On generalized Newton's aerodynamic problem

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This paper is dedicated to the memory of Anatoly M. Stepin

Abstract

We consider the generalized Newton's least resistance problem for convex bodies: minimize the functional $\iint_\Omega (1+|\nabla u(x,y)|^2)^{-1} dx\, dy$ in the class of concave functions $u : \Omega \to [0, M]$, where the domain $\Omega \subset \mathbb{R}^2$ is convex and bounded and $M > 0$. It has been known [1] that if u solves the problem then $|\nabla u(x, y)| \geq 1$ at all regular points (x, y) such that $u(x, y) < M$. We prove that if the upper level set $L = \{(x, y) : u(x, y) = M\}$ has nonempty interior, then for almost all points of its boundary $(\bar{x}, \bar{y}) \in \partial L$ one has $\lim_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ u(x,y)<M}} |\nabla u(x, y)| = 1$. As a by-product, we obtain a result concerning local properties of convex surfaces near ridge points.

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1 Introduction

In this paper we consider the following problem.

Problem 1. *Given $M > 0$ and a bounded convex set with nonempty interior $\Omega \subset \mathbb{R}^2$, minimize*

$$F(u) = \iint_\Omega \frac{1}{1 + |\nabla u(x, y)|^2}\, dxdy \tag{1}$$

in the class of concave functions $u : \Omega \to [0,\ M]$.

Note that Problem 1 is a generalization of the famous Newton's aerodynamic problem: Let $\Omega = \{(x, y) : x^2 + y^2 \leq 1\}$ be the unit circle centered at the origin; one needs to minimize the functional F given by (1) in the (narrower) class of concave radially symmetric functions, that is, functions u that can be represented as $u(x, y) = \phi(\sqrt{x^2 + y^2})$, with ϕ

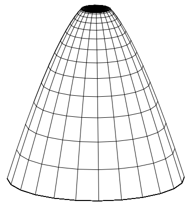

Figure 1: The solution of Newton's problem with $M = 2$.

being a concave monotone decreasing function of one variable. A solution of Newton's problem can be seen in Fig. 1.

Newton's aerodynamic problem and its generalization admit the following aerodynamic interpretation. Consider a convex body C moving with unit velocity through a highly rarified homogeneous medium composed of point particles at rest. The mutual interaction of particles is neglected. When colliding with the body, each particle is reflected elastically. As a result of collisions, there appears the drag force that acts on the body and slows down its motion.

Take a coordinate system with the coordinates x, y, z connected with the body such that the z-axis is parallel and co-directional to the velocity of the body. The upper part of the body's surface is the graph of a concave function $u = u_C : \Omega \to \mathbb{R}$, where $\Omega = \Omega_C$ is the projection of C on the xy-plane. Then the z-component of the drag force equals $-2\rho F(u)$, where ρ is the density of the medium and $F(u)$ is given by (1).

The generalized Problem 1 was stated in 1993 in the paper by Buttazzo and Kawohl [2] and still remains open. Several solutions to this problem obtained numerically in [7] can be seen in Fig. 2. In [1] (1995) it was proved, in particular, that $|\nabla u(x, y)| \geq 1$ for any regular point (x, y) such that $u(x, y) < M$. In the same paper it was conjectured that $|\nabla u(x, y)| \to 1$ as $u(x, y) \to M$.

This conjecture is in agreement with numerical simulation in the case when the (convex) set $L = \{(x, y) : u(x, y) = M\}$ has nonempty interior. On the other hand, if L is a line segment, it seems that $\inf_{u(x,y)<M} |\nabla u(x, y)| > 1$.[1]

In this paper we provide a sketch of the proof of the following theorem. The full proof will be published elsewhere.

Theorem 1. *Let u solve Problem 1 and let the upper level set $L = \{(x, y) : u(x, y) = M\}$ have nonempty interior. Then for almost all points $(\bar{x}, \bar{y}) \in \partial L$,*

$$\lim_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ (x,y)\in\Omega\setminus L}} |\nabla u(x, y)| = 1.$$

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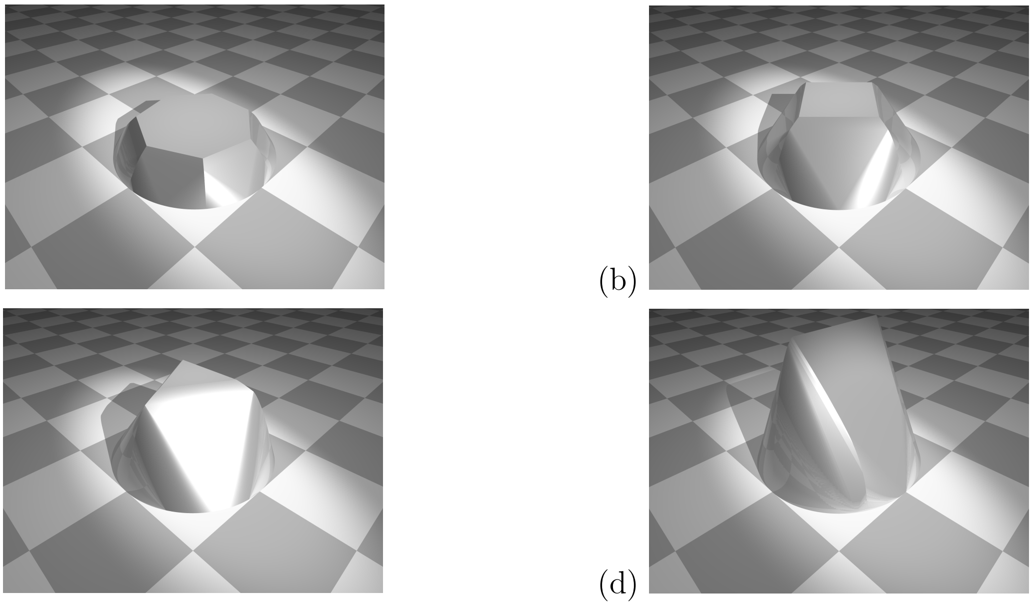
[1] G. Wachsmuth, personal communication.



(a) (b) (c) (d)

Figure 2: Numerical solutions of Problem 1 in the cases (a) $M = 0.4$; (b) $M = 0.7$; (c) $M = 1.0$; (d) $M = 1.5$.

2 Surface area measure

In the proof we use the representation of the integral in (1) in terms of *surface area measure.*

Let $C \subset \mathbb{R}^d$, $d \geq 2$, be a *convex body*, that is, a compact convex set with nonempty interior. Denote by n_r the outward normal to C at a regular point $r \in \partial C$.

Definition 1. *The* surface area measure *of C is the Borel measure ν_C in S^{d-1} defined by $\nu_C(A) := |\{r \in \partial C : n_r \in A\}|$ for any Borel set $A \subset S^{d-1}$.*

In the same way one can define the surface area measure induced by a Borel subset $B \subset \partial C$. In this case one only needs to replace ∂C with B in the definition.

By Alexandrov's theorem, the map $C \mapsto \nu_C$ is a one-to-one correspondence between the set of convex bodies in $\mathbb{R}^d$ with the geometric center at the origin and the set of measures on S^{d-1} satisfying the equation

$$\int_{S^{d-1}} n \, d\nu(n) = \vec{0} \tag{2}$$

and such that the linear span of $\operatorname{spt} \nu$ is $\mathbb{R}^d$.

Take the convex body $C = C_u$ associated with the function u, $C = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in \Omega,\ 0 \le z \le u(x, y)\}$. The functional F in (1) can be written as $F(u) = \mathcal{F}(\nu_C)$, where

$$\mathcal{F}(\nu) = \int_{S^2} f(n)\, d\nu(n);$$

here f is the function on S^2 defined by $f(x, y, z) = (z_+)^3$ and z_+ means the positive part of z, $z_+ := \max\{z, 0\}$ [3, 4].

Let us briefly mention the 2D version of Newton's problem: minimize $\int_0^1 (1 + u'^2(x))^{-1} dx$ in the class of concave functions $u : [0,\, 1] \to [0,\, M]$. It was considered in [2]. The solution is $u(x) = \min\{M,\, 1 - x\}$, if $M < 1$, and $u(x) = M(1 - x)$, if $M \ge 1$.

This problem can be reformulated in terms of surface area measure as follows:

$$\text{Minimize the integral} \qquad \int_{S^1} f(n)\, d\nu(n) \tag{3}$$

in the class of measures ν supported in the first quarter of the unit circumference $\{(x, z) \in \mathbb{R}^2 : x^2 + z^2 = 1,\, x \ge 0,\, z \ge 0\}$ and satisfying the relation $\int_{S^1} n\, d\nu(n) = (M, 1)$. (Here by slightly abusing the language we denote by f the function on $S^1 = \{x^2 + z^2 = 1\}$ defined by $f(x, z) = (z_+)^3$.)

The unique solution to problem (3) is the sum of two atoms concentrated at $(0, 1)$ and $\frac{1}{\sqrt{2}}(1, 1)$, if $M < 1$, and an atom concentrated at $\frac{1}{\sqrt{1+M^2}}(M, 1)$, if $M \ge 1$.

Let us formulate separately this statement in the particular case $M = 1$, which will be needed later on.

Proposition 1. *The minimum of the integral $\int_{S^1} f(n)\, d\nu(n)$ in the class of measures ν on S^1 satisfying the conditions*

(i) sptν lies in the quarter of the circumference $x^2 + z^2 = 1,\ x \ge 0,\ z \ge 0$;

(ii) $\int_{S^1} n\, d\nu(n) = \frac{1}{\sqrt{2}}(1, 1)$

equals $1/(2\sqrt{2})$, and the unique minimizer is the atom $\delta_{\frac{1}{\sqrt{2}}(1,1)}$.

3 Local properties of convex surfaces

Here we provide auxiliary results on surface area measure that will be needed later on.

Recall that a *convex body* is a convex compact set with nonempty interior.

Consider a convex body $C \subset \mathbb{R}^3$. A singular point r_0 of its boundary is called a *conical point*, if the tangent cone to C at r_0 is not degenerate (that is, does not contain straight lines), and a *ridge point*, if the tangent cone degenerates into a dihedral angle (see, e.g., [5]).

Take a ridge point $r_0 \in \partial C$ and let Π be a plane of support to C at r_0. Consider the part of ∂C containing r_0 cut off by a plane parallel to Π. We will study the limiting properties of this part of surface when the cutting plane approaches Π.

More precisely, let the tangent cone at r_0 be given by

$$(r - r_0,\, e_1) \le 0, \qquad (r - r_0,\, e_2) \le 0, \tag{4}$$

where e_1 and e_2 are non-collinear unit vectors, $e_1 \ne \pm e_2$. Here and in what follows, $(\cdot\,,\cdot)$ means the scalar product. The outward normals of all planes of support at r_0 form the curve

$$\Gamma = \Gamma_{e_1,e_2} = \{v = \mu_1 e_1 + \mu_2 e_2 : |v| = 1,\, \mu_1 \ge 0,\, \mu_2 \ge 0\} \subset S^2;$$

it is the minor arc of the great circle on S^2 through the points e_1 and e_2 bounded by these points.

Let e be a positive linear combination of e_1 and e_2, $e = \lambda_1 e_1 + \lambda_2 e_2$, $\lambda_1 > 0$, $\lambda_2 > 0$, $|e| = 1$. Denote by Π^t the plane of equation $(r - r_0,\, e) = -t$ and by Π_i $(i = 1, 2)$ the plane of equation $(r - r_0,\, e_i) = 0$. The point e lies on the curve Γ and does not coincide with its endpoints e_1 and e_2.

For $t \ge 0$ consider the convex body

$$C_t = C \cap \{r : (r - r_0,\, e) \ge -t\};$$

it is the piece of C cut off by the plane Π^t. It is assumed that the distance t between Π^t and r_0 is small. The body C_t is bounded by the planar domain

$$B_t = C \cap \{r : (r - r_0,\, e) = -t\} \subset \Pi^t$$

and the convex surface

$$S_t = \partial C \cap \{r : (r - r_0,\, e) \ge -t\} \subset \partial C; \tag{5}$$

that is, $\partial C_t = B_t \cup S_t$.

In what follows we denote by $|A|$ the 2-dimensional Hausdorff measure (area) of the Borel set A on the convex surface ∂C or on a plane. In particular, $|\square ABCD|$ means the area of the quadrangle $ABCD$. The same notation will be used for the length of a line segment or a curve; for instance, $|MN|$ means the length of the segment MN.

Let us define the normalized measure ν_t induced by the surface S_t as follows: for any Borel set $A \subset S^2$,

$$\nu_t(A) = \frac{1}{|B_t|} |\{r \in S_t :\, n_r \in A\}|.$$

The surface area measure of the convex body C_t equals $\nu_{C_t} = |B_t|\delta_{-e} + |B_t|\nu_t$, hence $\int_{S^2} n\, d\nu_{C_t}(n) = |B_t|(-e + \int_{S^2} n\, d\nu_t(n))$. Formula (2) applied to C_t results in

$$\int_{S^2} n\, d\nu_t(n) = e. \tag{6}$$

We say that ν_t weakly converges to ν_* as $t \to 0^+$, and use the notation $\nu_t \xrightarrow[t\to 0^+]{} \nu_*$, if for any continuous function f on S^2,

$$\lim_{t\to 0^+} \int_{S^2} f(n)\, d\nu_t(n) = \int_{S^2} f(n)\, d\nu_*(n).$$

Similarly, ν_* is called a weak partial limit of ν_t, if there exists a sequence $t_i > 0$, $i \in \mathbb{N}$ converging to zero such that for any continuous function f on S^2,

$$\lim_{i\to \infty} \int_{S^2} f(n)\, d\nu_{t_i}(n) = \int_{S^2} f(n)\, d\nu_*(n).$$

We are interested in studying the properties of the limiting measure (weak limit or partial weak limit) ν_*.

One of the properties is immediate: going to the limit $t \to 0^+$ or $t_i \to 0$ in formula (6), one obtains

$$\int_{S^2} n\, d\nu_*(n) = e. \tag{7}$$

Consider two examples.

Example 1. *Let r_0 be an interior point of an edge of a tetrahedron (see Fig. 3).*

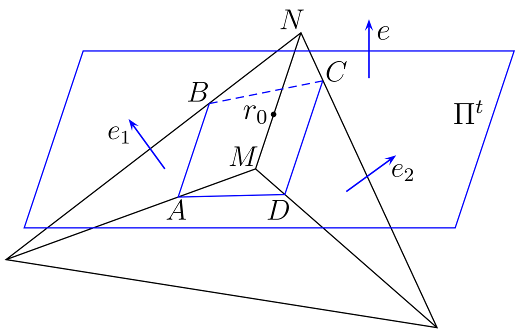


Figure 3: r_0 lies on the edge MN of the tetrahedron, and the section of the tetrahedron by the plane $\Pi^t : (r - r_0,\, e) = -t$ is the quadrilateral $ABCD$.

The surface S_t is composed of two quadrilaterals $MNBA$ and $MNCD$ and two triangles BCN and ADM. The outward normals to the quadrilaterals $MNBA$ and $MNCD$ are e_1 and e_2, respectively, and their areas are of the order of t, $|\square MNBA| = c_1 t + O(t^2)$, $|\square MNCD| = c_2 t + O(t^2)$, $c_1 > 0$, $c_2 > 0$. The areas of the triangles BCN and ADM are $O(t^2)$.

The planar surface B_t is the quadrilateral $ABCD$, the normal vector to it is $e = \lambda_1 e_1 + \lambda_2 e_2$, and its area is of the order of t, $|\square ABCD| = c_0 t + O(t^2)$, $c_0 > 0$. It follows that the corresponding measure ν_t weakly converges to the measure ν_* supported on the *two-point set* $\{e_1, e_2\}$, $\nu_* = \frac{c_1}{c_0}\delta_{e_1} + \frac{c_2}{c_0}\delta_{e_2}$. *Using formula* (7), *one finds that* $\frac{c_1}{c_0} = \lambda_1$, $\frac{c_2}{c_0} = \lambda_2$. *Thus, we have*

$$\nu_t \xrightarrow[t\to 0^+]{} \nu_*, \quad \textit{where} \quad \nu_* = \lambda_1\delta_{e_1} + \lambda_2\delta_{e_2}. \tag{8}$$

Example 2. *Let* C *be the part of a cylinder bounded by two planes,* $C = \{r = (x, y, z) : -z - 1 \le x \le z + 1,\ y^2 + z^2 \le 1\}$, *and take the ridge point* $r_0 = (0, 0, -1) \in \partial C$. *The outward vectors of the corresponding dihedral angle are* $e_1 = \frac{1}{\sqrt{2}}(-1, 0, -1)$ *and* $e_2 = \frac{1}{\sqrt{2}}(1, 0, -1)$. *We take* $e = (0, 0, -1) = \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_2$ *(see Fig. 4 (a)).*

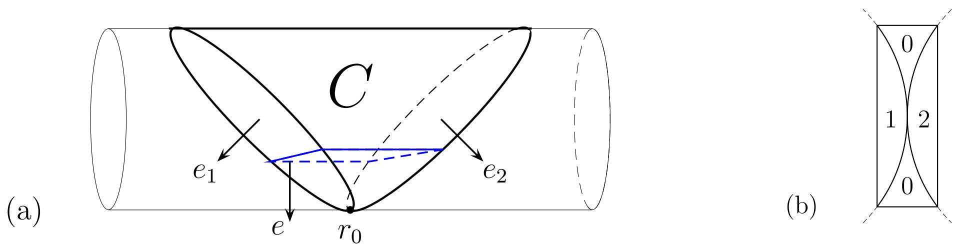


Figure 4: (a) C is the part of a cylinder bounded by two planes through r_0. (b) The projections of S_t^1, S_t^2, and S_t^0 on the xy-plane are marked by "1", "2", and "0", respectively.

We have $C_t = C \cap \{z \le -1 + t\}$, *and* B_t *is the rectangle* $-t \le x \le t$, $-\sqrt{2t - t^2} \le y \le \sqrt{2t - t^2}$ *in the plane* $z = -1 + t$. *The surface* S_t *is the union of three parts,* $S_t = S_t^1 \cup S_t^2 \cup S_t^0$, *where* S_t^1 *is the planar domain of equations* $x = -z - 1$, $x \ge -t$, $(x+1)^2 + y^2 \le 1$ *with the outward normal* e_1 *and* S_t^2 *is the planar domain of equations* $x = z + 1$, $x \le t$, $(x - 1)^2 + y^2 \le 1$ *with the outward normal* e_2. S_t^1 *and* S_t^2 *are segments of ellipses in the planes* $x = -z - 1$ *and* $x = z + 1$, *respectively. The surface* S_t^0 *is the graph of the function* $z(x, y) = -\sqrt{1 - y^2}$ *defined on the domain* $-(1 - \sqrt{1 - y^2}) \le x \le 1 - \sqrt{1 - y^2}$, $-\sqrt{2t - t^2} \le y \le \sqrt{2t - t^2}$; *see Fig. 4 (b). The outward normals to* S_t^0 *are contained in a neighborhood of* e *shrinking to* e *when* $t \to 0^+$.

The areas of the surfaces are easy to calculate, $|B_t| = 2t \cdot 2\sqrt{2t - t^2} = 4\sqrt{2}\, t^{3/2}(1 + o(1))$, $t \to 0$, $|S_t^0| = \frac{4\sqrt{2}}{3} t^{3/2}(1 + o(1))$, *and* $|S_t^1| = |S_t^2| = \frac{8}{3} t^{3/2}(1 + o(1))$, $t \to 0$. *It follows that* ν_t *converges to the measure*

$$\nu_* = \frac{\sqrt{2}}{3}\delta_{e_1} + \frac{\sqrt{2}}{3}\delta_{e_2} + \frac{1}{3}\delta_e$$

supported on the three-point set $\{e_1, e_2, e\} \subset \Gamma$.

The following Theorem describes the limiting behavior of ν_t in the general case.

Theorem 2. *The set of weak partial limits of* ν_t *as* $t \to 0^+$ *is nonempty, and each partial limit is supported on a subset of* Γ *containing* e_1 *and* e_2.

4 Sketch of the proof of Theorem 2

It is not difficult to show that the full measures $\nu_t(S^2)$ do not exceed a constant on a certain interval $(0, t_0)$, and therefore, there is at least one partial limit of ν_t as $t \to 0^+$. Further, the statement that each partial limit of ν_t is supported in Γ is a consequence of the fact that the graph of the subdifferential mapping of a convex function is closed (and the similar fact for concave functions); see Theorem 24.4 of the book [6]. It remains to prove that $\operatorname{spt} \nu_*$ contains e_1 and e_2.

Let us explain the underlying idea of the proof. It suffices to prove that $e_1 \in \operatorname{spt} \nu_*$; the proof for e_2 is the same. Let $l_1 = l_1^t$ and $l_2 = l_2^t$ be the parallel lines resulting from intersection of Π_1 and Π_2, correspondingly, with Π^t. In Fig. 5 the point r_0 is marked by A. Draw the plane through this point perpendicular to the edge of the dihedral angle, and let the segment $MN = M_tN_t$ be the intersection of this plane with B_t. We assume that the point M is closer to l_2 and N is closer to l_1. In the plane Π^t draw two lines of support to B_t orthogonal to l_1 and l_2.

Choose two points $C = C_t$ and $D = D_t$ in B_t that belong, respectively, to the first and second lines of support and assume, without loss of generality, that $\operatorname{dist}(D, MN) \geq \operatorname{dist}(C, MN)$. Draw the plane through the edge of the dihedral angle and the point D; let it intersect the line MN at the point $H = H_t$ (see Fig. 5).

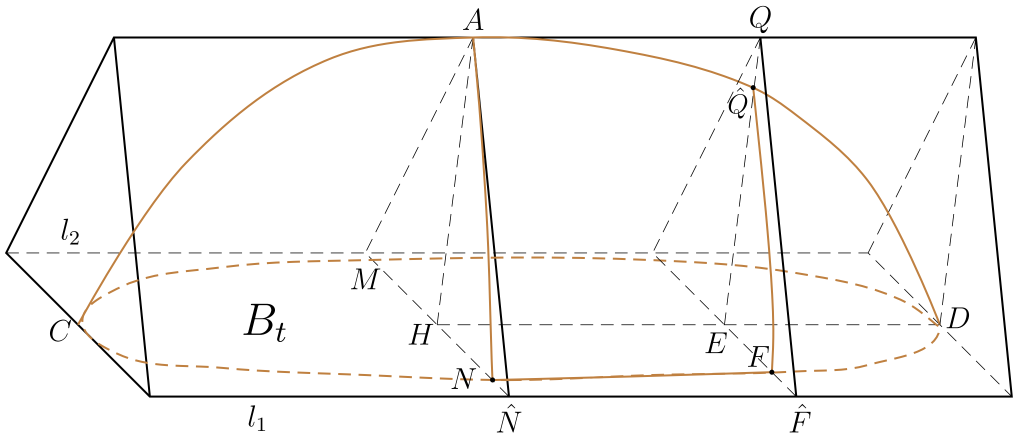


Figure 5: The convex body C_t in the dihedral angle and the corresponding notation.

Now fix $0 < \theta < 1$ and take the point $E = E_{t,\theta}$ on the segment HD so as $|HE| = \theta|HD|$. Further, we compare the surface area measures induced by the following two convex bodies. The first one is called $\mathrm{Prism}_t(\theta)$; it is the prism $HA\hat{N}EQ\hat{F}$ bounded by the 5 planes: Π_1, Π^t, the plane through the edge of the dihedral angle and through the points H and E, and the two planes orthogonal to that edge through the points H and E. The second body, called $C_t(\theta)$, is the intersection of this prism with C; in other words, it is the part of C contained in the prism and bounded by the 4 of the above planes (except for Π_1).

In Fig. 5 the point Q on the edge of the dihedral angle and the points $\hat{N}$ and $\hat{F}$ on l_1 are chosen so as the lines QE, $\hat{N}H$, $\hat{F}E$ are orthogonal to HD. The point $\hat{Q}$ is the intersection of the segment QE with ∂C. The surface area measure induced by the face $AQ\hat{F}\hat{N}$ of the prism is $\nu^0_{t,\theta} = |AQ\hat{F}\hat{N}|\delta_{e_1}$, and there is a partial limit of the normalized surface area measure $\frac{1}{|B_t|}\,\nu^0_{t,\theta}$ equal to $c\,\delta_{e_1}$ with a certain $c > 0$. The surface area measure induced by the part $A\hat{Q}FN$ of $\partial C_t(\theta)$ is denoted by $\nu_{A\hat{Q}FN}$, and the normalized surface area measure is $\frac{1}{|B_t|}\,\nu_{A\hat{Q}FN} =: \nu_{t,\theta}$.

We now compare the surface area measures of the prism $\mathrm{Prism}_t(\theta)$ and the body $C_t(\theta)$. Using the relation (2) for both bodies and comparing the areas of the four corresponding planar parts of the bodies, one finally comes to the conclusion that the set of partial limits of $\nu_{t,\theta}$ as $t \to 0^+$ is nonempty and for θ small and each partial limit $\nu_\theta = \lim_{i\to\infty} \nu_{t_i,\theta}$ (with $\lim_{i\to\infty} t_i = 0$), the angle between the vectors $\int_{S^2} n\, d\nu_\theta$ and $\int_{S^2} n\, d\nu^0_{t,\theta} = ce_1$ becomes small. More precisely, there is a nested family of open convex cones $\mathcal{U}_\theta(e_1)$ centered at 0 and shrinking to the ray $\{\lambda e_1,\ \lambda \ge 0\}$ as $\theta \to 0$ such that $\int_{S^2} n\, d\nu_\theta \in \mathcal{U}_\theta(e_1)$.

Let ν_* be a partial limit of ν_t, that is, $\nu_* = \lim_{i\to\infty} \nu_{t_i}$ with $\lim_{i\to\infty} t_i = 0$. Without loss of generality assume that the sequence of measures $\nu_{t_i,\theta}$ weakly converges to a measure ν_θ; otherwise choose a converging subsequence of this sequence. Since $\nu_\theta \le \nu_*$, we conclude that $\mathrm{spt}\,\nu_*$ has nonempty intersection with $\mathcal{U}_\theta(e_1) \cap \Gamma$. Since θ can be made arbitrarily small, $\mathrm{spt}\,\nu_*$ contains points arbitrarily close to e_1, and therefore, contains e_1. Theorem 2 is proved.

5 Proof of Theorem 1

Fix a regular point $(\bar{x}, \bar{y}) \in \partial L$ (note that the set of regular points of ∂L has full Lebesgue measure). The tangent cone to C with the vertex at $r_0 := (\bar{x}, \bar{y}, M)$ is a dihedral angle, with one face being horizontal and the other one having the slope $k \ge 1$. Assume that $k > 1$; our goal is to come to a contradiction.

Denote by (ϵ_1, ϵ_2) the outward normal to L at $(\bar{x}, \bar{y})$, by $r = (x, y, z)$ a generic point in $\mathbb{R}^3$, and let $e = \frac{1}{\sqrt{2}}(\epsilon_1, \epsilon_2, 1)$ and $\epsilon = (\epsilon_1, \epsilon_2, 0)$). The outward normals to the dihedral angle are $e_1 = (0, 0, 1)$ and $e_2 = \frac{1}{\sqrt{1+k^2}}(k\epsilon_1, k\epsilon_2, 1)$.

Take $t > 0$ and draw a plane with slope 1 parallel to the edge of the dihedral angle at the distance t from this edge. More precisely, the plane is given by the equation $(r - r_0,\, e) = -t$, which can be expanded to obtain $z = M - \sqrt{2}\,t + \epsilon_1(\bar{x} - x) + \epsilon_2(\bar{y} - y)$.

Now take the body $C^{(t)}$ obtained by cutting off a small part of $C = C_u$ by the plane. We have $C^{(t)} = \{(x, y, z) : (x, y) \in \Omega,\ 0 \le z \le u^{(t)}(x, y)\}$, where

$$u^{(t)}(x, y) = \min\{u(x, y),\ M - \sqrt{2}\,t + \epsilon_1(\bar{x} - x) + \epsilon_2(\bar{y} - y)\}.$$

We are going to prove that $F(u^{(t)}) < F(u)$ for a certain t, in contradiction with optimality of C.

Let B_t be the intersection of C with the cutting plane and S_t be the part of ∂C located above the plane, that is, $B_t = \{(x,y,z) :\ 0 \le z = M - \sqrt{2}\,t + \epsilon_1(\bar{x}-x) + \epsilon_2(\bar{y}-y) \le u(x,y)\}$ and $S_t = \{(x,y,z) :\ z = u(x,y) \ge M - \sqrt{2}\,t + \epsilon_1(\bar{x}-x) + \epsilon_2(\bar{y}-y)\}$.

Let ν_t be the normalized measure induced by S_t. Since the normalized measure induced by B_t is δ_e and $f(e) = 1/(2\sqrt{2})$, we have $\int_{S^2} f(n)\, d\delta_e(n) = 1/(2\sqrt{2})$, and

$$\frac{1}{|B_t|}\big(F(u) - F(u^{(t)})\big) = \frac{1}{|B_t|}\big(\mathcal{F}(\nu_C) - \mathcal{F}(\nu_{C^{(t)}})\big) = \int_{S^2} f(n)\, d\nu_t(n) - \frac{1}{2\sqrt{2}}.$$

By Theorem 2, there exists a weak partial limit $\nu_* = \lim_{i\to\infty} \nu_{t_i}$, and the support of ν_* is contained in the smaller arc of the big circle $\{x\epsilon + ze_1,\ x^2 + z^2 = 1\}$ bounded by the points e_1 and e_2 and contains these points. Thus,

(i) spt ν_* lies in the quarter of the circumference $\{x\epsilon + ze_1,\ x^2+z^2 = 1,\ x \ge 0,\ z \ge 0\}$;

(ii) by (7), using that $e = \frac{1}{\sqrt{2}}(\epsilon + e_1)$, we have $\int_{S^1} n\, d\nu_*(n) = \frac{1}{\sqrt{2}}(\epsilon + e_1)$,

and passing to the limit $i \to \infty$ one obtains

$$\lim_{i\to\infty} \frac{1}{|B_{t_i}|}\big(F(u) - F(u^{(t_i)})\big) = \int_{S^2} f(n)\, d\nu_*(n) - \frac{1}{2\sqrt{2}}. \tag{9}$$

According to Proposition 1, the infimum of $\int_{S^2} f(n)\, d\nu(n) - \frac{1}{2\sqrt{2}}$ in the class of measures ν satisfying (i) and (ii) is attained at the atomic measure δ_e and is equal to 0. The measure ν_* does not coincide with the minimizer, since its support contains two different points, therefore the expression in the right hand side of (9) is positive, in contradiction with optimality of C. Theorem 1 is proved.

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