hence, we now recall a few facts from the classical case and from the theory of perfect spaces and strong algebras. For more details on these spaces, we refer the reader to Gelfand et al.[20,21]

Starting from a decreasing family of Hilbert spaces $(\mathcal{H}_p, \|\cdot\|_{\mathcal{H}_p})_{p\in\mathbb{Z}}$, with increasing norms,

$$\dots \subseteq \mathcal{H}_2 \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_0 \subseteq \mathcal{H}_{-1} \subseteq \mathcal{H}_{-2} \dots ,$$

it is known that the intersection $\mathcal{F} = \cap_{p=0}^{\infty}\mathcal{H}_p$ is a Fréchet space. If, furthermore, $\mathcal{F}$ is perfect, then compactness is equivalent to being compact and bounded. In particular, this is ensured when for each p, there exists $q > p$ such that the injection map from $\mathcal{H}_q$ into $\mathcal{H}_p$ is compact. In the usual way, we identify $\mathcal{H}_p'$ with $\mathcal{H}_{-p}$. Then, $\mathcal{F}$ together with its dual $\mathcal{F}' := \cup_{p=0}^{\infty}\mathcal{H}_{-p}$ and $\mathcal{H}_0$ forms a Gel'fand triple $(\mathcal{F}, \Gamma(\mathcal{H}_0), \mathcal{F}')$. Indeed, the dual $\mathcal{F}'$ endowed with the strong topology defined in terms of the bounded sets of $\mathcal{F}$ is then locally convex. Furthermore, the strong topology coincides with the inductive limit topology. See Alpay and Salomon[22], section 3 for a discussion. Thus, the space of distributions $\mathcal{F}'$ is the dual of a Fréchet nuclear space.

We recall here two statements about compactness and convergence of sequences in $\mathcal{F}'$.

Proposition 3.3 (Gelfand and Shilov[20]). *A set is (weakly or strongly) compact in $\mathcal{F}'$ if and only if it is compact in one of the spaces $\mathcal{H}_{-p}$ in the corresponding norm.*

Proposition 3.4 (Gelfand and Shilov[20]). *Assume $\mathcal{F}'$ is perfect. Then, weak and strong convergence of sequences are equivalent, and a sequence converges (weakly or strongly) if and only if it converges in one of the spaces $\mathcal{H}_{-p}$ in the corresponding norm.*

Now, we are going to show that $\mathcal{F}$ can be made a topological algebra denoted $\mathfrak{S}_1$ where the product satisfies the so-called Våge inequality. This ensures that we can consider $\mathcal{F}'$ as an inductive limit of Hilbert spaces. This is used in the proof of Theorem 4.3.

A topological algebra is assumed to be separately continuous in each variable. It is not immediate but true that a strong algebra is jointly continuous in the two variables (see Bourbaki[23, IV.26, theorem 2] and also the discussion in Alpay and Salomon[24, pp. 215–216]).

In our case, we define

$$\mathcal{H}_p(\mathbf{c}) = \left\{ f = \sum_{\boldsymbol{\nu}\in\mathcal{I}} f_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\nu}} \in \overline{\mathcal{G}}_3^{(2)} \,\middle|\, \sum_{\boldsymbol{\nu}\in\mathcal{I}} |f_{\boldsymbol{\nu}}|^2 c_{\boldsymbol{\nu}}^{2p} < \infty \right\}, \tag{3.7}$$

with $p \in \mathbb{Z}$. The coefficients give rise a sequence $\mathbf{c} = (c_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{I}}$ of positive real numbers such that

$$c_{\boldsymbol{\nu}}c_{\boldsymbol{\mu}} \le c_{\boldsymbol{\gamma}}, \text{ for all } \quad \boldsymbol{\nu}, \boldsymbol{\mu} \in \mathcal{I} \text{ such that } \quad \boldsymbol{\nu} + \boldsymbol{\mu} = \boldsymbol{\gamma} \in \mathcal{I}, \tag{3.8}$$

and where

$$\sum_{\boldsymbol{\nu}\in\mathcal{I}} c_{\boldsymbol{\nu}}^{-2d} < \infty, \text{ for } \quad d = 1, 2, 3, \dots . \tag{3.9}$$

By construction, we have

$$\mathcal{H}_{-q}(\mathbf{c}) \subseteq \mathcal{H}_{-p}(\mathbf{c}),$$

if $p \geq q$.

From here on, we abbreviate $\mathcal{H}_{-p}(\mathbf{c})$ by $\mathcal{H}_{-p}$.

Definition 3.5. The norm $\|\cdot\|_{\mathcal{H}_{-p}}$ in $\mathcal{H}_{-p}$ is defined as

$$\|f\|_{\mathcal{H}_{-p}} := \sum_{\boldsymbol{\nu}\in\mathcal{I}} |f_{\boldsymbol{\nu}}|^2 c_{\boldsymbol{\nu}}^{-2p}.$$

Proposition 3.6. *If $c_{\boldsymbol{\nu}}c_{\boldsymbol{\mu}} = c_{\boldsymbol{\nu}+\boldsymbol{\mu}}$, then $c_{\mathbf{0}} = 1$.*

Proof. If so, then

$$c_{\mathbf{0}} c_{\boldsymbol{\mu}} = c_{\mathbf{0}+\boldsymbol{\mu}} = c_{\boldsymbol{\mu}}$$

so that $c_{\mathbf{0}} = 1$. □

Proposition 3.7. *Let* $\mathbf{c} = (c_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{I}}$ *be such that* $c_{\mathbf{0}} = 1$ *and* $c_{\boldsymbol{\nu}} > 1$*, for all* $\boldsymbol{\nu} \neq \mathbf{0}$*. Then,*

$$\lim_{p\to\infty} \|f\|_{\mathcal{H}_{-p}} = |f_{\mathbf{0}}|^2, \text{ for all } \quad f \in \mathcal{H}_{-p}.$$

Proof. Since $\lim_{p\to\infty} c_{\boldsymbol{\nu}}^{-2p} = 0$ for every $\boldsymbol{\nu} \neq \mathbf{0}$,

$$\lim_{p\to\infty} \|f\|_{\mathcal{H}_{-p}} = \lim_{p\to\infty} \sum_{\boldsymbol{\nu}\in\mathcal{I}} |f_{\boldsymbol{\nu}}|^2 c_{\boldsymbol{\nu}}^{-2p} = \sum_{\boldsymbol{\nu}\in\mathcal{I}} |f_{\boldsymbol{\nu}}|^2 \left(\lim_{p\to\infty} c_{\boldsymbol{\nu}}^{-2p}\right) = |f_{\mathbf{0}}|^2.$$

□

Definition 3.8. We consider the space

$$\mathfrak{S}_1 = \cap_{p\geq 0} \mathcal{H}_p \tag{3.10}$$

and its topological dual

$$\mathfrak{S}_{-1} = \cup_{p\geq 0} \mathcal{H}_{-p}, \tag{3.11}$$

which can be considered as analogs of the spaces $\mathcal{S}$ and $\mathcal{S}'$, respectively, in our setting.

The next theorem introduces a Våge-like inequality[25] which permits the analysis of stochastic processes to be done locally in a Hilbert space.

Theorem 3.9. *If* $f \in \mathcal{H}_{-q}$ *and* $g \in \mathcal{H}_{-p}$*, with* $p > q$*, then*

$$\|fg\|_{\mathcal{H}_{-p}} \leq C_{p-q} \|f\|_{\mathcal{H}_{-q}} \|g\|_{\mathcal{H}_{-p}}, \ \|gf\|_{\mathcal{H}_{-p}} \leq C_{p-q} \|f\|_{\mathcal{H}_{-q}} \|g\|_{\mathcal{H}_{-p}}, \tag{3.12}$$

with $C_{p-q} > 0$ *being a constant.*

Proof. Suppose $f \in \mathcal{H}_{-q}$ and $g \in \mathcal{H}_{-p}$. Applying Cauchy–Schwarz inequality, we get

$$\begin{aligned}
\|fg\|_{\mathcal{H}_{-p}}^2 &= \sum_{\boldsymbol{\gamma}\in\mathcal{I}} |(fg)_{\boldsymbol{\gamma}}|^2 c_{\boldsymbol{\gamma}}^{-2p} = \sum_{\boldsymbol{\gamma}\in\mathcal{I}} \left| \sum_{\substack{\boldsymbol{\nu},\boldsymbol{\mu}\in\mathcal{I} \\ \boldsymbol{\nu}+\boldsymbol{\mu}=\boldsymbol{\gamma}}} \sigma(\boldsymbol{\nu},\boldsymbol{\mu}) f_{\boldsymbol{\nu}} g_{\boldsymbol{\mu}} \right|^2 c_{\boldsymbol{\gamma}}^{-2p} \leq \sum_{\boldsymbol{\gamma}\in\mathcal{I}} \left(\sum_{\substack{\boldsymbol{\nu},\boldsymbol{\mu},\boldsymbol{\nu}',\boldsymbol{\mu}'\in\mathcal{I} \\ \boldsymbol{\nu}+\boldsymbol{\mu}=\boldsymbol{\nu}'+\boldsymbol{\mu}'=\boldsymbol{\gamma}}} |f_{\boldsymbol{\nu}}||g_{\boldsymbol{\mu}}||f_{\boldsymbol{\nu}'}||g_{\boldsymbol{\mu}'}| \right) c_{\boldsymbol{\gamma}}^{-2p} \\
&\leq \sum_{\boldsymbol{\gamma}\in\mathcal{I}} \left(\sum_{\substack{\boldsymbol{\nu},\boldsymbol{\mu},\boldsymbol{\nu}',\boldsymbol{\mu}'\in\mathcal{I} \\ \boldsymbol{\nu}+\boldsymbol{\mu}=\boldsymbol{\nu}'+\boldsymbol{\mu}'=\boldsymbol{\gamma}}} |f_{\boldsymbol{\nu}}| c_{\boldsymbol{\nu}}^{-p} |g_{\boldsymbol{\mu}}| c_{\boldsymbol{\mu}}^{-p} |f_{\boldsymbol{\nu}'}| c_{\boldsymbol{\nu}'}^{-p} |g_{\boldsymbol{\mu}'}| c_{\boldsymbol{\mu}'}^{-p} \right) \leq \sum_{\boldsymbol{\nu},\boldsymbol{\nu}'\in\mathcal{I}} |f_{\boldsymbol{\nu}}| c_{\boldsymbol{\nu}}^{-p} |f_{\boldsymbol{\nu}'}| c_{\boldsymbol{\nu}'}^{-p} \left(\sum_{\substack{\boldsymbol{\gamma}\in\mathcal{I}:\exists\boldsymbol{\mu},\boldsymbol{\mu}'\in\mathcal{I} \\ \boldsymbol{\nu}+\boldsymbol{\mu}=\boldsymbol{\nu}'+\boldsymbol{\mu}'=\boldsymbol{\gamma}}} |g_{\boldsymbol{\mu}}| c_{\boldsymbol{\mu}}^{-p} |g_{\boldsymbol{\mu}'}| c_{\boldsymbol{\mu}'}^{-p} \right) \\
&\leq \sum_{\boldsymbol{\nu},\boldsymbol{\nu}'\in\mathcal{I}} |f_{\boldsymbol{\nu}}| c_{\boldsymbol{\nu}}^{-p} |f_{\boldsymbol{\nu}'}| c_{\boldsymbol{\nu}'}^{-p} \left(\sum_{\substack{\boldsymbol{\gamma}\in\mathcal{I}:\exists\boldsymbol{\mu}\in\mathcal{I} \\ \boldsymbol{\nu}+\boldsymbol{\mu}=\boldsymbol{\gamma}}} |g_{\boldsymbol{\mu}}|^2 c_{\boldsymbol{\mu}}^{-2p} \right)^{\frac{1}{2}} \left(\sum_{\substack{\boldsymbol{\gamma}\in\mathcal{I}:\exists\boldsymbol{\mu}'\in\mathcal{I} \\ \boldsymbol{\nu}'+\boldsymbol{\mu}'=\boldsymbol{\gamma}}} |g_{\boldsymbol{\mu}'}|^2 c_{\boldsymbol{\mu}'}^{-2p} \right)^{\frac{1}{2}} \\
&\leq \sum_{\boldsymbol{\nu},\boldsymbol{\nu}'\in\mathcal{I}} |f_{\boldsymbol{\nu}}| c_{\boldsymbol{\nu}}^{-p} |f_{\boldsymbol{\nu}'}| c_{\boldsymbol{\nu}'}^{-p} \left(\sum_{\boldsymbol{\mu}\in\mathcal{I}} |g_{\boldsymbol{\mu}}|^2 c_{\boldsymbol{\mu}}^{-2p} \right)^{\frac{1}{2}} \left(\sum_{\boldsymbol{\mu}'\in\mathcal{I}} |g_{\boldsymbol{\mu}'}|^2 c_{\boldsymbol{\mu}'}^{-2p} \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{\boldsymbol{\nu}\in\mathcal{I}} |f_{\boldsymbol{\nu}}| c_{\boldsymbol{\nu}}^{-p} \right)^2 \|g\|_{\mathcal{H}_{-p}}^2 \leq \left(\sum_{\boldsymbol{\nu}\in\mathcal{I}} |f_{\boldsymbol{\nu}}| c_{\boldsymbol{\nu}}^{-q} c_{\boldsymbol{\nu}}^{-(p-q)} \right)^2 \|g\|_{\mathcal{H}_{-p}}^2 \leq \left(\sum_{\boldsymbol{\nu}\in\mathcal{I}} c_{\boldsymbol{\nu}}^{-2(p-q)} \right) \|f\|_{\mathcal{H}_{-q}}^2 \|g\|_{\mathcal{H}_{-p}}^2.
\end{aligned}$$

It remains to prove that there exists a sequence $\mathbf{c} = (c_{\boldsymbol{\nu}})$ such that $\sum_{\boldsymbol{\nu}\in\mathcal{I}} c_{\boldsymbol{\nu}}^{-2d} < \infty$ for all $d = 1, 2, \dots$. We assume this sequence to be given by

$$c_{\boldsymbol{\nu}} = e^{\sum_k \varphi(3^{k-1}\nu_k)} = e^{\varphi(\nu_1)+\varphi(3\nu_2)+\varphi(3^2\nu_3)+\dots},$$

where $\nu_k \in \{0, 1, 2\}$. We now look into the properties of such a function φ.

Bering in mind that $c_{\mathbf{0}} = 1$ and $c_{\boldsymbol{\nu}} c_{\boldsymbol{\mu}} = c_{\boldsymbol{\nu}+\boldsymbol{\mu}}$ for all $\boldsymbol{\nu}, \boldsymbol{\mu} \in \mathcal{I}$ such that $\boldsymbol{\nu} + \boldsymbol{\mu} \in \mathcal{I}$, we obtain

(i) $c_{\mathbf{0}} = e^{\sum_k \varphi(0)}$, leading to $\varphi(0) = 0$;
(ii) since $c_{\boldsymbol{\nu}} c_{\boldsymbol{\mu}} = c_{\boldsymbol{\nu}+\boldsymbol{\mu}}$ if $\boldsymbol{\nu} + \boldsymbol{\mu} \in \mathcal{I}$, we have that for a given position $k \in \mathbb{N}$, $\nu_k + \mu_k \neq 3, 4$;
(iii) $c_{\boldsymbol{\nu}} c_{\boldsymbol{\mu}} = c_{\boldsymbol{\nu}+\boldsymbol{\mu}}$ implies

$$e^{\sum_k \varphi(3^{k-1}\nu_k)} e^{\sum_j \varphi(3^{j-1}\mu_j)} = e^{\sum_k \varphi\left(3^{k-1}(\nu_k+\mu_k)\right)};$$

(iv) furthermore, for integers $k > j$ and $\nu_k, \mu_j \in \{1, 2\}$, we have $3^{k-1}\nu_k > 3^{j-1}\mu_j$ so that φ should be an increasing function, satisfying to $\varphi(a) + \varphi(b) = \varphi(a + b)$, for $a, b > 0$.

For example, consider $\varphi(x) = x$, $x > 0$. Hence,

$$\sum_{\boldsymbol{\nu}\in\mathcal{I}} c_{\boldsymbol{\nu}}^{-2d} = 1 + \sum_{\substack{\boldsymbol{\nu}\in\mathcal{I} \\ |\boldsymbol{\nu}\neq\mathbf{0}}} c_{\boldsymbol{\nu}}^{-2d} = 1 + \sum_{\substack{\boldsymbol{\nu}\in\mathcal{I} \\ |\boldsymbol{\nu}|\neq\mathbf{0}}} e^{-2d\sum\limits_k 3^{k-1}\nu_k}$$

Now, we split this sum in terms of the number $\#\boldsymbol{\nu}$ of nonzero entries in the sequence $\boldsymbol{\nu}$. Then,

$$\sum_{\boldsymbol{\nu}\in\mathcal{I}} c_{\boldsymbol{\nu}}^{-2d} = 1 + \sum_{m=1}^{\infty} \sum_{\substack{\boldsymbol{\nu}\in\mathcal{I} \\ \#\boldsymbol{\nu}=m}} e^{-2d\sum\limits_k 3^{k-1}\nu_k}.$$

We observe that for $m = 1$, we have

$$\begin{aligned}\sum_{\substack{\boldsymbol{\nu}\in\mathcal{I} \\ \#\boldsymbol{\nu}=1}} e^{-2d\sum_k 3^{k-1}\nu_k} &= \sum_{k=1}^{\infty} e^{-2d3^{k-1}\nu_k} \leq \sum_{k=1}^{\infty} e^{-2d3^{k-1}} \qquad (\text{recall:} \quad \nu_k = 1, 2) \\ &\leq \sum_{k=1}^{\infty} e^{-2dk} = e^{-2d}\frac{1}{1 - e^{-2d}} < \infty.\end{aligned}$$

Furthermore, remark that $0 < e^{-2d} < 1$, $d = 1, 2, \dots$ so that $\frac{1}{1-e^{-2d}} > 1$. Hence, we have

$$\begin{aligned} e^{-2d}\frac{1}{1 - e^{-2d}} < 1 &\Longleftrightarrow \quad 2e^{-2d} < 1 \\ &\Longleftrightarrow \quad -2d < -\ln 2 \\ &\Longleftrightarrow \quad d > \ln\sqrt{2}.\end{aligned}$$

Now, for $\#\boldsymbol{\nu} = m > 1$, we obtain

$$\begin{aligned}\sum_{\substack{\boldsymbol{\nu}\in\mathcal{I} \\ \#\boldsymbol{\nu}=m}} e^{-2d\sum\limits_k 3^{k-1}\nu_k} &= \sum_{\substack{\boldsymbol{\nu}\in\mathcal{I} \\ \#\boldsymbol{\nu}=m}} e^{-2d\nu_1} e^{-2d3\nu_2} \dots e^{-2d3^{m-1}\nu_m} \\ &\leq \left(\sum_{k=1}^{\infty} e^{-2dk}\right)^m = \left(\frac{e^{-2d}}{1 - e^{-2d}}\right)^m, \; m = 1, 2, 3, \dots\end{aligned}$$

so that

$$\sum_{\boldsymbol{\nu}\in\mathcal{I}} c_{\boldsymbol{\nu}}^{-2d} = 1 + \sum_{m=1}^{\infty} \sum_{\substack{\boldsymbol{\nu}\in\mathcal{I} \\ \#\boldsymbol{\nu}=m}} e^{-2d\sum_k 3^{k-1}\nu_k}$$
$$\leq 1 + \sum_{m=1}^{\infty}\left(\frac{e^{-2d}}{1-e^{-2d}}\right)^m$$
$$= \frac{1-e^{-2d}}{1-2e^{-2d}}.$$

Finally, we remark that although none of the Banach algebras $\mathcal{H}_{-p}$ is commutative, the second inequality in (3.12) holds with the same value of constant C_{p-q}. Indeed, due to the multiplication rules (2.14) and (2.15), we have that $f_{\boldsymbol{\nu}} g_{\boldsymbol{\mu}} \sigma(\boldsymbol{\nu},\boldsymbol{\mu})\mathbf{e}^{\boldsymbol{\nu}+\boldsymbol{\mu}} = g_{\boldsymbol{\mu}} f_{\boldsymbol{\nu}} \sigma(\boldsymbol{\mu},\boldsymbol{\nu})\mathbf{e}^{\boldsymbol{\mu}+\boldsymbol{\nu}}$, so that $\|fg\|_{\mathcal{H}_{-p}} = \|gf\|_{\mathcal{H}_{-p}}$. □

Proposition 3.10. *The space $\mathfrak{S}_{-1}$ equipped with the product induced by the coefficients is a strong algebra.*

Proof. Let us start by endowing $\mathfrak{S}_{-1}$ with the inductive topology. From Theorem 3.9, we get that the product of the algebra is separately continuous in every space $\mathcal{H}_{-p}$ which is the same as continuity in the inductive topology. Additionally, the product in $\mathfrak{S}_{-1}$ inherits associativity from our ternary Grassmann algebra $\mathcal{G}_3$. Therefore, $\mathfrak{S}_{-1}$ has a Banach algebra structure, and thus, we can consider it as the inductive limit of Banach spaces, which makes it a strong algebra. □

For more details on this proof, see Alpay and Salomon.[24] This also shows that the inductive topology is equivalent to the strong topology in $\mathfrak{S}_{-1}$. Furthermore, the product will also be associative in $\mathfrak{S}_{-1}$, and the multiplication is jointly continuous (Alpay and Salomon[24, p. 215, case (iv)] and also Bourbaki[23, IV.23, proposition 4]).

Then, by Alpay and Salomon,[22, theorem 3.7] we have $\mathfrak{S}_{-1}$ being nuclear and the dual of a perfect space.

Corollary 3.11. *Suppose $n \in \mathbb{N}$ and $f \in \mathcal{H}_{-p} \subseteq \mathcal{H}_{-p-2}$. Then,*

$$\|f^n\|_{\mathcal{H}_{-p-2}} \leq C_2^{n-1}\,\|f\|_{\mathcal{H}_{-p}}^n,$$

where $C_2 > 0$ is as in Theorem 3.9.

Proof. We have for every $f \in \mathcal{H}_{-p} \subseteq \mathcal{H}_{-p-2}$ that $\|f\|_{\mathcal{H}_{-p-2}} \leq \|f\|_{\mathcal{H}_{-p}}$. By Theorem 3.9,

$$\|f^n\|_{\mathcal{H}_{-p-2}} \leq C_2\|f\|_{\mathcal{H}_{-p}}\left\|f^{n-1}\right\|_{\mathcal{H}_{-p-2}}$$
$$\leq C_2^2\,\|f\|_{\mathcal{H}_{-p}}^2\,\left\|f^{n-2}\right\|_{\mathcal{H}_{-p-2}}$$
$$\leq C_2^{n-1}\,\|f\|_{\mathcal{H}_{-p}}^n.$$

□

Corollary 3.12. *Consider a power series*

$$F(\lambda) = \sum_{n\in\mathbb{N}_0} \alpha_n\lambda^n \tag{3.13}$$

absolutely convergent in the open disk with radius R, with $\alpha_n, \lambda \in \mathbb{C}$ and $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$.

For $f \in \mathcal{H}_{-p}$, we have that if

$$\|f\|_{\mathcal{H}_{-p}} < \frac{R}{C_2}, \tag{3.14}$$

then F(f) converges in $\mathcal{H}_{-p-2}$.

Proof. By assumption, if $|\lambda|<R$, power series (3.13) converges absolutely, that is,

$$\sum_{n\in\mathbb{N}_0} |\alpha_n\lambda^n| = \sum_{n\in\mathbb{N}_0} |\alpha_n|\,|\lambda^n| < \infty.$$

Applying Corollary 3.11, we obtain the absolute convergence of $F(f)$ in the space $\mathcal{H}_{-p-2}$ via

$$\begin{aligned}\sum_{n\in\mathbb{N}_0}\|\alpha_n f^n\|_{\mathcal{H}_{-p-2}} &= \sum_{n\in\mathbb{N}_0}|\alpha_n|^2\|f^n\|_{\mathcal{H}_{-p-2}}\\ &\leq \alpha_0 + C_2^{-1}\sum_{n\in\mathbb{N}}|\alpha_n|^2\left(C_2\|f\|_{\mathcal{H}_{-p}}\right)^n.\end{aligned}$$

Thus, $F(f)$ converges absolutely in $\mathcal{H}_{-p-2}$ if

$$C_2\|f\|_{\mathcal{H}_{-p}} < R$$

or, $\|f\|_{\mathcal{H}_{-p}} < \frac{R}{C_2}$. □

Corollary 3.13. *Suppose $F(\lambda)$ is power series like in the previous corollary (Corollary 3.12). Then, for all $f \in \mathfrak{S}_{-1}$ such that its scalar part $f_{\mathbf{0}}$ satisfies (3.14), we have that $F(f)$ converges in $\mathfrak{S}_{-1}$.*

Proof. Suppose $f \in \mathfrak{S}_{-1}$, then we know that there is $q_0 \in \mathbb{Z}$ with $f \in \mathcal{H}_{-q}$ for each $q \geq q_0$. Using Theorem 3.12, we have that to ensure convergence of $F(f)$, we need $\|f\|_{\mathcal{H}_{-q}} < R/C_2$, which in general is not valid. But, due to Proposition 3.7, this condition becomes

$$|f_{\mathbf{0}}|^2 < \frac{R}{C_2},$$

which gives us the statement of the corollary. □

Corollary 3.14. *Suppose $f \in \mathfrak{S}_{-1}$. Then, we have that f is invertible if and only if we have for its scalar part $f_{\mathbf{0}} \neq 0$.*

Proof. Let us assume that g is the inverse of f and its scalar part is denoted by $g_{\mathbf{0}}$, then we have $fg = 1$ implies $f_{\mathbf{0}}g_{\mathbf{0}} = 1$ and $f_{\mathbf{0}} \neq 0$.

To show the opposite direction, we suppose $f_{\mathbf{0}} \neq 0$, or with a convenient normalization $f_{\mathbf{0}} = 1$. From Corollary 3.13, we get that

$$F(f) = \sum_{n\in\mathbb{N}_0}(1-f)^n$$

converges when the scalar part of $1-f$ is less than C_2^{-1}. But $(1-f)_B = 0$, and we get that $g = F(f) \in \mathfrak{S}_{-1}$ and g is the inverse of f. □

3.2 | Berezin integration

We now look into a proper definition of path integration in the sense of Berezin. Berezin integrals are used in superspace theory as linear maps from polynomials in anti-commuting variables to elements of Grassmann algebras. As we aim to establish stochastic processes on generalized Grassmann algebras, it is necessary to construct a proper path integration on the arising infinite dimensional spaces of anti-commuting random variables. Such a path-integration theory was developed in Rogers[26] and DeWitt,[27] as a Fermionic counterpart of the quantum Bosonic case.

In what follows, we assume $f : \Omega \to \overline{\mathcal{G}}_3^{(2)}$, where $\Omega = \mathbb{R}$ or $\mathbb{C}$. We begin with the definition of the left multiplication operator acting on functions with values in $\overline{\mathcal{G}}_3^{(2)}$. For each $f \in \mathfrak{S}_{-1}$, we define the left multiplication operator M_f as

$$g \in \mathfrak{S}_{-1} \mapsto M_f g := fg \in \mathfrak{S}_{-1}. \tag{3.15}$$

Recall that since the inductive algebra $\mathfrak{S}_{-1}$ is a strong algebra, we have that the multiplication is jointly continuous. Using the basis elements of $\mathcal{G}_3$, we obtain

$$M_f g = \sum_{\boldsymbol{\nu}+\boldsymbol{\mu}\in\mathcal{I}} f_{\boldsymbol{\nu}} g_{\boldsymbol{\mu}}\sigma(\boldsymbol{\nu},\boldsymbol{\mu})\mathbf{e}^{\boldsymbol{\nu}+\boldsymbol{\mu}} := \sum_{\boldsymbol{\nu}+\boldsymbol{\mu}\in\mathcal{I}} f_{\boldsymbol{\nu}} g_{\boldsymbol{\mu}} M_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\mu}}.$$

Thus, the left multiplication operator M is defined by its action on the basis elements $\mathbf{e}^{\boldsymbol{\nu}}$ of $\mathcal{G}_3$,

$$M_{\boldsymbol{\nu}} \mapsto M_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\mu}} := \sigma(\boldsymbol{\nu},\boldsymbol{\mu})\mathbf{e}^{\boldsymbol{\nu}+\boldsymbol{\mu}}, \qquad \#(\boldsymbol{\nu}+\boldsymbol{\mu}) < \infty, \tag{3.16}$$

where $\sigma(\cdot,\cdot)$ is as in (2.15). We notice that for $f = \sum_{\boldsymbol{\nu}\in\mathcal{I}} f_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\nu}}$, we have

$$M_f 1 = \sum_{\boldsymbol{\nu}\in\mathcal{I}} f_{\boldsymbol{\nu}}\sigma(\boldsymbol{\nu},\mathbf{0})\mathbf{e}^{\boldsymbol{\nu}} = f,$$

as $\sigma(\boldsymbol{\nu},\mathbf{0}) = 1$.

An obvious problem that arises is that the left multiplication has a nontrivial kernel. To overcome this, we use the corresponding ℓ^2-inner product linked to the 2-norm of $\overline{\mathcal{G}}_3^{(2)}$,

$$\left\langle c_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\nu}}, c_{\boldsymbol{\eta}}\mathbf{e}^{\boldsymbol{\eta}}\right\rangle_2 := c_{\boldsymbol{\nu}}\bar{c}_{\boldsymbol{\eta}}\delta_{\boldsymbol{\nu},\boldsymbol{\eta}},\ c_{\boldsymbol{\nu}}, c_{\boldsymbol{\eta}} \in \mathbb{C},\ \boldsymbol{\nu},\boldsymbol{\eta} \in \{0,1,2\}^{\mathbb{N}}. \tag{3.17}$$

Also, we observe that $\mathbf{e}^{\mathbf{0}} = 1$ so that $M_{\mathbf{0}} = \mathrm{Id}$ is the identity operator, and $M_{\boldsymbol{\nu}} = M_{\nu_1} \ldots M_{\nu_d}$ for $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d)$.

Then,

$$\left\langle M_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\mu}}, \mathbf{e}^{\boldsymbol{\eta}}\right\rangle_2 = \sigma(\boldsymbol{\nu},\boldsymbol{\mu})\left\langle \mathbf{e}^{\boldsymbol{\nu}+\boldsymbol{\mu}}, \mathbf{e}^{\boldsymbol{\eta}}\right\rangle_2 = \sigma(\boldsymbol{\nu},\boldsymbol{\mu})\delta_{\boldsymbol{\nu}+\boldsymbol{\mu},\boldsymbol{\eta}},$$

where $\boldsymbol{\nu}+\boldsymbol{\mu} \in \{0,1,2\}^{\mathbb{N}}$.

For an arbitrary $\boldsymbol{\nu} \in \{0,1,2\}^{\mathbb{N}}$, we define the adjoint of $M_{\boldsymbol{\nu}}$, denoted by $M^*_{\boldsymbol{\nu}}$, as

$$\begin{aligned}\left\langle M^*_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\mu}}, \mathbf{e}^{\boldsymbol{\eta}}\right\rangle_2 &:= \left\langle \mathbf{e}^{\boldsymbol{\mu}}, M_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\eta}}\right\rangle_2\\ &= \overline{\sigma(\boldsymbol{\nu},\boldsymbol{\eta})}\left\langle \mathbf{e}^{\boldsymbol{\mu}}, \mathbf{e}^{\boldsymbol{\nu}+\boldsymbol{\eta}}\right\rangle_2\\ &= \overline{\sigma(\boldsymbol{\nu},\boldsymbol{\eta})}\delta_{\boldsymbol{\mu},\boldsymbol{\nu}+\boldsymbol{\eta}}\\ &= \overline{\sigma(\boldsymbol{\nu},\boldsymbol{\eta})}\delta_{\boldsymbol{\mu}-\boldsymbol{\nu},\boldsymbol{\eta}},\end{aligned}$$

leading to

$$M^*_{\boldsymbol{\nu}} \mapsto M^*_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\mu}} = \overline{\sigma(\boldsymbol{\nu},\boldsymbol{\mu}-\boldsymbol{\nu})}\mathbf{e}^{\boldsymbol{\mu}-\boldsymbol{\nu}},\ \boldsymbol{\mu}-\boldsymbol{\nu} \in \mathcal{I}, \tag{3.18}$$

with again $M^*_{\mathbf{0}} = \mathrm{Id}$ and $M^*_{\boldsymbol{\nu}} = M^*_{\nu_d}M^*_{\nu_{d-1}} \ldots M^*_{\nu_1}$ for $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d)$. Furthermore, for $f = \sum_{\boldsymbol{\nu}\in\mathcal{I}} f_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\nu}}, g = \sum_{\boldsymbol{\mu}\in\mathcal{I}} g_{\boldsymbol{\mu}}\mathbf{e}^{\boldsymbol{\mu}} \in \mathfrak{S}_{-1}$, we have $\left\langle M^*_f g, \mathbf{e}^{\boldsymbol{\eta}}\right\rangle_2 = \left\langle g, M_f\mathbf{e}^{\boldsymbol{\eta}}\right\rangle_2$ so that the adjoint becomes

$$M^*_f g = \sum_{\boldsymbol{\mu}-\boldsymbol{\nu}\in\mathcal{I}} \overline{f}_{\boldsymbol{\nu}} g_{\boldsymbol{\mu}}\overline{\sigma(\boldsymbol{\nu},\boldsymbol{\mu}-\boldsymbol{\nu})}\mathbf{e}^{\boldsymbol{\mu}-\boldsymbol{\nu}}.$$

Again, we notice that for $f = \sum_{\boldsymbol{\nu}\in\mathcal{I}} f_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\nu}}$, we have

$$M^*_f 1 = \sum_{\mathbf{0}-\boldsymbol{\nu}\in\mathcal{I}} \overline{f}_{\boldsymbol{\nu}}\overline{\sigma(\boldsymbol{\nu},-\boldsymbol{\nu})}\mathbf{e}^{\mathbf{0}-\boldsymbol{\nu}} = \overline{f}_{\mathbf{0}},$$

as $\mathbf{0}-\boldsymbol{\nu} \in \mathcal{I}$ if and only if $\boldsymbol{\nu} = \mathbf{0}$ and $\sigma(\mathbf{0},\mathbf{0}) = 1$.

We finalize the description of the operators $M_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\mu}}$ and $M^*_{\boldsymbol{\nu}}\mathbf{e}^{\boldsymbol{\mu}}$ with a table of the relevant pairs for each jth component of $\boldsymbol{\nu},\boldsymbol{\mu} \in \mathcal{I}$:

$\mu_j + \nu_j \in \{0,1,2\}$

ν_j \ μ_j	0	1	2
0	0	1	2
1	1	2	
2	2		

$\mu_j - \nu_j \in \{0,1,2\}$

ν_j \ μ_j	0	1	2
0	0	1	2
1		0	1
2			0

Remark that the $M_{\boldsymbol{\nu}}^*$ operator corresponds to a left derivative and it is analogous to the one traditionally defined in superanalysis and supersymmetry.[10,27-29] Hence, the Berezin integral can be defined in terms of $M_{\boldsymbol{\nu}}^*$ as

$$\int d\mathbf{e}^{\boldsymbol{\nu}} g := M_{\boldsymbol{\nu}}^* g = \sum_{\boldsymbol{\mu}-\boldsymbol{\nu}\in\mathcal{I}} g_{\boldsymbol{\mu}} \overline{\sigma(\boldsymbol{\nu}, \boldsymbol{\mu}-\boldsymbol{\nu})} \mathbf{e}^{\boldsymbol{\mu}-\boldsymbol{\nu}}, \tag{3.19}$$

for $g = \sum_{\boldsymbol{\mu}\in\mathcal{I}} g_{\boldsymbol{\mu}} \mathbf{e}^{\boldsymbol{\mu}} \in \mathfrak{S}_{-1}$, $\boldsymbol{\nu} \in \{0,1,2\}^{\mathbb{N}}$ and where $\#\boldsymbol{\nu} < \infty$.

We remark that in the particular case of $g = g_{\boldsymbol{\nu}} \mathbf{e}^{\boldsymbol{\nu}}$, then its Berezin integral becomes

$$\int d\mathbf{e}^{\boldsymbol{\nu}} (g_{\boldsymbol{\nu}} \mathbf{e}^{\boldsymbol{\nu}}) = g_{\boldsymbol{\nu}} M_{\boldsymbol{\nu}}^* \mathbf{e}^{\boldsymbol{\nu}} = g_{\boldsymbol{\nu}} \overline{\sigma(\boldsymbol{\nu}, \mathbf{0})} \mathbf{e}^{\mathbf{0}} = g_{\boldsymbol{\nu}} = \left\langle M_{g_{\boldsymbol{\nu}} \mathbf{e}^{\boldsymbol{\nu}}} 1, \mathbf{e}^{\boldsymbol{\nu}} \right\rangle_2.$$

Lemma 3.15. *Let $f, g \in \mathcal{H}_0$. Then, it holds*

$$\left\langle M_f 1, M_g 1 \right\rangle_2 = \langle f, g \rangle_2. \tag{3.20}$$

The lemma is immediate since we have

$$M_f 1 = f,$$

as seen above.

4 | STOCHASTIC PROCESSES AND THEIR DERIVATIVES

A second-order stochastic process indexed by a set S is a map f_t from S into some probability space $\mathbf{L}^2(\Omega, \mathcal{B}, P)$, and the covariance of the process is

$$k(t,s) = \int_\Omega \overline{f_t(w)} f_s(w) dP(w) \stackrel{\text{def.}}{=} \mathbb{E}_P \overline{f_t} f_s, \tag{4.1}$$

where $\mathbb{E}_P$ denotes the mathematical expectation with respect to P. Usually, in topological supersymmetry, one is interested not just in the total probability distribution P but also in the generalized probability distribution which consists the differential forms. For the sake of simplicity, we are restricting us here to the case of P with the consideration of the generalized probability distribution being done in a similar fashion than the classic case.[5,6] In order to define stochastic integrals, it is of interest to consider cases where the function $s \mapsto f_s$ is differentiable, possibly in a larger space than the original probability space (a space of stochastic distributions). Taking Hida's white noise space (see, e.g., Holden et al[16] and Kuo[17]) as probability space, this space of stochastic distributions, together with an underlying space of stochastic test functions, forms a Gel'fand triple, which allows to give useful models for stochastic processes and their derivatives and in which one can develop stochastic calculus.

There is more than one possible such Gel'fand triple. One particularly convenient space of stochastic distributions has been introduced by Yuri Kondratiev and has a special algebraic structure. It is a strong algebra, as defined above, and in fact Alpay and Salomon[30] got from it the inspiration and framework to define strong algebras.

Hida's white noise space is identified in a natural way with the Fock space associated to $\ell^2(\mathbb{N}_0, \mathbb{C})$, and this motivates the definition of stochastic processes as functions (or, as multiplication operators by functions) taking valued in the counterpart of the Fock space in various situations. This approach was also developed in Alpay et al,[31,32] in the setting of the grey noise space,[33] in the theory of noncommutative stochastic processes,[18] and in the setting of the Grassmann algebra.[11] The Wick product takes different forms in each of these cases, but they all satisfy Våge's inequality in an appropriately defined strong algebra; this allows to transfer the results from one setting to the other setting with the same proofs. We now introduce the counterpart of the Fock space in the present framework.

Definition 4.1. By analogy with the noncommutative setting, we define the 3-graded super Fock space as $\overline{\mathcal{G}}_3^{(2)}$.

We here explain the corresponding theory in our setting and first define what is meant by a stochastic process in the present framework. Let $(\xi_n)_{n\in\mathbb{N}}$ denote the system of normalized Hermite functions. They form an orthonormal basis of

$\mathbf{L}^2(\mathbb{R}, dx)$, and every element f in the latter can thus be written as

$$f(u) = \sum_{n=1}^{\infty} f_n \xi_n(u), \text{ with } \sum_{n=1}^{\infty} |f_n|^2 < \infty. \tag{4.2}$$

We define an isometric map

$$f \mapsto Xf = \sum_{n=1}^{\infty} f_n \mathbf{e}_n$$

from $\mathbf{L}^2(\mathbb{R}, dx)$ into $\overline{\mathcal{G}}_3^{(2)}$, and

$$M_{Xf} = \sum_{n=1}^{\infty} f_n M_{\mathbf{e}_n}. \tag{4.3}$$

Definition 4.2. A stochastic process indexed by a set S is a map $s \mapsto M_{Xf_s}$, where $f_s \in \mathbf{L}^2(\mathbb{R}, dx)$ for every $s \in S$. The covariance function of the process is defined by

$$\langle M_{Xf_s} 1, M_{Xf_t} 1 \rangle = \langle f_s, f_t \rangle_2. \tag{4.4}$$

In (4.4), the second inner product is the $\mathbf{L}^2(\mathbb{R}, dx)$ inner product, and the equality follows from (3.20). This equality plays a key role in the arguments. Counterparts of this equality hold in particular in the white noise space setting,[16] the Poisson noise setting,[16], (4.9.4), p. 204 the grey space noise setting,[33,34] and the free setting. In each case, the right-hand side stays the same, but the left-hand side can take quite different forms. Equation (4.4) allows to relate the underlying setting with the Lebesgue space.

In the free setting, the counterpart of the left-hand side of (4.4) is the trace of a C^*-algebra generated by the (real parts) of the creation operators. In the Grassmann setting, operators are also involved, to make contact with the Berezin integral. Here too, to define counterparts of the Berezin integrals, we introduced earlier multiplication operators, which can be seen as the analogs of the creation operators. For the discussion of stochastic processes themselves, we will not consider operators but directly functions.

We are interested in two special cases, namely, $S = \mathbb{R}$ (or a subinterval of it) and the real-valued Schwartz functions, here denoted by $\mathcal{S}(\mathbb{R})$. In the first, we consider covariance functions in (4.4) of the form

$$K_\sigma(t, s) = \int_{\mathbb{R}} \frac{(e^{iut} - 1)(e^{-ius} - 1)}{u^2} d\sigma(u), \tag{4.5}$$

where σ represents an increasing function such that the Stieltjes integral

$$\int_{\mathbb{R}} \frac{d\sigma(u)}{u^2 + 1} < \infty. \tag{4.6}$$

Such a family contains in particular the Brownian and the fractional Brownian motion.

Let us introduce an operator S_m in $\mathbf{L}^2(\mathbb{R})$ defined by

$$\widehat{S_m f}(u) = \sqrt{m(u)} \hat{f}(u), \tag{4.7}$$

with $\hat{f}$ denoting the Fourier transform of f. Keep in mind that in general, S_m is an unbounded operator. The domain of S_m is given by

$$\text{dom} \quad S_m = \left\{ f \in \mathbf{L}^2(\mathbb{R}) \middle| \int_{\mathbb{R}} m(u) |\hat{f}(u)|^2 du < \infty \right\},$$

which contains $\mathbf{1}_{[0,t]}$. We can now consider the action of the operator S_m on the function $\mathbf{1}_{[0,t]}$, that is,

$$f_m(t) = S_m \mathbf{1}_{[0,t]},$$

and via an application of Plancherel's identity, we get

$$\begin{aligned}\langle f_m(t), f_m(s)\rangle_{\mathbf{L}_2(\mathbb{R})} &= \frac{1}{2\pi}\left\langle \hat{f}_m(t), \hat{f}_m(s)\right\rangle_{\mathbf{L}_2(\mathbb{R})}\\ &= \frac{1}{2\pi}\left\langle \sqrt{m(u)}\widehat{\mathbf{1}}_{[0,t]}, \sqrt{m(u)}\widehat{\mathbf{1}}_{[0,s]}\right\rangle_{\mathbf{L}_2(\mathbb{R})}\\ &= \frac{1}{2\pi}\left\langle m(u)\frac{e^{-iut}-1}{u}, \frac{e^{-ius}-1}{u}\right\rangle_{\mathbf{L}_2(\mathbb{R})}\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\frac{(e^{iut}-1)(e^{-ius}-1)}{u^2}m(u)du.\end{aligned}$$

The $\overline{\mathcal{G}}_3^{(2)}$-valued process

$$XS_m\mathbf{1}_{[0,t]} = \sum_{n\in\mathbb{N}}\left(\int_0^t (S_m\xi_n)(u)du\right)\mathbf{e}_n$$

has covariance function equal to

$$\frac{1}{2\pi}\int_{\mathbb{R}}\frac{(e^{iut}-1)(e^{-ius}-1)}{u^2}m(u)du.$$

Theorem 4.3. *Let m be a positive measurable function, satisfying (4.8)*

$$m(u) \le \begin{cases} K|u|^{-b} & |u| \le 1,\\ K|u|^{2N} & |u| > 1,\end{cases} \tag{4.8}$$

with $b < 2$, $N \in \mathbb{N}_0$, and K represents a positive real constant and (4.6) (the latter for $d\sigma(t) = m(t)dt$). Then, $TS_m1_{0,t]}$is differentiable in $\mathfrak{S}_{-1}$, with continuous derivative there.

Theorem 4.4. *Let $s \mapsto f_s$ be a $\overline{\mathcal{G}}_3^{(2)}$-valued function such that the derivative $s \mapsto f_s'$ is continuous from [0, 1] into $\mathfrak{S}_{-1}$, and let $s \mapsto Y(s)$ be a continuous function from [0, 1] into $\mathfrak{S}_{-1}$. There is a $p \in \mathbb{N}$ such that the function $t \mapsto Y(t)f'(t)$ is continuous in $\mathcal{H}_{-p}$ and the corresponding Hilbert space integral $\int_0^1 Y(t)f_s'$ converges in $\mathfrak{S}_{-1}$.*

In the case of $S = \mathcal{S}(\mathbb{R})$, we consider a continuous positive operator A from $\mathcal{S}(\mathcal{R})$ into $\mathcal{S}'(\mathbb{R})$. On the one hand, applying the Bochner–Minlos theorem to the function $\exp(-\langle As, s\rangle)$ where the brackets denote the duality between $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$. We obtain a probability measure P_A on $\mathcal{S}'(\mathbb{R})$ such that

$$\mathbb{E}_{P_A} e^{-i\langle\cdot,s\rangle} = e^{-\langle As,s\rangle}, \tag{4.9}$$

and a centered Gaussian process indexed by $\mathcal{S}(\mathbb{R})$, defined by

$$Q_s(\omega) = \langle\omega, s\rangle, \tag{4.10}$$

with covariance function

$$\mathbb{E}_{P_A}(Q_{s_1}Q_{s_2}) = \langle As_1, s_2\rangle. \tag{4.11}$$

The operator A can be factored via $\mathbf{L}^2(\mathbb{R}, dx)$ as $A = T^*T$, where T is a continuous operator from $\mathcal{S}(\mathbb{R})$ into $\mathbf{L}^2(\mathbb{R}, dx)$ as $A = T^*T$, and therefore, (4.11) can be rewritten as

$$\mathbb{E}_{P_A}(Q_{s_1}Q_{s_2}) = \langle Ts_1, Ts_2\rangle_2, \tag{4.12}$$

where the latter brackets denote the inner product in $\mathbf{L}^2(\mathbb{R}, dx)$.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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