Method of nose stretching in Newton's problem of minimal resistance

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Abstract

We consider the problem $\inf \big\{ \iint_\Omega (1 + |\nabla u(x_1, x_2)|^2)^{-1} dx_1 dx_2 :$ the function $u : \Omega \to \mathbb{R}$ is concave and $0 \le u(x) \le M$ for all $x = (x_1, x_2) \in \Omega = \{|x| \le 1\} \big\}$ (Newton's problem) and its generalizations. In the paper by Brock, Ferone, and Kawohl (1996) it is proved that if a solution u is C^2 in an open set $\mathcal{U} \subset \Omega$ then $\det D^2 u = 0$ in $\mathcal{U}$. It follows that $\text{graph}(u)\rfloor_\mathcal{U}$ does not contain extreme points of the subgraph of u.

In this paper we prove a somewhat stronger result. Namely, there exists a solution u possessing the following property. If u is C^1 in an open set $\mathcal{U} \subset \Omega$ then $\text{graph}(u\rfloor_\mathcal{U})$ does not contain extreme points of the convex body $C_u = \{(x, z) : x \in \Omega,\ 0 \le z \le u(x)\}$. As a consequence, we have $C_u = \text{Conv}(\overline{\text{Sing}C_u})$, where $\text{Sing}C_u$ denotes the set of singular points of ∂C_u. We prove a similar result for a generalization of Newton's problem.

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1 Introduction

1.1 History of the problem

1.1.1 Isaac Newton in his *Principia* (1687) considered the following mechanical model. A solid body moves with constant velocity through a rarefied medium composed of point

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particles. The particles are initially at rest, and all collisions of particles with the body's surface are perfectly elastic.[1]

The medium is assumed to be extremely rarefied, so as mutual interaction of the medium particles can be neglected. In physical terms, this means that the free path length of particles is much larger than the size of the body. As a real-world application, one can imagine an artificial satellite with well polished surface moving around the Earth at low altitudes (between 100 and 1000 km) where the atmosphere is extremely thin (but is still present).

As a result of collisions of the body with the particles, the force of resistance is created, which acts on the body and slows down its velocity. Newton calculated the resistance of several geometrical shapes: a cylinder, a sphere, and a truncated cone. What is more important, he considered the following problem: minimize the resistance in the class of convex bodies that are rotationally symmetric with respect to a straight line parallel to the direction of motion and have fixed length along the direction of motion and fixed maximal width in the orthogonal direction.

Newton described the solution of this problem in geometrical terms (see, e. g., the book [49]). The solution looks like a truncated cone with slightly inflated lateral surface; see Fig. 1 for the case when the length is equal to the maximal width. The body in the

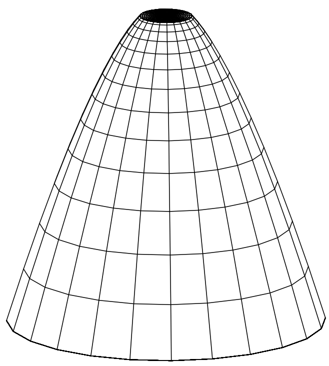

Figure 1: A solution to the rotationally symmetric Newton problem.

picture moves in the medium vertically upward; equivalently, one can assume that the body is at rest and there is an incident flow of particles moving vertically downward.

In modern terms the problem can be stated as follows. Choose a reference system with the coordinates x_1, x_2, z connected with the body so as the z-axis coincides with the body's axis of symmetry and is counter directional to the flow of particles. Let the front

[1]Actually, Newton considered a one-parameter family of reflection laws. Namely, a parameter $0 \le k \le 1$ is fixed, and in a reference system connected with the body, at each impact, the normal component of the particle's velocity of incidence is multiplied by $-k$, while the tangential component remains unchanged. In the case $k = 1$ we have the law of perfectly elastic (billiard) reflection.

part of the body be the graph of a radial function $z = \varphi(r)$, $r = \sqrt{x_1^2 + x_2^2}$, $0 \le r \le L$. Since the body is convex, the function φ is concave and monotone non-increasing.

The resistance equals $2\pi\rho v^2 R(\varphi)$, where ρ is the density of the medium, v is the scalar velocity of the body, and

$$R(\varphi) = \int_0^L \frac{1}{1 + \varphi'(r)^2}\, r\, dr.$$

The values ρ and v are fixed, so the problems is as follows: minimize $R(\varphi)$ in the class of concave monotone non-increasing functions $\varphi : [0, L] \to \mathbb{R}$ such that $0 \le \varphi(r) \le M$ for all r. Here M and $2L$ are, respectively, the fixed length and maximal width of the body.

The new life was given to the problem in 1993 when the paper by Buttazzo and Kawohl [10] was published. The authors considered the problem of minimal resistance in various classes, in particular in the class of convex (not necessarily symmetric) bodies and in the class of nonconvex bodies satisfying the so-called single impact condition. Since then, many research papers have been published in this area.

In the papers [11, 12, 13, 23, 25, 24, 14, 29], the problem in various classes of convex bodies is studied. In particular, in [24, 29] there are considered classes of bodies that are convex hulls of a certain pair of planar convex curves. In [10, 11, 31], the problem in some classes of (generally) nonconvex bodies are considered. In [17, 9], surveys of the current state of the problem are given.

The problem for rotationally symmetric bodies is studied under the additional conditions that the so-called arclength is fixed [6]; there is a friction in the course of body-particle interaction [21]; the thermal motion of the medium particles is present [48].

In [15, 16, 18, 40, 1, 39], the problem is studied in classes of nonconvex bodies satisfying the condition that each particle of the flow hits the body only once. Connection of this problem with Besicovitch's solution of the Kakeya needle problem [7] are found in [40, 39].

In [38], [37] the techniques of the theory of billiards are used to study the problem in classes of nonconvex bodies where multiple body-particle collisions are allowed. The problems of resistance optimization for bodies that perform both translational and rotational motion are studied in [44, 35, 34, 47, 46]. In these studies, methods of optimal mass transport theory [50, 32] are used, and applications to geometrical optics and mechanics (invisibility [2, 37, 45], Magnus effect [47], retroreflectors [5, 36], camouflaging [41]) are found.

1.1.2 Suppose that the body is convex and a reference system connected with the body with the coordinates x_1, x_2, z is chosen so as the z-axis is parallel and counter directional to the direction of the flow. Let the front part of the body's surface be the graph of a concave function $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^2$ is the projection of the body on the x_1x_2-plane. Then the vertical component of resistance is equal to $2\rho v^2 \mathcal{F}[u]$, where $\mathcal{F}[u] = \iint_\Omega (1 + |\nabla u(x_1, x_2)|^2)^{-1}\, dx_1 dx_2$. In what follows we assume that the numerical coefficient $2\rho v^2$ equals 1. The value $\mathcal{F}[u]$ is called *resistance*.

The earliest and the most direct, and perhaps the most difficult generalization of the original problem stated by Newton is as follows: find the body of minimal resistance in the class of convex bodies with fixed length along the direction of motion and fixed projection on the plane orthogonal to this direction [10]. The only difference as compared with the original problem is that the body in general is not rotationally symmetric (as a consequence, the orthogonal projection of the body on the x_1x_2-plane is not necessarily a circle). In terms of the function u, the problem reads as follows:

Generalized Newton's problem. *Given $M > 0$ and a convex body*[2] $\Omega \subset \mathbb{R}^2$,

$$\text{Minimize the functional} \quad \mathcal{F}[u] = \iint_\Omega \frac{1}{1 + |\nabla u(x_1, x_2)|^2} \, dx_1 \, dx_2 \tag{1}$$

in the class $\mathcal{C}_{\Omega,M}$ of concave functions $u : \Omega \to \mathbb{R}$ satisfying the condition $0 \le u(x) \le M$ for all $x = (x_1, x_2) \in \Omega$.

Along with this problem, we will also consider its more general version:

$$\text{Minimize the functional} \quad \mathcal{F}[u] = \iint_\Omega g(\nabla u(x_1, x_2)) \, dx_1 dx_2 \tag{2}$$

in the class $\mathcal{C}_{\Omega,M}$, where $g : \mathbb{R}^2 \to \mathbb{R}$ is a bounded continuous function.

Substituting $g(v) = 1/(1 + |v|^2)$ in (2), one comes to generalized Newton's problem (1).

The following is known about a solution u of problem (1).

$\mathbf{P_1}$. *There exists at least one solution* [11, 30]. *The same is true for the more general problem* (2) [13].

$\mathbf{P_2}$. *If ∇u exists at a point $x = (x_1, x_2) \in \Omega$ then either $|\nabla u(x)| \ge 1$, or $|\nabla u(x)| = 0$* [11]. *Moreover, if the upper level set $L = \{x : u(x) = M\}$ has a nonempty interior then for almost all $\bar{x} \in \partial L$, $\lim_{\substack{x \to \bar{x} \\ x \notin L}} |\nabla u(x)| = 1$* [43].

$\mathbf{P_3}$. *The upper level set L is not a point. Therefore, it is either a line segment or a planar convex body.*

The proof of statement P_3 is given in Section 4.3 of *Tesi di Laurea* of P. Guasoni (supervised by G. Buttazzo) [20].[3] Since the thesis is in Italian, for the reader's convenience we provide below the main idea of the proof.

Assume that L is a point and for $\varepsilon > 0$ denote $u_\varepsilon(x) = \min\{(1 + \varepsilon)u(x), M\}$. The function u_ε is concave, and in view of statement P_2, $u_\varepsilon < M$ outside the circle centered at L with the radius $M\varepsilon$. It follows that $\mathcal{F}[u_\varepsilon] \le \mathcal{F}[(1+\varepsilon)u] + \pi M^2\varepsilon^2$. Using again statement P_2, one shows that $\mathcal{F}[u] - \mathcal{F}[(1+\varepsilon)u] > c\varepsilon$ for a certain $c > 0$ and for ε sufficiently small, and therefore, $\mathcal{F}[u_\varepsilon] < \mathcal{F}[u]$ for ε sufficiently small, in contradiction with optimality of u.

[2] A convex body is a compact convex set with nonempty interior.

[3] Note in passing that in this thesis it was shown for the first time that Newton's radial solution (see Fig 1) does not solve generalized problem (1) for M sufficiently large.

P_4. *If u is C^2 in an open neighborhood $\mathcal{U}$ of a point $x \in \Omega$, then the matrix of the second derivatives $D^2u(x)$ has a zero eigenvalue* [12].

This statement implies that the gaussian curvature at each point of the surface $\{(x, u(x)) : x \in \mathcal{U}\}$ equals zero, and therefore, the surface is developable. Unfortunately, one cannot guarantee that the statement is applicable to the solution u, since it is not proved that u is C^2 in an open set.

P_5. *If all points of the curve $\partial\Omega$ are regular*[4] *(for example, Ω is a circle) then $u\rfloor_{\partial\Omega} = 0$* [42].

Some more results concerning the more general problem (2) are obtained by Lachand-Robert and Peletier in [23, 25].

P_6. *Let the function $g : \mathbb{R}^2 \to \mathbb{R}$ be strictly convex. Fix concave functions u_1 and u_2 on Ω such that $u_1 = u_2$ on $\partial\Omega$ and $u_1 < u_2$ in the interior of Ω, and consider the class $\mathcal{C}(u_1, u_2)$ of concave functions u on Ω satisfying $u_1 \le u \le u_2$. Then the unique solution of problem (2) in the class $\mathcal{C}(u_1, u_2)$ is u_1* [23].[5]

This statement is not applicable to generalized Newton's problem (1), since the function $g(v) = 1/(1 + |v|^2)$ is not convex.

P_7. *Assume that the function g satisfies the conditions (a) g is C^2 and the matrix $D^2g(v)$ has at least one negative eigenvalue for all $v \in \mathbb{R}^2$; (b) the relation $\sum_{i,j\in\{1,2\}} g''_{v_iv_j}(v)y_iy_j = 0 = \sum_{i,j,k\in\{1,2\}} g'''_{v_iv_jv_k}(v)y_iy_jy_k$ is impossible for $v \in \mathbb{R}^2$ and $y = (y_1, y_2) \neq (0, 0)$. If u is a solution of problem (2) then u is not strictly convex in any open convex subset $\mathcal{U} \subset \Omega$* [25].

Condition (b) in P_7 seems to be quite technical; we do not know if it is really necessary.

Note that the function $g(v) = 1/(1+|v|^2)$ satisfies conditions (a) and (b), and therefore, statement P_7 is applicable to generalized Newton's problem (1).

Statement P_7 means that there is a dense set of non-degenerate line segments in the graph of u. It can also be reformulated as follows. Let $C_u = \{(x, z) : x \in \Omega,\ 0 \le z \le u(x)\}$ and let $\mathrm{Ext}C_u$ denote the set of extreme points of ∂C_u; then $\partial C_u \setminus \mathrm{Ext}C_u$ is dense in ∂C_u. This statement can serve as a kind of substitute for P_4 if u happens to be not smooth enough.

Numerical study of problem (1) in the case when Ω is a circle has been carried out in the papers [22, 51].

Numerical results suggest that the graph of a solution u is a piecewise developable surface; more precisely, singular points of the graph form several curves, and the graph is the union of line segments with the endpoints on these curves. Statements P_4 and P_7 provide additional arguments in favor of this conjecture.

[4] A point $\xi \in \partial\Omega$ is called *regular*, if there is a unique line of support to Ω at ξ.

[5] The authors work with convex, rather than concave, functions u_1, u_2, u satisfying $u_2 \le u \le u_1$. With these changes, the statement remains the same.

1.1.3 In the present paper we prove that the graph of a certain solution u to problem (1) is formed by line segments and (possibly) planar triangles with the endpoints and vertices in $\overline{\text{Sing}(\text{graph}(u))} \cup \text{graph}(u\rfloor_{\partial\Omega})$, where $\overline{\text{Sing}(\text{graph}(u))}$ is the closure of the set of singular points of the graph of u[6] and $\text{graph}(u\rfloor_{\partial\Omega}) = \{(x, u(x)) : x \in \partial\Omega\}$ (Corollary 4). This means that the solution is defined uniquely by the set of its singular points and the values of u on the boundary of its domain.

Unfortunately, we still cannot guarantee that the set of singular points is closed. We cannot even affirm that this set is not dense in the graph. We believe that the following statement is true.

Conjecture. *The set of singular points of each solution to problem* (2) *is closed, and therefore, is nowhere dense.*

The main result of this paper is Theorem 2 stated in subsection 1.2.4 for a further generalization of Newton's problem and proved in Section 3. The method of the proof is new and is called *stretching the nose*. Corollary 4 is a simple consequence of Theorem 2. In general, we believe that it is fruitful to work with generalized versions of the problem, and that further progress can be achieved by using methods of convex geometry, including the notion of surface area measure.

1.2 Statement of the results and discussion

1.2.1 Let us imagine again an artificial satellite on a low-Earth orbit. It is known that the thermal velocity of molecules in the atmosphere is comparable with the satellite's velocity and therefore cannot be neglected. Moreover, the interaction of molecules with the satellite's surface by no means obeys the law of elastic reflection. This implies that formula (1) for resistance may not be valid, and additionally, not only the front part of the satellite's surface, but also its rear part should be taken in consideration.

1.2.2 In order to deal with the problem in this general setting, it is more natural and convenient to work with convex bodies, rather than with concave functions.

First introduce the notation. A *convex body* is a compact convex set with nonempty interior. In this paper, the letter C (also with some subscripts or superscripts) is always used to designate a convex body in $\mathbb{R}^3$. A point $\xi \in \partial C$ is called *singular* if there is more than one plane of support to C at ξ, and *regular* otherwise. The set of singular points of ∂C is denoted by Sing$C$. It is known that almost all points of ∂C are regular. The outward unit normal to C at a regular point $\xi \in \partial C$ is denoted by n_ξ. If a plane of support at ξ is unique (and therefore ξ is regular), it is called the *tangent plane* at ξ.

A point $x \in C$ is an *extreme point* of C, if it is not a convex combination $x = \lambda a + (1-\lambda)b,\ a, b \in C,\ a \neq b,\ 0 < \lambda < 1$. The set of extreme points of C is denoted as Ext$C$. The convex hull of a set A is denoted as Conv$A$. A plane of support to a convex body is always assumed to be oriented by the outward normal vector.

[6] A point $(x, u(x))$ of the graph is singular, if u is not differentiable at x.

In the sequel we will use the notion of *surface area measure*. The definition we give below slightly differs from the traditional one.

Definition 1. *Let $D \subset \partial C$ be a Borel set. The* surface area measure *of D is the measure ν_D on the unit sphere S^2 defined by*

$$\nu_D(A) = Leb\big(\{\xi \in D : \ \xi \text{ is a regular point of } \partial C \text{ and } n_\xi \in A\}\big)$$

for any Borel set $A \subset S^2$. Here Leb means the standard Lebesgue measure (area) on ∂C. In the particular case when $D = \partial C$, the corresponding surface area measure is denoted as $\nu_{\partial C}$.

If D is not a planar set, the measure ν_D does not depend on the choice of the convex body C whose boundary contains D. If D is planar, the measure depends on the choice of the normal to D (n or $-n$); in what follows it will always be clear, which normal should be chosen.

The notion of surface area measure and the related operation between convex bodies called *Blaschke addition* go back to Minkowski [33]. The map $C \mapsto \nu_{\partial C}$ between the set of convex bodies in $\mathbb{R}^3$ and the set of measures ν on S^2 that are not supported in any large circle and satisfy the relation $\int_{S^2} n \, d\nu(n) = \vec{0}$ is surjective, and $\nu_{C_1} = \nu_{C_2}$ *iff* C_1 is a translation of C_2 [3]. The Blaschke sum of two convex bodies C_1 and C_2 is the (defined up to a translation) convex body C such that $\nu_{\partial C} = \nu_{\partial C_1} + \nu_{\partial C_2}$.

An easy-to-read survey on Blaschke addition, also containing some new results, can be found in [4]. A characterization of Blaschke addition is given in [19]. A computer realization of Blaschke addition is provided in [52]. For possible generalizations of surface area measure see, e. g., [26, 27, 28].

1.2.3 Let $f : S^2 \to \mathbb{R}$ be a continuous function, and let $D \subset \partial C$ be a Borel subset of a convex body $C \subset \mathbb{R}^3$. We define the functional

$$F(D) = \int_D f(n_\xi) \, d\xi,$$

where $d\xi$ denotes the standard 2-dimensional Lebesgue measure on ∂C. Making a change of variable, one can write this functional as

$$F(D) = \int_{S^2} f(n) \, d\nu_D(n).$$

If D is not a planar set, $F(D)$ does not depend on the choice of C. If D is planar, the choice of the normal to D, and therefore the value $F(D)$, will always be uniquely defined.

Let two compact convex sets $C_1 \subset C_2 \subset \mathbb{R}^3$ be fixed. We consider the problem:

$$\textit{Minimize } F(\partial C) = \int_{\partial C} f(n_\xi) \, d\xi \ \textit{ in the class of convex bodies } C_1 \subset C \subset C_2. \qquad (3)$$

It is known that for any continuous function f and each pair of sets C_1 and C_2, this problem has at least one solution; see the paper of Buttazzo and Guasoni [13].

This statement of the problem can be interpreted as follows. In the real-life case, the pressure of the flow at a point $\xi \in \partial C$ depends only on the slope of the surface at ξ, that is, equals $p(n_\xi)\, n_\xi$, where p is a certain function on S^2 determined, e.g., by the character of the thermal motion of atmospheric particles and by the body-particle interaction. The projection of the drag force on the z-axis equals $F(\partial C) = \int_{\partial C} f(n_\xi)\, d\xi$, where $f(n) = p(n)\, n_3$ for $n = (n_1, n_2, n_3) \in S^2$.

The condition $C_1 \subset C \subset C_2$ can also be reasonably interpreted. Suppose we are given a metal body occupying the domain $C_2 \setminus C_1$ and are going to remove a part of material of the body to produce the optimal streamlined shape when moving in a certain direction. The resulting shape is $C \setminus C_1$, where C satisfies the above condition.

Remark 1. *Consider the cylinder $C_2 = \Omega \times [0,\, M]$ and its bottom $C_1 = \Omega \times \{0\}$, where $\Omega \in \mathbb{R}^2$ is a convex body and $M > 0$. Then each body C satisfying the condition $C_1 \subset C \subset C_2$ is bounded below by C_1 and above by the graph of a concave function $u : \Omega \to \mathbb{R}$ satisfying the inequalities $0 \le u(x) \le M$ for all x, and therefore, is defined by*

$$C = C_u = \{(x, z) :\, x \in \Omega,\ 0 \le z \le u(x)\}.$$

The body's boundary ∂C is the union of the disc C_1, a part of the cylindrical boundary $\partial\Omega \times [0,\, M]$, and the graph of u.

Consider the function f in the form $f(n) = p(n)\, n_3$, where p is a continuous function on S^2. Then the integral $\int f(n_\xi)\, d\xi$ over the cylindrical boundary is zero, the integral over the disc C_1 is constant, and the integral over the graph of u (after making the change of variable $\xi \rightsquigarrow x_1, x_2$ and taking into account that $d\xi = \sqrt{1 + u'^2_{x_1} + u'^2_{x_2}}\, dx_1 dx_2$ and the outward normal at a point of the graph of u is $n = (n_1, n_2, n_3) = (-u'_{x_1}, -u'_{x_2}, 1)/\sqrt{1 + u'^2_{x_1} + u'^2_{x_2}}$) equals

$$\mathcal{F}[u] = \iint_\Omega g(\nabla u(x_1, x_2))\, dx_1 dx_2,$$

where

$$g(\upsilon_1, \upsilon_2) = p\Big(\frac{1}{\sqrt{1 + \upsilon_1^2 + \upsilon_2^2}}(-\upsilon_1, -\upsilon_2, 1)\Big). \tag{4}$$

Thus, problem (3) *is reduced to problem* (2).

Inversely, let g be a bounded continuous function; then problem (2) *amounts to problem* (3) *with*

$$f(n) = \begin{cases} g\big(-\frac{n_1}{n_3}, -\frac{n_2}{n_3}\big) n_3, & \textit{if } n_3 > 0; \\ 0 & \textit{if } n_3 = 0; \\ \textit{arbitrary}, & \textit{if } n_3 < 0. \end{cases}$$

Remark 2. *In the model where the absolute temperature of the medium is 0 and the body-particle collisions are perfectly elastic (billiard-like), the pressure equals* $p(n) = ((n_3)_+)^2$, *where* $z_+ = \max\{0, z\}$ *means the positive part of* z. *This is the sine-squared law, well known in aerodynamics. The problem of minimal resistance in the class of bodies* $C_1 \subset C \subset C_2$ *here takes the form* (3) *with* $f(n) = p(n)n_3 = ((n_3)_+)^3$.

In the particular case when C_2 *is the cylinder and* C_1 *is its bottom,* $C_2 = \Omega \times [0, M]$, $C_1 = \Omega \times \{0\}$, *the problem amounts to problem* (2), *where by formula* (4) *we have* $g(v) = 1/(1 + |v|^2)$; *in other words, one comes to generalized Newton's problem* (1).

Remark 3. *In general a solution to problem* (3) *may not be unique. For example, suppose that* $C_1 = \Omega \times \{0\}$, $C_2 = \Omega \times [0, 1]$, $f > 0$ *in a small neighborhood of* $(0, 0, M)$, *and* $f = 0$ *outside this neighborhood. Then the minimal value of the functional equals 0 and is attained at a family of bodies* $Conv(C_1 \cup \{(a, b, c)\})$, $(a, b) \in \Omega$, *with* c *being sufficiently large (but smaller than* M*).*

Remark 4. *In generalized Newton's problem* (1) *with the circular base,* $\Omega = \{|x| \le 1\}$, *the numerical study [22, 51] seems to indicate that there exists a sequence of values* $+\infty = M_1 > M_2 > M_3 > \ldots$ *converging to zero such that for* $M_k < M < M_{k-1}$, $k = 2, 3, \ldots$ *the solution is unique (up to a rotation about the* z*-axis) and the top level set* $\{u(x) = M\}$ *is a regular* k*-gon, and for each value* $M = M_k$ *there are two distinct solutions with the top level sets being a regular* k*-gon and a regular* $(k + 1)$*-gon.*

1.2.4 Denote by $D^2u(x)$ the matrix of second derivatives (whenever it exists),

$$D^2u(x) = \begin{bmatrix} u''_{x_1x_1}(x) & u''_{x_1x_2}(x) \\ u''_{x_2x_1}(x) & u''_{x_2x_2}(x) \end{bmatrix}.$$

Let us formulate statement P_4 in a more general form.

Theorem 1. *Let* g *be of class* C^2 *and the matrix* D^2g *have at least one negative eigenvalue for all values of the argument. Assume that* u *is a solution of problem* (2) *and* $u \in C^2(\mathcal{U})$ *for an open set* $\mathcal{U} \subset \Omega$. *Then* $\det D^2u(x) = 0$ *for all* $x \in \mathcal{U}$.

Statement P_4 corresponds to the particular case of this theorem when $g(v) = 1/(1 + |v|^2)$ related to generalized Newton's problem; it was proved in [12] (Theorem 2.1 and Remark 3.4). The proof of Theorem 1 is basically the same. For the reader's convenience, it is provided in Section 2.

The statement of Theorem 1 implies that the gaussian curvature at each point of the surface $\{(x, u(x)) : x \in \mathcal{U}\}$ equals zero, and therefore, the surface is developable. It follows that no point of this surface is an extreme point of the body

$$C_u = \{(x, z) : x \in \Omega, \; 0 \le z \le u(x)\}.$$

In other words, we have the following

Corollary 1. *Under the assumptions of Theorem 1 we have*

$$\mathrm{Ext} C_u \cap \{(x, u(x)) : x \in \mathcal{U}\} = \emptyset.$$

The following question still remains. Suppose that an open subset of the lateral boundary of an optimal body does not contain singular points. Is it true that it does not contain extreme points (and therefore, is developable)?

Note that in this question only C^1 (rather than C^2) smoothness of the surface is a priori assumed.

We shall prove that the answer to this question is positive, even in the case of more general problem (3), for at least one solution of the problem.

The main result of this paper is the following theorem.

Theorem 2. *There is a solution* $\widehat{C}$ *of problem* (3) *such that*

$$\mathrm{Ext}\widehat{C} \subset \partial C_1 \cup \partial C_2 \cup \overline{\mathrm{Sing}\widehat{C}}. \tag{5}$$

Here and in what follows, the bar means closure.

Moreover, if a solution C *does not satisfy* (5) *then the set of solutions is extremely degenerate; namely, there is a family of solutions* $\{C(\vec{s}) : \ \vec{s} = (s_i)_{i\in\mathbb{N}} \in [0,\, 1]^\infty\}$*, where* $\vec{0} = (0,\, 0, \ldots)$*,* $\vec{1} = (1,\, 1, \ldots)$*, such that* $C(\vec{0}) = C$ *and* $\widehat{C} = C(\vec{1})$ *satisfies* (5)*. The corresponding family of surface area measures is linear and infinite dimensional; that is, there is a linearly independent set of signed measures* ν_i*,* $i \in \mathbb{N}$ *such that for all* $\vec{s} \in [0,\, 1]^\infty$*,*

$$\nu_{\partial C(\vec{s})} = \nu_{\partial C} + \sum_{i=1}^{\infty} s_i \nu_i.$$

This theorem will be proved in Section 3.

Remark 5. *An example of extremely degenerate case is provided by problem* (3) *with* $C_1 = \Omega \times \{0\}$*,* $C_2 = \Omega \times [0,\, M]$*, and the function*

$$f(n) = \begin{cases} \langle n, e\rangle, & \text{if } n_3 \geq 0; \\ \text{arbitrary}, & \text{if } n_3 < 0, \end{cases}$$

where $e = (e_1, e_2, e_3)$ *is a fixed vector and* $\langle\cdot\,,\cdot\rangle$ *means the scalar product. Using the well-known equation* $\int_{\partial C} n_\xi \, d\xi = \vec{0}$ *and taking into account that the outward normal to* ∂C *at each interior point of* C_1 *equals* $(0, 0, -1)$*, one obtains* $\int_{\partial C \setminus C_1} n_\xi \, d\xi = -\int_{C_1} n_\xi \, d\xi = |C_1|\,(0, 0, 1)$*, and therefore,*

$$F(\partial C) = \int_{C_1} f(0,0,-1)\, d\xi + \int_{\partial C\setminus C_1} \langle n_\xi, e\rangle \, d\xi = |C_1|\, f(0,0,1) + |C_1|\, e_3.$$

This means that all admissible bodies are solutions of the problem.

Note that, according to Remark 1, this problem is reduced to generalized Newton's problem (2) *with* $g(v_1, v_2) = -e_1 v_1 - e_2 v_2 + e_3$.

Can degeneracy appear in a less trivial case when $f(n)$ *does not coincide, even locally, with a function of the kind* $\langle n, e \rangle$*? This question is open, and it seems to be difficult. Let us explain why. Indeed, let* C_1 *and* C_2 *be as above and let* f *be a generic* C^2 *function. According to Remark 1, the problem can be reduced to problem* (2) *with the corresponding* C^2 *function* g *defined by* (4)*. Let* u *be a solution to the corresponding problem* (2).

In the "nice" case the domain Ω *is divided by curves into several smaller domains where* u *is* C^2 *smooth and* $D^2 g$ *does not have zero eigenvalues. If in a subdomain the matrix* $D^2 g$ *has at least one negative eigenvalue then, according to Corollary 1, the corresponding part of the graph of* u *does not contain extreme points. If, otherwise, in a subdomain* $D^2 g$ *is striclty positive definite then, applying statement* P_7*, one finds that the restriction of* u *on this subdomain is the smallest concave function coinciding with* u *on the boundary of the subdomain, and therefore, again, the corresponding part of the graph of* u *does not contain extreme points. Thus,* u *cannot induce degeneracy.*

This argument shows that, if degeneracy really exists in nontrivial cases, it should be related either with a very special behavior of the function g*, or with a "strange" solution* u *that is* C^1 *but is not* C^2 *on an open set.*

Remark 6. *Notice that problem* (3) *can be expressed as a minimization problem for a linear functional on a set of measures*

$$\inf_{\Upsilon(C_1,C_2)} \int_{S^2} f(n)\, d\nu(n), \quad \text{where } \Upsilon(C_1, C_2) = \{\nu_{\partial C} : C_1 \subset C \subset C_2\}.$$

This fact was first observed by Carlier and Lachand-Robert in [14]. Theorem 2 implies that if a solution ν *is not the surface area measure of a convex body satisfying* (5)*, then the set of solutions contains an infinite-dimensional cube* $\{\nu + \sum_{i=1}^{\infty} s_i \nu_i,\ s_i \in [0,\, 1]\}$.

Using the Krein-Milman theorem, one immediately obtains the following corollary of Theorem 2.

Corollary 2. *There is at least one solution* C *to problem* (3) *satisfying the equation*

$$C = \mathrm{Conv}\Big(\overline{\mathrm{Sing}C} \cup \big(\partial C \cap \partial C_1\big) \cup \big(\partial C \cap \partial C_2\big)\Big).$$

As applied to problem (2), we obtain the following statement (cf. Corollary 1).

Corollary 3. *There exists a solution* u *to problem* (2) *possessing the following property. If* $u \in C^1(\mathcal{U})$ *and* $u < M$ *in an open set* $\mathcal{U} \subset \Omega$*, then*

$$\mathrm{Ext}C_u \cap \{(x, u(x)) : x \in \mathcal{U}\} = \emptyset.$$

Notice that no assumptions on smoothness of the (bounded and continuous) function g are imposed here. Besides, it is a priori assumed that u is C^1, rather than C^2.

The following statement concerns generalized Newton's problem.

Corollary 4. *There exists a solution u of problem* (1) *such that*

$$C_u = \text{Conv}(\overline{\text{Sing}C_u}). \tag{6}$$

It follows that the boundary of C_u is composed of line segments and planar triangles with the endpoints and vertices in the closure of the set of singular points. Yet, we cannot exclude the possibility that the set of singular points is dense in the graph of u, and therefore, all these segments and triangles degenerate to singletons.

Proof of Corollary 4. We have seen in Remark 2 that generalized Newton's problem (1) is equivalent to problem (3) with $f(n) = ((n_3)_+)^3$. According to Theorem 2, there is a solution $C = C_u$ of this problem such that

$$\text{Ext}C_u \subset \partial C_1 \cup \partial C_2 \cup \overline{\text{Sing}C_u};$$

in other words, each extreme point of C_u lies either in $\overline{\text{Sing}C_u}$, or in $(\partial C_1 \cup \partial C_2) \cap \partial C_u$.

We are going to prove that each extreme point of C_u contained in $(\partial C_1 \cup \partial C_2) \cap \partial C_u$, is also contained in $\overline{\text{Sing}C_u}$. It will follow that

$$\text{Ext}C_u \subset \overline{\text{Sing}C_u},$$

and by the Krein-Milman theorem, we will have

$$C_u = \text{Conv}\big(\text{Ext}C_u\big) \subset \text{Conv}(\overline{\text{Sing}C_u}).$$

The inverse inclusion $\text{Conv}(\overline{\text{Sing}C_u}) \subset C_u$ is obvious.

The set $(\partial C_1 \cup \partial C_2) \cap \partial C_u = \partial C_2 \cap \partial C_u$ is the union of the base $\Omega \times \{0\}$, the cylindric surface $\{(x, z) : x \in \partial\Omega,\ 0 \le z \le u(x)\}$ (which may degenerate to a curve), and the upper level set $L = \{(x, M) : u(x) = M\}$. All extreme points of C_u that belong to $\partial C_2 \cap \partial C_u$ are contained in the union of the curves $\partial\Omega \times \{0\}$, $\text{graph}(u\rfloor_{\partial\Omega}) = \{(x, u(x)) : x \in \partial\Omega\}$, and ∂L (the two former curves may coincide and the latter one may degenerate to a line segment); see Fig. 2.

Each point of the curve $\partial\Omega \times \{0\}$ is singular, since there are at least two, horizontal and vertical, planes of support through it. Each point of ∂L is singular, since, in view of statement P_2, there are at least two planes of support through it: one of them is horizontal and the other one has the slope ≥ 1. It remains to show that each extreme point of the curve $\text{graph}(u\rfloor_{\partial\Omega}) = \{(x, u(x)) : x \in \partial\Omega\}$ belongs to $\overline{\text{Sing}C_u}$.

Each point $\xi \in \partial C_u$ is a convex combination of at most three extreme points of C_u: $\xi = \lambda_1\xi_1 + \lambda_2\xi_2 + \lambda_3\xi_3$; $\xi_1, \xi_2, \xi_3 \in \text{Ext}C_u$, $\lambda_1 \ge 0$, $\lambda_2 \ge 0$, $\lambda_3 \ge 0$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$. (It may happen that some of the points coincide.) Let ξ be a point of the graph of u,

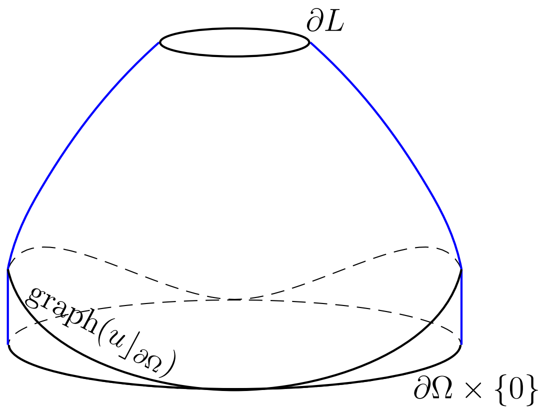

Figure 2: The set C_u and the curves $\partial\Omega \times \{0\}$, $\text{graph}(u\rfloor_{\partial\Omega})$, and ∂L.

$\xi = (x, u(x))$, with x in the interior of Ω; then the corresponding points ξ_1, ξ_2, ξ_2 also lie on the graph, $\xi_i = (x_i, u(x_i))$, $x_i \in \Omega$, $i = 1, 2, 3$. Let us show that for all i, $\xi_i \in \overline{\text{Sing}C_u}$.

Indeed, if x_i is in the interior of Ω and $u(x_i) < M$, then by Theorem 2, the point ξ_i lies in $\overline{\text{Sing}C_u}$. If $u(x_i) = M$, then by statement P_2, $\xi_i \in \text{Sing}C_u$. Finally, if $x_i \in \partial\Omega$, there are at least two planes of support at ξ_i; one of them is vertical, and the other one contains the line through ξ and ξ_i, and so, $\xi_i \in \text{Sing}C_u$.

Now take an extreme point $\xi = (x, u(x))$ with $x \in \partial\Omega$, and consider a sequence of points $\xi_k = (x_k, u(x_k))$ converging to ξ, with x_k in the interior of Ω. Each of these points is the convex combination of three extreme points (some of them may coincide), $\xi_k = \lambda_{1k}\xi_{1k} + \lambda_{2k}\xi_{2k} + \lambda_{3k}\xi_{3k}$, $\lambda_{1k} \geq 0$, $\lambda_{2k} \geq 0$, $\lambda_{3k} \geq 0$, $\lambda_{1k} + \lambda_{2k} + \lambda_{3k} = 1$. We have proved that $\xi_{ik} \in \overline{\text{Sing}C_u}$ for all i and k. Without loss of generality assume that there exist the limits $\lim_{k\to\infty} \lambda_{ik} = \lambda_i^*$, $\lim_{k\to\infty} \xi_{ik} = \xi_i^*$. It follows that $\xi = \lambda_1^*\xi_1^* + \lambda_2^*\xi_2^* + \lambda_3^*\xi_3^*$ and $\xi_i^* \in \overline{\text{Sing}C_u}$ for $i = 1, 2, 3$. Since the point ξ is extreme, this convex combination is degenerate and ξ coincides with one of the points ξ_i^*, and therefore, belongs to $\overline{\text{Sing}C_u}$. This finishes the proof of Corollary 4. □

Remark 7. *Concerning generalized Newton's problem with* $\Omega = \{|x| \leq 1\}$*, one can draw the following conclusion. If the numerical observation that there are only one or two distinct (up to a rotation) solutions is true, then each solution* u *satisfies* (6)*.*

In fact, numerical study seems to indicate that (at least for $M < 1.5$*) the optimal body is indeed the convex hull of the union of several curves composed of singular points: the circumference* $\partial\Omega \times \{0\}$*, the boundary of a regular polygon in the horizontal plane* $\{z = M\}$*, and several convex curves joining each vertex of the polygon with a point of the circumference.*

2 Proof of Theorem 1

Assume the contrary, that is, there is a point $x_0 \in \mathcal{U}$ such that $D^2u(x_0) > 0$. It is not an interior point of the set $\{u = M\}$, hence there is a sequence of points $x^{(i)} \in \mathcal{U}$, with $u(x^{(i)}) < M$, converging to x_0. (If $u(x_0) < M$, one can take the constant sequence $x^{(i)} = x_0$). Fix the value i sufficiently large, so as $D^2u(x^{(i)}) > 0$.

Changing, if necessary, the orthogonal system of coordinates, one can put $x^{(i)} = (0, 0)$. For $\delta > 0$ and $c > 0$ sufficiently small one has $D^2u(x) \ge c$ and $u''_{x_1x_1}(x) \le -c$ when $|x| \le \delta$, and additionally, the circle $|x| \le \delta$ is contained in $\mathcal{U}$.

Take a C^2 function $h : \mathbb{R}^2 \to \mathbb{R}$ equal to zero outside the circle $|x| \le \delta$. For $|t|$ sufficiently small, $D^2\big(u(x) + th(x)\big) > 0$ and $u''_{x_1x_1}(x) + th''_{x_1x_1}(x) < 0$, and therefore, the function $u + th$ is concave. Besides, taking $|t|$ sufficiently small, one can ensure that $0 < u(x) + th(x) < M$ for all x.

Since u minimizes the functional $\mathcal{F}$, we have

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{F}[u + th] = \frac{d^2}{dt^2}\Big|_{t=0} \iint_\Omega g(\nabla u(x_1, x_2) + t\nabla h(x_1, x_2))\, dx_1dx_2$$

$$= \frac{1}{2} \iint_{\mathbb{R}^2} \nabla h(x)^T D^2g(\nabla u(x))\nabla h(x)\, dx_1dx_2 \ge 0$$

(we represent the gradient as a row vector, $\nabla h = (h_x, h_y)$).

Now taking $h(x) = \phi(x/\varepsilon)$, where $0 < \varepsilon < 1$ and ϕ is a C^2 function vanishing outside the circle $|x| \le \delta$, and making the change of variable $x = \varepsilon y$, one obtains

$$\iint_{\mathbb{R}^2} \nabla\phi(y)D^2g(\nabla u(\varepsilon y))\nabla\phi(y)^T dy_1dy_2 \ge 0.$$

Passing to the limit $\varepsilon \to 0$ one gets

$$\iint_{\mathbb{R}^2} \nabla\phi(y)D^2g(\nabla u(0))\nabla\phi(y)^T dy_1dy_2 \ge 0.$$

Take the change of variables $y = \Lambda\chi$, with Λ being an orthogonal matrix with $\det\Lambda = 1$ diagonalizing the matrix $D^2g(\nabla u(0))$, that is,

$$\Lambda^T D^2g(\nabla u(0))\Lambda = \begin{bmatrix} a & 0 \\ 0 & -b \end{bmatrix} \quad \text{with} \quad b > 0.$$

Denoting $\psi(\chi) = \phi(\Lambda\chi)$ and taking into account that $\nabla\psi(\chi) = \nabla\phi(\Lambda\chi)\Lambda$, one comes to the inequality

$$\iint_{\mathbb{R}^2} \nabla\psi(\chi) \begin{bmatrix} a & 0 \\ 0 & -b \end{bmatrix} \nabla\psi(\chi)^T d\chi_1d\chi_2$$

$$= a\iint_{\mathbb{R}^2} \psi'_{\chi_1}(\chi)^2\, d\chi_1d\chi_2 - b\iint_{\mathbb{R}^2} \psi'_{\chi_2}(\chi)^2\, d\chi_1d\chi_2 \ge 0. \tag{7}$$

Now let $\psi(\chi) = \psi(\chi, \tau) = \gamma(\chi_1)\gamma(\chi_2)\sin(\chi_2/\tau)$, where $\gamma : \mathbb{R} \to \mathbb{R}$ is a smooth function vanishing outside a small neighborhood of 0 and $\tau \neq 0$. The former integral is uniformly bounded over all τ; indeed,

$$\iint_{\mathbb{R}^2} (\psi'_{\chi_1}(\chi))^2\, d\chi_1 d\chi_2 \le \int \gamma'^2(\chi_1)\, d\chi_1 \int \gamma^2(\chi_2)\, d\chi_2,$$

whereas the latter one goes to infinity as $\tau \to 0$,

$$\iint_{\mathbb{R}^2} (\psi'_{\chi_2}(\chi))^2\, d\chi_1 d\chi_2 = \frac{1}{\tau^2} \int \gamma^2(\chi_1)\, d\chi_1 \int \gamma^2(\chi_2) \cos^2(\chi_2/\tau)\, d\chi_2 + O(1/\tau)$$

$$\to +\infty \quad \text{as} \quad \tau \to 0.$$

It follows that the left hand side in (7) is negative for $|\tau|$ sufficiently small. The contradiction finishes the proof.

3 Proof of Theorem 2

The main idea of the proof consists in the procedure which is called *stretching the nose* and is described in Lemma 5. To the best of our knowledge, this procedure is new.

Namely, suppose that the body C is a solution to problem (3) but does not satisfy (5). Take a point $\xi \in \text{Ext}C \setminus (\partial C_1 \cup \partial C_2 \cup \overline{\text{Sing}C})$, then choose a point O outside C sufficiently close to ξ, and define a 1-parameter family of bodies $C(s)$, $0 \le s \le 1$ with the endpoints at $C(0) = C$ and $C(1) = \text{Conv}(C \cup \{O\})$; see Fig. 5. The corresponding family of measures $\nu_{\partial C(s)}$ is a line segment. We prove that all bodies $C(s)$ are solutions to problem (3). This is the main point in the proof of Theorem 2.

We also prove that the set $\text{Ext}C \setminus (\partial C_1 \cup \partial C_2 \cup \overline{\text{Sing}C})$ does not have isolated points, and therefore, is infinite (Lemma 1 and Corollary 5). Using this fact, in Lemma 4 we find an infinite sequence $\xi_1, \xi_2, \ldots$ of points dense in this set and a sequence of points $O_1, O_2, \ldots$ outside C (with each point O_j being sufficiently close to ξ_j) so as the "noses" $\text{Conv}(C \cup \{O_j\}) \setminus C$ are mutually disjoint and the body $\widehat{C} = \text{Conv}(C \cup \{O_1, O_2, \ldots\})$ satisfies (5). We define a family of intermediate convex bodies $C(\vec{s})$, $\vec{s} \in [0, 1]^\infty$, with $C(\vec{0}) = C$ and $C(\vec{1}) = \widehat{C}$, such that $C(\underbrace{0, \ldots, 0, 1}_{j}, 0, \ldots) = \text{Conv}(C \cup \{O_j\})$ and the corresponding family of surface area measures is linear, and prove that all bodies of the family are solutions to problem (3).

Denote by $B_r(a)$ the open ball with radius r and with the center at a. Let C be a convex body.

Lemma 1. *The set* $\text{Ext}C \setminus \overline{\text{Sing}C}$ *does not have isolated points.*

Proof. Assume the contrary and let ξ be an isolated point of $\mathrm{Ext}C \setminus \overline{\mathrm{Sing}C}$; then for some $\varepsilon > 0$, the punctured neighborhood $B_\varepsilon(\xi) \setminus \{\xi\}$ does not intersect $\mathrm{Ext}C \cup \overline{\mathrm{Sing}C}$.

By Minkowski's Theorem, $C = \mathrm{Conv}(\mathrm{Ext}C)$. Further, we have

$$\mathrm{Ext}C \subset \big(C \setminus B_\varepsilon(\xi)\big) \cup \{\xi\} \subset \mathrm{Conv}\big(C \setminus B_\varepsilon(\xi)\big) \cup \{\xi\},$$

hence

$$C = \mathrm{Conv}(\mathrm{Ext}C) \subset \mathrm{Conv}\Big[\mathrm{Conv}\big(C \setminus B_\varepsilon(\xi)\big) \cup \{\xi\}\Big],$$

and therefore, we have the equality

$$C = \mathrm{Conv}\Big[\mathrm{Conv}\big(C \setminus B_\varepsilon(\xi)\big) \cup \{\xi\}\Big]. \tag{8}$$

Since the point ξ lies outside $\mathrm{Conv}\big(C \setminus B_\varepsilon(\xi)\big)$, it is a singular point of the convex body in the right hand side of (8). This contradicts the assumption that ξ is not a singular point of ∂C. $\square$

Consider three convex bodies C, C_1, and C_2. Taking into account that the sets ∂C_1 and ∂C_2 are closed, we immediately obtain the following corollary of Lemma 1.

Corollary 5. *The set* $\mathrm{Ext}C \setminus (\partial C_1 \cup \partial C_2 \cup \overline{\mathrm{Sing}C})$ *does not have isolated points.*

The following Lemmas 2 and 3 are technical; they will be used in the proofs of Lemmas 4 and 5.

Lemma 2. *Let* $\{O_1, O_2, \ldots\}$ *be a finite or countable set of points outside* C *such that for all* $i \neq j$ *the intersection of the open line interval* (O_i, O_j) *with* C *is not empty. Denote*

$$\tilde{C} = \mathrm{Conv}(C \cup \{O_1, O_2, \ldots\}).$$

Then

(a)

$$\tilde{C} = \cup_i \mathrm{Conv}(C \cup \{O_i\}).$$

(b) If the set of points O_i *is finite,* $\xi \in \mathrm{Ext}C \setminus \overline{\mathrm{Sing}C}$*, and for all* i *the intersection of the interval* (O_i, ξ) *with* C *is not empty, then* $\xi \in \mathrm{Ext}\tilde{C} \setminus \overline{\mathrm{Sing}\tilde{C}}$.

(c)

$$\{O_1, O_2, \ldots\} \subset \mathrm{Ext}\tilde{C} \subset \mathrm{Ext}C \cup \{O_1, O_2, \ldots\}.$$

(d) Assume that for all i *and all* $\xi \in \mathrm{Sing}C$ *the intersection of the interval* (O_i, ξ) *with* C *is not empty. Then*

$$\mathrm{Sing}C \cup \{O_1, O_2, \ldots\} = \mathrm{Sing}\tilde{C}.$$

Proof. (a) Take a point $x \in \tilde{C}$. One needs to prove that for some j, $x \in \operatorname{Conv}(C \cup \{O_j\})$. Two cases are possible: either $x \in C$, or x is a convex combination

$$x = \lambda x_0 + \sum_{i=1}^{m} \lambda_i O_i, \quad m \geq 1, \tag{9}$$

where $x_0 \in C$, $\lambda \geq 0$, $\lambda_i > 0$ for all i, and $\lambda + \sum_1^m \lambda_i = 1$. In the former case there is nothing to prove. In the latter case assume, without loss of generality, that the convex combination in the right hand side of (9) is the shortest one, that is, m cannot be made smaller. Let us show that $m = 1$.

Indeed, suppose that $m \geq 2$. Since the interval (O_1, O_2) intersects C, we have $\mu O_1 + (1-\mu)O_2 = \hat{x} \in C$ for some $0 < \mu < 1$. Assume without loss of generality that $\lambda_1/\lambda_2 \geq \mu/(1-\mu)$; then

$$\lambda_1 O_1 + \lambda_2 O_2 = \frac{\lambda_2}{1-\mu}\,\hat{x} + \lambda_2\Big(\frac{\lambda_1}{\lambda_2} - \frac{\mu}{1-\mu}\Big)O_1,$$

and the convex combination in (9) can be shortened,

$$x = \tilde{\lambda}\tilde{x} + \tilde{\lambda}_1 O_1 + \sum_{i=3}^{m} \lambda_i O_i,$$

where

$$\tilde{\lambda} = \lambda + \frac{\lambda_2}{1-\mu}, \quad \tilde{x} = \frac{\lambda(1-\mu)}{\lambda(1-\mu)+\lambda_2}\,x_0 + \frac{\lambda_2}{\lambda(1-\mu)+\lambda_2}\,\hat{x} \in C, \quad \tilde{\lambda}_1 = \lambda_2\Big(\frac{\lambda_1}{\lambda_2} - \frac{\mu}{1-\mu}\Big).$$

(If $m = 2$, the sum $\sum_3^m$ equals zero.)

This contradiction shows that $m = 1$; that is, for some j,

$$x = \lambda x_0 + \lambda_j O_j \in \operatorname{Conv}(C \cup \{O_j\}) \quad \text{with } \lambda \geq 0,\ \lambda_j > 0,\ \lambda + \lambda_j = 1.$$

Claim (a) is proved.

(b) Assume that ξ is not an extreme point of $\tilde{C}$. This means that ξ is an interior point of a line segment $[\xi_1, \xi_2] \subset \tilde{C}$. We are going to prove that each of the semiopen intervals $[\xi_1, \xi)$ and $[\xi_2, \xi)$ contains a point of C, and therefore, ξ is not an extreme point of C, in contradiction with the assumption that $\xi \in \operatorname{Ext}C$.

Suppose that $\xi_1 \notin C$. Since the point ξ_1 is in $\tilde{C}$, then by claim (a), for some i we have $\xi_1 \in [O_i, x_1)$, where $x_1 \in C$. On the other hand, by the hypothesis of the lemma, a point $x \in (O_i, \xi)$ lies in C.

If the triangle $\xi O_i x_1$ is non-degenerate then the segments $[x, x_1]$ and (ξ, ξ_1) intersect at a point ξ_1'. Since x and x_1 lie in C, ξ_1' also belongs to C.

If, otherwise, the points ξ, O_i, x_1 are collinear then $\mathrm{Conv}(\xi, x, x_1)$ is a segment on the line $O_i\xi$ and is contained in C (since the points ξ, x, x_1 lie in C). The point $\xi_1(\notin C)$ lies on this line between the segment and the point O_i; hence $\xi_1 \in [O_i,\, \xi)$. The same inclusion holds for x; it follows that $x \in (\xi,\, \xi_1)$.

Thus, in any case a point of the segment $[\xi_1,\, \xi)$ (either ξ_1, or ξ_1', or x) belongs to C. The same argument holds for the segment $[\xi_2,\, \xi)$. Hence we have $\xi \notin \mathrm{Ext}C$. This contradiction proves that $\xi \in \mathrm{Ext}\tilde{C}$.

It remains to prove that $\xi \notin \overline{\mathrm{Sing}\tilde{C}}$. Indeed, assume the contrary; then ξ is the limit of a sequence $\xi_i \in \mathrm{Sing}\tilde{C}$. Since the set of points O_j is finite, there is a value j such that infinitely many points ξ_i are contained in $\mathrm{Conv}(C \cup \{O_j\})$. Additionally, for i sufficiently large, ξ_i do not coincide with O_j.

There is a neighborhood of ξ that does not contain singular points of ∂C. If a regular point of ∂C belongs to $\partial\tilde{C}$, then it is also a regular point of $\partial\tilde{C}$. It follows that for i sufficiently large, ξ_i are not contained in ∂C.

Thus, without loss of generality one can assume that all points ξ_i lie in $\mathrm{Conv}(C \cup \{O_j\}) \setminus (C \cup \{O_j\})$, and therefore, for some $x_i \in C$, $\xi_i \in (x_i,\, O_j)$. Taking if necessary a subsequence, we assume that x_i converge to a certain $x \in C$, and therefore, $\xi \in [x,\, O_j)$. By the hypothesis of the lemma, there is a point $x' \in C$ contained in $(\xi,\, O_j)$. We have $\xi \in [x,\, x') \subset C$. Since ξ is an extreme point of C, we conclude that $\xi = x$.

Each plane of support to $\tilde{C}$ at ξ_i is also a plane of support to $\tilde{C}$ at each point of the segment $[x_i,\, O_j]$, and in particular, at x_i. It is also a plane of support to C at x_i. Since $\xi_i \in \mathrm{Sing}\tilde{C}$, there are more than one such plane, and therefore, $x_i \in \mathrm{Sing}C$. It follows that $\xi = \lim_{i\to\infty} x_i \in \overline{\mathrm{Sing}C}$. The obtained contradiction proves claim (b).

(c) The set $\partial\tilde{C}$ is the union of (i) a part of the boundary ∂C, (ii) open segments of the form $(O_i,\, \xi)$, where $\xi \in \partial C$ and the segment lies in a plane of support to C through O_i, and (iii) the points O_i.

If a point $\xi \in \mathrm{Ext}\tilde{C}$ belongs to ∂C then it is also an extreme point of C. Open segments of the form $(O_i,\, \xi)$ obviously do not contain extreme points of $\tilde{C}$, and all points O_i are extreme points of $\tilde{C}$. Claim (c) is proved.

(d) Obviously, each point O_i is a singular point of $\partial\tilde{C}$.

Take a point $\xi \in \mathrm{Sing}C$ and take a plane of support to C at ξ. Since for all i, $(O_i,\, \xi) \cap C \neq \emptyset$, we conclude that the body C and all points O_i lie in the same half-space bounded by the plane. It follows that this plane is also a plane of support for the body $\tilde{C} = \mathrm{Conv}(C \cup \{O_1,\, O_2, \ldots\})$.

Thus, each plane of support to C at ξ is also a plane of support to $\tilde{C}$. Since such a plane is not unique, we conclude that $\xi \in \mathrm{Sing}\tilde{C}$. Hence $\mathrm{Sing}C \cup \{O_1,\, O_2, \ldots\} \subset \mathrm{Sing}\tilde{C}$.

Now suppose that $x \in \mathrm{Sing}\tilde{C}$. There may be three cases: (i) $x \in \partial C$; (ii) x is contained in an open segment $(O_i,\, \xi)$, where $\xi \in \partial C$ and $(O_i,\, \xi) \cap C = \emptyset$; (iii) x coincides with a point O_i.

In the case (i) each plane of support to $\tilde{C}$ at x is also a plane of support to C at x.

Since it is not unique, we have $x \in \text{Sing}C$.

The case (ii) is impossible. Indeed, otherwise each plane of support to $\tilde{C}$ at x is also a plane of support to C at ξ. Since it is not unique, we have a contradiction with the hypothesis in item (d) of Lemma 2 stating that $(O_i,\, \xi) \cap C \neq \emptyset$.

It follows that $\text{Sing}C \cup \{O_1,\, O_2, \ldots\} \supset \text{Sing}\tilde{C}$. Claim (d) is proved. $\square$

In what follows, *dist* means the Euclidean distance between two points.

Lemma 3. *Consider a point $\xi \in \text{Ext}C \setminus \overline{\text{Sing}C}$ and a closed set $A \subset C$ that does not contain ξ. Then for any $\varepsilon > 0$ there exists a point O outside C such that*

(a) dist$(\xi, O) < \varepsilon$;

(b) for any $x \in A$, the intersection of the open segment $(O,\, x)$ with the interior of C is not empty.

Proof. The convex hull $\text{Conv}(C \setminus B_\varepsilon(\xi))$ does not contain ξ. Making if necessary ε sufficiently small, we can assume that the set $A \cup \overline{\text{Sing}C}$ does not intersect $B_\varepsilon(\xi)$, and therefore, is contained in $\text{Conv}(C \setminus B_\varepsilon(\xi))$.

Take a plane Π that separates the point ξ and the set $\text{Conv}(C \setminus B_\varepsilon(\xi))$. Let this plane be given by $\langle x,\, n\rangle = c$, with ξ being contained in the half-space $\langle x,\, n\rangle > c$ and $\text{Conv}(C \setminus B_\varepsilon(\xi))$, in the complementary half-space $\langle x,\, n\rangle < c$. Here and in what follows, $\langle\cdot\,,\cdot\rangle$ means the scalar product. See Fig. 3.

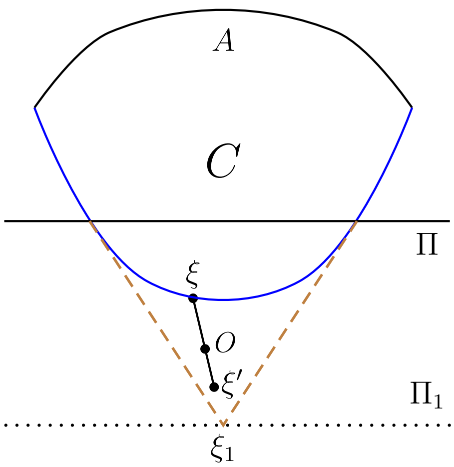

Figure 3: The plane Π separates ξ and A. The set K is bounded above by Π and below by the dashed line. The part of C below Π is contained in the ε-neighborhood of ξ.

Draw the tangent planes to C through all points of $\partial C \cap \Pi$ (which are regular); the intersection of the half-space $\langle x,\, n\rangle \geq c$ and all closed half-spaces bounded by these planes and containing C is a closed convex set containing ξ. Let it be denoted by K.

Let us show that $K \setminus C$ is not empty. If K is unbounded, this is obvious. If K is bounded, draw the plane of support to K with the outward normal n and denote it by Π_1. Thus, K is contained between the planes Π and Π_1; see Fig. 3.

Take a point ξ_1 in the intersection $\Pi_1 \cap \partial K$. There is at least one more plane of support to K through a point of $\partial C \cap \Pi$ that contains ξ_1. It follows that ξ_1 is a singular point of ∂K. Hence it does not belong to C, since otherwise it is also a singular point of ∂C. Thus, $\xi_1 \in K \setminus C$.

Take a point ξ' in the interior of $K \setminus C$. Draw the line segment $[\xi, \xi']$ and find a point O on it that lies outside C and belongs to $B_\varepsilon(\xi)$. Thus, condition (a) is satisfied, due to the choice of O.

Take a point $x \in A$. The point x lies in the intersection of closed half-spaces bounded by the tangent planes to C through all points of $\partial C \cap \Pi$ and containing C, and O lies in the intersection of the corresponding open half-spaces. It follows that the point of intersection of the interval (O, x) with the plane Π lies in the interior of the planar set $C \cap \Pi$, and therefore, belongs to the interior of C. Thus, condition (b) is also satisfied. $\square$

Assume that we are given three convex bodies $C_1 \subset C \subset C_2$.

Lemma 4. *Suppose that the set $E := \mathrm{Ext}C \setminus (\partial C_1 \cup \partial C_2 \cup \overline{\mathrm{Sing}C})$ is not empty and choose a point $\hat{\xi} \in E$. Then there exists an infinite sequence of points $\{O_j,\, j \in J \subset \mathbb{N}\}$ in $C_2 \setminus C$ such that*

(a) for all $i \neq j$ and all $\xi \in C_1 \cup \overline{\mathrm{Sing}C} \cup \{\hat{\xi}\}$, the intersections of the open segments (O_i, O_j) and (O_i, ξ) with the interior of C are not empty;

(b) for $\widehat{C} = \mathrm{Conv}(C \cup \{O_j,\, j \in J\})$ one has

$$\mathrm{Ext}\widehat{C} \subset \partial C_1 \cup \partial C_2 \cup \overline{\mathrm{Sing}\widehat{C}}.$$

Proof. Take a sequence of positive values ε_n converging to 0. Choose a finite sequence of open sets (for example, open balls) $D_1, \ldots, D_{j_1}$ in $\mathbb{R}^3$, each set of diameter less than ε_1, such that the union of the sets contains E. Next we define a finite sequence of open sets $D_{j_1+1}, \ldots, D_{j_2}$, each set of diameter less than ε_2, such that their union contains E. Continuing this process, we obtain infinite sequences of integers $0 = j_0 < j_1 < j_2 < \ldots$ and sets $D_1, D_2, \ldots$ such that for each $n \geq 1$, the diameter of each of the domains $D_{j_{n-1}+1}, \ldots, D_{j_n}$ is less than ε_n and

$$E \subset \bigcup_{i=j_{n-1}+1}^{j_n} D_i.$$

We are going to define inductively an infinite set of natural numbers $J \subset \mathbb{N}$ and a sequence of points O_j, $j \in J$, in $C_2 \setminus C$ satisfying condition (a). Denote $\{1, \ldots, m\}' := J \cap \{1, \ldots, m\}$. Define the sets

$$\widehat{C}^m := \mathrm{Conv}\big(C \cup \big\{O_j,\, j \in \{1, \ldots, m\}'\big\}\big) \ \text{ and } \ E_m := \mathrm{Ext}\widehat{C}^m \setminus (\partial C_1 \cup \partial C_2 \cup \overline{\mathrm{Sing}\widehat{C}^m})$$

(in particular, $\widehat{C}^0 = C$ and $E_0 = E$); we additionally require that the sets E_m contain $\hat{\xi}$ and are nested, that is, for $m_1 \le m_2$ we have $E_{m_2} \subset E_{m_1} \subset E$.

For $m = 0$ the set of points O_i is empty, and therefore, condition (a) is trivially satisfied, and $\hat{\xi} \in E = E_0$. Now suppose that for a certain integer $m \ge 0$, the set $\{1, \dots, m\}'$ is defined, the points O_j, $j \in \{1, \dots, m\}'$ in $C_2 \setminus C$ satisfying condition (a) are chosen, and the inclusions $\hat{\xi} \in E_m \subset \ldots \subset E_1 \subset E_0 = E$ take place. If $E_m \cap D_{m+1} = \emptyset$, let $m + 1 \notin J$. In this case the statement of induction for $m + 1$ is trivially satisfied.

If, otherwise, the set $E_m \cap D_{m+1}$ is not empty, let $m + 1 \in J$ and take a point ξ_{m+1} from this set distinct from $\hat{\xi}$. (Such a point exists, since by Corollary 5, the set E_m does not have isolated points, and therefore, $E_m \cap D_{m+1}$ is not a singleton.) Let n be such that $m + 1 \in \{j_{n-1} + 1, \dots, j_n\}$. Applying Lemma 3 to the convex body $\widehat{C}^m$, the point ξ_{m+1}, and the set $A = C_1 \cup \overline{\mathrm{Sing}\widehat{C}^m} \cup \{\hat{\xi}\}$, take a point O_{m+1} in $C_2 \setminus \widehat{C}^m$ such that (a) $\mathrm{dist}(\xi_{m+1}, O_{m+1}) < \varepsilon_n$; (b) for all $x \in C_1 \cup \overline{\mathrm{Sing}\widehat{C}^m} \cup \{\hat{\xi}\}$, the intersections of the open segment $(O_{m+1},\, x)$ with the interior of C is not empty.

By the hypothesis of induction, for $i \in \{1, \dots, m\}'$ and $\xi \in \mathrm{Sing}C$, the intersection of the interval $(O_i,\, \xi)$ with the interior of C is not empty. Hence by claim (d) of Lemma 2,

$$\mathrm{Sing}C \cup \big\{O_i,\, i \in \{1, \dots, m\}'\big\} = \mathrm{Sing}\widehat{C}^m.$$

It follows that for any $\xi \in C_1 \cup \overline{\mathrm{Sing}C} \cup \{\hat{\xi}\}$, the intersection of the open segment $(O_{m+1},\, \xi)$ with the interior of C is not empty, and for any $i \in \{1, \dots, m\}'$, the intersection of $(O_{m+1},\, O_i)$ with the interior of C is not empty. Thus, condition (a) is satisfied for the extended sequence of points O_i, $i \in \{1, \dots, m + 1\}' = \{1, \dots, m\}' \cup \{m + 1\}$.

By claim (c) of Lemma 2, $\mathrm{Ext}\widehat{C}^{m+1} \subset \mathrm{Ext}\widehat{C}^m \cup \{O_{m+1}\}$, and by claim (d) of the same lemma, $\mathrm{Sing}\widehat{C}^{m+1} = \mathrm{Sing}\widehat{C}^m \cup \{O_{m+1}\}$. It follows that

$$E_{m+1} = \mathrm{Ext}\widehat{C}^{m+1} \setminus (\partial C_1 \cup \partial C_2 \cup \overline{\mathrm{Sing}\widehat{C}^{m+1}}) \subset \mathrm{Ext}\widehat{C}^m \setminus (\partial C_1 \cup \partial C_2 \cup \overline{\mathrm{Sing}\widehat{C}^m}) = E_m.$$

Further, since $\hat{\xi} \in E_m \subset \mathrm{Ext}\widehat{C}^m \setminus \overline{\mathrm{Sing}\widehat{C}^m}$ and the intersection of the interval $(O_{m+1},\, \hat{\xi})$ with C is non-empty, making use of claim (b) of Lemma 2 we conclude that $\hat{\xi} \in \mathrm{Ext}\widehat{C}^{m+1} \setminus \overline{\mathrm{Sing}\widehat{C}^{m+1}}$. It follows that $\hat{\xi} \in E_{m+1}$. The statement of induction is completely proved for $m + 1$.

In each subsequence $\{j_{n-1} + 1, \dots, j_n\}$ there is a number m such that D_m contains $\hat{\xi}$, and therefore, $E_{m-1} \cap D_m$ is not empty. It follows that $m \in J$; hence the parameter set J is infinite.

We have proved that the sequence of points O_j, $j \in J$ satisfies claim (a) of Lemma 4.

Consider the set $\widehat{C} = \mathrm{Conv}(C \cup \{O_j,\, j \in J\})$. For each $m = 0, 1, 2, \ldots$ we have $\widehat{C} = \mathrm{Conv}(\widehat{C}^m \cup \{O_j,\, j(\in J) \ge m + 1\})$. Applying claims (c) and (d) of Lemma 2 to the convex bodies $\widehat{C}$ and $\widehat{C}^m$ and to the set of points $\{O_j,\, j(\in J) \ge m + 1\}$, we obtain

$$\mathrm{Ext}\widehat{C} \subset \mathrm{Ext}\widehat{C}^m \cup \{O_j,\, j(\in J) \ge m+1\} \quad \text{and} \quad \mathrm{Sing}\widehat{C} = \mathrm{Sing}\widehat{C}^m \cup \{O_j,\, j(\in J) \ge m+1\},$$

hence

$$E_\infty := \mathrm{Ext}\widehat{C} \setminus (\partial C_1 \cup \partial C_2 \cup \overline{\mathrm{Sing}\widehat{C}}) \subset \mathrm{Ext}\widehat{C}^m \setminus (\partial C_1 \cup \partial C_2 \cup \overline{\mathrm{Sing}\widehat{C}^m}) = E_m \subset E.$$

It remains to prove that E_∞ is empty.

Assume the contrary and take $\xi \in E_\infty$. For any natural n there is $m_n \in \{j_{n-1} + 1, \dots, j_n\}$ such that D_{m_n} contains ξ. It follows that the set $E_{m_n-1} \cap D_{m_n} \supset E_\infty \cap D_{m_n} \ni \xi$ is non-empty, and therefore, $m_n \in J$. For the points $\xi_{m_n} \in D_{m_n}$ and O_{m_n} chosen above we have

$$\mathrm{dist}(O_{m_n}, \xi) \le \mathrm{dist}(O_{m_n}, \xi_{m_n}) + \mathrm{dist}(\xi_{m_n}, \xi) < 2\varepsilon_n.$$

It follows that the sequence O_{m_n} converges to ξ as $n \to \infty$. Since by claim (d) of Lemma 2, $\{O_j,\, j \in J\} \subset \mathrm{Sing}\widehat{C}$, we have $\xi \in \overline{\mathrm{Sing}\widehat{C}}$, and therefore, $\xi \notin E_\infty$. The obtained contradiction proves claim (b) of Lemma 4. ☐

The method we use in the following lemma can be called *stretching the nose*.

Let O be a point outside C. For $0 \le s \le 1$ define the set

$$C(s) = \bigcup_{\sqrt{1-s} \le \lambda \le 1} \big(\lambda C + (1-\lambda)O\big). \tag{10}$$

In particular, $C(0) = C$ and $C(1) = \mathrm{Conv}(C \cup \{O\})$.

It is easy to see that $C(s)$ is a convex body.

The sets $C(0)$, $C(1)$, and $C(s)$ with $0 < s < 1$ are shown light gray in Figs. 4(a), 4(b), and 5(a), respectively.

Lemma 5. *Let C be a solution to problem* (3) *and let a point $O \in C_2 \setminus C$ be such that for any $x \in C_1 \cup \overline{\mathrm{Sing}C}$, the intersection of the open segment $(O,\, x)$ with the interior of C is not empty. Then*

(a) all convex bodies $C(s)$, $0 \le s \le 1$ given by (10) *are also solutions to problem* (3)*;*

(b) the measures $\nu_{\partial C(s)}$, $s \in [0,\, 1]$ form a linear segment: $\nu_{\partial C(s)} = \nu_{\partial C} + s\nu_0$, where ν_0 is a signed measure on S^2.

This lemma is the main point in the proof of Theorem 2. In turn, the main point in the proof of this lemma is extension of the family of admissible convex bodies $C(s)$ to negative values of s and the statement that the composite function $F(\partial C(s))$ is linear for $s \in [0,\, 1]$ and differentiable at $s = 0$.

Proof. The sets $C(s)$ can be defined in another way. Draw all the rays with vertex at O that intersect C. The union of these rays is a closed convex cone. Denote by A and A' the initial (closer to O) and the final points of intersection of a generic ray with C. If the ray is tangent then its intersection with C is the line segment $[A, A']$ (which may degenerate to a point if $A = A'$). Otherwise, the intersection is the 2-point set $\{A,\, A'\}$.

Denote by C_- the union of the segments OA' of all rays, and by $\partial_+ C$ the union of the corresponding points A'. Denote by V the surface composed of the segments $[O, A']$ contained in the tangent rays. The boundary of C_- is the union $\partial_+ C \cup V$.

For each ray OA, denote by AA' the ray contained in OA with the vertex at A. Denote by C_+ the union of the rays AA', and by $\partial_- C$ the union of the points A corresponding to all rays and the segments $[A, A']$ contained in the tangent rays. The boundary of C_+ is the union of $\partial_- C$ and the rays with the vertices at the points A' contained in the tangent rays OA'.

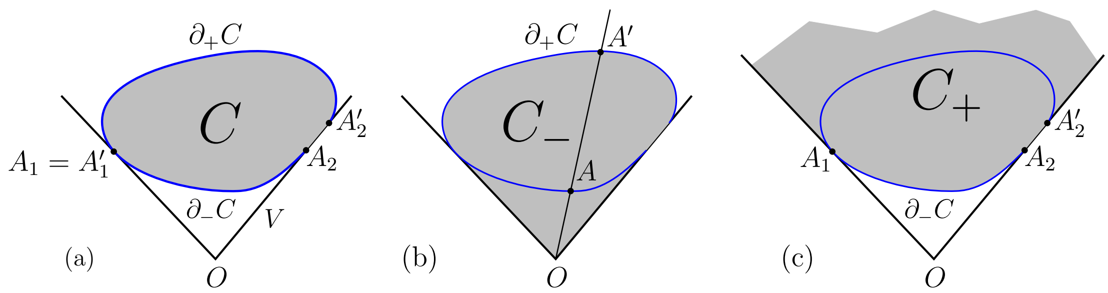

Figure 4: The sets C, C_-, and C_+ are shown in figures (a), (b), and (c), respectively. The upper curve $A_1 A_2'$ is $\partial_+ C$, and the lower curve $A_1 A_2 A_2'$ is $\partial_- C$.

We have $C = C_- \cap C_+$; see Fig. 4.

Place the origin at the point O; then tC designates the dilation of C with center O and ratio t. The set $C(s)$ now takes the form (see Fig. 5)

$$C(s) = \begin{cases} C_- \cap \sqrt{1-s}\, C_+, & \text{if } s < 1 \\ C_-, & \text{if } s = 1. \end{cases} \tag{11}$$

In particular, $C(0) = C_- \cap C_+ = C$ and $C(1) = C_- = \mathrm{Conv}(C \cup \{O\})$. Since $C_1 \subset C \subset C(s) \subset \mathrm{Conv}(C \cup \{O\}) \subset C_2$, the bodies $C(s)$, $0 \le s \le 1$ are admissible.

Note that formula (11) defines the body $C(s)$ also for the values $s < 0$.

All segments $[O,\, A]$ and the segments $[O,\, A']$ contained in the tangent rays do not intersect the interior of C, hence by the hypothesis of the lemma, no point of these intervals belongs to $C_1 \cup \overline{\mathrm{Sing} C}$. The set $\partial_- C$ is contained in the union of these intervals, therefore $\partial_- C$ does not intersect $C_1 \cup \overline{\mathrm{Sing} C}$. Since both sets, $\partial_- C$ and $C_1 \cup \overline{\mathrm{Sing} C}$, are compact, for $|s|$ sufficiently small the dilated set $\sqrt{1-s}\ \partial_- C$ also does not intersect $C_1 \cup \overline{\mathrm{Sing} C}$, and therefore in particular, $C_1 \subset C(s) \subset C_2$, that is, $C(s)$ is admissible.

Let us now study the composite function $F(\partial C(s))$. For $0 \le s \le 1$ this function is linear. Indeed, $\partial C(s)$ is composed of the surfaces $\partial_+ C$, $\sqrt{1-s}\, \partial_- C$, and $V \setminus \sqrt{1-s}\, V$ (note that $\sqrt{1-s}\, V \subset V$). The surface area measure of $\partial C(s)$ is $\nu_{\partial C(s)} = \nu_{\partial_+ C} + (1 - s)\nu_{\partial_- C} + s\nu_V$. It can be represented as

$$\nu_{\partial C(s)} = \nu_{\partial C} + s\nu_0, \quad s \in [0,\, 1], \quad \text{where} \ \ \nu_0 = \nu_V - \nu_{\partial_- C}.$$

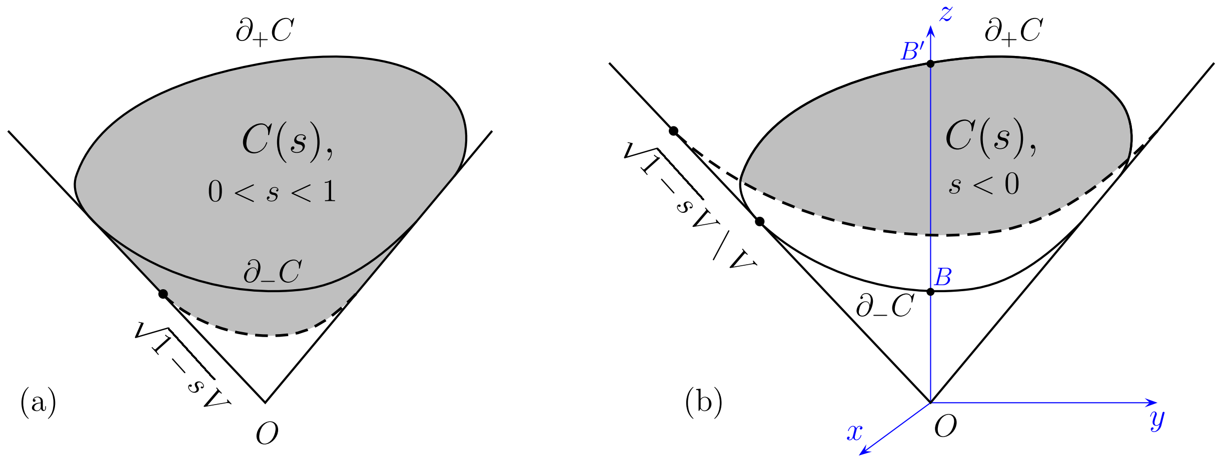

Figure 5: The set $C(s)$ (a) for $0 < s < 1$ and (b) for $s < 0$. The surface $\sqrt{1-s}\,\partial_- C$ is shown as dashed line.

Thus, claim (b) of the lemma is proved.

For $0 \le s \le 1$ we have $F(\sqrt{1-s}\,\partial_- C) = (1-s)F(\partial_- C)$ and $F(V \setminus \sqrt{1-s}\,V) = F(V) - (1-s)F(V) = sF(V)$, therefore

$$F(\partial C(s)) = F(\partial_+ C) + (1-s)F(\partial_- C) + sF(V).$$

Let us now show that the derivative $\frac{d}{ds}\Big\rfloor_{s=0} F(\partial C(s))$ exists.

The calculation of the right derivative is straightforward,

$$\frac{d}{ds}\Big\rfloor_{s=0^+} F(\partial C(s)) = \lim_{s\to 0^+} \frac{F(\partial C(s)) - F(\partial C)}{s} = F(V) - F(\partial_- C).$$

If $s < 0$, the boundary of the convex body $C(s)$ is composed of parts of the surfaces $\partial_+ C$ and $\sqrt{1-s}\,\partial_- C$. Namely,

$$\partial C(s) = (\partial_+ C \cap \sqrt{1-s}\,C_+) \cup (\sqrt{1-s}\,\partial_- C \cap C_-),$$

and the complementary parts of these surfaces, $\partial_+ C \setminus \sqrt{1-s}\,C_+$ and $\sqrt{1-s}\,\partial_- C \setminus C_-$, do not take part of the boundary. Therefore we have

$$\begin{aligned} F(\partial C(s)) &= F(\partial_+ C) + (1-s)F(\partial_- C) - \big[F(\partial_+ C \setminus \sqrt{1-s}\,C_+) + F(\sqrt{1-s}\,\partial_- C \setminus C_-)\big] \\ &= F(\partial_+ C) + (1-s)F(\partial_- C) + sF(V) \\ &\quad - \big[sF(V) + F(\partial_+ C \setminus \sqrt{1-s}\,C_+) + F(\sqrt{1-s}\,\partial_- C \setminus C_-)\big]. \end{aligned}$$

Therefore, the left derivative (if exists) equals

$$\frac{d}{ds}\Big\rfloor_{s=0^-} F(\partial C(s)) = \lim_{s\to 0^-} \frac{F(\partial C(s)) - F(\partial C)}{s} = F(V) - F(\partial_- C) - \lim_{s\to 0^+} \frac{R(s)}{s},$$

where

$$R(s) = sF(V) + F(\partial_+ C \setminus \sqrt{1-s}\, C_+) + F(\sqrt{1-s}\, \partial_- C \setminus C_-)$$
$$= F(\partial_+ C \setminus \sqrt{1-s}\, C_+) + F(\sqrt{1-s}\, \partial_- C \setminus C_-) - F(\sqrt{1-s}\, V \setminus V).$$

Let us prove that $R(s) = o(s)$ as $s \to 0^-$; it will follow that the derivative $\frac{d}{ds}\big\rfloor_{s=0} F(\partial C(s))$ exists and is equal to $F(V) - F(\partial_- C)$.

Draw a straight line through O intersecting the interior of C. Let B and B' be the points of intersection of this line with ∂C, so as the open segment OB is outside C.

Introduce an ortogonal coordinate system with the coordinates x, y, z so as the origin is at O and the z-axis coincides with the axis OB; see Fig. 5 (b). Let D_s, D_s^+, D_s^-, $s < 0$, be the orthogonal projections of $\sqrt{1-s}\, V \setminus V$, $\partial_+ C \setminus \sqrt{1-s}\, C_+$, $\sqrt{1-s}\, \partial_- C \setminus C_-$, respectively, on the xy-plane. The area of D_s equals $-ks$, where k is the area of the corresponding projection of V. For $-s$ sufficiently small, the domains D_s^+ and D_s^- have disjoint interiors and $D_s^+ \cup D_s^- = D_s$, hence

$$\text{Area}(D_s^+) + \text{Area}(D_s^-) = \text{Area}(D_s) = -ks.$$

Denote by $n(x,y) = (n_1(x,y),\, n_2(x,y),\, n_3(x,y))$ the outward normal to $\sqrt{1-s}\, V \setminus V$ at the pre-image of $(x,y) \in D_s$ under the projection. Similarly, let $n^+(x,y) = (n_1^+(x,y),\, n_2^+(x,y),\, n_3^+(x,y))$ and $n^-(x,y) = (n_1^-(x,y),\, n_2^-(x,y),\, n_3^-(x,y))$ be the outward normals to $\partial_+ C \setminus \sqrt{1-s}\, C_+$ and $\sqrt{1-s}\, \partial_- C \setminus C_-$, respectively. The third components of these vectors, $n_3(x,y)$, $n_3^+(x,y)$, and $n_3^-(x,y)$, are negative for $-s$ sufficiently small.

The function $n(x,y)$ is continuous in D_s and is constant in the radial direction. The function $n^+(x,y)$ coincides with $n(x,y)$ on the inner boundary of D_s; in other words, for any $(x,y) \in D_s^+$ there exists $0 < c \le 1$ such that $(cx, cy) \in D_s^+$ and $n(x,y) = n(cx,cy) = n^+(cx,cy)$. The function $n^+(x,y)$ is continuous (and therefore uniformly continuous) in the closure of D_s for $-s$ sufficiently small. Thus, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $0 < -s < \delta$, for all $(x,y) \in D_s^+$, and for a suitable positive $c = c(x,y) \in (0, 1]$ we have $(cx,cy) \in D_s^+$, $n(cx,cy) = n^+(cx,cy)$, and $|n^+(x,y) - n^+(cx,cy)| < \varepsilon$; it follows that for any $(x,y) \in D_s^+$, $|n^+(x,y) - n(x,y)| < \varepsilon$.

A similar reasoning holds for the function $n^-(x,y)$ with $(x,y) \in D_s^-$. As a result we have

$$\sup_{(x,y)\in D_s^+} |n^+(x,y) - n(x,y)| + \sup_{(x,y)\in D_s^-} |n^-(x,y) - n(x,y)| \to 0 \quad \text{as } s \to 0^-.$$

For $-s$ sufficiently small the function $p(n) = \frac{f(n)}{|n_3|}$ is well defined, and therefore is uniformly continuous, in the closure of the set $\{n(x,y) : (x,y) \in D_s\} \cup \{n^+(x,y) : (x,y) \in D_s^+\} \cup \{n^-(x,y) : (x,y) \in D_s^-\} \subset S^2$, hence

$$\sup_{(x,y)\in D_s^+} |p(n(x,y)) - p(n^+(x,y))| + \sup_{(x,y)\in D_s^-} |p(n(x,y)) - p(n^-(x,y))| =: \alpha(s) \to 0 \text{ as } s \to 0^-.$$

We have

$$R(s) = \int_{\partial_+ C \setminus \sqrt{1-s}\, C_+} f(n_\xi)\, d\xi + \int_{\sqrt{1-s}\, \partial_- C \setminus C_-} f(n_\xi)\, d\xi - \int_{\sqrt{1-s}\, V \setminus V} f(n_\xi)\, d\xi.$$

Making the change of variable $\xi \rightsquigarrow x, y$ in these integrals and taking into account that $d\xi = \frac{dx\, dy}{|n_3^+(x,y)|}$, $d\xi = \frac{dx\, dy}{|n_3^-(x,y)|}$, and $d\xi = \frac{dx\, dy}{|n_3(x,y)|}$ in the first, second, and third integrals, respectively, we get

$$R(s) = \int_{D_s^+} p(n^+(x,y))\, dx\, dy + \int_{D_s^-} p(n^-(x,y))\, dx\, dy - \int_{D_s} p(n(x,y))\, dx\, dy$$

$$= \int_{D_s^+} \big(p(n^+(x,y)) - p(n(x,y))\big)\, dx\, dy + \int_{D_s^-} \big(p(n^-(x,y)) - p(n(x,y))\big)\, dx\, dy,$$

and so,

$$|R(s)| \le \int_{D_s^+} |(p(n^+(x,y)) - p(n(x,y)))|\, dx\, dy + \int_{D_s^-} |(p(n^-(x,y)) - p(n(x,y)))|\, dx\, dy$$

$$\le \alpha(s)\, k|s| = o(s) \ \text{ as } \ s \to 0^-.$$

It follows that there exists the derivative

$$\frac{d}{ds}\Big\rfloor_{s=0} F(\partial C(s)) = F(V) - F(\partial_- C).$$

Since $F(\partial C(s))$ takes the minimal value at $s = 0$, we have $\frac{d}{ds}\rfloor_{s=0} F(\partial C(s)) = 0$, therefore $F(\partial C(s))$ is constant for $0 \le s \le 1$. Thus, all bodies $C(s)$, $0 \le s \le 1$ are solutions of problem (3). Claim (a) of the lemma is also proved. □

Let us finish the proof of the theorem.

Let C be a solution of problem (3). Assuming that the set $\text{Ext}C \setminus (\partial C_1 \cup \partial C_2 \cup \overline{\text{Sing}C})$ is not empty, we use Lemma 4 to obtain an infinite sequence of points $O_1, O_2, \ldots$ in $C_2 \setminus C$ such that (a) for all $i \neq j$ and all $\xi \in C_1 \cup \overline{\text{Sing}C}$, the intersections of the open segments (O_i, O_j) and (O_i, ξ) with the interior of C are not empty; (b) for $\widehat{C} = \text{Conv}(C \cup \{O_1, O_2, \ldots\})$ holds

$$\text{Ext}\widehat{C} \subset \partial C_1 \cup \partial C_2 \cup \overline{\text{Sing}\widehat{C}}.$$

For any $\vec{s} = (s_1, s_2, \ldots) \in [0, 1]^\infty$ define

$$C(\vec{s}) = \cup_i C_i(s_i), \ \text{ where } \ C_i(s) = \bigcup_{\sqrt{1-s} \le \lambda \le 1} \big(\lambda C + (1-\lambda) O_i\big).$$

One has $C(\vec{0}) = C$ and by claim (a) of Lemma 2, $C(\vec{1}) = \cup_i \text{Conv}(C \cup \{O_i\}) = \widehat{C}$. Denote $\vec{e}_i = (\underbrace{0, \ldots, 0, 1}_{i}, 0, \ldots)$; then we have $C(\vec{e}_i) = \text{Conv}(C \cup \{O_j\})$.

In Appendix 1 we prove that each set $C(\vec{s})$ is convex. We have $C_1 \subset C \subset C(\vec{s}) \subset C_2$, hence $C(\vec{s})$ belongs to the class of admissible bodies.

Fix i and consider all rays with vertex at O_i intersecting C. Denote by A and A' the initial (closer to O_i) and the final points of intersection of a generic ray of this kind with C. Let $\partial_i^- C$ be the union of the points A corresponding to all rays and the segments $[A,\, A']$ contained in the rays tangent to C. Let V_i be the union of all segments $[O_i,\, A']$ contained in the tangent rays.

In Appendix 2 we prove that the closed sets bounded by the surfaces V_i and $\partial_i^- C$ are mutually disjoint. (Note that every such set is the union of all segments $[O_i,\, A]$ and the segments $[O_i,\, A']$ corresponding to the tangent rays.)

The boundary of C is the disjoint union of all surfaces $\partial_i^- C$ and the remaining part of the boundary, $\partial C \setminus (\cup_i \partial_i^- C)$,

$$\partial C = \cup_i \partial_i^- C \bigcup \Big(\partial C \setminus (\cup_i \partial_i^- C) \Big).$$

Denote by $T_i(k)$ the dilation with center O_i and ratio k and consider the convex body $C(s\vec{e}_i)$, $0 \le s \le 1$. Its boundary is the disjoint union

$$\partial C(s\vec{e}_i) = \big(\partial C \setminus \partial_i^- C \big) \cup \Big(T_i(\sqrt{1-s})(\partial_i^- C) \cup \big(V_i \setminus T_i(\sqrt{1-s})(V_i) \big) \Big).$$

Correspondingly, its surface measure is

$$\nu_{\partial C(s\vec{e}_i)} = (\nu_{\partial C} - \nu_{\partial_i^- C}) + (\nu_{T_i(\sqrt{1-s})(\partial_i^- C)} + \nu_{V_i} - \nu_{T_i(\sqrt{1-s})(V_i)})$$

$$= (\nu_{\partial C} - \nu_{\partial_i^- C}) + \big((1-s)\nu_{\partial_i^- C} + \nu_{V_i} - (1-s)\nu_{V_i} \big) = \nu_{\partial C} + s\nu_i,$$

where $\nu_i = \nu_{V_i} - \nu_{\partial_i^- C}$, and

$$F(\partial C(s\vec{e}_i)) = F(\nu_{\partial C}) + s(F(V_i) - F(\partial_i^- C).$$

Since by Lemma 5, every convex body $C(s\vec{e}_i)$ is a solution of problem (3), we have

$$F(V_i) - F(\partial_i^- C) = 0. \tag{12}$$

In general, the boundary of $C(\vec{s})$, $\vec{s} \in [0,\, 1]^\infty$ is the disjoint union of $\partial C \setminus (\cup_i \partial_i^- C)$ and the surfaces $T_i(\sqrt{1-s_i})(\partial_i^- C)$ and $V_i \setminus T_i(\sqrt{1-s_i})(V_i)$ for all values of i,

$$\partial C(\vec{s}) = \bigcup_i \Big(T_i(\sqrt{1-s_i})(\partial_i^- C) \cup \big(V_i \setminus T_i(\sqrt{1-s_i})(V_i) \big) \Big) \bigcup \Big(\partial C \setminus (\cup_i \partial_i^- C) \Big);$$

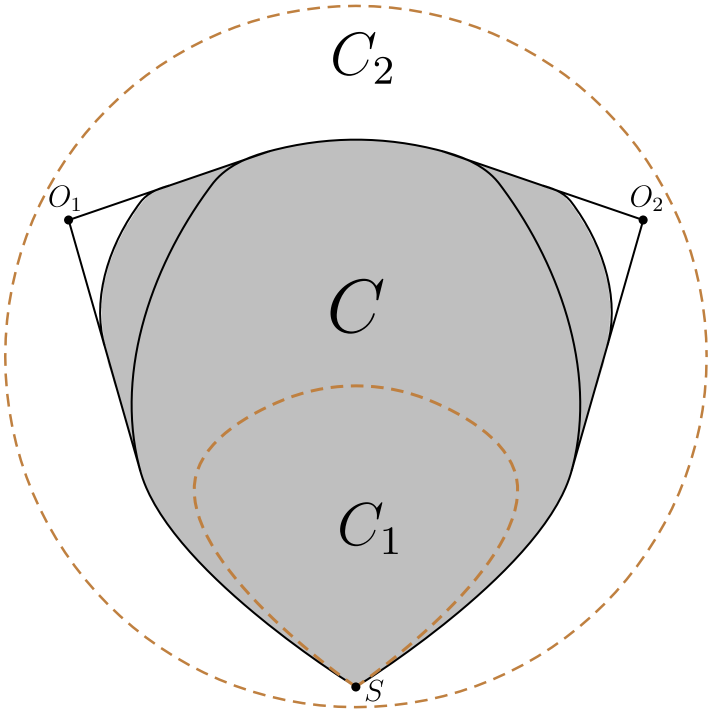

Figure 6: The body $C(\vec{s})$ with $\vec{s} = (\frac{3}{4}, \frac{3}{4}, 0, 0, \dots)$ is shown in gray. The body $C(\vec{0})$ coincides with C. The body $C(1, 1, 0, 0, \dots) = \mathrm{Conv}(C \cup \{O_1, O_2\})$ is bounded by the closed curve O_1O_2S. The bodies C_1 and C_2 are bounded by dashed lines. The set Sing$C$ is represented here by the point S.

see Fig. 6 for the case when $\vec{s} = (\frac{3}{4}, \frac{3}{4}, 0, 0, \dots)$. Correspondingly, the surface measure is

$$\nu_{\partial C(\vec{s})} = \sum_i \Big(\nu_{T_i(\sqrt{1-s_i})(\partial_i^- C)} + \nu_{V_i} - \nu_{T_i(\sqrt{1-s_i})(V_i)}\Big) + \Big(\nu_{\partial C} - \sum_i \nu_{\partial_i^- C}\Big)$$

$$= \sum_i \Big((1-s_i)\nu_{\partial_i^- C} + \nu_{V_i} - (1-s_i)\nu_{V_i}\Big) + \Big(\nu_{\partial C} - \sum_i \nu_{\partial_i^- C}\Big) = \nu_{\partial C} + \sum_i s_i(\nu_{V_i} - \nu_{\partial_i^- C}).$$

Hence we have

$$\nu_{\partial C(\vec{s})} = \nu_{\partial C} + \sum_i s_i \nu_i \quad \text{with} \quad \nu_i = \nu_{V_i} - \nu_{\partial_i^- C}$$

and

$$F(\partial C(\vec{s})) = F(\partial C) + \sum_i s_i(F(V_i) - F(\partial_i^- C)).$$

Using (12), one obtains $F(\partial C(\vec{s})) = F(\partial C)$, that is, for every $\vec{s}$ the convex body $C(\vec{s})$ is a solution to problem (3). Theorem 2 is proved.

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Appendix 1

Since all sets $C_i(s_i)$ are convex, it suffices to show that for all $i \neq j$, $x_i \in C_i(s_i)$ and $x_j \in C_j(s_j)$, the segment $[x_i,\, x_j]$ lies in $C_i(s_i) \cup C_j(s_j)$.

We have

$$x_i = \mu_1 \hat{x}_i + (1-\mu_1) O_i \quad \text{and} \quad x_j = \mu_2 \hat{x}_j + (1-\mu_2) O_j \tag{13}$$

for some $\hat{x}_i,\, \hat{x}_j \in C$, $\sqrt{1-s_i} \le \mu_1 \le 1$, $\sqrt{1-s_j} \le \mu_2 \le 1$. If $\mu_1 = 1$ or $\mu_2 = 1$ then the segment $[x_i,\, x_j]$ is contained in $C_j(s_j)$ or $C_i(s_i)$, respectively. If, otherwise, both μ_1 and μ_2 are not equal to 1 then we make use of the fact that the open segment $(O_i,\, O_j)$ intersects C, and therefore, for some $0 < \lambda < 1$ the point $x_0 = \lambda O_i + (1-\lambda) O_j$ lies in C.

Take the point $\bar{x} = \tilde{\lambda} x_i + (1-\tilde{\lambda}) x_j$, where

$$\tilde{\lambda} = \frac{\frac{\lambda}{1-\mu_1}}{\frac{\lambda}{1-\mu_1} + \frac{1-\lambda}{1-\mu_2}} \quad \text{and hence,} \quad 1 - \tilde{\lambda} = \frac{\frac{1-\lambda}{1-\mu_2}}{\frac{\lambda}{1-\mu_1} + \frac{1-\lambda}{1-\mu_2}}.$$

Using formula (13), one sees that $\bar{x}$ is a convex combination of the points x_0, $\hat{x}_i$, $\hat{x}_j$, and therefore, lies in C.

Thus, the segment $[x_i,\, x_j]$ is divided by the point $\bar{x}$ into two parts $[x_i,\, \bar{x}]$ and $[\bar{x},\, x_j]$, with $x_i \in C_i(s_i)$, $x_j \in C_j(s_j)$, $\bar{x} \in C$. Hence the former segment belongs to $C_i(s_i)$ and the latter one belongs to $C_j(s_j)$.

Thus, $C(\vec{s})$ is convex.

Appendix 2

The closed set bounded by the surfaces V_i and $\partial_i^- C$ is the union of segments $[O_i,\, A_i]$ contained in rays from O_i intersecting C and segments $[O_i,\, A_i']$ contained in tangent rays from O_i. A generic segment of this kind ($[O_i,\, A_i]$ or $[O_i,\, A_i']$) will be denoted as $[O_i,\, B_i]$.

It suffices to show that for $i \neq j$, the generic segments $[O_i,\, B_i]$ and $[O_j,\, B_j]$ are disjoint.

Assume the contrary: $[O_i,\, B_i]$ and $[O_j,\, B_j]$ intersect at a point ξ, and therefore,

$$\xi = \lambda_i O_i + (1-\lambda_i) B_i = \lambda_j O_j + (1-\lambda_j) B_j$$

for $0 \le \lambda_i \le 1$, $0 \le \lambda_j \le 1$. We know that a point of the interval $(O_i,\, O_j)$ belongs to the interior of C; let it be $O = \mu O_i + (1-\mu) O_j$, $0 < \mu < 1$. We also know that the points B_i

and B_j lie in C and that the segments $[O_i,\, B_i]$ and $[O_j,\, B_j]$ do not contain interior points of C.

Suppose that $\lambda_i \neq 0$ and $\lambda_j \neq 0$. Take the values

$$\tilde\lambda = \frac{\frac{\mu}{\lambda_i}}{\frac{\mu}{\lambda_i} + \frac{1-\mu}{\lambda_j}}, \qquad 1 - \tilde\lambda = \frac{\frac{1-\mu}{\lambda_j}}{\frac{\mu}{\lambda_i} + \frac{1-\mu}{\lambda_j}}.$$

We have

$$\xi = \tilde\lambda\big[\lambda_i O_i + (1-\lambda_i) B_i\big] + (1-\tilde\lambda)\big[\lambda_j O_j + (1-\lambda_j) B_j\big] = \frac{1}{\frac{\mu}{\lambda_i} + \frac{1-\mu}{\lambda_j}}\, O + \tilde\lambda(1-\lambda_i)\, B_i + (1-\tilde\lambda)(1-\lambda_j)\, B_j;$$

that is, ξ is a convex combination of O, B_i, and B_j, with nonzero coefficient at O. Hence ξ lies in the interior of C, in contradiction with the condition that no point of $[O_i,\, B_i]$ belongs to the interior of C.

It remains to consider the cases $\lambda_i = 0$ and $\lambda_j = 0$. Let, for example, $\lambda_i = 0$; then we have $B_i = \lambda_j O_j + (1-\lambda_j) B_j$. If $\lambda_j \neq 0$, denote $\tilde\mu = \frac{\lambda_j}{(1-\mu)(1-\lambda_j)+\lambda_j}$ and take the point

$$\xi = \tilde\mu O + (1-\tilde\mu) B_j = \tilde\mu\mu O_i + \frac{1-\mu}{(1-\mu)(1-\lambda_j)+\lambda_j}\, B_i \in [O_i,\, B_i],$$

Since O is in the interior of C, $B_j \in C$, and $\tilde\mu \neq 0$, the point ξ lies in the interior of C. We have again a contradiction with the condition that no point of $[O_i,\, B_i]$ belongs to the interior of C.

If $\lambda_i = 0$ and $\lambda_j = 0$, we have $B_i = B_j$. The points O_i, B_i, and O_j are not collinear, since otherwise the segment $(O_i,\, O_j) = (O_i,\, B_i) \cup [B_i,\, O_j)$ does not contain interior points of C. Using that the segment $[O,\, B_i]$ is contained in C and no point of the segments $[O_i,\, B_i]$ and $[B_i,\, O_j]$ is contained in the interior of C, one concludes that the intersection of C with the plane $O_i O_j B_i$ is a planar convex body contained in the angle $O_i B_i O_j$, and both lines $O_i B_i$ and $O_j B_i$ are lines of support to this planar body. Since B_i is a regular point of C, the plane $O_i O_j B_i$ is tangent to C, in contradiction with the assumption that $O \in (O_i,\, O_j)$ is an interior point of C.

Thus, the segments $[O_i,\, B_i]$ and $[O_j,\, B_j]$ are disjoint.

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