

Research Article

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Synchronization of Caputo fractional neural networks with bounded time variable delays

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Abstract: One of the main problems connected with neural networks is synchronization. We examine a model of a neural network with time-varying delay and also the case when the connection weights (the influential strength of the j th neuron to the i th neuron) are variable in time and unbounded. The rate of change of the dynamics of all neurons is described by the Caputo fractional derivative. We apply Lyapunov functions and the Razumikhin method to obtain some sufficient conditions to ensure synchronization in the model. These sufficient conditions are explicitly expressed in terms of the parameters of the system, and hence, they are easily verifiable. We illustrate our theory with a particular nonlinear neural network.

Keywords: nonlinear neural networks, delay, synchronization, Lyapunov functions

MSC 2020: 34-xx, 39-xx, 44-xx

1 Introduction

Nowadays, the synchronization of fractional-order delayed neural networks has attracted more and more attention. Some synchronization results have been obtained, for instance, in [1–3], where the authors studied the synchronization of fractional-order memristor-based neural networks with delay. The problem of synchronization of the fractional neural network with delay is studied in [4]. In [5,6], the synchronization of fractional-order neural networks with multiple time delays is investigated. Note that in most of the known literature models, the connection weights are constants and the controllers are proportional to the error with a constant.

Motivated by the above discussions, the main goal of this paper is to study synchronization of neural networks with delay and with the Caputo fractional derivative. Note that the studies in this paper are more general than the existing ones in the literature. We study the general case of time-varying self-regulating parameters of all units and also time-varying functions of the connection between two neurons in the network. For example, in [4] the model is considered in the case all the connection weights are constants, the self-inhibition rate is described by a constant, and for a very special output depending on the Lipschitz constants of the activation functions. Also, we are applying Mittag-Leffler function with one parameter, which is deeply connected with the applied fractional derivative, and on the other side is a generalization of the exponential function applied in [1,5,7]. The study in this paper is based on the application of Lyapunov functions. To deal with the presence of the delay, we apply the Razumikhin method. By constructing an appropriate Lyapunov function, applying a special type of its derivative (as introduced in [8]), and

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a comparison principle with time delay, we obtain some sufficient conditions to ensure synchronization in the model. The theoretical results are illustrated in an example.

The main contributions of this paper could be summarized as follows:

- We include in the model the Mittag-Leffler function in both the self-inhibition rate and the connection weights, which is more appropriate when we are modeling with a Caputo fractional derivative.
- We consider the variable coefficients in both the self-inhibition rate and the connection weights.
- We study the bounded variable in time delay.
- We consider output coupling controller depending on the delay and with variable in time input matrices.
- We apply the Razumikhin method and Dini fractional derivatives of Lyapunov functions, which are connected with both the presence of the delay and Caputo fractional derivative in the system.

2 System description

We will consider the model based on an analog circuit, consisting of capacitors, resistors, and amplifiers. The input–output characteristics of the amplifiers in the circuits are modeled by some functions known as activation functions. In the literature, these functions are assumed to satisfy a wide variety of assumptions, mainly they are Lipschitz functions.

Initially, we will give some basic definitions from fractional calculus. We fix a fractional order $q \in (0, 1)$ everywhere in the paper, as it is usually done in many applications in science and engineering (for some physical interpretation of the fractional order see, for example, [9]).

1. The Riemann-Liouville fractional derivative of order $q \in (0, 1)$ of $m(\cdot)$ is given by ([10])

$${}^{RL}D_t^q m(t) = \frac{1}{\Gamma(1 - q)} \frac{d}{dt} \int_0^t (t - s)^{-q} m(s) ds, \quad t \geq t_0,$$

where $\Gamma(\cdot)$ denotes the Gamma function.

2. The Caputo fractional derivative of order $q \in (0, 1)$ is defined by ([10])

$${}^C D_t^q m(t) = \frac{1}{\Gamma(1 - q)} \int_0^t (t - s)^{-q} m'(s) ds, \quad t \geq 0.$$

We recall that the Mittag-Leffler function with one parameter is defined as

$$E_q(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(1 + qi)}, \quad q > 0, z \in \mathbb{C}. \tag{1}$$

Definition 1. [11] Function $m \in C^q(\mathbb{R}_+, \mathbb{R}^n)$ if $m(\cdot)$ is differentiable on \mathbb{R}_+ and the Caputo derivative ${}^C D_t^q$ exists, for all $t \in \mathbb{R}_+$.

In our further investigations, we will use the following notations: $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$ is the Euclidean norm of the vector $x = (x_1, x_2, \dots, x_n)$, the set $C(J, \mathbb{R}^n)$ of all continuous functions $u : J \rightarrow \mathbb{R}^n, J \subset \mathbb{R}_+$ and for any function $\phi \in C([-r, 0], \mathbb{R}^n)$, we define $\|\phi\|_0 = \sup_{s \in [-r, 0]} \|\phi(s)\|$, where $r > 0$ is a given number (it will be connected with the delay).

Consider the general model of neural networks with time variable bounded delay involving the Mittag-Leffler function with one parameter $E_q(\cdot)$ defined above:

$${}^C D_t^q x_i(t) = E_q(pt^q) \left(-c_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n b_{ij}(t)g_j(x_j(\tau(t))) \right) + I_i(t), \quad \text{for } t \geq 0, \tag{2}$$

for $i = 1, 2, \dots, n$, where

- n represents the number of neurons in the network,
- $x_i(t)$ is the pseudostate variable denoting the average membrane potential of the i th neuron at time t ,
- $x(t) = (x_1(t), x_2(t), \dots, x_n(t)) \in \mathbb{R}^n$, and $p > 0, q \in (0, 1)$ are constants,
- $\tau \in C(\mathbb{R}_+, [-r, \infty))$, $-r \leq \tau(t) \leq t$ for $t \geq 0$ denotes the communication delay of the neuron,
- $f_j(x_j(t)), g_j(x_j(\tau(t)))$ denote the activation functions of the neurons at time t and $\tau(t)$, respectively, and they represent the response of the j th neuron to its membrane potential and define $f(x) = (f_1(x_1), f_2(x_2), \dots, f_n(x_n))$ and $g(x) = (g_1(x_1), g_2(x_2), \dots, g_n(x_n))$,
- $I = (I_1, I_2, \dots, I_n)$ is an external bias vector, $C(t) = \text{diag}(c_1(t), c_2(t), \dots, c_n(t))$, and $c_i(t)E_q(pt^q)$ is the self-inhibition rate of i th neuron (the rate at which the i th neuron returns to a resting state without being connected to the neural network),
- matrix $A(t) = \{a_{ij}(t)\}_{n \times n}$ and $a_{ij}(t)E_q(pt^q)$ denote the connection weights (the influential strength of the j th neuron to the i th neuron) at the current time t ,
- matrix $B(t) = \{b_{ij}(t)\}_{n \times n}$ and $b_{ij}(t)E_q(pt^q)$ indicate the delayed connection weights (the influential strength of the j th neuron to the i th neuron at delay time $\tau(t)$).

We set up the following initial condition to model (2):

$$x_i(t) = \phi_i(t) \quad t \in [-r, 0], \quad i = 1, 2, \dots, n, \tag{3}$$

where $\phi_i \in C([-r, 0], \mathbb{R}^n)$, and let $\phi = (\phi_1, \phi_2, \dots, \phi_n)$.

Note that model (2) is more general than the existing ones in the literature. For example, the synchronization of model (2) is studied in [4] in the case all the connection weights are constants, the self-inhibition rate is described by a constant, and for a very special output depending on the Lipschitz constants of the activation functions.

A similar model is studied in [13], but the connection weights are bounded, which is not satisfied in this case because of the presence of Mittag-Leffler function.

System (2) is considered as the driven system, and the corresponding response system (slave system) is described by the following fractional-order differential equations:

$${}_0^C D_t^q y_i(t) = E_q(pt^q) \left(-c_i(t)y_i(t) + \sum_{j=1}^n a_{ij}(t)f_j(y_j(t)) + \sum_{j=1}^n b_{ij}(t)g_j(y_j(\tau(t))) + I_i(t) + u_i(t) \right), \quad \text{for } t \geq 0, \tag{4}$$

with initial condition

$$y_i(s) = \varphi_i(s), \quad s \in [-r, 0], \tag{5}$$

for $i = 1, 2, \dots, n$, where $y = (y_1, y_2, \dots, y_n)$ denotes the state variable of the response system (4) and $u = (u_1, u_2, \dots, u_n)$ indicates the synchronous controller to be designed.

Definition 2. The master delay Caputo fractional system (2) and the slave delay Caputo fractional system (4) are globally asymptotically synchronized if, for any initial functions $\phi, \varphi \in C([-r, 0], \mathbb{R}^n)$ with $\|\phi - \varphi\|_0 < \infty$, the limit

$$\lim_{t \rightarrow \infty} \|x(t; \phi) - y(t; \varphi)\| = 0$$

holds, where $x(t; \phi)$ and $y(t; \varphi)$ are solutions of the master delay Caputo fractional system (2) with initial condition (3) and the slave delay Caputo fractional system (4) with initial condition (5), respectively.

The main goal of the paper is to implement appropriate controllers $u_i(t), i = 1, 2, \dots, n$ for the response system, such that the controlled response system (2) and (3) could be synchronized with the drive system (4).

One approach to study synchronization problems is based on applying Lyapunov-like functions. Thus, we start by defining what is a Lyapunov function and then its derivative among the fractional equation.

Definition 3. [8] Let $J \subset \mathbb{R}_+$ be a given interval. We will say that the function $V : J \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ belongs to the class $\Lambda(J, \mathbb{R}^n)$ if V is continuous at every $(t, x) \in J \times \mathbb{R}^n$ and it is locally Lipschitzian with respect to its second argument.

In connection with the Caputo fractional derivative, it is necessary to define in an appropriate way the derivative of the Lyapunov functions among a nonlinear Caputo fractional differential equation

$${}_0^C D_t^q x(t) = \mathcal{G}(t, x(t), x(\tau(t))), \quad t \geq 0, \tag{6}$$

where the function $\mathcal{G} : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n)$.

The studied model is a partial case of the nonlinear system (6). Some existence and uniqueness results of global solutions of Caputo fractional differential equations are given in [12].

In this paper, we will use Dini fractional derivative of the Lyapunov function $V \in \Lambda(J, \mathbb{R}^n)$ among the fractional system (6), introduced in [8] by

$$D_{(6)}^+ V(t, \psi) = \limsup_{h \rightarrow 0} \frac{1}{h} \left[V(t, \psi(0)) + \sum_{s=1}^{\lceil \frac{t}{h} \rceil} (-1)^s {}_q C_s V(t - sh, \psi(0) - h^q \mathcal{G}(t, \psi(0), \psi(\tau(0)))) \right], \tag{7}$$

where $\psi \in C([-r, 0], \mathbb{R}^n)$, $\lceil \frac{t}{h} \rceil$ denotes the integer part of the fraction $\frac{t}{h}$, and ${}_q C_s = \frac{q(q-1)\dots(q-s+1)}{s!}$.

We will illustrate the application of the above defined Dini fractional derivative among the fractional system (6) to a particular Lyapunov function.

Example 1. Consider the particular Lyapunov function

$$V(t, x) = E_q(pt^q) \sum_{i=1}^n x_i^2,$$

with $x = (x_1, x_2, \dots, x_n)$ and the Mittag-Leffler function with one parameter $E_q(pt^q)$, defined by (1).

Let $\psi \in C([-r, 0], \mathbb{R}^n)$ with $\psi = (\psi_1, \psi_2, \dots, \psi_n)$. Then, the Dini fractional derivative of the Lyapunov function V among the fractional system (6) is

$$\begin{aligned} D_{(6)}^+ V(t, \psi) &= \limsup_{h \rightarrow 0} \frac{1}{h^q} \left[E_q(pt^q) \sum_{j=1}^n (\psi_j(0))^2 \right. \\ &\quad \left. - \sum_{s=1}^{\lceil \frac{t}{h} \rceil} (-1)^{s+1} {}_q C_s E_q(p(t - sh)^q) \sum_{i=1}^n (\psi_i(0) - h^q \mathcal{G}_i(t, \psi(0), \psi(\tau(0))))^2 \right] \\ &= \limsup_{h \rightarrow 0} \frac{1}{h^q} \left[E_q(pt^q) \sum_{i=1}^n ((\psi_i(0))^2 - (\psi_i(0) - h^q \mathcal{G}_i(t, \psi(0), \psi(\tau(0))))^2) \right. \\ &\quad \left. + \sum_{i=1}^n (\psi_i(0) - h^q \mathcal{G}_i(t, \psi(0), \psi(\tau(0))))^2 \sum_{s=0}^{\lceil \frac{t}{h} \rceil} (-1)^s {}_q C_s E_q(p(t - sh)^q) \right] \\ &= 2E_q(pt^q) \sum_{i=1}^n \psi_i(0) \mathcal{G}_i(t, \psi(0), \psi(\tau(0))) + \sum_{i=1}^n (\psi_i(0))^2 {}_0^{RL} D^q (E_q(pt^q)). \end{aligned} \tag{8}$$

Using the formula

$${}_0^{RL} D_t^q (E_q(pt^q)) = \frac{t^{-q}}{\Gamma(1 - q)} + pE_q(pt^q)$$

in (8), we obtain

$$D_{(6)}^+ V(t, \psi) = 2E_q(pt^q) \sum_{i=1}^n \psi_i(0) \mathcal{G}_i(t, \psi(0), \psi(\tau(0))) + \left(\frac{t^{-q}}{\Gamma(1-q)} + pE_q(pt^q) \right) \sum_{i=1}^n (\psi_i(0))^2, \quad t \geq 0. \quad (9)$$

In the sequel, we will use the following comparison result:

Lemma 1. [8] *Assume:*

1. Function $x \in C^q(\mathbb{R}_+, \mathbb{R}^n)$ is a solution of (6) with initial condition $x(s) = \phi(s)$, $s \in [-r, 0]$, where $\phi \in C([-r, 0], \mathbb{R}^n)$.
2. Function $V \in \Lambda([-r, \infty), \mathbb{R}^n)$ is such that, for any point $t \in \mathbb{R}_+$ with

$$V(t + \Theta, \psi(\Theta)) \leq V(t, \psi(0)), \quad \Theta \in [-r, 0],$$

the inequality

$$D_{(6)}^+ V(t, \psi) \leq 0 \quad (10)$$

holds, where $\psi(\Theta) = x(t + \Theta)$, $\Theta \in [-r, 0]$.

Then,

$$V(t, x(t; \phi)) \leq \max_{\Theta \in [-r, 0]} V(\Theta, \phi(\Theta)), \quad \text{for } t \geq 0.$$

3 Synchronization with output coupling controller depending on delay

The controller with time delay in the response system can be taken as output coupling in the following way:

$$u_i(t) = E_q(pt^q) \left(\sum_{j=1}^n \xi_{ij}(t) (f_j(y_j(t)) - f_j(x_j(t))) + \sum_{j=1}^n \eta_{ij}(t) (g_j(y_j(\tau(t))) - g_j(x_j(\tau(t)))) \right), \quad i = 1, 2, \dots, n, \quad (11)$$

where $E_q(pt^q)\Xi(t) = \{E_q(pt^q)\xi_{ij}(t)\}$ and $E_q(pt^q)\Upsilon(t) = \{E_q(pt^q)\eta_{ij}(t)\}$ denote the control income matrices at current time t and delay time $\tau(t)$, respectively.

Define the error function $e(t) = x(t) - y(t)$, where $x(\cdot), y(\cdot)$ are solutions of the initial value problems for the driven system (2), (3) and for the response system (4), respectively, with controller defined by (11). Then, the synchronization of system (4) is equivalent to asymptotic stability of the zero solution of the system for the error functions given by

$${}_0^C D_t^q e_i(t) = E_q(pt^q) \left(-c_i(t) e_i(t) + \sum_{j=1}^n (a_{ij}(t) - \xi_{ij}(t)) F_j(e_j(t)) + \sum_{j=1}^n (b_{ij}(t) - \eta_{ij}(t)) G_j(e_j(\tau(t))) \right), \quad (12)$$

for $t \geq 0, i = 1, 2, \dots, n,$

$$e_i(s) = \phi_i(s) - \varphi_i(s), \quad s \in [-r, 0],$$

where

$$F_j(e_j(t)) = F_j(y_j(t) - x_j(t)) = f_j(y_j(t)) - f(x_j(t)) \quad \text{and} \quad G_j(e_j(t)) = G_j(y_j(t) - x_j(t)) = g_j(y_j(t)) - g(x_j(t)).$$

We assume the following:

Assumption A1. The neuron activation functions are Lipschitz, i.e., there exist positive numbers L_i, M_i , $i = 1, 2, \dots, n$, such that

$$|f_i(u) - f_i(v)| \leq L_i |u - v| \quad \text{and} \quad |g_i(u) - g_i(v)| \leq M_i |u - v|,$$

for $i = 1, 2, \dots, n$, and for all $u, v \in \mathbb{R}$.

Assumption A2. Functions $a_{i,j}, b_{i,j}, \xi_{i,j}, \eta_{i,j} \in C(\mathbb{R}_+, \mathbb{R})$ and there exist positive constants $H_{ij}, K_{ij}, P_{ij}, Q_{ij}, i, j = 1, 2, \dots, n$, such that

$$|a_{ij}(t) - \xi_{ij}(t)| \leq \frac{H_{ij}}{E_q(pt^q) + Q_{ij}}, \quad t \geq 0,$$

and

$$|b_{ij}(t) - \eta_{ij}(t)| \leq \frac{K_{ij}}{E_q(pt^q) + P_{ij}}, \quad t \geq 0.$$

Assumption A3. Functions $c_i \in C(\mathbb{R}_+, (0, \infty)), i = 1, 2, \dots, n$, are such that $c_i(t) \geq C(t) > 0$, for $t \geq 0$, where $C \in C([0, \infty), (0, \infty))$ and

$$C(t) \geq 0.5 \left(p + t^{-q} + \max_{i=1,2,\dots,n} \sum_{j=1}^n H_{ij}L_j + \sum_{i=1}^n \max_{j=1,2,\dots,n} H_{ij}L_j + \max_{i=1,2,\dots,n} \sum_{j=1}^n K_{ij}M_j + E_q(pr^q) \sum_{i=1}^n \max_{j=1,2,\dots,n} K_{ij}M_j \right), \quad (13)$$

where the constants $L_i, M_i, i = 1, 2, \dots, n$, and $H_{ij}, K_{ij}, i, j = 1, 2, \dots, n$ are defined in Assumptions A1 and A2, respectively.

Remark 1. Assumption A1 means the activation functions are Lipschitz. Assumption A2 gives bounds between the connection weights and the controllers at the current time and the delayed time. Assumption A3 guarantees small enough Lipschitz constants and bounds in the above conditions A1 and A2.

Remark 2. If Assumption A1 is satisfied, then functions F, G in (12) satisfy

$$|F_j(u)| \leq L_j|u| \quad \text{and} \quad |G_j(u)| \leq M_j|u|,$$

for $j = 1, 2, \dots, n$ and for any $u \in \mathbb{R}$.

Theorem 1. Let Assumptions A1–A3 be satisfied. Then, the master delay Caputo fractional system (2), (3) and the slave delay Caputo fractional system (4) are globally asymptotically synchronized.

Proof. Consider the Lyapunov function

$$V(t, x) = \begin{cases} E_q(pt^q)x^T x, & \text{for } x \in \mathbb{R}^n, \quad t \geq 0, \\ x^T x, & \text{for } x \in \mathbb{R}^n, \quad t \in [-r, 0]. \end{cases}$$

Let the function $e(\cdot) = e(t), t \geq 0$ be a solution of the initial value problem (12) and let $t \geq 0$ be a point such that

$$V(t + s, \psi(s)) < V(t, \psi(0)), \quad s \in [-r, 0), \quad (14)$$

where $\psi(s) = e(t + s), s \in [-r, 0]$.

If inequality (14) holds, then for $t \geq r$ the inequality

$$E_q(p(t + s)^q) \sum_{i=1}^n \psi_i^2(s) < E_q(pt^q) \sum_{i=1}^n \psi_i^2(0), \quad s \in [-r, 0)$$

holds, and, for $t \in [0, r)$, the inequality

$$\sum_{i=1}^n \psi_i^2(s) < \sum_{i=1}^n \psi_i^2(0), \quad s \in [-r, 0)$$

holds.

From Example 1 and equation (9) with

$$\mathcal{G}_i(t, x, y) = E_q(pt^q) \left(-c_i(t)x_i + \sum_{j=1}^N (a_{ij}(t) - \xi_{ij}(t))F_j(x_j) + \sum_{j=1}^N (b_{ij}(t) - \eta_{ij}(t))G_j(y_j) \right),$$

we get for the Dini fractional derivative of the Lyapunov function V among the fractional system (12) at the point t and the function ψ defined above:

$$\begin{aligned} D_{(12)}^+ V(t, \psi) &= 2E_q(pt^q) \sum_{i=1}^n \psi_i(0) \mathcal{G}_i(t, \psi(0), \psi(\tau(0))) + \left(\frac{t^{-q}}{\Gamma(1-q)} + pE_q(pt^q) \right) \sum_{i=1}^n (\psi_i(0))^2 \\ &= 2E_q^2(pt^q) \sum_{i=1}^n \psi_i(0) \left[-c_i(t)\psi_i(0) + \sum_{j=1}^n (a_{ij}(t) - \xi_{ij}(t))F_j(\psi_j(0)) \right. \\ &\quad \left. + \sum_{j=1}^n (b_{ij}(t) - \eta_{ij}(t))G_j(\psi_j(\tau(0))) \right] + \left(\frac{t^{-q}}{\Gamma(1-q)} + pE_q(pt^q) \right) \sum_{i=1}^n (\psi_i(0))^2 \\ &\leq -2E_q^2(pt^q) \sum_{i=1}^n \psi_i^2(0)c_i(t) + 2E_q^2(pt^q) \sum_{i=1}^n \psi_i(0) \sum_{j=1}^n (a_{ij}(t) - \xi_{ij}(t))F_j(\psi_j(0)) \\ &\quad + 2E_q^2(pt^q) \sum_{i=1}^n \psi_i(0) \sum_{j=1}^n (b_{ij}(t) - \eta_{ij}(t))G_j(\psi_j(\tau(0))) + \left(\frac{t^{-q}}{E_q(pt^q)\Gamma(1-q)} + p \right) V(t, \psi(0)). \end{aligned} \tag{15}$$

Applying the formula

$$\frac{E_q(pt^q)}{E_q(pt^q) + \Xi} \leq 1, \quad t \geq 0,$$

with $\Xi = P_{ij}$ and $\Xi = Q_{ij}$, Remark 2 and Assumptions A1–A2, we obtain

$$\begin{aligned} D_{(12)}^+ V(t, \psi) &\leq V(t, \psi(0)) \left(p + \frac{t^{-q}}{E_q(pt^q)\Gamma(1-q)} \right) - 2E_q^2(pt^q) \sum_{i=1}^n c_i(t)\psi_i^2(0) \\ &\quad + 2E_q^2(pt^q) \sum_{i=1}^n |\psi_i(0)| \sum_{j=1}^n |a_{ij}(t) - \xi_{ij}(t)|L_j|\psi_j(0)| + 2E_q^2(pt^q) \sum_{i=1}^n |\psi_i(0)| \sum_{j=1}^n |b_{ij}(t) - \eta_{ij}(t)|M_j|\psi_j(\tau(0))| \\ &\leq V(t, \psi(0)) \left(p + \frac{t^{-q}}{E_q(pt^q)\Gamma(1-q)} - 2C(t) \right) + E_q^2(pt^q) \sum_{i=1}^n \sum_{j=1}^m |a_{ij}(t) - \xi_{ij}(t)|L_j[\psi_i^2(0) + \psi_j^2(0)] \\ &\quad + E_q^2(pt^q) \sum_{i=1}^n \sum_{j=1}^n |b_{ij}(t) - \eta_{ij}(t)|M_j[\psi_i^2(0) + \psi_j^2(\tau(0))] \\ &\leq V(t, \psi(0)) (p + t^{-q} - 2C(t)) + E_q(pt^q) \left(\max_{i=1,2,\dots,n} \sum_{j=1}^n \frac{H_{ij}}{E_q(pt^q) + Q_{ij}} L_j \right) V(t, \psi(0)) \\ &\quad + E_q(pt^q) \left(\sum_{i=1}^n \max_{j=1,2,\dots,n} \frac{H_{ij}}{E_q(pt^q) + Q_{ij}} L_j \right) V(t, \psi(0)) \\ &\quad + E_q(pt^q) \left(\max_{i=1,2,\dots,n} \sum_{j=1}^n \frac{K_{ij}}{E_q(pt^q) + P_{ij}} M_j \right) V(t, \psi(0)) \\ &\quad + E_q^2(pt^q) \left(\sum_{i=1}^n \max_{j=1,2,\dots,n} \frac{K_{ij}}{E_q(pt^q) + P_{ij}} M_j \right) \sum_{j=1}^n \psi_j^2(\tau(0)) \\ &\leq V(t, \psi(0)) \left(p + t^{-q} - 2C(t) + \max_{i=1,2,\dots,n} \sum_{j=1}^n H_{ij}L_j + \sum_{i=1}^n \max_{j=1,2,\dots,n} H_{ij}L_j + \max_{i=1,2,\dots,n} \sum_{j=1}^n K_{ij}M_j \right) \\ &\quad + E_q(pt^q) \left(\sum_{i=1}^m \max_{j=1,2,\dots,n} K_{ij}M_j \right) \sum_{j=1}^n \psi_j^2(\tau(0)). \end{aligned} \tag{16}$$

If $t \in [0, r]$, then

$$\sum_{i=1}^n \psi_i^2(\tau(0)) < \sum_{i=1}^n \psi_i^2(0)$$

and, from inequality (16), we obtain

$$D_{(12)}^+ V(t, \psi) \leq V(t, \psi(0)) \left(p + t^{-q} - 2C(t) + \max_{i=1,2,\dots,n} \sum_{j=1}^N H_{ij} L_j + \sum_{i=1}^n \max_j H_{ij} L_j + \max_{i=1,2,\dots,n} \sum_{j=1}^N K_{ij} M_j + \sum_{i=1}^n \max_{j=1,2,\dots,n} K_{ij} M_j \right). \tag{17}$$

If $t > r$, then applying

$$\frac{E_q(pt^q)}{E_q(p(t + \tau(0))^q)} \leq E_q(pr^q)$$

and the inequality

$$E_q(p(t + \tau(0))^q) \sum_{i=1}^n \psi_i^2(\tau(0)) < E_q(pt^q) \sum_{i=1}^n \psi_i^2(0),$$

we obtain

$$D_{(12)}^+ V(t, \psi) \leq V(t, \psi(0)) \left(p + t^{-q} - 2C(t) + \max_{i=1,2,\dots,n} \sum_{j=1}^N H_{ij} L_j + \sum_{i=1}^n \max_{j=1,2,\dots,n} H_{ij} L_j + \max_{i=1,2,\dots,n} \sum_{j=1}^N K_{ij} M_j \right) + \frac{E_q(pt^q)}{E_q(p(t + \tau(0))^q)} \left(\sum_{i=1}^m \max_{j=1,2,\dots,n} K_{ij} M_j \right) V(t, \psi(0)) \leq V(t, \psi(0)) \left(p + t^{-q} - 2C(t) + \max_{i=1,2,\dots,n} \sum_{j=1}^N H_{ij} L_j + \sum_{i=1}^n \max_{j=1,2,\dots,n} H_{ij} L_j + \max_{i=1,2,\dots,n} \sum_{j=1}^N K_{ij} M_j + E_q(pr^q) \sum_{i=1}^m \max_{j=1,2,\dots,n} K_{ij} M_j \right). \tag{18}$$

From inequalities (17), (18), and Lemma 1, it follows that

$$\|e(t)\|^2 = \sum_{i=1}^n e_i^2(t) = \frac{1}{E_q(pt^q)} V(t, e(t)) \leq \frac{1}{E_q(pt^q)} \max_{s \in [-r, 0]} V(s, \phi_i(s) - \varphi_i(s)) = \frac{1}{E_q(pt^q)} \|\phi - \varphi\|_0^2, \quad t \geq 0. \tag{19}$$

If the initial functions $\phi, \varphi \in C([-r, 0], \mathbb{R}^n)$ are such that $\|\phi - \varphi\|_0 < \infty$, then

$$\lim_{t \rightarrow \infty} \|x(t; \phi) - y(t; \varphi)\| = 0$$

and, therefore, the master delay Caputo fractional system (2) and the slave delay Caputo fractional system (4) are globally asymptotically synchronized. □

4 An application

As an example, we will study a partial case of the driven system (2) and the corresponding slave system (4) with the controller with time delay taken as output coupling given by (11).

Consider the following master delay Caputo fractional system with three agents, i.e., $n = 3$:

$$\begin{aligned}
 {}_0^C D_t^{0.5} x_i(t) &= E_{0.5}(2t^{0.5}) \left(-(2 + 0.5E_{0.5}(2) + 0.5t^{-0.5})x_i(t) + \sum_{j=1}^3 a_{ij}(t) \tanh(x_j(t)) \right. \\
 &\quad \left. + \sum_{j=1}^3 b_{ij}(t) \tanh(x_j(t - |\sin(t)|)) \right), \quad \text{for } t \geq 0, \\
 x_i(s) &= \phi_i(s), \quad \text{for } s \in [-1, 0],
 \end{aligned} \tag{20}$$

and the slave system described by the following fractional-order differential equations:

$$\begin{aligned}
 {}_0^C D_t^{0.5} y_i(t) &= E_{0.5}(2t^{0.5}) \left(-(2 + 0.5E_{0.5}(2) + 0.5t^{-0.5})y_i(t) + \sum_{j=1}^3 a_{ij}(t) \tanh(y_j(t)) \right. \\
 &\quad \left. + \sum_{j=1}^3 b_{ij}(t) \tanh(y_j(t - |\sin(t)|)) \right) + \sum_{j=1}^n \xi_{ij}(t) (\tanh(y_j(t)) - \tanh(x_j(t))) \\
 &\quad + \sum_{j=1}^n \eta_{ij}(t) (\tanh(y_j(t - |\sin(t)|)) - \tanh(x_j(t - |\sin(t)|))), \quad \text{for } t \geq 0, \\
 y_i(s) &= \varphi_i(s), \quad \text{for } s \in [-1, 0],
 \end{aligned} \tag{21}$$

for $i = 1, 2, 3$, where $\phi_1(s) = 1$, $\phi_2(s) = 0.1$, $\phi_3(s) = 0.5$, $\varphi_1(s) = 0.5$, $\varphi_2(s) = 1.5$, and $\varphi_3(s) = 2$. Matrices $A = \{a_{ij}(t)\}$ of the connection weights at the present time and $B = \{b_{ij}(t)\}$, $i, j = 1, 2, 3$, $t \geq 0$ of the connection weights at the delay time are given by

$$A = \begin{pmatrix} 0 & -2 & 1.3 \\ 1 & 0 & 0.2 \\ 0.4 & -2.2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & -2.1 \\ 0.3 & 0 & 2.1 \\ 1.2 & 1 & 0 \end{pmatrix}. \tag{22}$$

Controllers' matrices $\Xi = \{\xi_{ij}(t)\}$, $E = \{\eta_{ij}(t)\}$, $i, j = 1, 2, 3$, $t \geq 0$ at the current time and the delayed time, respectively, are defined by

$$\Xi = \begin{pmatrix} \frac{1}{2E_{0.5}(2t^{0.5}) + 0.001} & -2 - \frac{1}{2E_{0.5}(2t^{0.5}) + 0.001} & 1.3 + \frac{1}{2E_{0.5}(2t^{0.5}) + 0.001} \\ 1 + \frac{1}{2E_{0.5}(2t^{0.5}) + 0.001} & \frac{1}{2E_{0.5}(2t^{0.5}) + 0.001} & 0.2 + \frac{1}{2E_{0.5}(2t^{0.5}) + 0.001} \\ 0.4 - \frac{1}{2E_{0.5}(2t^{0.5}) + 0.001} & -2.2 + \frac{1}{2E_{0.5}(2t^{0.5}) + 0.001} & \frac{1}{2E_{0.5}(2t^{0.5}) + 0.001} \end{pmatrix}, \tag{23}$$

$$E = \begin{pmatrix} \frac{1}{E_{0.5}(2t^{0.5}) + 0.001} & 1 - \frac{1}{E_{0.5}(2t^{0.5}) + 0.001} & -2.1 + \frac{1}{E_{0.5}(2t^{0.5}) + 0.001} \\ 0.3 + \frac{1}{E_{0.5}(2t^{0.5}) + 0.001} & \frac{1}{E_{0.5}(2t^{0.5}) + 0.1} & 2.1 + \frac{1}{E_{0.5}(2t^{0.5}) + 0.001} \\ 1.2 + \frac{1}{E_{0.5}(2t^{0.5}) + 0.001} & 1 - \frac{1}{E_{0.5}(2t^{0.5}) + 0.001} & \frac{1}{E_{0.5}(2t^{0.5}) + 0.001} \end{pmatrix}. \tag{24}$$

Note that Assumption A1 is satisfied with $L_i = M_i = 1$, $i = 1, 2, 3$. It is easy to check that Assumptions A2 and A3 are also satisfied with $B_{ij} = K_{ij} = 1$, $Q_{ij} = P_{ij} = 0.001$. Therefore, according to Theorem 1, the master delay Caputo fractional system (20) and the slave delay Caputo fractional system (21) are globally asymptotically synchronized. The solutions of both systems are graphed in Figures 1–3. Also, in Figure 4, the differences between components of both solutions are graphed. As it can be observed, these differences approach 0 very fast.

Remark 3. The synchronization of Caputo delay neural networks was also studied in [14]. But, in that case, the connection weights (the influential strength of the j th neuron to the i th neuron) at the current time as well

as at the delayed time are bounded from above. Now, because of the Mittag-Leffler function, this condition is not satisfied and the results from [14] could not be applied to the master delay Caputo fractional system (20) and the slave delay Caputo fractional system (21).

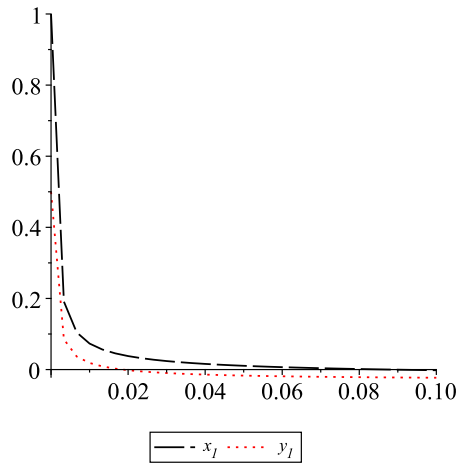


Figure 1: Graphs of first components of the solutions of both the slave and the master delay Caputo fractional system (21), (20).

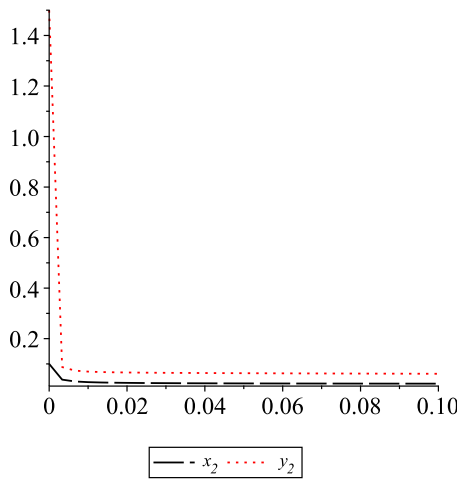


Figure 2: Graphs of second components of the solutions of both the slave and the master delay Caputo fractional system (21), (20).

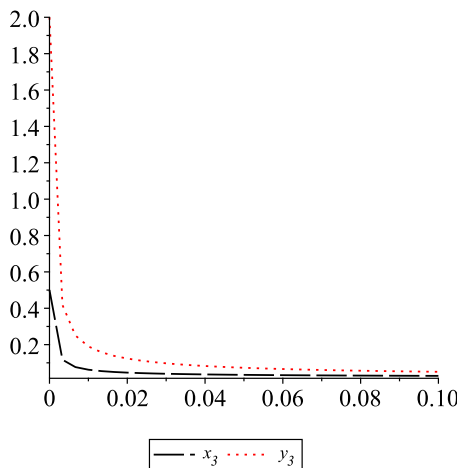


Figure 3: Graphs of third components of the solutions of both the slave and the master delay Caputo fractional system (21), (20).

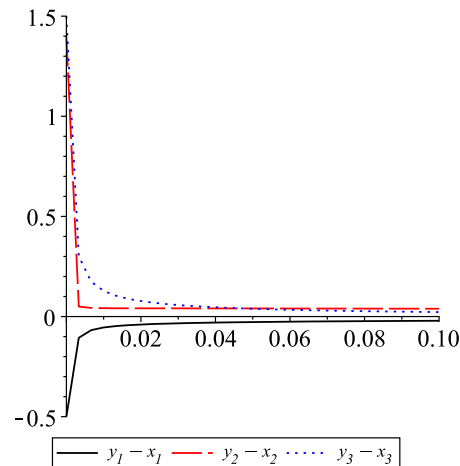


Figure 4: Graphs of the differences between the solutions of the master system (20) and the slave system (21).

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