Time-fractional telegraph equation of distributed order in higher dimensions*

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Abstract

In this work, the Cauchy problem for the time-fractional telegraph equation of distributed order in $\mathbb{R}^n \times \mathbb{R}^+$ is considered. By employing the technique of the Fourier, Laplace and Mellin transforms, a representation of the fundamental solution of this equation in terms of convolutions involving the Fox H-function is obtained. Some particular choices of the density functions in the form of elementary functions are studied. Fractional moments of the fundamental solution are computed in the Laplace domain. Finally, by application of the Tauberian theorems we study the asymptotic behaviour of the second-order moment (variance) in the time domain.

Keywords: Time-fractional telegraph equation; Distributed order; Laplace, Fourier and Mellin transforms; Fox H-functions; Fractional moments; Tauberian theorems.

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1 Introduction

Fractional partial differential equations (FPDE) with distributed order have been studied over the past decades (see e.g. [2,3,9,20,21,25,31]). One reason of the interest is the relation of these equations with physical processes involving times-scales, for example, fractional kinetics, the Cauchy problem of time-fractional diffusion-wave, generalized time-fractional diffusion, time-fractional reaction-diffusion, fractional sub-diffusion equations, and continuous random walk processes (see [16, 24] and references therein indicated). For a general overview of fractional pseudo-differential equations of distributed order we refer to the work of Umarov and Gorenflo [33]. The idea of fractional derivative of distributed order goes back to Caputo in [11] to study anomalous diffusion in viscoelasticity. He introduced the integro-differential operator (also called distributed order fractional derivative) in the form

$$\left(D^{p(\beta)} f\right)(t) = \int_0^2 p\left(\beta\right) \left(D^{\beta} f\right)(t) \; d\beta,$$

D^β being the Caputo fractional derivative of order β and $p(\beta)$ a non-negative weight/density function or non-negative generalized function. The integral over the order parameter of fractional differentiation is used to sum

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the contributions of the variable order model over the physical domain. First results about general solution of linear distributed order differential equations were obtained by Caputo in [9] and Bagley and Torvik in [7,8]. Later, Naber [25], Umarov et al [33], and Kochubei [19] studied linear distributed order FPDE with temporal fractional derivatives and give some existence and uniqueness results for the initial-value problems for these equations.

The time-fractional diffusion equation of distributed order was already studied in the literature in the one-dimensional case. The research work of Mainardi and his collaborators found important connections with Fox H-functions via the use of integral transforms such as Laplace, Fourier and Mellin transforms (see [21–23]). Chechkin et al [10, 12] discussed the properties of diffusion equations with fractional derivatives of distributed order for the description of anomalous relaxation and diffusion phenomena getting less anomalous in the course of time, called, respectively, accelerating subdiffusion and decelerating superdiffusion. Fundamental solutions for time-fractional diffusion equations of distributed order were presented in [3] and [15] for the one-dimensional case that is, one space variable. It was proved in [15] that in the cases of the time-fractional diffusion and wave equations of distributed order the first fundamental solution can be interpreted as a spatial probability density function evolving in time. Boundary value problems for the generalized time-fractional diffusion equation of distributed order were studied in [20] and maximum principles for such equation were presented in [2]. More recently, we can find the work of Sandeva et al [30] where it was investigated the solution of generalized distributed order diffusion equations with composite time-fractional derivative by using the Fourier–Laplace transform method. There are also works dealing with numerical methods for solving these equations but our focus is on the analytical analysis for FPDE of distributed order.

Time-fractional diffusion-wave equations of distributed order are also studied in the literature by different methods. In [4–6] the authors obtained the solution via the resolution of an integral equation of Volterra type, while in [3, 15] the authors use appropriate joint integral transforms to obtain the solution. The simultaneous presence of distributed order differentiation $0 < \alpha \leq 1$ and $1 < \beta \leq 2$ generalizes the single order time-fractional diffusion-wave equation, to the so-called telegraphic equation. In this paper, our aim is to study the following multidimensional time-fractional telegraph equation of distributed order

$$\int_1^2 b_2(\beta) \left[{}_{0^+}^{C}\partial_t^{\beta} u(x,t) \right] d\beta + a \int_0^1 b_1(\alpha) \left[{}_{0^+}^{C}\partial_t^{\alpha} u(x,t) \right] d\alpha - c^2 \, \Delta_x u(x,t) + d^2 \, u(x,t) = q(x,t), \qquad (1)$$

(see (17), in Section 3, for more details). In most of the works presented in the literature, the analytical resolution of equation (1) is based essentially on the application of integral transforms (see [21]). In our work, we use also integral transforms, more precisely by using the combination of Laplace, Fourier and Mellin transforms, we find the explicit solution of equation (1) with appropriate initial and boundary conditions, in terms of convolutions involving Fox H-functions. The key points to obtain our main result is the use of the classical Titchmarsh's Theorem to invert the Laplace transform, and the use of the Mellin transform to invert the Fourier transform. The combination of these tools were used in [23] and [3] for the one-dimensional case and for special cases of the equation (1). In our case, we deal with two order-density functions $b_1(\alpha)$ and $b_2(\beta)$ and the complete equation in several space variables, which results in more elaborate computations. As a byproduct the first and second fundamental solutions are obtained as Fox H-functions. For some choices of the density functions we were able to compute explicit solutions for equation (1). Using our computations we manage to compute the fractional moments of arbitrary order of the first fundamental solution in the Laplace domain. In fact, the knowledge of the second-order moment (variance) is important to classify the type of diffusion. By the Tauberian theorems we study in more detail the asymptotic behaviour near zero and near infinity of the second-order moment for the cases of fast/slow-diffusion and super fast/slow-diffusion.

The structure of the paper reads as follows: in the Preliminaries's section we recall some basic facts about fractional derivatives, integral transforms, and special functions, which are necessary for the development of this work. In Section 3 we obtain a representation of the solution of equation (1) via convolution integrals involving Fox H-functions. In the following section, we consider some particular cases of the density functions $b_1(\alpha)$ and $b_2(\beta)$, such as constant functions, linear functions, sinusoidal functions, and exponential functions. Moreover, since the choices of the density functions $b_1(\alpha)$ and $b_2(\beta)$ are independent it is possible to have different particular cases of equation (1) by considering different expressions for the density functions. In Section 5 we obtain the expression of the fractional moments of arbitrary order in the Laplace domain of the first fundamental solution of (1). Making use of the Tauberian theorems we study the asymptotic behaviours of the second-order moment in the time domain for $t \to 0^+$ and $t \to +\infty$ from the asymptotic behaviours of the corresponding second-order moment in the Laplace domain for $\mathbf{s} \to +\infty$ and $\mathbf{s} \to 0^+$, respectively. In the final part of the paper we present and analyse some plots of the second-order moment for some particular cases. As it will be shown the graphical representations support the analytical conclusions obtained via the Tauberian theorems.

2 Preliminaries

Let $a, b \in \mathbb{R}$ with $a < b$ and $\alpha > 0$. The left Riemann-Liouville fractional integral I^{γ}_{a+} of order $\gamma > 0$ is given by (see [17])

$$\left(I^{\gamma}_{a+} f\right)(x) = \frac{1}{\Gamma(\gamma)} \int_a^x \frac{f(t)}{(x-t)^{1-\gamma}}\, dt, \qquad x > a.$$

Let ${}^C D^{\gamma}_{a+}$ denote the left Caputo fractional derivative of order $\gamma > 0$ on $[a, b] \subset \mathbb{R}$, which is defined by (see [17])

$$\left({}^C D^{\gamma}_{a+} f\right)(x) = \left(I^{m-\gamma}_{a+} D^m f\right)(x) = \frac{1}{\Gamma(m-\gamma)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\gamma-m+1}}\, dt, \qquad x > a, \tag{2}$$

where $m = [\gamma] + 1$ and $[\gamma]$ means the integer part of γ. The previous definitions of fractional integrals and derivatives can be naturally extended to $\mathbb{R}^n$ considering partial fractional integrals and derivatives (see Chapter 5 in [29]). For instance, in (1), ${}^C_{0+}\partial^{\beta}_t u(x, t)$ denotes the partial left Caputo fractional derivative with respect to t with starting point $t = 0$.

In this work some integral transforms are used, namely, the Laplace, the Fourier and the Mellin transforms. The Laplace transform of a real valued function $f(t)$ is defined by (see [17])

$$\mathcal{L}\{f(t)\}(\mathbf{s}) = \widetilde{f}(\mathbf{s}) = \int_0^{+\infty} e^{-\mathbf{s}t} f(t)\, dt, \quad \mathrm{Re}(\mathbf{s}) \in \mathbb{C}$$

and when it is applied to (2) leads to (see formula (5.3.3) in [17])

$$\mathcal{L}\left\{{}^C D^{\gamma}_{a+} f(t)\right\}(\mathbf{s}) = \mathbf{s}^{\gamma} \widetilde{f}(\mathbf{s}) - \sum_{j=0}^{m-1} f^{(j)}(a)\, \mathbf{s}^{\gamma-j-1}, \qquad m = [\gamma] + 1. \tag{3}$$

Concerning the inverse Laplace transform of functions involving a branch point, we have the theorem from Titchmarsh (see [32]).

Theorem 2.1 *Let $\widetilde{f}(\mathbf{s})$ be an analytic function which has a branch cut on the real negative semiaxis, which has the following properties*

$$\widetilde{f}(\mathbf{s}) = O(1), \quad |\mathbf{s}| \to +\infty, \qquad\qquad \widetilde{f}(\mathbf{s}) = O\left(\frac{1}{|\mathbf{s}|}\right), \quad |\mathbf{s}| \to 0,$$

for any sector $|\arg(\mathbf{s})| < \pi - \eta$, where $0 < \eta < \pi$. Then the inverse Laplace transform of $\widetilde{f}(\mathbf{s})$ is given by

$$f(t) = \mathcal{L}^{-1}\left\{\widetilde{f}(\mathbf{s})\right\}(t) = -\frac{1}{\pi} \int_0^{+\infty} e^{-rt}\, \mathrm{Im}\left(\widetilde{f}(re^{i\pi})\right) dr.$$

The Laplace convolution operator of two functions is defined by the integral

$$(f *_t g)(t) = \int_0^t f(t-w)\, g(w)\, dw, \qquad t \in \mathbb{R}^+ \tag{4}$$

and the application of the Convolution Theorem to (4) leads to

$$\mathcal{L}\{(f *_t g)(t)\}(\mathbf{s}) = \mathcal{L}\{f\}(\mathbf{s})\, \mathcal{L}\{g\}(\mathbf{s}). \tag{5}$$

The n-dimensional Fourier transform of a function $f(x)$ of $x \in \mathbb{R}^n$ is defined by (see [17])

$$\mathcal{F}\{f(x)\}(\kappa) = \widehat{f}(\kappa) = \int_{\mathbb{R}^n} e^{i\kappa\cdot x} f(x)\, dx, \qquad \kappa \in \mathbb{R}^n,$$

while the corresponding inverse Fourier transform is given by the formula

$$f(x) = \mathcal{F}^{-1}\left\{\widehat{f}(\kappa)\right\}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ix\cdot\kappa}\, f(\kappa)\, d\kappa, \qquad x \in \mathbb{R}^n. \tag{6}$$

The Fourier convolution operator of two functions is defined by the integral

$$(f *_x g)(x) = \int_{\mathbb{R}^n} f(x-z)\, g(z)\, dz, \qquad x \in \mathbb{R}^n \tag{7}$$

and the application of the Convolution Theorem to (7) leads to

$$\mathcal{F}\{(f *_x g)(x)\}(\kappa) = \mathcal{F}\{f\}(\kappa)\, \mathcal{F}\{g\}(\kappa). \tag{8}$$

For the n-dimensional Laplace operator $\Delta_x = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ we have (see formula (1.3.32) in [17])

$$\mathcal{F}\{\Delta f(x)\}(\kappa) = -|\kappa|^2\, \mathcal{F}\{f(x)\}(\kappa). \tag{9}$$

Another integral transform that we use in this work is the Mellin transform. For f locally integrable on $]0,+\infty[$ it is defined by (see [17])

$$\mathcal{M}\{f(w)\}(s) = f^*(s) = \int_0^{+\infty} w^{s-1} f(w)\, dw, \qquad s \in \mathbb{C}, \tag{10}$$

and the inverse Mellin transform is given by

$$f(w) = \mathcal{M}^{-1}\{f(s)\}(w) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} w^{-s} f(s)\, ds, \qquad w > 0, \quad \gamma = \mathrm{Re}(s). \tag{11}$$

The condition for the existence of (10) is that $-p < \gamma < -q$ (called the fundamental strip), where p, q, are the order of f at the origin and ∞, respectively. The integration in (11) is performed along the imaginary axis and the result does not depend on the choice of γ inside the fundamental strip. For more information about this transform and its properties, see e.g. [17, 27]. The Mellin convolution between two functions is defined by

$$(f *_{\mathcal{M}} g)(x) = \int_0^{+\infty} f\left(\frac{x}{u}\right)\, g(u)\, \frac{du}{u}, \tag{12}$$

and satisfies the Mellin Convolution Theorem (see formula (1.4.40) in [17])

$$\mathcal{M}\{f *_{\mathcal{M}} g\}(s) = \mathcal{M}\{f\}(s)\ \mathcal{M}\{g\}(s).$$

The following relation holds (see (1.4.30) in [17])

$$\mathcal{M}\left\{f\left(\frac{1}{x}\right)\right\}(s) = \mathcal{M}\{f\}(-s), \tag{13}$$

and the Mellin transform of the Bessel function is given by (see formula (8.4.19.2) in [27])

$$\mathcal{M}\left\{J_\nu\left(\frac{2}{\sqrt{x}}\right)\right\}(s) = \frac{\Gamma\left(\frac{\nu}{2} - s\right)}{\Gamma\left(s + \frac{\nu}{2} + 1\right)}, \qquad -\frac{3}{4} < \mathrm{Re}(s) < \frac{\nu}{2}. \tag{14}$$

The solution of the time-fractional telegraph equation of distributed order obtained in this work involves the Fox H-function $H_{p,q}^{m,n}$, which is defined, via a Mellin-Barnes type integral in the form (see [18]), by

$$H_{p,q}^{m,n}\left[\, z \,\middle|\, \begin{matrix} (a_1,\ \alpha_1), \ldots, (a_p,\ \alpha_p) \\ \\ (b_1,\ \beta_1), \ldots, (b_q,\ \beta_q) \end{matrix} \right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma(b_j + \beta_j s)\ \prod_{i=1}^{n}\Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i + \alpha_i s)\ \prod_{j=m+1}^{q}\Gamma(1 - b_j - \beta_j s)}\ z^{-s}\, ds, \tag{15}$$

where $a_i, b_j \in \mathbb{C}$, and $\alpha_i, \beta_j \in \mathbb{R}^+$, for $i = 1, \ldots, p$ and $j = 1, \ldots, q$, and $\mathcal{L}$ is a suitable contour in the complex plane separating the poles of the two factors in the numerator (see [18]).

The particular cases studied in Section 4 involve the use of the two-argument inverse tangent $\arctan(x, y)$. This function computes the principal value of the argument function of the complex number $z = x + yi$. In terms of the standard inverse tangent, whose range is $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, it can be expressed as follows

$$\arctan(x,\, y) \;=\; \begin{cases} \arctan\left(\dfrac{y}{x}\right), & \text{if } x > 0 \\ \dfrac{\pi}{2} - \arctan\left(\dfrac{x}{y}\right), & \text{if } y > 0 \\ -\dfrac{\pi}{2} - \arctan\left(\dfrac{x}{y}\right), & \text{if } y < 0 \\ \arctan\left(\dfrac{y}{x}\right) \pm \pi, & \text{if } x < 0 \\ \text{undefined}, & \text{if } x = 0 \text{ and } y = 0 \end{cases} . \tag{16}$$

Throughout the paper, we assume that all the involved functions are Laplace and Fourier transformable, in order to be possible to obtain a general solution of (1).

3 Time-fractional telegraph equation of distributed order

Let us consider the following time-fractional telegraph equation of distributed order

$$\int_1^2 b_2(\beta) \left[{}_{0^+}^{\,C}\partial_t^{\beta} u(x,t) \right] d\beta + a \int_0^1 b_1(\alpha) \left[{}_{0^+}^{\,C}\partial_t^{\alpha} u(x,t) \right] d\alpha - c^2\, \Delta_x u(x,t) + d^2\, u(x,t) \;=\; q(x,t)\,, \tag{17}$$

for given order-density functions $b_2(\beta) > 0$ and $b_1(\alpha) > 0$, subject to the following initial and boundary conditions

$$u(x,0) = f(x)\,, \quad \frac{\partial u}{\partial t}(x,0) = g(x)\,, \quad \lim_{|x|\to+\infty} u(x,t) = 0, \quad \int_1^2 b_2(\beta)\, d\beta = C_2, \quad \int_0^1 b_1(\alpha)\, d\alpha = C_1, \tag{18}$$

where $(x,\, t) \in \mathbb{R}^n \times \mathbb{R}^+$, Δ_x is the classical Laplace operator in $\mathbb{R}^n$, the partial time-fractional derivatives of order $\beta \in]1, 2]$ and $\alpha \in]0, 1]$ are in the Caputo sense and given by (2), $a \in \mathbb{R}_0^+$, $c, d \in \mathbb{R}$, and $C_1, C_2 \in \mathbb{R}^+$. The positive constants C_1 and C_2 can be taken as 1 if we want to assume the normalization condition for the integral. In order to obtain the solution of (17)-(18) we extend the ideas presented in [25] and [3] to find the fundamental solution related to the generic order-density functions $b_2(\beta)$ and $b_1(\alpha)$.

3.1 Solution in the Fourier-Laplace Domain

In order to analytically determine the solution of (17)-(18) in the space-time domain we start applying the Fourier and Laplace transforms to (17). After that , there are two alternative strategies related to the order in carrying out the inversions of the Fourier and Laplace transforms are performed (see [21]):

(S1) invert the Fourier transform, giving $\widetilde{u}(x, s)$, and then invert the remaining Laplace transform.

(S2) invert the Laplace transform, giving $\widehat{u}(\kappa, t)$, and then invert the remaining Fourier transform.

In this work we consider (S2) where the inversion of the Laplace transform is performed via the classical Titchmarsh's Theorem, and the inversion of the Fourier transform is performed via the Mellin transform.

Let us start applying in (17) the Laplace transform with respect to the variable $t \in \mathbb{R}^+$ and the $n-$dimensional Fourier transform with respect to the variable $x \in \mathbb{R}^n$. Taking into account relations (3) and (9), and the initial conditions in (18), we obtain

$$\begin{aligned} &\widehat{\widetilde{u}}(\kappa, \mathbf{s}) \int_1^2 b_2(\beta)\, \mathbf{s}^{\beta}\, d\beta - \widehat{f}(\kappa) \int_1^2 b_2(\beta)\, \mathbf{s}^{\beta-1}\, d\beta - \widehat{g}(\kappa) \int_1^2 b_2(\beta)\, \mathbf{s}^{\beta-2}\, d\beta \\ &\quad + a\, \widehat{\widetilde{u}}(\kappa, \mathbf{s}) \int_0^1 b_1(\alpha)\, \mathbf{s}^{\alpha}\, d\alpha - a\, \widehat{f}(\kappa) \int_0^1 b_1(\alpha)\, \mathbf{s}^{\alpha-1}\, d\alpha + c^2\, |\kappa|^2\, \widehat{\widetilde{u}}(\kappa, \mathbf{s}) + d^2\, \widehat{\widetilde{u}}(\kappa, \mathbf{s}) = \widehat{\widetilde{q}}(\kappa, \mathbf{s}), \end{aligned}$$

which is equivalent to

$$\widehat{\widetilde{u}}(\kappa,\mathbf{s}) = \frac{\widehat{f}(\kappa)\ \left[\mathbf{s}\,B_2\left(\mathbf{s}\right)+\mathbf{s}\,B_1\left(\mathbf{s}\right)-\mathbf{s}\,\frac{d^2}{c^2}\right]+\widehat{g}(\kappa)\ \left[B_2\left(\mathbf{s}\right)-\frac{d^2}{c^2}\right]}{\mathbf{s}^2\ \left[B_2\left(\mathbf{s}\right)+B_1\left(\mathbf{s}\right)+|\kappa|^2\right]}+\frac{\widehat{\widetilde{q}}(\kappa,\mathbf{s})}{c^2\ \left[B_2\left(\mathbf{s}\right)+B_1\left(\mathbf{s}\right)+|\kappa|^2\right]}, \tag{19}$$

where $\widehat{f}\,(\kappa)$ and $\widehat{g}\,(\kappa)$ are the Fourier transforms of the functions $f\,(x)$ and $g\,(x)$, respectively, and

$$B_2\left(\mathbf{s}\right) = \frac{1}{c^2}\left[\int_1^2 b_2(\beta)\,\mathbf{s}^\beta\,d\beta+d^2\right], \tag{20}$$

$$B_1\left(\mathbf{s}\right) = \frac{a}{c^2}\int_0^1 b_1(\alpha)\,\mathbf{s}^\alpha\,d\alpha. \tag{21}$$

3.2 Solution in the time-space domain

In this section we perform the inversion of the Laplace and Fourier transforms in order to obtain our solution in the time-space domain. Let us consider the following auxiliar functions in the Laplace domain

$$\widehat{\widetilde{u}}_1\left(\kappa,\mathbf{s}\right) = \frac{B_2\left(\mathbf{s}\right)+B_1\left(\mathbf{s}\right)}{\mathbf{s}^p\ \left(B_2\left(\mathbf{s}\right)+B_1\left(\mathbf{s}\right)+|\kappa|^2\right)}, \tag{22}$$

$$\widehat{\widetilde{u}}_2\left(\kappa,\mathbf{s}\right) = \frac{1}{\mathbf{s}^p\ \left(B_2\left(\mathbf{s}\right)+B_1\left(\mathbf{s}\right)+|\kappa|^2\right)}, \tag{23}$$

$$\widehat{\widetilde{u}}_3\left(\kappa,\mathbf{s}\right) = \frac{B_2\left(\mathbf{s}\right)}{\mathbf{s}^p\ \left(B_2\left(\mathbf{s}\right)+B_1\left(\mathbf{s}\right)+|\kappa|^2\right)}, \tag{24}$$

with $p \geq 0$. Supposing that these functions are in the conditions of Theorem 2.1, which happens for the particular cases we consider in Section 4, we have

$$\widehat{u}_1\left(\kappa,t\right) = -\frac{1}{\pi}\int_0^{+\infty} e^{-rt}\,\mathrm{Im}\left(\widehat{\widetilde{u}}_1\left(\kappa,re^{i\pi}\right)\right)\,dr, \tag{25}$$

$$\widehat{u}_2\left(\kappa,t\right) = -\frac{1}{\pi}\int_0^{+\infty} e^{-rt}\,\mathrm{Im}\left(\widehat{\widetilde{u}}_2\left(\kappa,re^{i\pi}\right)\right)\,dr, \tag{26}$$

$$\widehat{u}_3\left(\kappa,t\right) = -\frac{1}{\pi}\int_0^{+\infty} e^{-rt}\,\mathrm{Im}\left(\widehat{\widetilde{u}}_3\left(\kappa,re^{i\pi}\right)\right)\,dr. \tag{27}$$

In order to simplify (25), (26), and (27), we need to evaluate the imaginary parts of the functions $\widehat{\widetilde{u}}_j\left(\kappa,re^{i\pi}\right)$, $j=1,2,3$, along the ray $\mathbf{s}=re^{i\pi}$, with $r>0$. In this sense, by writing

$$B_2\left(re^{i\pi}\right) = \rho_2\left(\cos\left(\gamma_2\pi\right)+i\,\sin\left(\gamma_2\pi\right)\right) \Longrightarrow \begin{cases}\rho_2 = \rho_2\left(r\right) = \left|B_2\left(re^{i\pi}\right)\right| \\ \gamma_2 = \gamma_2\left(r\right) = \dfrac{1}{\pi}\arg\left(B_2\left(re^{i\pi}\right)\right)\end{cases}, \tag{28}$$

$$B_1\left(re^{i\pi}\right) = \rho_1\left(\cos\left(\gamma_1\pi\right)+i\,\sin\left(\gamma_1\pi\right)\right) \Longrightarrow \begin{cases}\rho_1 = \rho_1\left(r\right) = \left|B_1\left(re^{i\pi}\right)\right| \\ \gamma_1 = \gamma_1\left(r\right) = \dfrac{1}{\pi}\arg\left(B_1\left(re^{i\pi}\right)\right)\end{cases}, \tag{29}$$

we obtain, after straightforward calculations, the following expressions for the imaginary part of the functions $\widehat{\widetilde{u}}_j$, $j=1,2,3$

$$\mathrm{Im}\left\{\widehat{\widetilde{u}}_1\left(\kappa,re^{i\pi}\right)\right\} = K_1\left(p,|\kappa|,r\right) = \frac{B\,|\kappa|^2}{(-r)^p\ \left[\left(A+|\kappa|^2\right)^2+B^2\right]}, \tag{30}$$

$$\mathrm{Im}\left\{\widehat{\widetilde{u}}_2\left(\kappa,re^{i\pi}\right)\right\} = K_2\left(p,|\kappa|,r\right) = \frac{-B}{(-r)^p\ \left[\left(A+|\kappa|^2\right)^2+B^2\right]}, \tag{31}$$

$$\mathrm{Im}\left\{\widehat{\widetilde{u}}_3\left(\kappa,re^{i\pi}\right)\right\} = K_3\left(p,|\kappa|,r\right) = \frac{\rho_2\left[\rho_1\sin\left(\gamma_1\pi-\gamma_2\pi\right)+|\kappa|^2\sin\left(\gamma_2\pi\right)\right]}{(-r)^p\ \left[\left(A+|\kappa|^2\right)^2+B^2\right]}, \tag{32}$$

where

$$A=\rho_2\cos\left(\gamma_2\pi\right)+\rho_1\cos\left(\gamma_1\pi\right)\qquad\text{and}\qquad B=\rho_2\sin\left(\gamma_2\pi\right)+\rho_1\sin\left(\gamma_1\pi\right). \tag{33}$$

Applying the inverse Laplace transform to (19) and taking into account Theorem 2.1, and expressions (25), (26), (27), (30), (31), and (32), we obtain

$$\begin{aligned}
\widehat{u}\left(\kappa,t\right) = &-\frac{\widehat{f}\left(\kappa\right)}{\pi}\int_0^{+\infty} e^{-rt}\left[K_1\left(1,\left|\kappa\right|,r\right)-\frac{d^2}{c^2}\,K_2\left(1,\left|\kappa\right|,r\right)\right]\,dr\\
&-\frac{\widehat{g}\left(\kappa\right)}{\pi}\int_0^{+\infty} e^{-rt}\left[K_3\left(2,\left|\kappa\right|,r\right)-\frac{d^2}{c^2}\,K_2\left(2,\left|\kappa\right|,r\right)\right]\,dr\\
&-\frac{\widehat{q}\left(\kappa,t\right)}{\pi c^2}\ast_t\int_0^{+\infty} e^{-rt}\,K_2\left(0,\left|\kappa\right|,r\right)\,dr,
\end{aligned} \tag{34}$$

where $\ast_t$ is given by (4) and in the last term me made use of (5). For the inversion of the Fourier transform, taking into account (6) and (8), we obtain

$$\begin{aligned}
u\left(x,t\right) = &-f\left(x\right)\ast_x\mathcal{F}^{-1}\left\{\frac{1}{\pi}\int_0^{+\infty} e^{-rt}\left[K_1\left(1,\left|\kappa\right|,r\right)-\frac{d^2}{c^2}\,K_2\left(1,\left|\kappa\right|,r\right)\right]\,dr\right\}\left(x,t\right)\\
&-g\left(x\right)\ast_x\mathcal{F}^{-1}\left\{\frac{1}{\pi}\int_0^{+\infty} e^{-rt}\left[K_3\left(2,\left|\kappa\right|,r\right)-\frac{d^2}{c^2}\,K_2\left(2,\left|\kappa\right|,r\right)\right]\,dr\right\}\left(x,t\right)\\
&-q\left(x,t\right)\ast_t\ast_x\mathcal{F}^{-1}\left\{\frac{1}{\pi c^2}\int_0^{+\infty} e^{-rt}\,K_2\left(0,\left|\kappa\right|,r\right)\,dr\right\}\left(x\right).
\end{aligned} \tag{35}$$

Making use of the following formula presented in [29] for the inverse Fourier transform

$$\frac{1}{\left(2\pi\right)^n}\int_{\mathbb{R}^n} e^{-ix\cdot\kappa}\,\varphi\left(\left|\kappa\right|\right)\,d\kappa=\frac{\left|x\right|^{1-\frac{n}{2}}}{\left(2\pi\right)^{\frac{n}{2}}}\int_0^{+\infty}\varphi\left(w\right)\,w^{\frac{n}{2}}\,J_{\frac{n}{2}-1}\left(\left|x\right|w\right)\,dw, \tag{36}$$

and since we are dealing with radial functions in κ, (35) can be rewritten as

$$\begin{aligned}
u\left(x,t\right) = &-\frac{1}{\pi}\,f\left(x\right)\ast_x\left[\frac{\left|x\right|^{1-\frac{n}{2}}}{\left(2\pi\right)^{\frac{n}{2}}}\int_0^{+\infty}\int_0^{+\infty} e^{-rt}\left[K_1\left(1,w,r\right)-\frac{d^2}{c^2}\,K_2\left(1,w,r\right)\right]\,dr\,w^{\frac{n}{2}}\,J_{\frac{n}{2}-1}\left(\left|x\right|w\right)\,dw\right]\\
&-\frac{1}{\pi}\,g\left(x\right)\ast_x\left[\frac{\left|x\right|^{1-\frac{n}{2}}}{\left(2\pi\right)^{\frac{n}{2}}}\int_0^{+\infty}\int_0^{+\infty} e^{-rt}\left[K_3\left(2,w,r\right)-\frac{d^2}{c^2}\,K_2\left(2,w,r\right)\right]\,dr\,w^{\frac{n}{2}}\,J_{\frac{n}{2}-1}\left(\left|x\right|w\right)\,dw\right]\\
&-\frac{1}{\pi c^2}\,q\left(x,t\right)\ast_t\ast_x\left[\frac{\left|x\right|^{1-\frac{n}{2}}}{\left(2\pi\right)^{\frac{n}{2}}}\int_0^{+\infty}\int_0^{+\infty} e^{-rt}\,K_2\left(0,w,r\right)\,dr\,w^{\frac{n}{2}}\,J_{\frac{n}{2}-1}\left(\left|x\right|w\right)\,dw\right]\\
= &-\frac{1}{\pi}\,f\left(x\right)\ast_x\left[\int_0^{+\infty} e^{-rt}\,\underbrace{\frac{\left|x\right|^{1-\frac{n}{2}}}{\left(2\pi\right)^{\frac{n}{2}}}\int_0^{+\infty}\left[K_1\left(1,w,r\right)-\frac{d^2}{c^2}\,K_2\left(1,w,r\right)\right]\,w^{\frac{n}{2}}\,J_{\frac{n}{2}-1}\left(\left|x\right|w\right)\,dw\,dr}_{\mathbf{I_1}}\right]\\
&-\frac{1}{\pi}\,g\left(x\right)\ast_x\left[\int_0^{+\infty} e^{-rt}\,\underbrace{\frac{\left|x\right|^{1-\frac{n}{2}}}{\left(2\pi\right)^{\frac{n}{2}}}\int_0^{+\infty}\left[K_3\left(2,w,r\right)-\frac{d^2}{c^2}\,K_2\left(2,w,r\right)\right]\,w^{\frac{n}{2}}\,J_{\frac{n}{2}-1}\left(\left|x\right|w\right)\,dw\,dr}_{\mathbf{I_2}}\right]\\
&-\frac{1}{\pi c^2}\,q\left(x,t\right)\ast_t\ast_x\left[\int_0^{+\infty} e^{-rt}\,\underbrace{\frac{\left|x\right|^{1-\frac{n}{2}}}{\left(2\pi\right)^{\frac{n}{2}}}\int_0^{+\infty} K_2\left(0,w,r\right)\,w^{\frac{n}{2}}\,J_{\frac{n}{2}-1}\left(\left|x\right|w\right)\,dw\,dr}_{\mathbf{I_3}}\right].
\end{aligned} \tag{37}$$

To compute explicitly $\mathbf{I_1}$, $\mathbf{I_2}$, and $\mathbf{I_3}$ in (37) we are going to use the Mellin transform. First, we rewrite these integrals as a Mellin convolution (12). In fact considering the following auxiliar functions

$$g_1(w) = K_1(1,w,r) - \frac{d^2}{c^2} K_2(1,w,r) \qquad g_2(w) = K_3(2,w,r) - \frac{d^2}{c^2} K_2(2,w,r)$$

$$g_3(w) = K_2(0,w,r) \qquad f(w) = \frac{1}{(2\pi)^{\frac{n}{2}} \, |x|^n \, w^{\frac{n}{2}+1}} J_{\frac{n}{2}-1}\left(\frac{1}{w}\right)$$

we have

$$\begin{aligned}
\mathbf{I_1} &= \mathcal{M}\{g_1 *_{\mathcal{M}} f\}\left(\frac{1}{|x|}\right) \\
&= \int_0^{+\infty} g_1(w)\, f\left(\frac{1}{|x|\, w}\right) \frac{dw}{w} \\
&= \int_0^{+\infty} \left[K_1(1,w,r) - \frac{d^2}{c^2} K_2(1,w,r)\right] \frac{w^{\frac{n}{2}+1} |x|^{\frac{n}{2}+1}}{(2\pi)^{\frac{n}{2}} |x|^n} J_{\frac{n}{2}-1}(|x|\, w)\, \frac{dw}{w} \\
&= \frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \int_0^{+\infty} \left[K_1(1,w,r) - \frac{d^2}{c^2} K_2(1,w,r)\right] w^{\frac{n}{2}} J_{\frac{n}{2}-1}(|x|\, w)\, dw
\end{aligned}$$

and analogously,

$$\mathbf{I_2} = \mathcal{M}\{g_2 *_{\mathcal{M}} f\}\left(\frac{1}{|x|}\right) \qquad \text{and} \qquad \mathbf{I_3} = \mathcal{M}\{g_3 *_{\mathcal{M}} f\}\left(\frac{1}{|x|}\right). \tag{38}$$

From the relations (13) and (12) we have for $\mathbf{I_1}$

$$\mathcal{M}\{\mathbf{I_1}\}(s) = \mathcal{M}\left\{g_1 *_{\mathcal{M}} f\left(\frac{1}{|x|}\right)\right\}(s) = \mathcal{M}\{g_1\}(-s)\, \mathcal{M}\{f\}(-s),$$

which is equivalent to

$$\mathcal{M}\{\mathbf{I_1}\}(-s) = \mathcal{M}\{g_1\}(s)\, \mathcal{M}\{f\}(s). \tag{39}$$

In a similar way we obtain

$$\mathcal{M}\{\mathbf{I_2}\}(-s) = \mathcal{M}\{g_2\}(s)\, \mathcal{M}\{f\}(s), \tag{40}$$

and

$$\mathcal{M}\{\mathbf{I_3}\}(-s) = \mathcal{M}\{g_3\}(s)\, \mathcal{M}\{f\}(s). \tag{41}$$

Let us now compute the Mellin transforms that appear in (39), (40), and (41). Taking into account (10), we obtain

$$\mathcal{M}\{f\}(s) = \frac{1}{(2\pi)^{\frac{n}{2}} \, |x|^{\frac{n}{2}}} \int_0^{+\infty} w^{s-1}\, w^{-\frac{n}{2}-1} J_{\frac{n}{2}-1}\left(\frac{1}{w}\right) dw.$$

Considering the change of variables $\frac{1}{w} = \frac{2}{\sqrt{z}}$ and taking into account (10), (14), and the duplication formula for the Gamma function (see [1])

$$\Gamma(z) = \frac{\sqrt{\pi}}{2^{2z-1}} \frac{\Gamma(2z)}{\Gamma\left(z+\frac{1}{2}\right)} \tag{42}$$

we get

$$\begin{aligned}
\mathcal{M}\{f\}(s) &= \frac{1}{\pi^{\frac{n}{2}} \, |x|^2 \, 2^s} \int_0^{+\infty} z^{\frac{s}{2}-\frac{n}{4}-1} J_{\frac{n}{2}-1}\left(\frac{2}{\sqrt{z}}\right) dz \\
&= \frac{1}{\pi^{\frac{n}{2}} \, |x|^2 \, 2^s} \mathcal{M}\left\{J_{\frac{n}{2}-1}\left(\frac{2}{\sqrt{z}}\right)\right\}\left(\frac{s}{2} - \frac{n}{4} - \frac{1}{2}\right) \\
&= \frac{1}{\pi^{\frac{n}{2}} \, |x|^2 \, 2^s} \frac{\Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}, \qquad -\frac{3}{4} < \operatorname{Re}(s) < \frac{n}{4} - \frac{1}{2} \\
&= \frac{1}{\pi^{\frac{n-1}{2}} \, |x|^n \, 2^{n-1}} \frac{\Gamma(n-s)}{\Gamma\left(\frac{n+1-s}{2}\right) \Gamma\left(\frac{s}{2}\right)}.
\end{aligned} \tag{43}$$

Now, we calculate the Mellin transform of the function g_1. Taking into account (10), (30), (31), and (33), we get

$$\begin{aligned}\mathcal{M}\{g_1\}(s) &= \int_0^{+\infty} w^{s-1}\, K_1(1,w,r)\; dw - \frac{d^2}{c^2}\int_0^{+\infty} w^{s-1}\, K_2(1,w,r)\; dw \\ &= -\frac{B}{r}\int_0^{+\infty} \frac{w^{s+1}}{\left(A+w^2\right)^2+B^2}\, dw - \frac{d^2 B}{c^2 r}\int_0^{+\infty} \frac{w^{s-1}}{\left(A+w^2\right)^2+B^2}\, dw. \qquad (44)\end{aligned}$$

Considering the change of variables $w^2 = z$ in (44) we obtain

$$\mathcal{M}\{g_1\}(s) = -\frac{B}{2r}\underbrace{\int_0^{+\infty}\frac{z^{\frac{s}{2}}}{z^2+2Az+A^2+B^2}\, dz}_{\mathbf{I_4}} - \frac{d^2 B}{2c^2 r}\underbrace{\int_0^{+\infty}\frac{z^{\frac{s}{2}-1}}{z^2+2Az+A^2+B^2}\, dz}_{\mathbf{I_5}}. \qquad (45)$$

Taking into account formula (2.2.9.36) in [28]

$$\int_0^{+\infty}\frac{x^{\alpha-1}}{ax^2+bx+c}\, dx \;=\frac{\pi\,\sin\left[(1-\alpha)\,\psi\right]\, c^{\frac{\alpha}{2}-1}}{a^{\frac{\alpha}{2}}\,\sin(\psi)\,\sin(\alpha\pi)}, \qquad ac > b^2, \qquad \psi = \arccos\left(\frac{b}{\sqrt{ac}}\right), \qquad (46)$$

with

$$\alpha = \frac{s}{2}+1 \text{ for } \mathbf{I_4}, \quad \alpha = \frac{s}{2} \text{ for } \mathbf{I_5}, \quad a = 1, \qquad b = A, \quad c = A^2+B^2, \quad \psi = \arccos\left(\frac{A}{\sqrt{A^2+B^2}}\right), \qquad (47)$$

and making use of the following property of the Gamma function (see [1])

$$\csc(\pi z) \;= \frac{1}{\pi}\,\Gamma(1-z)\;\Gamma(z) \qquad (48)$$

we compute the integrals $\mathbf{I_4}$ and $\mathbf{I_5}$, getting

$$\begin{aligned}\mathbf{I_4} &= \int_0^{+\infty}\frac{z^{\frac{s}{2}}}{z^2+2Az+A^2+B^2}\, dw \\ &= \frac{\pi}{\sin(\psi)}\,\frac{\sin\left(-\frac{s\psi}{2}\right)}{\sin\left(\left(\frac{s}{2}+1\right)\pi\right)}\,\left(A^2+B^2\right)^{\frac{s}{4}-\frac{1}{2}} \\ &= -\frac{\pi}{\sin(\psi)}\,\frac{\sin\left(\frac{s\psi}{2}\right)}{\sin\left(-\frac{s\pi}{2}\right)}\,\left(A^2+B^2\right)^{\frac{s}{4}-\frac{1}{2}} \\ &= -\frac{\pi}{\sin(\psi)}\,\frac{\csc\left(-\frac{s\pi}{2}\right)}{\csc\left(\frac{s\psi}{2}\right)}\,\left(A^2+B^2\right)^{\frac{s}{4}-\frac{1}{2}} \\ &= -\frac{\pi}{\sin(\psi)}\,\frac{\Gamma\left(1+\frac{s}{2}\right)\,\Gamma\left(1-\left(1+\frac{s}{2}\right)\right)}{\Gamma\left(\frac{s\psi}{2\pi}\right)\,\Gamma\left(1-\frac{s\psi}{2\pi}\right)}\,\left(A^2+B^2\right)^{\frac{s}{4}-\frac{1}{2}}, \qquad (49)\end{aligned}$$

and

$$\mathbf{I_5} \;= \int_0^{+\infty}\frac{z^{\frac{s}{2}-1}}{z^2+2Az+A^2+B^2}\, dw \;= -\frac{\pi}{\sin(\psi)}\,\frac{\Gamma\left(\frac{s}{2}\right)\,\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(\frac{\psi}{\pi}\left(\frac{s}{2}-1\right)\right)\,\Gamma\left(1-\frac{\psi}{\pi}\left(\frac{s}{2}-1\right)\right)}\,\left(A^2+B^2\right)^{\frac{s}{4}-1}. \qquad (50)$$

Hence, from (49) and (50) we conclude that (45) takes the form

$$\begin{aligned}\mathcal{M}\{g_1\}(s) &= \frac{B\pi}{2r\sin(\psi)}\,\frac{\Gamma\left(1+\frac{s}{2}\right)\,\Gamma\left(1-\left(1+\frac{s}{2}\right)\right)}{\Gamma\left(\frac{s\psi}{2\pi}\right)\,\Gamma\left(1-\frac{s\psi}{2\pi}\right)}\,\left(A^2+B^2\right)^{\frac{s}{4}-\frac{1}{2}} \\ &\quad + \frac{d^2 B\pi}{2c^2 r\sin(\psi)}\,\frac{\Gamma\left(\frac{s}{2}\right)\,\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(\frac{\psi}{\pi}\left(\frac{s}{2}-1\right)\right)\,\Gamma\left(1-\frac{\psi}{\pi}\left(\frac{s}{2}-1\right)\right)}\,\left(A^2+B^2\right)^{\frac{s}{4}-1}. \qquad (51)\end{aligned}$$

Let us now calculate the Mellin transform of g_2. Taking into account (10), (31), (32), and (33), we get

$$\begin{aligned}
\mathcal{M}\left\{g_2\right\}(s) &= \int_0^{+\infty} w^{s-1}\, K_3\left(2,w,r\right)\, dw - \frac{d^2}{c^2}\int_0^{+\infty} w^{s-1}\, K_2\left(2,w,r\right)\, dw \\
&= \frac{\rho_2}{r^2}\int_0^{+\infty} w^{s-1}\,\frac{\left(A+w^2\right)\sin\left(\gamma_2\pi\right) - B\,\cos\left(\gamma_2\pi\right)}{\left(A+w^2\right)^2+B^2}\, dw + \frac{d^2 B}{c^2 r^2}\int_0^{+\infty}\frac{w^{s-1}}{\left(A+w^2\right)^2+B^2}\, dw \\
&= \frac{\rho_2\sin\left(\gamma_2\pi\right)}{r^2}\int_0^{+\infty}\frac{w^{s-1}\,\left(A+w^2\right)}{\left(A+w^2\right)^2+B^2}\, dw - \frac{B\rho_2\cos\left(\gamma_2\pi\right)}{r^2}\int_0^{+\infty}\frac{w^{s-1}}{\left(A+w^2\right)^2+B^2}\, dw \\
&+ \frac{d^2 B}{c^2 r^2}\int_0^{+\infty}\frac{w^{s-1}}{\left(A+w^2\right)^2+B^2}\, dw \\
&= \frac{A\rho_2\sin\left(\gamma_2\pi\right)}{r^2}\int_0^{+\infty}\frac{w^{s-1}}{\left(A+w^2\right)^2+B^2}\, dw + \frac{\rho_2\sin\left(\gamma_2\pi\right)}{r^2}\int_0^{+\infty}\frac{w^{s+1}}{\left(A+w^2\right)^2+B^2}\, dw \\
&+ \left[-\frac{B\rho_2\cos\left(\gamma_2\pi\right)}{r^2} + \frac{d^2 B}{c^2 r^2}\right]\int_0^{+\infty}\frac{w^{s-1}}{\left(A+w^2\right)^2+B^2}\, dw \\
&= \frac{\rho_2\sin\left(\gamma_2\pi\right)}{r^2}\int_0^{+\infty}\frac{w^{s+1}}{\left(A+w^2\right)^2+B^2}\, dw \\
&+ \frac{Ac^2\rho_2\sin\left(\gamma_2\pi\right) - Bc^2\rho_2\cos\left(\gamma_2\pi\right) + d^2 B}{c^2 r^2}\int_0^{+\infty}\frac{w^{s-1}}{\left(A+w^2\right)^2+B^2}\, dw. \qquad (52)
\end{aligned}$$

Considering the change of variables $w^2 = z$ in (52), we obtain

$$\begin{aligned}
\mathcal{M}\left\{g_2\right\}(s) = \frac{\rho_2\sin\left(\gamma_2\pi\right)}{2r^2}\int_0^{+\infty}\frac{z^{\frac{s}{2}}}{z^2+2Az+A^2+B^2}\, dz \\
+ \frac{Ac^2\rho_2\sin\left(\gamma_2\pi\right) - Bc^2\rho_2\cos\left(\gamma_2\pi\right) + d^2 B}{2c^2 r^2}\int_0^{+\infty}\frac{z^{\frac{s}{2}-1}}{z^2+2Az+A^2+B^2}\, dz. \qquad (53)
\end{aligned}$$

The two integrals in (53) correspond to the integrals $\mathbf{I_4}$ and $\mathbf{I_5}$. Hence, from (49) and (50) we arrive to

$$\begin{aligned}
\mathcal{M}\left\{g_2\right\}(s) = -\frac{\pi\rho_2\sin\left(\gamma_2\pi\right)}{2r^2\sin\left(\psi\right)}\,\frac{\Gamma\left(1+\frac{s}{2}\right)\,\Gamma\left(1-\left(1+\frac{s}{2}\right)\right)}{\Gamma\left(\frac{s\psi}{2\pi}\right)\,\Gamma\left(1-\frac{s\psi}{2\pi}\right)}\,\left(A^2+B^2\right)^{\frac{s}{4}-\frac{1}{2}} \\
-\frac{\pi\left(Ac^2\rho_2\sin\left(\gamma_2\pi\right) - Bc^2\rho_2\cos\left(\gamma_2\pi\right) + d^2 B\right)}{2c^2 r^2\sin\left(\psi\right)}\,\frac{\Gamma\left(\frac{s}{2}\right)\,\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(\frac{\psi}{\pi}\left(\frac{s}{2}-1\right)\right)\,\Gamma\left(1-\frac{\psi}{\pi}\left(\frac{s}{2}-1\right)\right)}\,\left(A^2+B^2\right)^{\frac{s}{4}-1}. \qquad (54)
\end{aligned}$$

We finally calculate the Mellin transform of g_3. Taking into account (10), (31), and (33), we get

$$\begin{aligned}
\mathcal{M}\left\{g_3\right\}(s) &= \int_0^{+\infty} w^{s-1}\, K_2\left(0,w,r\right)\, dw \\
&= -B\int_0^{+\infty} w^{s-1}\,\frac{1}{\left(A+w^2\right)^2+B^2}\, dw \\
&= -B\int_0^{+\infty}\frac{w^{s-1}}{\left(A+w^2\right)^2+B^2}\, dw. \qquad (55)
\end{aligned}$$

Considering the change of variables $w^2 = z$ in (55), we obtain

$$\mathcal{M}\left\{g_3\right\}(s) = -\frac{B}{2}\int_0^{+\infty}\frac{z^{\frac{s}{2}-1}}{z^2+2Az+A^2+B^2}\, dz. \qquad (56)$$

The integral in (56) corresponds to the integral $\mathbf{I_5}$. Hence, from (50) we arrive to

$$\mathcal{M}\left\{g_3\right\}(s) = \frac{B\pi}{2\sin\left(\psi\right)}\,\frac{\Gamma\left(\frac{s}{2}\right)\,\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(\frac{\psi}{\pi}\left(\frac{s}{2}-1\right)\right)\,\Gamma\left(1-\frac{\psi}{\pi}\left(\frac{s}{2}-1\right)\right)}\,\left(A^2+B^2\right)^{\frac{s}{4}-1}. \qquad (57)$$

Now, using the inverse Mellin transform (11) applied to (39), (40), and (41) we obtain, respectively, the representation of integrals $\mathbf{I_1}$, $\mathbf{I_2}$, and $\mathbf{I_3}$ in terms of Mellin-Barnes integrals and, consequently, as Fox H-functions. For the integral $\mathbf{I_1}$, taking into account (11), (33), (51), and (43), we obtain

$$\mathbf{I_1} = \frac{B\left(A^2+B^2\right)^{-\frac{1}{2}}}{r\,\pi^{\frac{n-3}{2}}\left(2\,|x|\right)^n\,\sin(\psi)}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(1+\frac{s}{2}\right)\,\Gamma(n-s)\,\Gamma\left(-\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\,\Gamma\left(\frac{\psi s}{2\pi}\right)\,\Gamma\left(\frac{n+1}{2}-\frac{s}{2}\right)\,\Gamma\left(1-\frac{\psi s}{2\pi}\right)}\left(\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\right)^{-s}ds$$

$$+\frac{Bd^2\left(A^2+B^2\right)^{-1}}{rc^2\,\pi^{\frac{n-3}{2}}\left(2\,|x|\right)^n\,\sin(\psi)}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(n-s)\,\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(-\frac{\psi}{\pi}+\frac{\psi s}{2\pi}\right)\,\Gamma\left(\frac{n+1}{2}-\frac{s}{2}\right)\,\Gamma\left(1+\frac{\psi}{\pi}-\frac{\psi s}{2\pi}\right)}\left(\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\right)^{-s}ds$$

which is equivalent, by (15), to the following expression in terms of Fox H-functions

$$\mathbf{I_1} = \frac{B\left(A^2+B^2\right)^{-\frac{1}{2}}}{r\,\pi^{\frac{n-3}{2}}\left(2\,|x|\right)^n\,\sin(\psi)}\,H^{1,2}_{4,3}\left[\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\,\middle|\,\begin{matrix}(1-n,\,1)\,,\left(1,\,\frac{1}{2}\right),\left(0,\,\frac{1}{2}\right),\left(0,\,\frac{\psi}{2\pi}\right)\\ \left(1,\,\frac{1}{2}\right),\left(\frac{1-n}{2},\,\frac{1}{2}\right),\left(0,\,\frac{\psi}{2\pi}\right)\end{matrix}\right]$$

$$+\frac{Bd^2\left(A^2+B^2\right)^{-1}}{rc^2\,\pi^{\frac{n-3}{2}}\left(2\,|x|\right)^n\,\sin(\psi)}\,H^{0,2}_{3,2}\left[\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\,\middle|\,\begin{matrix}(1-n,\,1)\,,\left(0,\,\frac{1}{2}\right),\left(-\frac{\psi}{\pi},\,\frac{\psi}{2\pi}\right)\\ \left(\frac{1-n}{2},\,\frac{1}{2}\right),\left(-\frac{\psi}{\pi},\,\frac{\psi}{2\pi}\right)\end{matrix}\right]. \qquad (58)$$

For the integral $\mathbf{I_2}$, taking into account (11), (33), (54), and (43), we obtain

$$\mathbf{I_2} = \frac{\rho_2\,\sin(\gamma_2\pi)\left(A^2+B^2\right)^{-\frac{1}{2}}}{r^2\,\pi^{\frac{n-3}{2}}\left(2\,|x|\right)^n\,\sin(\psi)}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(1+\frac{s}{2}\right)\,\Gamma(n-s)\,\Gamma\left(-\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\,\Gamma\left(\frac{\psi s}{2\pi}\right)\,\Gamma\left(\frac{n+1}{2}-\frac{s}{2}\right)\,\Gamma\left(1-\frac{\psi s}{2\pi}\right)}\left(\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\right)^{-s}ds$$

$$-\frac{\left(Ac^2\rho_2\,\sin(\gamma_2\pi)-Bc^2\rho_2\,\cos(\gamma_2\pi)+Bd^2\right)\left(A^2+B^2\right)^{-1}}{r^2c^2\,\pi^{\frac{n-3}{2}}\left(2\,|x|\right)^n\,\sin(\psi)}$$

$$\times\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(n-s)\,\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(-\frac{\psi}{\pi}+\frac{\psi s}{2\pi}\right)\,\Gamma\left(\frac{n+1}{2}-\frac{s}{2}\right)\,\Gamma\left(1+\frac{\psi}{\pi}-\frac{\psi s}{2\pi}\right)}\left(\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\right)^{-s}ds$$

which is equivalent, by (15), to the following expression in terms of Fox H-functions

$$\mathbf{I_2} = \frac{\rho_2\,\sin(\gamma_2\pi)\left(A^2+B^2\right)^{-\frac{1}{2}}}{r^2\,\pi^{\frac{n-3}{2}}\left(2\,|x|\right)^n\,\sin(\psi)}\,H^{1,2}_{4,3}\left[\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\,\middle|\,\begin{matrix}(1-n,\,1)\,,\left(1,\,\frac{1}{2}\right),\left(0,\,\frac{1}{2}\right),\left(0,\,\frac{\psi}{2\pi}\right)\\ \left(1,\,\frac{1}{2}\right),\left(\frac{1-n}{2},\,\frac{1}{2}\right),\left(0,\,\frac{\psi}{2\pi}\right)\end{matrix}\right]$$

$$-\frac{\left(Ac^2\rho_2\,\sin(\gamma_2\pi)-Bc^2\rho_2\,\cos(\gamma_2\pi)+Bd^2\right)\left(A^2+B^2\right)^{-1}}{r^2c^2\,\pi^{\frac{n-3}{2}}\left(2\,|x|\right)^n\,\sin(\psi)}$$

$$\times H^{0,2}_{3,2}\left[\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\,\middle|\,\begin{matrix}(1-n,\,1)\,,\left(0,\,\frac{1}{2}\right),\left(-\frac{\psi}{\pi},\,\frac{\psi}{2\pi}\right)\\ \left(\frac{1-n}{2},\,\frac{1}{2}\right),\left(-\frac{\psi}{\pi},\,\frac{\psi}{2\pi}\right)\end{matrix}\right]. \qquad (59)$$

Finally, for the integral $\mathbf{I_3}$, taking into account (11), (33), (57), and (43), we obtain

$$\mathbf{I_3} = \frac{B\left(A^2+B^2\right)^{-1}}{\pi^{\frac{n-3}{2}}\left(2\,|x|\right)^n\,\sin(\psi)}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(n-s)\,\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(-\frac{\psi}{\pi}+\frac{\psi s}{2\pi}\right)\,\Gamma\left(\frac{n+1}{2}-\frac{s}{2}\right)\,\Gamma\left(1+\frac{\psi}{\pi}-\frac{\psi s}{2\pi}\right)}\left(\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\right)^{-s}ds$$

which is equivalent, by (15), to the following expression in terms of Fox H-functions

$$\mathbf{I_3} = \frac{B\left(A^2+B^2\right)^{-1}}{\pi^{\frac{n-3}{2}}\left(2\left|x\right|\right)^n \sin\left(\psi\right)} H^{0,2}_{3,2}\left[\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{\left|x\right|}\left|\begin{array}{c}\left(1-n,\,1\right),\,\left(0,\,\frac{1}{2}\right),\,\left(-\frac{\psi}{\pi},\,\frac{\psi}{2\pi}\right)\\ \left(\frac{1-n}{2},\,\frac{1}{2}\right),\,\left(-\frac{\psi}{\pi},\,\frac{\psi}{2\pi}\right)\end{array}\right.\right]. \tag{60}$$

From (58), (59), and (60) we conclude that the representation (37) of the solution $u\left(x,t\right)$ of (17)-(18) corresponds to the sum of convolution integrals involving Fox H-functions.

In the next subsection we summarize our calculations in the main result of the paper.

3.3 Main result and corollary

Taking into account (37), (58), (59), and (60) we obtain our main result.

Theorem 3.1 *The solution of the time-fractional telegraph equation of distributed order* (17) *subject to the conditions* (18) *is given, in terms of convolution integrals, by*

$$u\left(x,t\right) = \int_{\mathbb{R}^n} f\left(z\right) G_1\left(x-z,\,t\right)\,dz + \int_{\mathbb{R}^n} g\left(z\right) G_2\left(x-z,\,t\right)\,dz + \int_{\mathbb{R}^n}\int_0^t q\left(z,\,w\right) G_3\left(x-z,\,t-w\right)\,dw\,dz, \tag{61}$$

where the fundamental solutions G_1, G_2, and G_3 are given by

$$G_1\left(x,t\right) = \frac{-1}{\pi^{\frac{n-1}{2}}\left(2\left|x\right|\right)^n}\int_0^{+\infty}\frac{B\left(A^2+B^2\right)^{-\frac{1}{2}}\,e^{-rt}}{r\,\sin\left(\psi\right)}$$
$$\times\left[\mathcal{H}\left(\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{\left|x\right|}\right) + \frac{d^2}{c^2}\left(A^2+B^2\right)^{-\frac{1}{2}}\mathcal{H}^*\left(\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{\left|x\right|}\right)\right]dr,$$

$$G_2\left(x,t\right) = \frac{-1}{\pi^{\frac{n-1}{2}}\left(2\left|x\right|\right)^n}\int_0^{+\infty}\frac{\left(A^2+B^2\right)^{-\frac{1}{2}}\,e^{-rt}}{r^2\,\sin\left(\psi\right)}\left[-\rho_2\sin\left(\gamma_2\pi\right)\mathcal{H}\left(\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{\left|x\right|}\right)\right.$$
$$\left.-\frac{1}{c^2}\left(Ac^2\rho_2\,\sin\left(\gamma_2\pi\right) - Bc^2\rho_2\,\cos\left(\gamma_2\pi\right) + Bd^2\right)\left(A^2+B^2\right)^{-\frac{1}{2}}\mathcal{H}^*\left(\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{\left|x\right|}\right)\right]dr,$$

$$G_3\left(x,t\right) = \frac{-1}{c^2\,\pi^{\frac{n-1}{2}}\left(2\left|x\right|\right)^n}\int_0^{+\infty}\frac{B\left(A^2+B^2\right)^{-1}\,e^{-rt}}{\sin\left(\psi\right)}\mathcal{H}^*\left(\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{\left|x\right|}\right)dr,$$

where ρ_2 and γ_2, A and B, and ψ are given, respectively, by relations (28), (33), *and* (47), *and the functions $\mathcal{H}$ and $\mathcal{H}^*$ are expressed in terms of the following Fox H-functions*

$$\mathcal{H}\left(\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{\left|x\right|}\right) = H^{1,2}_{4,3}\left[\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{\left|x\right|}\left|\begin{array}{c}\left(1-n,\,1\right),\,\left(1,\,\frac{1}{2}\right),\,\left(0,\,\frac{1}{2}\right),\,\left(0,\,\frac{\psi}{2\pi}\right)\\ \left(1,\,\frac{1}{2}\right),\,\left(\frac{1-n}{2},\,\frac{1}{2}\right),\,\left(0,\,\frac{\psi}{2\pi}\right)\end{array}\right.\right],$$

$$\mathcal{H}^*\left(\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{\left|x\right|}\right) = H^{0,2}_{3,2}\left[\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{\left|x\right|}\left|\begin{array}{c}\left(1-n,\,1\right),\,\left(0,\,\frac{1}{2}\right),\,\left(-\frac{\psi}{\pi},\,\frac{\psi}{2\pi}\right)\\ \left(\frac{1-n}{2},\,\frac{1}{2}\right),\,\left(-\frac{\psi}{\pi},\,\frac{\psi}{2\pi}\right)\end{array}\right.\right].$$

Remark 3.2 *If we consider*

$$f\left(x\right) = \delta\left(x\right) = \prod_{i=1}^{n}\delta\left(x_i\right), \qquad g\left(x\right) = q\left(x,t\right) = 0, \qquad a = c = 1, \qquad d = \sqrt{\lambda}$$

with $\lambda \in \mathbb{R}^+$ in (17)-(18)*, then the solution $u(x,t)$ given by* (61) *corresponds to the eigenfunction of the time-fractional telegraph equation of distributed order in $\mathbb{R}^n \times \mathbb{R}^+$. Moreover, if additionally $b_2(\beta) = 0$ (resp. $b_1(\alpha) = 0$) we obtain the representation of the eigenfunctions of the time-fractional diffusion (resp. wave) equation of distributed order in $\mathbb{R}^n \times \mathbb{R}^+$.*

Considering $a = 0$ in Theorem 3.1, we have the following simplifications

$$B_1(s) = 0, \qquad A = \rho\cos(\gamma\pi), \qquad B = \rho\sin(\gamma\pi), \qquad A^2 + B^2 = \rho^2, \qquad \psi = \gamma\pi,$$

which give the following result.

Corollary 3.3 *The solution of the time-fractional wave equation of distributed order*

$$\int_1^2 b_2(\beta) \left[{}_{0^+}^{C}\partial_t^\beta u(x,t) \right] d\beta - c^2 \Delta_x u(x,t) + d^2 u(x,t) = q(x,t)$$

for given order-density function $b_2(\beta)$, subject to the following initial and boundary conditions

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad \lim_{|x|\to\infty} u(x,t) = 0, \qquad \int_1^2 b_2(\beta)\, d\beta = 1,$$

is given, in terms of convolution integrals, by

$$u(x,t) = \int_{\mathbb{R}^n} f(z)\, G_1(x-z,\, t)\, dz + \int_{\mathbb{R}^n} g(z)\, G_2(x-z,\, t)\, dz + \int_{\mathbb{R}^n} \int_0^t q(z,\, w)\, G_3(x-z,\, t-w)\, dw\, dz,$$

where the fundamental solutions G_1, G_2, and G_3 are given by

$$G_1(x,t) = \frac{-1}{\pi^{\frac{n-1}{2}}(2|x|)^n} \int_0^{+\infty} \frac{e^{-rt}}{r} \left[\mathcal{H}\left(\frac{1}{|x|\sqrt{\rho}}\right) + \frac{d^2}{c^2\rho}\, \mathcal{H}^*\left(\frac{1}{|x|\sqrt{\rho}}\right) \right] dr,$$

$$G_2(x,t) = \frac{-1}{\pi^{\frac{n-1}{2}}(2|x|)^n} \int_0^{+\infty} \frac{e^{-rt}}{r^2} \left[-\mathcal{H}\left(\frac{1}{|x|\sqrt{\rho}}\right) - \frac{d^2}{c^2\rho}\, \mathcal{H}^*\left(\frac{1}{|x|\sqrt{\rho}}\right) \right] dr,$$

$$G_3(x,t) = \frac{-1}{c^2\,\pi^{\frac{n-1}{2}}(2|x|)^n} \int_0^{+\infty} \frac{e^{-rt}}{\rho}\, \mathcal{H}^*\left(\frac{1}{|x|\sqrt{\rho}}\right) dr$$

with ρ and γ given by relations (28)*, and the functions $\mathcal{H}$ and $\mathcal{H}^*$ are expressed in terms of the following Fox H-functions*

$$\mathcal{H}\left(\frac{1}{|x|\sqrt{\rho}}\right) = H_{3,2}^{0,2}\left[\frac{1}{|x|\sqrt{\rho}} \,\middle|\, \begin{array}{c} (1-n,\, 1),\, \left(1,\, \frac{1}{2}\right),\, \left(0,\, \frac{\gamma}{2}\right) \\ \left(\frac{1-n}{2},\, \frac{1}{2}\right),\, \left(0,\, \frac{\gamma}{2}\right) \end{array} \right],$$

$$\mathcal{H}^*\left(\frac{1}{|x|\sqrt{\rho}}\right) = H_{3,2}^{0,2}\left[\frac{1}{|x|\sqrt{\rho}} \,\middle|\, \begin{array}{c} (1-n,\, 1),\, \left(0,\, \frac{1}{2}\right),\, \left(-\gamma,\, \frac{\gamma}{2}\right) \\ \left(\frac{1-n}{2},\, \frac{1}{2}\right),\, \left(-\gamma,\, \frac{\gamma}{2}\right) \end{array} \right].$$

Remark 3.4 *If we consider the one dimensional case in the previous result, i.e., when $n = 1$ in Corollary 3.3 we obtain the same results presented in [3] for the time-fractional Klein-Gordon equation of distributed order where we corrected some missprints. Moreover, we observe that the definition of the Fox H-function considered in [3] is different from the one considered in this work, more precisely, the authors considered the change of variable $s \mapsto -s$ in* (15)*.*

The numerical implementation of (61) is possible, however, depends substantially on the study of the asymptotic behaviour of the fundamental solutions G_1, G_2, and G_3 through the study of the asymptotic behaviour of the associated H-functions. This is not the subject of this work and is left for future work. We would like to remark also that (61) is a very general solution, but for particular cases of the dimension, the fractional parameters, and/or the density functions, it is possible to get simpler expressions.

4 Particular cases

In this section we consider the cases where the density functions b_1 and b_2 are constant functions, linear functions, sinusoidal functions, and exponential functions. We start pointing out that it is not possible to obtain simple/pratical expressions for ρ_1, ρ_2, γ_1, and γ_2 in (20) and (21), and therefore for A and B in (33), for arbitrary functions $b_2(\beta)$ and $b_1(\alpha)$. For this particular choices of b_1 and b_2 we obtain, when possible, the correspondent expressions of ρ_1, ρ_2, γ_1, γ_2. We observe that the quantities A and B depend on ρ_1, ρ_2, γ_1, γ_2 and they appear in Theorem 3.1. Due to the independence of the choice of b_1 and b_2 it is possible to obtain different versions of Theorem 3.1 and Corollary 3.3. For example, it is possible to obtain an explicit expression of the solution of (17)-(18) when b_1 is a linear function (see Subsection 4.4) and b_2 is an exponential function (see Subsection 4.7).

4.1 Single order case

Here we consider the case of the time-fractional telegraph equation in $\mathbb{R}^n \times \mathbb{R}^+$ with single order fractional-derivative. Putting

$$b_1(\alpha) = \delta(\alpha - \alpha_1), \qquad 0 < \alpha_1 \leq 1, \qquad b_2(\beta) = \delta(\beta - \beta_1), \qquad 1 < \beta_1 \leq 2,$$

in (17) then we get

$$B_1(\mathbf{s}) = \frac{a}{c^2}\,\mathbf{s}^{\alpha_1}, \qquad\qquad B_2(\mathbf{s}) = \frac{1}{c^2}\,\mathbf{s}^{\beta_1}. \tag{62}$$

From (62) we have that

$$B_1\left(re^{i\pi}\right) = \frac{a}{c^2}\,r^{\alpha_1}\,e^{i\alpha_1\pi}, \qquad\qquad B_2\left(re^{i\pi}\right) = \frac{1}{c^2}\,r^{\beta_1}\,e^{i\beta_1\pi}$$

and relations (28) and (29) become

$$\begin{cases} \rho_1(r) = \dfrac{a}{c^2}\,r^{\alpha_1} \\ \gamma_1(r) = \alpha_1\,(\text{const.}) \end{cases}, \qquad\qquad \begin{cases} \rho_2(r) = \dfrac{1}{c^2}\,r^{\beta_1} \\ \gamma_2(r) = \beta_1\,(\text{const.}) \end{cases}.$$

In these conditions, (19) becomes

$$\widehat{\widetilde{u}}(\kappa, \mathbf{s}) = \frac{\widehat{f}(\kappa)\left[\mathbf{s}^{\beta_1-1} + a\,\mathbf{s}^{\alpha_1-1}\right] + \widehat{g}(\kappa)\;\mathbf{s}^{\beta_1} + \widehat{\widetilde{q}}(\kappa, \mathbf{s})}{\mathbf{s}^{\beta_1} + a\,\mathbf{s}^{\alpha_1} + c^2\,|\kappa|^2}. \tag{63}$$

If we additionally consider in (63)

$$f(x) = \delta(x) = \prod_{j=1}^{n} \delta(x_j), \qquad\qquad g(x) = 0, \qquad\qquad q(x,t) = 0,$$

it becomes

$$\widehat{\widetilde{u}}(\kappa, \mathbf{s}) = \frac{\mathbf{s}^{\beta_1-1} + a\,\mathbf{s}^{\alpha_1-1}}{\mathbf{s}^{\beta_1} + a\,\mathbf{s}^{\alpha_1} + c^2\,|\kappa|^2}$$

which corresponds to the Laplace-Fourier transform of the first fundamental solution of the time-fractional telegraph equation of single-order deduced in [14] (see expression (4.1) in Section 4), and therefore there is a consistency in the obtained results.

4.2 Multi-order case

We now consider the case of the time-fractional telegraph equation in $\mathbb{R}^n \times \mathbb{R}^+$ for multi-order time-fractional derivatives, i.e., let

$$b_1(\alpha) = \sum_{j=1}^{q} a_j\,\delta(\alpha - \alpha_j), \qquad 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_q < 1, \quad a_j \in \mathbb{R},$$

$$b_2(\beta) = \sum_{j=1}^{p} b_j\,\delta(\beta - \beta_j), \qquad 1 < \beta_1 < \beta_2 < \cdots < \beta_p < 2, \quad b_j \in \mathbb{R},$$

in (17), so that

$$B_1(\mathbf{s}) = \frac{a}{c^2} \sum_{j=1}^{q} a_j \, \mathbf{s}^{\alpha_j}, \qquad B_2(\mathbf{s}) = \frac{1}{c^2} \sum_{j=1}^{p} b_j \, \mathbf{s}^{\beta_j}.$$

In this case we were not able to compute the correspondent expressions (28) and (29) for this case.

4.3 Uniform distributed case

Here we consider the case where the density functions $b_2(\beta)$ and $b_1(\alpha)$ are constant, i.e., let us consider in (17) $b_1(\alpha) = \kappa_1$ and $b_2(\beta) = \kappa_2$, with $\kappa_2, \kappa_1 \in \mathbb{R}^+$, which implies from (20) and (21) that

$$B_1(s) = \frac{a\,\kappa_1}{c^2} \frac{s-1}{\ln(s)}, \qquad B_2(s) = \frac{\kappa_2}{c^2} \frac{s^2 - s}{\ln(s)}. \tag{64}$$

Let us now obtain the expressions for ρ_1 and γ_1. Taking into account (29), (64), and the definition of the complex logarithm, we have that

$$B_1\left(re^{i\pi}\right) = \frac{a\,\kappa_1}{c^2} \frac{re^{i\pi} - 1}{\ln\left(re^{i\pi}\right)} = \frac{a\,\kappa_1}{c^2} \frac{-r-1}{\ln(r) + i\pi} = \frac{-a\,\kappa_1(r+1)}{c^2} \frac{1}{\ln(r) + i\pi}. \tag{65}$$

Taking into account (16), we have the following representation in terms of complex exponentials:

$$z_1 = \frac{-a\,\kappa_1(r+1)}{c^2} = \frac{a\,\kappa_1(r+1)}{c^2} e^{i\pi}, \tag{66}$$

$$z_2 = \frac{1}{\ln(r) + i\pi} = \frac{\ln(r) - i\,\pi}{\ln^2(r) + \pi^2} = \frac{1}{\sqrt{\ln^2(r) + \pi^2}} \exp\left(-i\left(\frac{\pi}{2} + \arctan\left(-\frac{\ln(r)}{\pi}\right)\right)\right). \tag{67}$$

From (66) and (67), expression (65) becomes

$$B_1\left(re^{i\pi}\right) = \frac{a\,\kappa_1(r+1)}{c^2\sqrt{\ln^2(r) + \pi^2}} \exp\left(i\left(\frac{\pi}{2} - \arctan\left(-\frac{\ln(r)}{\pi}\right)\right)\right),$$

and hence

$$\begin{cases} \rho_1 = \rho_1(r) = \dfrac{a\,\kappa_1}{c^2} \dfrac{r+1}{\sqrt{\ln^2(r) + \pi^2}} \\ \gamma_1 = \gamma_1(r) = \dfrac{1}{2} - \dfrac{1}{\pi} \arctan\left(-\dfrac{\ln(r)}{\pi}\right) \end{cases}.$$

Now we pass to the deduction of the expressions of ρ_2 and γ_2. From (64) we have the following relation

$$B_2(s) = \frac{1}{c^2} \frac{s^2 - s}{\ln(s)} = \frac{s\,\kappa_2}{a\,\kappa_1} B_1(s)$$

which implies that

$$B_2\left(re^{i\pi}\right) = \frac{\kappa_2\, re^{i\pi}}{a} B_1\left(re^{i\pi}\right) = \frac{\kappa_2\left(r^2 + r\right)}{c^2\sqrt{\ln^2(r) + \pi^2}} \exp\left(-i\left(\frac{\pi}{2} + \arctan\left(-\frac{\ln(r)}{\pi}\right)\right)\right),$$

and hence

$$\begin{cases} \rho_2 = \rho_2(r) = \dfrac{\kappa_2}{c^2} \dfrac{r^2 + r}{\sqrt{\ln^2(r) + \pi^2}} \\ \gamma_2 = \gamma_2(r) = -\left(\dfrac{1}{2} + \dfrac{1}{\pi} \arctan\left(-\dfrac{\ln(r)}{\pi}\right)\right) \end{cases}.$$

4.4 Linear density functions

Here we consider the case where the density functions $b_1(\alpha)$ and $b_2(\beta)$ are linear functions, i.e., let us consider in (17)

$$b_1(\alpha) = 2\alpha, \qquad 0 < \alpha \leq 1; \qquad\qquad b_2(\beta) = -2(\beta - 2), \qquad 1 < \beta \leq 2.$$

Considering the change of variables $\mathbf{s}^\alpha = t$ and $\mathbf{s}^\beta = t$ and making an integration by parts, we have from (21) and (20) that

$$B_1(\mathbf{s}) = \frac{2a}{c^2}\frac{\mathbf{s}\ln(\mathbf{s}) - \mathbf{s} + 1}{\ln^2(\mathbf{s})}, \qquad\qquad B_2(\mathbf{s}) = -\frac{2}{c^2}\frac{\mathbf{s}\ln(\mathbf{s}) - \mathbf{s}^2 + \mathbf{s}}{\ln^2(\mathbf{s})}. \tag{68}$$

Let us now obtain the expressions for ρ_1 and γ_1. From (29) and (68) we have that

$$B_1\left(re^{i\pi}\right) = \frac{2a}{c^2}\frac{re^{i\pi}\ln\left(re^{i\pi}\right) - re^{i\pi} + 1}{\ln^2\left(re^{i\pi}\right)}. \tag{69}$$

Since

$$\begin{aligned}
z_1 &= re^{i\pi}\ln\left(re^{i\pi}\right) - re^{i\pi} + 1 = -r\ln(r) + r + 1 - i\,r\pi \\
&= \sqrt{(r + 1 - r\ln(r))^2 + r^2\pi^2}\,\exp\left(-i\left(\frac{\pi}{2} + \arctan\left(\frac{r\ln(r) - r - 1}{r\pi}\right)\right)\right), \\
z_2 &= \ln^2\left(re^{i\pi}\right) = \ln^2(r) - \pi^2 + i\,2\pi\ln(r) \\
&= \left(\ln^2(r) + \pi^2\right)\exp\left(i\arctan\left(\ln^2(r) - \pi^2,\, 2\pi\ln(r)\right)\right),
\end{aligned}$$

then (69) takes the form

$$\begin{aligned}
B_1\left(re^{i\pi}\right) = \frac{2a}{c^2}&\frac{\sqrt{(r + 1 - r\ln(r))^2 + r^2\pi^2}}{\ln^2(r) + \pi^2} \\
&\times\exp\left(-i\left(\frac{\pi}{2} + \arctan\left(\frac{r\ln(r) - r - 1}{r\pi}\right) + \arctan\left(\ln^2(r) - \pi^2,\, 2\pi\ln(r)\right)\right)\right),
\end{aligned}$$

and hence

$$\begin{cases}
\rho_1(r) = \dfrac{2a}{c^2}\sqrt{\dfrac{(r + 1 - r\ln(r))^2 + r^2\pi^2}{\ln^2(r) + \pi^2}} \\[2ex]
\gamma_1(r) = -1 - \dfrac{1}{\pi}\arctan\left(\dfrac{r\ln(r) - r - 1}{r\pi}\right) - \dfrac{1}{\pi}\arctan\left(\ln^2(r) - \pi^2,\, 2\pi\ln(r)\right)
\end{cases}.$$

Now we obtain the expressions for ρ_2 and γ_2 in a similar way. From (28) and (68) we have that

$$B_2\left(re^{i\pi}\right) = -\frac{2}{c^2}\frac{re^{i\pi}\ln\left(re^{i\pi}\right) - \left(re^{i\pi}\right)^2 + re^{i\pi}}{\ln^2\left(re^{i\pi}\right)}.$$

Since

$$\begin{aligned}
z_1 &= -\frac{2}{c^2} = \frac{2}{c^2}\exp(i\pi) \\
z_2 &= re^{i\pi}\ln\left(re^{i\pi}\right) - \left(re^{i\pi}\right)^2 + re^{i\pi} = -r(\ln(r) + r + 1) - i\,r\pi \\
&= r\sqrt{(\ln(r) + r + 1)^2 + \pi^2}\,\exp\left(-i\left(\frac{\pi}{2} + \arctan\left(\frac{\ln(r) + r + 1}{\pi}\right)\right)\right), \\
z_3 &= \ln^2\left(re^{i\pi}\right) = \ln^2(r) - \pi^2 + i\,2\pi\ln(r) \\
&= \left(\ln^2(r) + \pi^2\right)\exp\left(i\arctan\left(\ln^2(r) - \pi^2,\, 2\pi\ln(r)\right)\right),
\end{aligned}$$

then

$$B_2\left(re^{i\pi}\right) = \frac{2r}{c^2}\,\frac{\sqrt{\left(\ln(r)+r+1\right)^2+\pi^2}}{\ln^2(r)+\pi^2}$$
$$\times\exp\left(i\left(\frac{\pi}{2}-\arctan\left(\frac{\ln(r)+r+1}{\pi}\right)-\arctan\left(\ln^2(r)-\pi^2,\,2\pi\ln(r)\right)\right)\right),$$

and hence

$$\begin{cases}\rho_2 = \rho_2(r) = \dfrac{2r}{c^2}\sqrt{\dfrac{\left(\ln(r)+r+1\right)^2+\pi^2}{\left(\ln^2(r)+\pi^2\right)^2}} \\ \gamma_2 = \gamma_2(r) = \dfrac{1}{2}-\dfrac{1}{\pi}\arctan\left(\dfrac{\ln(r)+r+1}{\pi}\right)-\dfrac{1}{\pi}\arctan\left(\ln^2(r)-\pi^2,\,2\pi\ln(r)\right)\end{cases}.$$

4.5 Sinusoidal density functions

Here we consider the case where the density functions $b_1(\alpha)$ and $b_2(\beta)$ are sinusoidal functions, i.e., let us consider in (17)

$$b_1(\alpha) = \frac{\pi}{2}\sin(\alpha\pi), \qquad 0<\alpha\le 1; \qquad\qquad b_2(\beta) = -\frac{\pi}{2}\sin(\beta\pi), \qquad 1<\beta\le 2.$$

Integrating by parts, we have from (21) and (20) that

$$B_1(\mathbf{s}) = \frac{a\pi}{2c^2}\,\frac{\mathbf{s}+1}{\ln^2(\mathbf{s})+\pi^2}, \qquad\qquad B_2(\mathbf{s}) = \frac{\pi^2}{2c^2}\,\frac{\mathbf{s}(\mathbf{s}+1)}{\ln^2(\mathbf{s})+\pi^2}. \tag{70}$$

Now we deduce the expressions for ρ_1 and γ_1. From (29) and (70) we have that

$$B_1\left(re^{i\pi}\right) = \frac{a\pi}{2c^2}\,\frac{re^{i\pi}+1}{\ln^2\left(re^{i\pi}\right)+\pi^2}. \tag{71}$$

Since

$$z_1 = re^{i\pi}+1 = 1-r\sqrt{(1-r)^2}\arctan(1-r,\,0),$$
$$z_2 = \ln^2\left(re^{i\pi}\right)+\pi^2 = \ln(r)\left(\ln(r)+i\,2\pi\right)$$
$$= \left(\ln^2(r)\left(\ln^2(r)+4\pi^2\right)\right)^{\frac{1}{2}}\exp\left(i\left(\frac{\pi}{2}-\arctan\left(\frac{\ln(r)}{2\pi}\right)\right)\right),$$

then (71) becomes

$$B_1\left(re^{i\pi}\right) = \frac{a\pi^2}{2c^2}\sqrt{(1-r)^2\ln^2\left(\left(\ln^2(r)+4\pi^2\right)\right)}\exp\left(i\left(\arctan(1-r,\,0)-\frac{\pi}{2}+\arctan\left(\frac{\ln(r)}{2\pi}\right)\right)\right),$$

and hence

$$\begin{cases}\rho_1 = \rho_1(r) = \dfrac{a\pi^2}{2c^2}\sqrt{(1-r)^2\ln^2(r)\left(\ln^2(r)+4\pi^2\right)} \\ \gamma_1 = \gamma_1(r) = -\dfrac{1}{2}+\dfrac{1}{\pi}\arctan(1-r,\,0)+\dfrac{1}{\pi}\arctan\left(\dfrac{\ln(r)}{2\pi}\right)\end{cases}. \tag{72}$$

Now we deduce the expression for ρ_2 and γ_2. From (70) we have the following relation

$$B_2\left(re^{i\pi}\right) = \frac{1}{a}re^{i\pi}B_1\left(re^{i\pi}\right). \tag{73}$$

Therefore, from (73) and (72) we immediately conclude that

$$B_2\left(re^{i\pi}\right) = \frac{r\pi^2}{2c^2}\sqrt{(1-r)^2\ln^2(r)\left(\ln^2(r)+4\pi^2\right)}\exp\left(i\left(\frac{\pi}{2}+\arctan(1-r,\,0)+\arctan\left(\frac{\ln(r)}{2\pi}\right)\right)\right),$$

and hence

$$\begin{cases} \rho_2 = \rho_2(r) = \dfrac{r\pi^2}{2c^2}\sqrt{(1-r)^2\ln^2(r)\left(\ln^2(r)+4\pi^2\right)} \\ \\ \gamma_2 = \gamma_2(r) = \dfrac{1}{2} + \dfrac{1}{\pi}\arctan(1-r,\,0) + \dfrac{1}{\pi}\arctan\left(\dfrac{\ln(r)}{2\pi}\right) \end{cases}.$$

4.6 Sinusoidal density functions II

Here, we consider another case where the density functions $b_1(\alpha)$ and $b_2(\beta)$ are also sinusoidal functions. Let us consider in (17)

$$b_1(\alpha) = \frac{\pi}{4}\sin\left(\frac{\alpha\pi}{2}\right), \qquad 0 < \alpha \le 1; \qquad\qquad b_2(\beta) = \frac{\pi}{4}\sin\left(\frac{\beta\pi}{2}\right), \qquad 1 < \beta \le 2,$$

where b_1 and b_2 are such that

$$\int_0^1 b_1(\alpha)\,d\alpha + \int_1^2 b_2(\beta)\,d\beta = 1.$$

Integrating by parts, we have from (21) and (20) that

$$B_1(\mathbf{s}) = \frac{a\pi}{2c^2}\frac{2\mathbf{s}\,\ln(\mathbf{s})+\pi}{4\ln^2(\mathbf{s})+\pi^2}, \qquad\qquad B_2(\mathbf{s}) = \frac{\pi}{2c^2}\frac{\mathbf{s}\,(\pi\mathbf{s}-2\ln(\mathbf{s}))}{4\ln^2(\mathbf{s})+\pi^2}. \tag{74}$$

From (29) and (74) we have that

$$B_1\left(re^{i\pi}\right) = \frac{a\pi}{2c^2}\frac{2re^{i\pi}\,\ln\left(re^{i\pi}\right)+\pi}{4\ln^2\left(re^{i\pi}\right)+\pi^2}. \tag{75}$$

Since

$$\begin{aligned} z_1 &= 2re^{i\pi}\,\ln\left(re^{i\pi}\right)+\pi = \pi - 2r\,\ln(r) - i\,2r\pi \\ &= \left((\pi-2r\,\ln(r))^2+4r^2\pi^2\right)^{\frac{1}{2}}\exp\left(-i\left(\frac{\pi}{2}+\arctan\frac{2r\,\ln(r)-\pi}{2r\pi}\right)\right), \\ z_2 &= 4\ln^2\left(re^{i\pi}\right)+\pi^2 = 4\ln^2(r)-3\pi^2+i\,8\pi\ln(r) \\ &= \left(\left(4\ln^2(r)-3\pi^2\right)^2+64\pi^2\ln^2(r)\right)^{\frac{1}{2}}\exp\left(i\,\arctan\left(4\ln^2(r)-3\pi^2,\,8\pi\ln(r)\right)\right), \end{aligned}$$

then expression (75) becomes

$$\begin{aligned} B_1\left(re^{i\pi}\right) = {}& \frac{a\pi}{2c^2}\sqrt{\frac{(\pi-2r\,\ln(r))^2+4r^2\pi^2}{\left(4\ln^2(r)-3\pi^2\right)^2+64\pi^2\ln^2(r)}} \\ & \times\exp\left(-i\left(\frac{\pi}{2}+\arctan\left(\frac{2r\,\ln(r)-\pi}{2r\pi}\right)+\arctan\left(4\ln^2(r)-3\pi^2,\,8\pi\ln(r)\right)\right)\right), \end{aligned}$$

and hence

$$\begin{cases} \rho_1 = \rho_1(r) = \dfrac{a\pi}{2c^2}\sqrt{\dfrac{(\pi-2r\,\ln(r))^2+4r^2\pi^2}{\left(4\ln^2(r)-3\pi^2\right)^2+64\pi^2\ln^2(r)}} \\ \\ \gamma_1 = \gamma_1(r) = -\dfrac{1}{2} - \dfrac{1}{\pi}\arctan\left(\dfrac{2r\,\ln(r)-\pi}{2r\pi}\right) - \dfrac{1}{\pi}\arctan\left(4\ln^2(r)-3\pi^2,\,8\pi\ln(r)\right) \end{cases}.$$

Let us now obtain the expressions for ρ_2 and γ_2. From (28) and (74) we have that

$$B_2\left(re^{i\pi}\right) = \frac{\pi}{2c^2}\frac{re^{i\pi}\left(\pi\,re^{i\pi}-2\ln\left(re^{i\pi}\right)\right)}{4\ln^2\left(re^{i\pi}\right)+\pi^2}. \tag{76}$$

Since

$$z_1 = re^{i\pi}\left(\pi\, re^{i\pi} - 2\ln\left(re^{i\pi}\right)\right) \; = r\left(r\pi + 2\ln(r) + i\,2\pi\right)$$
$$= r\left(\left(r\pi + 2\ln(r)\right)^2 + 4\pi^2\right)^{\frac{1}{2}} \exp\left(i\left(\frac{\pi}{2} - \arctan\left(\frac{r\pi + 2\ln(r)}{2\pi}\right)\right)\right),$$
$$z_2 = \left(\left(4\ln^2(r) - 3\pi^2\right)^2 + 64\pi^2\ln^2(r)\right)^{\frac{1}{2}} \exp\left(i\,\arctan\left(4\ln^2(r) - 3\pi^2,\, 8\pi\ln(r)\right)\right),$$

then expression (76) takes the form

$$B_2\left(re^{i\pi}\right) = \frac{r\pi}{2c^2}\sqrt{\frac{\left(r\pi + 2\ln(r)\right)^2 + 4\pi^2}{\left(4\ln^2(r) - 3\pi^2\right)^2 + 64\pi^2\ln^2(r)}}$$
$$\times \exp\left(i\left(\frac{\pi}{2} - \arctan\left(\frac{r\pi + 2\ln(r)}{2\pi}\right) - \arctan\left(4\ln^2(r) - 3\pi^2,\, 8\pi\ln(r)\right)\right)\right),$$

and hence

$$\begin{cases} \rho_2 = \rho_2(r) = \dfrac{r\pi}{2c^2}\sqrt{\dfrac{\left(r\pi + 2\ln(r)\right)^2 + 4\pi^2}{\left(4\ln^2(r) - 3\pi^2\right)^2 + 64\pi^2\ln^2(r)}} \\ \gamma_2 = \gamma_2(r) = \dfrac{1}{2} - \dfrac{1}{\pi}\arctan\left(\dfrac{r\pi + 2\ln(r)}{2\pi}\right) - \dfrac{1}{\pi}\arctan\left(4\ln^2(r) - 3\pi^2,\, 8\pi\ln(r)\right) \end{cases}.$$

4.7 Exponential density functions

Here we consider the case where the density functions $b_1(\alpha)$ and $b_2(\beta)$ are exponential functions, i.e., let us consider in (17)

$$b_1(\alpha) = \frac{1}{e-1}\,e^{\alpha}, \qquad 0 < \alpha \le 1; \qquad\qquad b_2(\beta) = \frac{e^2}{e-1}\,e^{-\beta}, \qquad 1 < \beta \le 2.$$

From (21) and (20), we have that

$$B_1(\mathbf{s}) = \frac{a}{e^2(e-1)}\,\frac{e\mathbf{s} - 1}{1 + \ln(\mathbf{s})}, \qquad\qquad B_2(\mathbf{s}) = \frac{1}{c^2(e-1)}\,\frac{\mathbf{s}(\mathbf{s} - e)}{\ln(\mathbf{s}) - 1}. \tag{77}$$

From (29) and (77) we have that

$$B_1\left(re^{i\pi}\right) = \frac{a}{e^2(e-1)}\,\frac{e\,re^{i\pi} - 1}{1 + \ln\left(re^{i\pi}\right)}. \tag{78}$$

Since

$$z_1 = e\,re^{i\pi} - 1 = (er + 1)\,e^{i\pi},$$
$$z_2 = 1 + \ln\left(re^{i\pi}\right) = 1 + \ln(r) + i\pi = \left(\left(\ln^2(r) + 1\right)^2 + \pi^2\right)^{\frac{1}{2}} \exp\left(i\left(\frac{\pi}{2} - \arctan\left(\frac{\ln(r) + 1}{\pi}\right)\right)\right),$$

then (78) takes the form

$$B_1\left(re^{i\pi}\right) = \frac{a}{c^2(e-1)}\,\frac{(er + 1)}{\sqrt{\left(\ln^2(r) + 1\right)^2 + \pi^2}} \exp\left(i\left(\frac{\pi}{2} + \arctan\left(\frac{\ln(r) + 1}{\pi}\right)\right)\right),$$

and hence

$$\begin{cases} \rho_1 = \rho_1(r) = \dfrac{a}{c^2(e-1)}\,\dfrac{(er + 1)}{\sqrt{\left(\ln^2(r) + 1\right)^2 + \pi^2}} \\ \gamma_1 = \gamma_1(r) = \dfrac{1}{2} + \dfrac{1}{\pi}\arctan\left(\dfrac{\ln(r) + 1}{\pi}\right) \end{cases}.$$

Let us now obtain the expression for ρ_2 and γ_2. From (28) and (77) we have that

$$B_2\left(re^{i\pi}\right) = \frac{1}{c^2(e-1)} \frac{re^{i\pi}\left(re^{i\pi}-e\right)}{\ln\left(re^{i\pi}\right)-1}. \tag{79}$$

Since

$$z_1 = re^{i\pi}\left(re^{i\pi}-e\right) = r(r+e),$$

$$z_2 = \ln\left(re^{i\pi}\right)-1 = \ln(r)-1+i\pi = \left((\ln(r)-1)^2+\pi^2\right)^{\frac{1}{2}} \exp\left(i\left(\frac{\pi}{2}-\arctan\left(\frac{\ln(r)-1}{\pi}\right)\right)\right),$$

then (79) takes the form

$$B_2\left(re^{i\pi}\right) = \frac{1}{c^2(e-1)} \frac{r(r+e)}{\sqrt{(\ln(r)-1)^2+\pi^2}} \exp\left(-i\left(\frac{\pi}{2}-\arctan\left(\frac{\ln(r)-1}{\pi}\right)\right)\right),$$

and hence

$$\begin{cases} \rho_2 = \rho_2(r) = \dfrac{1}{c^2(e-1)} \dfrac{r(r+e)}{\sqrt{(\ln(r)-1)^2+\pi^2}} \\ \gamma_2(r) = -\dfrac{1}{2}+\dfrac{1}{\pi}\arctan\left(\dfrac{\ln(r)-1}{\pi}\right) \end{cases}.$$

5 Moments

In this section we obtain the expression for some fractional moments of the first fundamental solution G_1 of the time-fractional telegraph equation of distributed order (17) in the Laplace domain, and we apply the Tauberian theorems to study the asymptotic behaviour of the second-order moment in the time domain for $t \to 0^+$ and $t \to +\infty$ knowing the asymptotic behaviour of second-order moment in the Laplace domain for $\mathbf{s} \to +\infty$ and $\mathbf{s} \to 0$, respectively.

It is well known that the Mellin transform (10) can be interpreted as the fractional moment of order $s-1$ of the function f (see [13]). Therefore, we can calculate the fractional moments of arbitrary order $\gamma > 0$ of $\widetilde{G}_1$, where $\widetilde{G}_1$ denotes the Laplace transform of G_1. Denoting by $\mathbf{s}$ the variable in the Laplace domain and by r the radial quantity $|x|$, we have, from the definition of the Mellin transform, that

$$\widetilde{\mathbf{M}}^{\gamma}(\mathbf{s}) = \int_0^{+\infty} r^{\gamma}\, \widetilde{G}_1(r,\mathbf{s})\, dr = \int_0^{+\infty} r^{\gamma-n+1-1}\, r^n \widetilde{G}_1(r,\mathbf{s})\, dr = \mathcal{M}\left\{r^n\, \widetilde{G}_1(r,\mathbf{s})\right\}(\gamma-n+1). \tag{80}$$

Recalling that for the case of the first fundamental solution G_1 we assume in (17) that

$$f(x) = \delta(x) = \prod_{j=1}^{n} \delta(x_j), \qquad g(x) = q(x,t) = 0,$$

we have, from (19), that

$$G_1(r,t) = \mathcal{L}^{-1}\left\{\mathcal{F}^{-1}\left\{\widehat{\widetilde{u}}_1(\kappa,\mathbf{s})\Big|_{p=1} - \frac{d^2}{c^2}\widehat{\widetilde{u}}_2(\kappa,\mathbf{s})\Big|_{p=1}\right\}(r,\mathbf{s})\right\}(r,t)$$

which is equivalent to

$$\widetilde{G}_1(r,\mathbf{s}) = \mathcal{L}\{G_1(r,t)\}(r,\mathbf{s}) = \mathcal{F}^{-1}\left\{\widehat{\widetilde{u}}_1(\kappa,\mathbf{s})\Big|_{p=1} - \frac{d^2}{c^2}\widehat{\widetilde{u}}_2(\kappa,\mathbf{s})\Big|_{p=1}\right\}(r,\mathbf{s}).$$

Let us now calculate the inverse Fourier transform that appears in the previous expression. As it was done in Section 3 we make use of the Mellin transform to calculate the integral. In fact, taking into account (36), we have that

$$\begin{aligned} &\mathcal{F}^{-1}\left\{\widehat{\widetilde{u}}_1(\kappa,\mathbf{s})\Big|_{p=1} - \frac{d^2}{c^2}\widehat{\widetilde{u}}_2(\kappa,\mathbf{s})\Big|_{p=1}\right\}(r,\mathbf{s}) \\ &= \frac{r^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \int_0^{+\infty} \left[\widehat{\widetilde{u}}_1(w,\mathbf{s})\Big|_{p=1} - \frac{d^2}{c^2}\widehat{\widetilde{u}}_2(w,\mathbf{s})\Big|_{p=1}\right] w^{\frac{n}{2}}\, J_{\frac{n}{2}-1}(|x|\, w)\, dw \qquad (81) \\ &= \mathcal{M}\{g_4 *_{\mathcal{M}} f_4\}\left(\frac{1}{r}\right), \end{aligned}$$

where $*_{\mathcal{M}}$ denotes the Mellin convolution given by (12) at the point $\frac{1}{r}$ with

$$g_4(w) = \widehat{\widehat{u}}_1(w,\mathbf{s})\Big|_{p=1} - \frac{d^2}{c^2}\,\widehat{\widehat{u}}_2(w,\mathbf{s})\Big|_{p=1} \qquad \text{and} \qquad f_4(w) = \frac{1}{(2\pi)^{\frac{n}{2}}\ |x|^n\ w^{\frac{n}{2}+1}}\, J_{\frac{n}{2}-1}\left(\frac{1}{w}\right).$$

Denoting by $\mathbf{I_6}$ the integral in (81), we have, by relations (13) and (12), that

$$\mathcal{M}\{\mathbf{I_6}\}(s) = \mathcal{M}\left\{g_4 *_{\mathcal{M}} f_4\left(\frac{1}{r}\right)\right\}(s) = \mathcal{M}\{g_4\}(-s)\,\mathcal{M}\{f_4\}(-s)$$

which is equivalent to

$$\mathcal{M}\{\mathbf{I_6}\}(-s) = \mathcal{M}\{g_4\}(s)\,\mathcal{M}\{f_4\}(s). \tag{82}$$

Since f_4 is equal to f in (43), we have that

$$\mathcal{M}\{f_4\}(s) = \frac{1}{\pi^{\frac{n-1}{2}}\ |x|^n\ 2^{n-1}}\,\frac{\Gamma(n-s)}{\Gamma\left(\frac{n+1-s}{2}\right)\Gamma\left(\frac{s}{2}\right)}. \tag{83}$$

Now we calculate the Mellin transform of function g_4. Taking into account (10), and (22), (23) with $p=1$, we get

$$\begin{aligned}\mathcal{M}\{g_4\}(s) &= \int_0^{+\infty} w^{s-1}\left[\widehat{\widehat{u}}_1(w,\mathbf{s})\Big|_{p=1} - \frac{d^2}{c^2}\,\widehat{\widehat{u}}_2(w,\mathbf{s})\Big|_{p=1}\right] dw \\ &= \frac{c^2\,B_2(\mathbf{s}) + c^2\,B_1(\mathbf{s}) - d^2}{c^2\,\mathbf{s}}\int_0^{+\infty}\frac{w^{s-1}}{B_2(\mathbf{s}) + B_1(\mathbf{s}) + w^2}\,dw.\end{aligned}$$

Taking into account formula (46) with

$$\alpha = s, \qquad a = 1, \qquad b = 0, \qquad c = B_2(\mathbf{s}) + B_1(\mathbf{s}), \qquad \psi = \frac{\pi}{2},$$

and making use of (48), we conclude that

$$\begin{aligned}\mathcal{M}\{g_4\}(s) &= \frac{\pi\left(c^2\,B_2\left(\mathbf{s} + c^2\,B_1\left(\mathbf{s} - d^2\right)\right)\right)}{c^2\,\mathbf{s}}\,\frac{\csc(s\pi)}{\csc\left((1-s)\,\frac{\pi}{2}\right)}\left(B_2(\mathbf{s}) + B_1(\mathbf{s})\right)^{\frac{s}{2}-1} \\ &= \frac{\pi\left(c^2\,B_2\left(\mathbf{s} + c^2\,B_1\left(\mathbf{s} - d^2\right)\right)\right)}{c^2\,\mathbf{s}}\,\frac{\Gamma(1-s)\,\Gamma(s)}{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}\left(B_2(\mathbf{s}) + B_1(\mathbf{s})\right)^{\frac{s}{2}-1}.\end{aligned} \tag{84}$$

From (83), (84), (82), and (80) we conclude that

$$\begin{aligned}&\mathcal{M}\left\{r^n\,\mathcal{F}^{-1}\left\{\widehat{\widehat{u}}_1(\kappa,\mathbf{s})\Big|_{p=1} - \frac{d^2}{c^2}\,\widehat{\widehat{u}}_2(\kappa,\mathbf{s})\Big|_{p=1}\right\}(r,\mathbf{s})\right\}(\gamma-n+1,\mathbf{s}) \\ &= \frac{\left(c^2\,B_2(\mathbf{s}) + c^2\,B_1(\mathbf{s}) - d^2\right)\left(B_2(\mathbf{s}) + B_1(\mathbf{s})\right)^{-\frac{s}{2}-1}}{\pi^{\frac{n-3}{2}}\,2^{n-1}c^2\mathbf{s}}\,\frac{\Gamma(n+s)\,\Gamma(1+s)\,\Gamma(-s)}{\Gamma\left(\frac{n+1+s}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}\Bigg|_{s=\gamma-n+1}.\end{aligned} \tag{85}$$

By the duplication formula of the Gamma function (42), we have the following equalities for the Gamma functions that appear in (85)

$$\frac{\Gamma(n+s)}{\Gamma\left(\frac{1}{2}+\frac{n+s}{2}\right)} = \frac{2^{n+s-1}}{\sqrt{\pi}}\,\Gamma\left(\frac{n+s}{2}\right), \tag{86}$$

$$\frac{\Gamma(1+s)}{\Gamma\left(\frac{1+s}{2}\right)} = \frac{2^s}{\sqrt{\pi}}\,\Gamma\left(1+\frac{s}{2}\right), \tag{87}$$

$$\frac{\Gamma(-s)}{\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)} = \frac{2^{-s-1}}{\sqrt{\pi}}\,\Gamma\left(-\frac{s}{2}\right). \tag{88}$$

Taking into account (86), (87), and (88), expression (85) simplifies to

$$\begin{aligned}&\mathcal{M}\left\{r^n\,\mathcal{F}^{-1}\left\{\widehat{\widehat{u}}_1(\kappa,\mathbf{s})\Big|_{p=1} - \frac{d^2}{c^2}\,\widehat{\widehat{u}}_2(\kappa,\mathbf{s})\Big|_{p=1}\right\}(r,\mathbf{s})\right\}(\gamma-n+1,\mathbf{s}) \\ &= \frac{\left(c^2\,B_2(\mathbf{s}) + c^2\,B_1(\mathbf{s}) - d^2\right)\left(B_2(\mathbf{s}) + B_1(\mathbf{s})\right)^{-\frac{s}{2}-1}}{\pi^{\frac{n}{2}}\,c^2\mathbf{s}}\,2^{s-1}\,\Gamma\left(\frac{n+s}{2}\right)\Gamma\left(1+\frac{s}{2}\right)\Bigg|_{s=\gamma-n+1},\end{aligned}$$

and consequently the fractional moments of arbitrary order γ in the Laplace domain are given by

$$\widetilde{\mathbf{M}}^{\gamma}(\mathbf{s}) = \frac{\left(c^2 B_2(\mathbf{s}) + c^2 B_1(\mathbf{s}) - d^2\right)\left(B_2(\mathbf{s}) + B_1(\mathbf{s})\right)^{\frac{-\gamma+n-3}{2}}}{\pi^{\frac{n}{2}} c^2 \mathbf{s}} 2^{\gamma-n}\,\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(\frac{3+\gamma-n}{2}\right). \tag{89}$$

If we restrict (89) to the diffusion case studied in [13], i.e., if we consider $d=0$, $b_2(\beta)=0$ and $b_1(\alpha)=\delta(\alpha-\alpha_1)$, with $0<\alpha_1\leq 1$, we have that $B_2(\mathbf{s})=0$ and $B_1(\mathbf{s})=\frac{1}{c^2}\mathbf{s}^{\alpha_1}$, and (89) becomes equal to

$$\widetilde{\mathbf{M}}^{\alpha_1,\gamma}(\mathbf{s}) = \frac{\left(4c^2\right)^{\frac{\gamma-n+1}{2}}}{2\pi^{\frac{n}{2}}}\mathbf{s}^{-\frac{\alpha_1(\gamma-n+1)}{2}-1}\,\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(\frac{3+\gamma-n}{2}\right). \tag{90}$$

Taking into account the following formula for the inverse Laplace transform (see (2.1.1.1) in [26])

$$\mathcal{L}^{-1}\left\{\frac{1}{\mathbf{s}^{\nu}}\right\}(t) = \frac{t^{\nu-1}}{\Gamma(\nu)}, \qquad \nu>0, \tag{91}$$

we conclude that

$$\mathbf{M}^{\alpha_1,\gamma}(t) = \frac{\left(4c^2\right)^{\frac{\gamma-n+1}{2}}}{2\pi^{\frac{n}{2}}}\,\frac{\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(\frac{3+\gamma-n}{2}\right)}{\Gamma\left(1+\frac{\alpha_1(\gamma-n+1)}{2}\right)}\,t^{\frac{\alpha_1(\gamma-n+1)}{2}}$$

which coincides with the expression (68) in [13], and shows consistency in the obtained expression. Let us now analyse expression (89). We start pointing out the following special cases:

- When $\gamma=n-2k-3$, with $n>2k+3$ and $k\in\mathbb{N}_0$, the correspondent moments in the Laplace domain become infinite.
- When $\gamma=1$ (mean value), we have

$$\widetilde{\mathbf{M}}^{1}(\mathbf{s}) = \frac{\left(c^2 B_2(\mathbf{s}) + c^2 B_1(\mathbf{s}) - d^2\right)\left(B_2(\mathbf{s}) + B_1(\mathbf{s})\right)^{\frac{n}{2}-2}}{\pi^{\frac{n}{2}} c^2 \mathbf{s}} 2^{1-n}\,\Gamma\left(2-\frac{n}{2}\right) \tag{92}$$

  which becomes infinite when $n=4+2k$, with $k\in\mathbb{N}_0$.
- When $\gamma=2$ (variance), we have

$$\widetilde{\mathbf{M}}^{2}(\mathbf{s}) = \frac{\left(c^2 B_2(\mathbf{s}) + c^2 B_1(\mathbf{s}) - d^2\right)\left(B_2(\mathbf{s}) + B_1(\mathbf{s})\right)^{\frac{n-5}{2}}}{\pi^{\frac{n-1}{2}} c^2 \mathbf{s}} 2^{1-n}\,\Gamma\left(\frac{5-n}{2}\right) \tag{93}$$

  which becomes infinite when $n=5+2k$, with $k\in\mathbb{N}_0$.
- When $\gamma=3$ (3rd moment), we have

$$\widetilde{\mathbf{M}}^{3}(\mathbf{s}) = \frac{\left(c^2 B_2(\mathbf{s}) + c^2 B_1(\mathbf{s}) - d^2\right)\left(B_2(\mathbf{s}) + B_1(\mathbf{s})\right)^{\frac{n}{2}-3}}{\pi^{\frac{n}{2}} c^2 \mathbf{s}} 2^{3-n}\,\Gamma\left(3-\frac{n}{2}\right) \tag{94}$$

  which becomes infinite when $n=6+2k$, with $k\in\mathbb{N}_0$.

5.1 Tauberian analysis for the second-order moment (variance)

In this subsection we make use of the Tauberian theorems to derive, from (93) with $d=0$, the asymptotic behaviour of $\mathbf{M}^2(t)$, for $t\to 0^+$ and $t\to+\infty$ knowing the asymptotic behaviours of $\widetilde{\mathbf{M}}^2(\mathbf{s})$ for $\mathbf{s}\to+\infty$ and $\mathbf{s}\to 0$, respectively. To better understand the diffusion and wave cases, we perform a separate analysis considering particular choices of $b_1(\alpha)$ and $b_2(\beta)$. Let us recall some necessary Laplacian inversion formulas that can be found in [26]:

- Formula (2.5.1.12):

$$\mathcal{L}^{-1}\left\{\frac{1}{\mathbf{s}^{\nu}}\ln^{n}(\mathbf{s})\right\}(t) = \left(-\frac{d}{d\mu}\right)^{n}\left[\frac{t^{\mu-1}}{\Gamma(\mu)}\right]\Bigg|_{\mu=\nu}, \qquad n\in\mathbb{N}. \tag{95}$$

- Formula (2.5.6.5):

$$\mathcal{L}^{-1}\left\{\frac{1}{\mathbf{s}^{\nu}}\ln^{\mu}(a\mathbf{s})\right\}(t) = \frac{a^{\nu-1}}{\Gamma(-\mu)}\int_0^{+\infty}\frac{w^{-\mu-1}}{\Gamma(\nu+w)}\left(\frac{t}{a}\right)^{w+\nu-1}dw, \qquad \mathrm{Re}(\mu)<0,\ a>0,\ \mathrm{Re}(\mathbf{s})>0. \tag{96}$$

5.1.1 The diffusion case

Here we consider $b_2(\beta) = 0 = B_2(\mathbf{s}) = 0$, and some particular choices of $b_1(\alpha)$ in (93). For $B_2(\mathbf{s}) = 0$ the second-order moment in the Laplace domain becomes

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{2^{1-n}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}}\,\frac{(B_1(\mathbf{s}))^{\frac{n-3}{2}}}{\mathbf{s}}, \quad n \neq 5+2k,\ k \in \mathbb{N}_0. \tag{97}$$

From (97) we immediately see that for $n = 3$, the moment does not depends on $B_1(\mathbf{s})$ and it is given by

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{1}{4\pi\,\mathbf{s}}, \qquad \text{so that} \qquad \mathbf{M}^2(t) = \frac{1}{4\pi}.$$

Therefore, in the following particular cases we omit the analysis for the dimension $n = 3$.

- *Slow-diffusion*: Let us consider

$$b_1(\alpha) = \kappa_1\,\delta(\alpha-\alpha_1) + \kappa_2\,\delta(\alpha-\alpha_2), \qquad 0<\alpha_1<\alpha_2<1,\ \ \kappa_1,\kappa_2>0,\ \ \kappa_1+\kappa_2=1$$

which implies that

$$B_1(\mathbf{s}) = \frac{a\kappa_1}{c^2}\,\mathbf{s}^{\alpha_1} + \frac{a\kappa_2}{c^2}\,\mathbf{s}^{\alpha_2}. \tag{98}$$

Considering (98) in (97) we have the following behaviour of $\widetilde{\mathbf{M}}^2(\mathbf{s})$ when $\mathbf{s} \to 0^+$

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{2^{1-n}\,a^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\frac{(\kappa_1\,\mathbf{s}^{\alpha_1}+\kappa_2\,\mathbf{s}^{\alpha_2})^{\frac{n-3}{2}}}{\mathbf{s}} \sim \frac{2^{1-n}\,(a\,\kappa_1)^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\mathbf{s}^{\frac{\alpha_1(n-3)}{2}-1}.$$

Concerning the symbol $\sim$ used in the previous and subsequent expression, it must be understood in the following sense: given two functions $f(w)$ and $g(w)$, we say that f and g are said to be asymptotically equivalent as $w\to\infty$ (resp. as $w\to 0$), i.e. $f \sim g$, if and only if $\lim_{w\to\infty}\frac{f(w)}{g(w)} = 1$ (resp. $\lim_{w\to 0}\frac{f(w)}{g(w)} = 1$).

Making use of (91) to invert the Laplace transform, we obtain for $t \to +\infty$

$$\mathbf{M}^2(t) \sim \begin{cases} \dfrac{c^2}{a\,\kappa_1}\,\dfrac{t^{\alpha_1}}{\Gamma(1+\alpha_1)}, & n=1 \\[2ex] \dfrac{c}{4\sqrt{a\,\kappa_1}}\,\dfrac{t^{\frac{\alpha_1}{2}}}{\Gamma\left(1+\frac{\alpha_1}{2}\right)}, & n=2 \\[2ex] \dfrac{2^{1-n}\,(a\,\kappa_1)^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\dfrac{t^{\frac{-\alpha_1(n-3)}{2}}}{\Gamma\left(1-\frac{\alpha_1(n-3)}{2}\right)}, & 0<\alpha_1<\min\left\{1,\frac{2}{n-3}\right\} \ \wedge\ n=4+2k,\ k\in\mathbb{N}_0 \end{cases}. \tag{99}$$

In the *normal diffusion* process, corresponding to $\alpha_1 = 1$, we have that $\frac{\mathbf{M}^2(t)}{t} \to c > 0$, as $t \to +\infty$. However, in the fractional case it holds $\frac{\mathbf{M}^2(t)}{t} \to 0$, as $t\to+\infty$, for all the dimensions. This corresponds to a *slow-diffusion* process, as was observed in [23] for the case n=1. Also, we note a different behaviour of $\mathbf{M}^2(t)$ along the dimensions described, and the restriction of the parameter α_1 for dimensions $n = 6+2k$, $k \in \mathbb{N}_0$.

From (98) we have the following behaviour of (97) when $\mathbf{s} \to +\infty$

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{2^{1-n}\,a^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\frac{(\kappa_1\,\mathbf{s}^{\alpha_1}+\kappa_2\,\mathbf{s}^{\alpha_2})^{\frac{n-3}{2}}}{\mathbf{s}} \sim \frac{2^{1-n}\,(a\,\kappa_2)^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\mathbf{s}^{\frac{\alpha_2(n-3)}{2}-1}.$$

Making use of (91) to invert the Laplace transform, we obtain for $t \to 0^+$

$$\mathbf{M}^2(t) \sim \begin{cases} \dfrac{c^2}{a\,\kappa_2}\,\dfrac{t^{\alpha_2}}{\Gamma(1+\alpha_2)}, & n=1 \\[2ex] \dfrac{c}{4\sqrt{a\,\kappa_2}}\,\dfrac{t^{\frac{\alpha_2}{2}}}{\Gamma\left(1+\frac{\alpha_2}{2}\right)}, & n=2 \\[2ex] \dfrac{2^{1-n}\,(a\,\kappa_2)^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\dfrac{t^{\frac{-\alpha_2(n-3)}{2}}}{\Gamma\left(1-\frac{\alpha_2(n-3)}{2}\right)}, & 0<\alpha_2<\min\left\{1,\frac{2}{n-3}\right\} \ \wedge\ n=4+2k,\ k\in\mathbb{N}_0 \end{cases}. \tag{100}$$

- *Super slow-diffusion I*: Let us consider

$$b_1(\alpha) = \kappa_1 \,(const.), \qquad 0 < \alpha \le 1,$$

which implies that

$$B_1(\mathbf{s}) = \frac{a\,\kappa_1}{c^2}\,\frac{\mathbf{s}-1}{\ln(\mathbf{s})}. \tag{101}$$

From (101) we have the following behaviour of (97) when $\mathbf{s} \to 0^+$

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{2^{1-n}\,(a\,\kappa_1)^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\frac{(-\ln(\mathbf{s}))^{\frac{3-n}{2}}}{(1-\mathbf{s})^{\frac{3-n}{2}}\,\mathbf{s}} \sim \frac{2^{1-n}\,(a\,\kappa_1)^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\frac{(-\ln(\mathbf{s}))^{\frac{3-n}{2}}}{\mathbf{s}}.$$

Making use of (95) (for $n=1$) and (96) (for $n = 4+2k$, $k \in \mathbb{N}_0$) to invert the Laplace transform, we obtain for $t \to +\infty$

$$\mathbf{M}^2(t) \sim \begin{cases} \dfrac{c^2}{a\,\kappa_1}\,\ln(t), & n=1 \\ \dfrac{c}{4\sqrt{a\,\kappa_1}}\,\mathcal{L}^{-1}\left\{\dfrac{(-\ln(\mathbf{s}))^{\frac{1}{2}}}{\mathbf{s}}\right\}(t), & n=2 \\ \dfrac{2^{1-n}\,(a\,\kappa_1)^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\dfrac{(-1)^{\frac{3-n}{2}}}{\Gamma\left(\frac{n-3}{2}\right)}\displaystyle\int_0^{+\infty}\frac{w^{\frac{n-5}{2}}\,t^w}{\Gamma(1+w)}\,dw, & n = 4+2k,\ k\in\mathbb{N}_0 \end{cases}. \tag{102}$$

For $n=1$, we see that $\frac{\mathbf{M}^2(t)}{t} \to 0$, as $t \to +\infty$ and the decay turns out to be slower in comparison with the previous case of slow diffusion. Therefore, this case corresponds to a *super slow-diffusion* process. The decay turns out be different along the dimensions, although, we don't have explicit formulas of the asymptotic behaviour of $\mathbf{M}^2(t)$ at infinity to confirm it.

From (101) we have the following behaviour of (97) when $\mathbf{s} \to +\infty$

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{2^{1-n}\,(a\,\kappa_1)^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\frac{(\ln(\mathbf{s}))^{\frac{3-n}{2}}}{(\mathbf{s}-1)^{\frac{3-n}{2}}\,\mathbf{s}} \sim \frac{2^{1-n}\,(a\,\kappa_1)^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\frac{(\ln(\mathbf{s}))^{\frac{3-n}{2}}}{\mathbf{s}^{\frac{5-n}{2}}}.$$

Making use of (95) (for $n=1$), and (96) (for $n = 4+2k$, $k \in \mathbb{N}_0$) to invert the Laplace transform, we obtain for $t \to 0^+$

$$\mathbf{M}^2(t) \sim \begin{cases} -\dfrac{c^2}{a\,\kappa_1}\,t\,\ln(t), & n=1 \\ \dfrac{c}{4\sqrt{a\,\kappa_1}}\,\mathcal{L}^{-1}\left\{\dfrac{(-\ln(\mathbf{s}))^{\frac{1}{2}}}{\mathbf{s}^{\frac{3}{2}}}\right\}(t), & n=2 \\ \dfrac{2^{1-n}\,(a\,\kappa_1)^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\dfrac{1}{\Gamma\left(\frac{n-3}{2}\right)}\displaystyle\int_0^{+\infty}\frac{w^{\frac{n-5}{2}}\,t^{\frac{3+n}{2}+w}}{\Gamma\left(\frac{5-n}{2}+w\right)}\,dw, & n = 4+2k,\ k\in\mathbb{N}_0 \end{cases}.$$

- *Super slow-diffusion II*: Let us consider

$$b_1(\alpha) = 2\alpha, \qquad 0 < \alpha \le 1,$$

which implies that

$$B_1(\mathbf{s}) = \frac{2a}{c^2}\,\frac{\mathbf{s}\,\ln(\mathbf{s}) - s + 1}{\ln^2(\mathbf{s})}. \tag{103}$$

From (103) we have the following behaviour of (97) when $\mathbf{s} \to 0^+$

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{a^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{2^{\frac{n+1}{2}}\,\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\frac{(\mathbf{s}\,\ln(\mathbf{s}) - \mathbf{s} + 1)^{\frac{n-3}{2}}}{(\ln(\mathbf{s}))^{n-3}\,\mathbf{s}} \sim \frac{a^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{2^{\frac{n+1}{2}}\,\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\frac{(\ln(\mathbf{s}))^{3-n}}{\mathbf{s}}.$$

Making use of (95) (for $n = 1, 2$) and (96) (for $n = 4 + 2k$, $k \in \mathbb{N}_0$) to invert the Laplace transform, we obtain for $t \to +\infty$

$$\mathbf{M}^2(t) \sim \begin{cases} \dfrac{c^2}{2a} \ln^2(t), & n = 1 \\ -\dfrac{c}{4\sqrt{2a}} \ln(t), & n = 2 \\ \dfrac{a^{\frac{n-3}{2}} \Gamma\left(\frac{5-n}{2}\right)}{2^{\frac{n+1}{2}} \pi^{\frac{n-1}{2}} c^{n-3}} \dfrac{1}{\Gamma(n-3)} \displaystyle\int_0^{+\infty} \frac{w^{n-4}\, t^w}{\Gamma(1+w)}\, dw, & n = 4 + 2k,\ k \in \mathbb{N}_0 \end{cases} . \tag{104}$$

As explained before this case corresponds also to a *super slow-diffusion* process since $\frac{\mathbf{M}^2(t)}{t} \to 0$ as $t \to +\infty$ as can be easily observed for dimensions $n = 1$ and $n = 2$.

From (103) we have the following behaviour of (97) when $\mathbf{s} \to +\infty$

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{a^{\frac{n-3}{2}} \Gamma\left(\frac{5-n}{2}\right)}{2^{\frac{n+1}{2}} \pi^{\frac{n-1}{2}} c^{n-3}} \frac{(\mathbf{s}\ln(\mathbf{s}) - \mathbf{s} + 1)^{\frac{n-3}{2}}}{(\ln(\mathbf{s}))^{n-3}\, \mathbf{s}} \sim \frac{a^{\frac{n-3}{2}} \Gamma\left(\frac{5-n}{2}\right)}{2^{\frac{n+1}{2}} \pi^{\frac{n-1}{2}} c^{n-3}} \frac{(\ln(\mathbf{s}))^{\frac{3-n}{2}}}{\mathbf{s}^{\frac{5-n}{2}}}.$$

Making use of (95) (for $n = 1$) and (96) (for $n = 4 + 2k$, $k \in \mathbb{N}_0$) to invert the Laplace transform, we obtain for $t \to 0^+$

$$\mathbf{M}^2(t) \sim \begin{cases} -\dfrac{c^2}{2a}\, t \ln(t), & n = 1 \\ \dfrac{c}{4\sqrt{a\,\kappa_1}} \mathcal{L}^{-1}\left\{ \dfrac{(\ln(\mathbf{s}))^{\frac{1}{2}}}{\mathbf{s}^{\frac{3}{2}}} \right\}(t), & n = 2 \\ \dfrac{a^{\frac{n-3}{2}} \Gamma\left(\frac{5-n}{2}\right)}{2^{\frac{n+1}{2}} \pi^{\frac{n-1}{2}} c^{n-3}} \dfrac{1}{\Gamma\left(\frac{n-3}{2}\right)} \displaystyle\int_0^{+\infty} \frac{w^{\frac{n-5}{2}}\, t^{\frac{3+n}{2}+w}}{\Gamma\left(\frac{5-n}{2}+w\right)}\, dw, & n = 4 + 2k,\ k \in \mathbb{N}_0 \end{cases} .$$

- *Super slow-diffusion III*: Let us consider

$$b_1(\alpha) = \frac{\pi}{2} \sin(\alpha\pi), \qquad 0 < \alpha \leq 1$$

which implies that

$$B_1(\mathbf{s}) = \frac{a\pi^2}{2c^2} \frac{\mathbf{s}+1}{\pi^2 + \ln^2(\mathbf{s})}. \tag{105}$$

From (105) we have the following behaviour of (97) when $\mathbf{s} \to 0^+$

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{2^{\frac{5-3n}{2}} a^{\frac{n-3}{2}} \Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{5-n}{2}} c^{n-3}} \frac{(\mathbf{s}+1)^{\frac{n-3}{2}}}{\left(\pi^2 + \ln^2(\mathbf{s})\right)^{\frac{n-3}{2}} \mathbf{s}} \sim \frac{2^{\frac{5-3n}{2}} a^{\frac{n-3}{2}} \Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{5-n}{2}} c^{n-3}} \frac{\left(\pi^2 + \ln^2(\mathbf{s})\right)^{\frac{3-n}{2}}}{\mathbf{s}}.$$

Making use of (91) and (95) (for $n = 1$) to invert the Laplace transform, we obtain for $t \to +\infty$

$$\mathbf{M}^2(t) \sim \begin{cases} \dfrac{2c^2}{a\,\pi^2} \ln^2(t), & n = 1 \\ \dfrac{c}{2\pi\sqrt{2a}} \mathcal{L}^{-1}\left\{ \dfrac{\left(\pi^2 + \ln^2(\mathbf{s})\right)^{\frac{1}{2}}}{\mathbf{s}} \right\}(t), & n = 2 \\ \dfrac{2^{\frac{5-3n}{2}} a^{\frac{n-3}{2}} \Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{5-n}{2}} c^{n-3}} \mathcal{L}^{-1}\left\{ \dfrac{\left(\pi^2 + \ln^2(\mathbf{s})\right)^{\frac{3-n}{2}}}{\mathbf{s}} \right\}(t), & n = 4 + 2k,\ k \in \mathbb{N}_0 \end{cases} . \tag{106}$$

Again, we can see that this case corresponds to a *super slow-diffusion* process, although only for $n = 1$ we have an explicit expression of the asymptotic behaviour of $\mathbf{M}^2(t)$ at infinity.

From (105) we have the following behaviour of (97) when $\mathbf{s} \to +\infty$

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{2^{\frac{5-3n}{2}} a^{\frac{n-3}{2}} \Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{5-n}{2}} c^{n-3}} \frac{(\mathbf{s}+1)^{\frac{n-3}{2}}}{\left(\pi^2 + \ln^2(\mathbf{s})\right)^{\frac{n-3}{2}} \mathbf{s}} \sim \frac{2^{\frac{5-3n}{2}} a^{\frac{n-3}{2}} \Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{5-n}{2}} c^{n-3}} \frac{\left(\pi^2 + \ln^2(\mathbf{s})\right)^{\frac{3-n}{2}}}{\mathbf{s}^{\frac{5-n}{2}}}.$$

Making use of (91) and (95) (for $n=1$) to invert the Laplace transform, we obtain for $t \to 0^+$

$$\mathbf{M}^2(t) \sim \begin{cases} \dfrac{2c^2}{a\pi^2}\, t \ln^2(t)\,, & n=1 \\ \dfrac{c}{2\pi\sqrt{2a}}\, \mathcal{L}^{-1}\left\{ \dfrac{\left(\pi^2 + \ln^2(\mathbf{s})\right)^{\frac{1}{2}}}{\mathbf{s}^{\frac{3}{2}}} \right\}(t)\,, & n=2 \\ \dfrac{2^{\frac{5-3n}{2}}\, a^{\frac{n-3}{2}}\, \Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{5-n}{2}}\, c^{n-3}}\, \mathcal{L}^{-1}\left\{ \dfrac{\left(\pi^2 + \ln^2(\mathbf{s})\right)^{\frac{3-n}{2}}}{\mathbf{s}^{\frac{5-n}{2}}} \right\}(t)\,, & n = 4+2k,\ k \in \mathbb{N}_0 \end{cases}.$$

5.1.2 The wave case

Here we consider $b_1(\alpha) = 0 = B_1(\mathbf{s}) = 0$, and some particular choices of $b_2(\beta)$ in (93). For $B_1(\mathbf{s}) = 0$ the expression for the second-order moment becomes

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{2^{1-n}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}}\, \frac{\left(B_2(\mathbf{s})\right)^{\frac{n-3}{2}}}{\mathbf{s}}, \qquad n \neq 5+2k,\ k \in \mathbb{N}_0. \tag{107}$$

From (107) we immediately see that for $n=3$, the moment does not depend on $B_2(\mathbf{s})$ and it is given by

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{1}{4\pi\,\mathbf{s}}, \qquad \text{so that} \qquad \mathbf{M}^2(t) = \frac{1}{4\pi}.$$

Therefore, in the following particular cases we omit the analysis for the dimension $n=3$.

- *Super fast-diffusion*: Let us consider

$$b_2(\beta) = \kappa_1\,\delta(\beta - \beta_2) + \kappa_2\,\delta(\beta - \beta_2)\,, \qquad 1 < \beta_1 < \beta_2 \leq 2, \quad \kappa_1, \kappa_2 > 0, \quad \kappa_1 + \kappa_2 = 1$$

which implies that

$$B_2(\mathbf{s}) = \frac{\kappa_1}{c^2}\,\mathbf{s}^{\beta_1} + \frac{\kappa_2}{c^2}\,\mathbf{s}^{\beta_2}. \tag{108}$$

From (108) we have the following behaviour of (107) when $\mathbf{s} \to 0^+$

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{2^{1-n}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\, c^{n-3}}\, \frac{\left(\kappa_1\,\mathbf{s}^{\beta_1} + \kappa_2\,\mathbf{s}^{\beta_2}\right)^{\frac{n-3}{2}}}{\mathbf{s}} \sim \frac{2^{1-n}\,\kappa_1^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\, c^{n-3}}\, \mathbf{s}^{\frac{\beta_1(n-3)}{2}-1}.$$

Making use of (91) (for $n=1,\,2,\,4$) to invert the Laplace transform, we obtain for $t \to +\infty$

$$\mathbf{M}^2(t) \sim \begin{cases} \dfrac{c^2}{\kappa_1}\, \dfrac{t^{\beta_1}}{\Gamma(1+\beta_1)}, & n=1 \\ \dfrac{c}{4\sqrt{\kappa_1}}\, \dfrac{t^{\frac{\beta_1}{2}}}{\Gamma\left(1+\frac{\beta_1}{2}\right)}, & n=2 \\ \dfrac{\sqrt{\kappa_1}}{8c\pi}\, \dfrac{t^{-\frac{\beta_1}{2}}}{\Gamma\left(1-\frac{\beta_1}{2}\right)}, & n=4 \ \wedge\ 1<\beta_1<2 \\ \dfrac{2^{1-n}\,\kappa_1^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\, c^{n-3}}\, \mathcal{L}^{-1}\left\{ \mathbf{s}^{\frac{\beta_1(n-3)}{2}-1} \right\}(t)\,, & n = 6+2k,\ k \in \mathbb{N}_0 \end{cases}. \tag{109}$$

For $n=1,\,2$ we can see that $\frac{\mathbf{M}^2(t)}{t} \to +\infty$, which corresponds to a *super fast-diffusion* process. However, for $n=4$ (and also $n=3$) we have that $\frac{\mathbf{M}^2(t)}{t} \to 0$, which shows a *slow-diffusion* process. Therefore, we conclude that in the wave case the type of process vary from dimension 1 to higher dimensions.

From (108) we have the following behaviour of (107) when $\mathbf{s} \to +\infty$

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{2^{1-n}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\, c^{n-3}}\, \frac{\left(\kappa_1\,\mathbf{s}^{\beta_1} + \kappa_2\,\mathbf{s}^{\beta_2}\right)^{\frac{n-3}{2}}}{\mathbf{s}} \sim \frac{2^{1-n}\,\kappa_2^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\, c^{n-3}}\, \mathbf{s}^{\frac{\beta_2(n-3)}{2}-1}.$$

Making use of (91) (for $n = 1, 2, 4$) to invert the Laplace transform, we obtain for $t \to 0^+$

$$\mathbf{M}^2(t) \sim \begin{cases} \dfrac{c^2}{\kappa_2} \dfrac{t^{\beta_2}}{\Gamma(1+\beta_2)}, & n = 1 \\ \dfrac{c}{4\sqrt{\kappa_2}} \dfrac{t^{\frac{\beta_2}{2}}}{\Gamma\left(1+\frac{\beta_2}{2}\right)}, & n = 2 \\ \dfrac{\sqrt{\kappa_2}}{8c\pi} \dfrac{t^{-\frac{\beta_2}{2}}}{\Gamma\left(1-\frac{\beta_2}{2}\right)}, & n = 4 \ \wedge \ 1 < \beta_2 < 2 \\ \dfrac{2^{1-n}\,\kappa_2^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}} \mathcal{L}^{-1}\left\{\mathbf{s}^{\frac{\beta_2(n-3)}{2}-1}\right\}(t), & n = 6+2k,\ k \in \mathbb{N}_0 \end{cases}. \tag{110}$$

- *Fast-diffusion I*: Let us consider

$$b_2(\beta) = \kappa_2 \,(const.), \qquad 1 < \beta \le 2,$$

which implies that

$$B_2(\mathbf{s}) = \frac{\kappa_2}{c^2} \frac{\mathbf{s}^2 - s}{\ln(\mathbf{s})}. \tag{111}$$

From (111) we have the following behaviour of (107) when $\mathbf{s} \to 0^+$

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{2^{1-n}\,\kappa_2^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}} \frac{(-\ln(\mathbf{s}))^{\frac{3-n}{2}}}{(1-\mathbf{s})^{\frac{3-n}{2}}\,\mathbf{s}^{\frac{5-n}{2}}} \sim \frac{2^{1-n}\,\kappa_2^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}} \frac{(-\ln(\mathbf{s}))^{\frac{3-n}{2}}}{\mathbf{s}^{\frac{5-n}{2}}}.$$

Making use of (95) (for $n = 1$) and (96) (for $n = 4 + 2k$, $k \in \mathbb{N}_0$) to invert the Laplace transform, we obtain for $t \to +\infty$

$$\mathbf{M}^2(t) \sim \begin{cases} \dfrac{c^2}{\kappa_2}\, t \ln(t), & n = 1 \\ \dfrac{c}{4\sqrt{\kappa_2}} \mathcal{L}^{-1}\left\{\dfrac{(-\ln(\mathbf{s}))^{\frac{1}{2}}}{\mathbf{s}^{\frac{3}{2}}}\right\}(t), & n = 2 \\ \dfrac{2^{1-n}\,\kappa_2^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}} \dfrac{(-1)^{\frac{3-n}{2}}}{\Gamma\left(\frac{n-3}{2}\right)} \displaystyle\int_0^{+\infty} \dfrac{w^{\frac{n-5}{2}}\, t^{\frac{3-n}{2}+w}}{\Gamma\left(\frac{5-n}{2}+w\right)}\, dw, & n = 4+2k,\ k \in \mathbb{N}_0 \end{cases}. \tag{112}$$

In this case, we only could obtain an explicit formula for the asymptotic behaviour of $\mathbf{M}^2(t)$ at infinity for the dimension $n = 1$. It shows a *fast-diffusion* process since $\frac{\mathbf{M}^2(t)}{t} \to +\infty$, as $t \to +\infty$. Since the growth is not so pronounced when compared with the super fast-diffusion process of the previous example, we called it only *fast-diffusion* process. For dimensions higher we cannot present any conclusion.

From (111) we have the following behaviour of (107) when $\mathbf{s} \to +\infty$

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{2^{1-n}\,\kappa_2^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}} \frac{(\ln(\mathbf{s}))^{\frac{3-n}{2}}}{(\mathbf{s}-1)^{\frac{3-n}{2}}\,\mathbf{s}^{\frac{5-n}{2}}} \sim \frac{2^{1-n}\,\kappa_2^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}} \frac{(\ln(\mathbf{s}))^{\frac{3-n}{2}}}{\mathbf{s}^{4-n}}.$$

Making use of (95) (for $n = 1$) and (96) (for $n = 4 + 2k$, $k \in \mathbb{N}_0$) to invert the Laplace transform, we obtain for $t \to 0^+$

$$\mathbf{M}^2(t) \sim \begin{cases} \dfrac{c^2}{\kappa_2} \dfrac{t^2}{2} \ln\left(\dfrac{1}{t}\right), & n = 1 \\ \dfrac{c}{4\sqrt{\kappa_2}} \mathcal{L}^{-1}\left\{\dfrac{(\ln(\mathbf{s}))^{\frac{1}{2}}}{\mathbf{s}^2}\right\}(t), & n = 2 \\ \dfrac{2^{1-n}\,\kappa_2^{\frac{n-3}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}} \dfrac{1}{\Gamma\left(\frac{n-3}{2}\right)} \displaystyle\int_0^{+\infty} \dfrac{w^{\frac{n-5}{2}}\, t^{3-n+w}}{\Gamma(4-n+w)}\, dw, & n = 4+2k,\ k \in \mathbb{N}_0 \end{cases}.$$

- *Fast-diffusion II*: Let us consider

$$b_2(\beta) = -2(\beta - 2), \qquad 1 < \beta \le 2$$

which implies that

$$B_2(\mathbf{s}) = -\frac{2}{c^2}\,\frac{\mathbf{s}\,\ln(\mathbf{s}) - \mathbf{s}^2 + \mathbf{s}}{\ln^2(\mathbf{s})}. \tag{113}$$

From (113) we have the following behaviour of (107) when $\mathbf{s} \to 0^+$

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{\Gamma\left(\frac{5-n}{2}\right)}{2^{\frac{n+1}{2}}\,\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\frac{\left(-\left(\mathbf{s}\,\ln(\mathbf{s}) - \mathbf{s}^2 + \mathbf{s}\right)\right)^{\frac{n-3}{2}}}{\left(\ln^2(\mathbf{s})\right)^{\frac{n-3}{2}}\,\mathbf{s}} \sim \frac{\Gamma\left(\frac{5-n}{2}\right)}{2^{\frac{n+1}{2}}\,\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\frac{\left(-\ln(\mathbf{s})\right)^{\frac{3-n}{2}}}{\mathbf{s}^{\frac{5-n}{2}}}.$$

Making use of (95) (for $n=1$) and (96) (for $n = 4+2k$, $k \in \mathbb{N}_0$) to invert the Laplace transform, we obtain for $t \to +\infty$

$$\mathbf{M}^2(t) \sim \begin{cases} \dfrac{c^2}{2}\,t\,\ln(t)\,, & n=1 \\[2ex] \dfrac{c}{4\sqrt{2}}\,\mathcal{L}^{-1}\left\{\dfrac{\left(-\ln(\mathbf{s})\right)^{\frac{1}{2}}}{\mathbf{s}^{\frac{3}{2}}}\right\}(t)\,, & n=2 \\[2ex] \dfrac{\Gamma\left(\frac{5-n}{2}\right)}{2^{\frac{n+1}{2}}\,\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\dfrac{(-1)^{\frac{n-3}{2}}}{\Gamma\left(\frac{n-3}{2}\right)}\displaystyle\int_0^{+\infty}\dfrac{w^{\frac{n-5}{2}}\,t^{\frac{3-n}{2}+w}}{\Gamma\left(\frac{5-n}{2}+w\right)}\,dw, & n = 4+2k,\ k\in\mathbb{N}_0 \end{cases}. \tag{114}$$

For $n=1$ we see that $\frac{\mathbf{M}^2(t)}{t} \to +\infty$, as $t \to +\infty$ and corresponds to a *fast-diffusion* process. For higher dimensions we cannot present any conclusion.

From (113) we have the following behaviour of (107) when $\mathbf{s} \to +\infty$

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{\Gamma\left(\frac{5-n}{2}\right)}{2^{\frac{n+1}{2}}\,\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\frac{\left(-\left(\mathbf{s}\,\ln(\mathbf{s}) - \mathbf{s}^2 + \mathbf{s}\right)\right)^{\frac{n-3}{2}}}{\left(\ln^2(\mathbf{s})\right)^{\frac{n-3}{2}}\,\mathbf{s}} \sim \frac{\Gamma\left(\frac{5-n}{2}\right)}{2^{\frac{3n-5}{2}}\,\pi^{\frac{n-1}{2}}\,c^{n-3}}\,\frac{\left(\ln(\mathbf{s})\right)^{3-n}}{\mathbf{s}^{4-n}}.$$

Making use of (95) (for $n = 1, 2$), and (96) (for $n = 4+2k$, $k \in \mathbb{N}_0$) to invert the Laplace transform, we obtain for $t \to 0^+$

$$\mathbf{M}^2(t) \sim \begin{cases} 2c^2\,\dfrac{t^2}{2}\,\ln(t)\,, & n=1 \\[2ex] \dfrac{c}{2\sqrt{2}}\,t\,\ln\left(\dfrac{1}{t}\right), & n=2 \\[2ex] \dfrac{\Gamma\left(\frac{5-n}{2}\right)}{2^{\frac{3n-5}{2}}\,\pi^{\frac{n-1}{2}}\,c^{n-3}}\displaystyle\int_0^{+\infty}\dfrac{w^{n-4}\,t^{3+n+w}}{\Gamma(4-n+w)}\,dw, & n = 4+2k,\ k\in\mathbb{N}_0 \end{cases}.$$

- *Fast-diffusion III*: Let us consider

$$b_2(\beta) = -\frac{\pi}{2}\,\sin(\beta\pi)\,, \qquad 1 < \beta \le 2$$

which implies that

$$B_2(\mathbf{s}) = \frac{\pi^2}{2c^2}\,\frac{\pi\,\mathbf{s}\,(\mathbf{s}+1)}{\pi^2 + \ln^2(\mathbf{s})}. \tag{115}$$

From (115) we have the following behaviour of (107) when $\mathbf{s} \to 0^+$

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{2^{\frac{5-3n}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{5-n}{2}}\,c^{n-3}}\,\frac{\left(s\,(\mathbf{s}+1)\right)^{\frac{n-3}{2}}}{\left(\pi^2 + \ln^2(\mathbf{s})\right)^{\frac{n-3}{2}}\,\mathbf{s}} \sim \frac{2^{\frac{5-3n}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{5-n}{2}}\,c^{n-3}}\,\frac{\left(\pi^2 + \ln^2(\mathbf{s})\right)^{\frac{3-n}{2}}}{\mathbf{s}^{\frac{5-n}{2}}}.$$

Making use of (91) and (95) (for $n = 1$) to invert the Laplace transform, we obtain for $t \to +\infty$

$$\mathbf{M}^2(t) \sim \begin{cases} \dfrac{2c^2}{\pi^2}\, t \ln^2(t)\,, & n = 1 \\ \dfrac{c}{2\pi\sqrt{2}}\, \mathcal{L}^{-1}\left\{ \dfrac{\left(\pi^2 + \ln^2(\mathbf{s})\right)^{\frac{1}{2}}}{\mathbf{s}^{\frac{3}{2}}} \right\}(t)\,, & n = 2 \\ \dfrac{2^{\frac{5-3n}{2}}\, a^{\frac{n-3}{2}}\, \Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{5-n}{2}}\, c^{n-3}}\, \mathcal{L}^{-1}\left\{ \dfrac{\left(\pi^2 + \ln^2(\mathbf{s})\right)^{\frac{3-n}{2}}}{\mathbf{s}^{\frac{5-n}{2}}} \right\}(t)\,, & n = 4 + 2k,\ k \in \mathbb{N}_0 \end{cases} . \tag{116}$$

For $n = 1$ we see that $\frac{\mathbf{M}^2(t)}{t} \to +\infty$, as $t \to +\infty$ and corresponds to a *fast-diffusion* process. For higher dimensions we cannot present any conclusion.

From (115) we have the following behaviour of (107) when $\mathbf{s} \to +\infty$

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{2^{\frac{5-3n}{2}}\, \Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{5-n}{2}}\, c^{n-3}}\, \frac{\left(s\,(\mathbf{s}+1)\right)^{\frac{n-3}{2}}}{\left(\pi^2 + \ln^2(\mathbf{s})\right)^{\frac{n-3}{2}}\, \mathbf{s}} \sim \frac{2^{\frac{5-3n}{2}}\, \Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{5-n}{2}}\, c^{n-3}}\, \frac{\left(\pi^2 + \ln^2(\mathbf{s})\right)^{\frac{3-n}{2}}}{\mathbf{s}^{4-n}}.$$

Making use of (91) and (95) (for $n = 1$) to invert the Laplace transform, we obtain for $t \to 0^+$

$$\mathbf{M}^2(t) \sim \begin{cases} \dfrac{c^2}{\pi^2}\, t^2 \ln^2(t)\,, & n = 1 \\ \dfrac{c}{2\pi\sqrt{2}}\, \mathcal{L}^{-1}\left\{ \dfrac{\left(\pi^2 + \ln^2(\mathbf{s})\right)^{\frac{1}{2}}}{\mathbf{s}^{2}} \right\}(t)\,, & n = 2 \\ \dfrac{2^{\frac{5-3n}{2}}\, a^{\frac{n-3}{2}}\, \Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{5-n}{2}}\, c^{n-3}}\, \mathcal{L}^{-1}\left\{ \dfrac{\left(\pi^2 + \ln^2(\mathbf{s})\right)^{\frac{3-n}{2}}}{\mathbf{s}^{4-n}} \right\}(t)\,, & n = 4 + 2k,\ k \in \mathbb{N}_0 \end{cases} .$$

5.1.3 Graphical representations

In this section we present and analyse the graphical representation of the asymptotic behaviour of $\mathbf{M}^2(t)$, for some of the cases studied previously separating the diffusion and the wave cases.

The diffusion case: In Figure 1 we have the graphical representation of (99), (102), (104), and (106) for $n = 1$ and using a logarithmic scale.

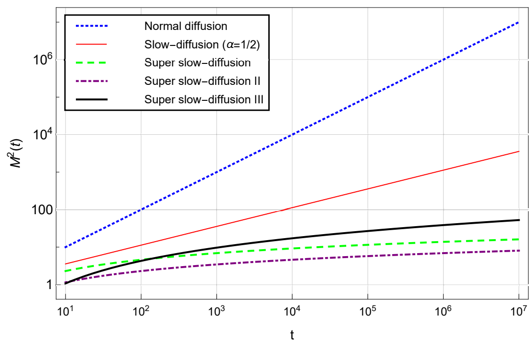

Figure 1: Representation for $n = 1$ of the *normal diffusion, slow-diffusion, and super slow-diffusion* processes.

Looking at the plot we see that, for large values of t, the transition from the *normal diffusion* to *slow-diffusion* and then to *supper slow-diffusion* is characterized by slower growth of the variance. It is also seen that for the different *super-slow diffusion* processes that we exhibit, the behaviour of $\mathbf{M}^2(t)$ at initial times is not the same at large values of t, justifying the necessity of the logarithmic scale. In the following figure we present a graphical representation of (100) (on the left) and (99) (on the right) for $\alpha_1 = 0.5, 0.75$, and different values of the dimension.

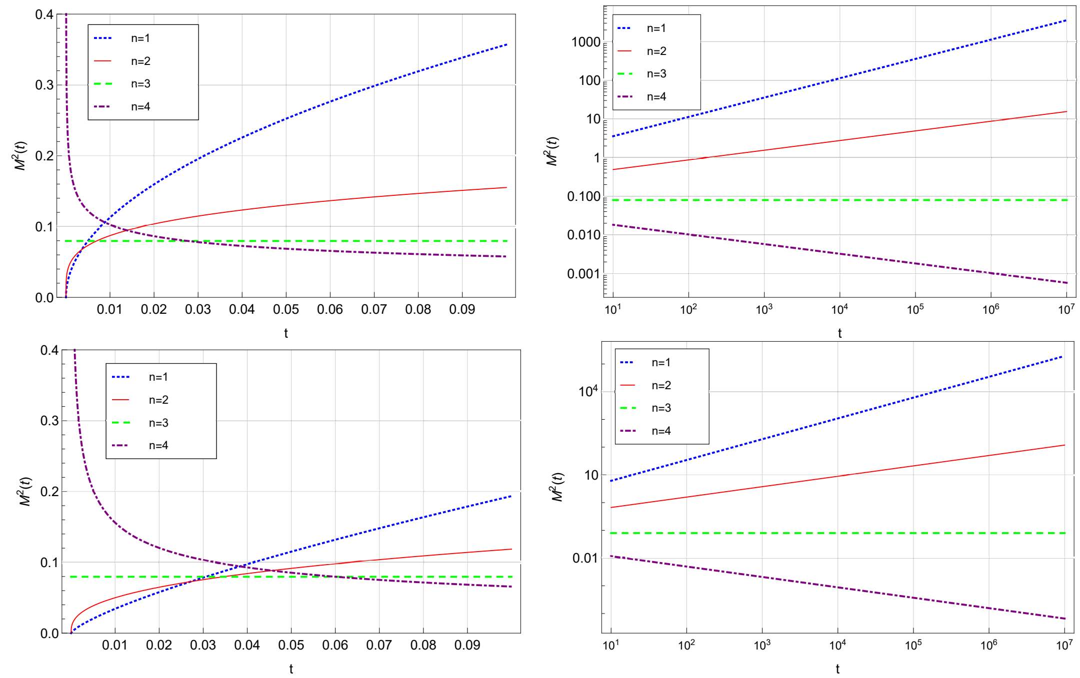

Figure 2: Representation of the *slow-diffusion* process for $t \to 0^+$ (left) and for $t \to +\infty$ (right), for $\alpha_1 = 0.5, 0.75$ (1st and 2nd lines respectively), and different values of n

From these plots we see that the diffusion along the dimensions is different and has a transition of the behaviour at $n = 3$. In fact, for $n = 1$ and $n = 2$ the variance increases for large values of t with different slope, for $n = 3$ the variance is constant, while for $n = 4$ (and also $n = 4 + 2k$, $k \in \mathbb{N}$) the variance decreases for large values of t. This is also consequence of a different behaviour of the variance for small values of t, where the diffusion is faster when the dimension increases (as we can see in the plots on the left). Finally, we present a graphical representation in logarithmic scale of (99) for $n = 1$, 2, 4, and different values of α_1.

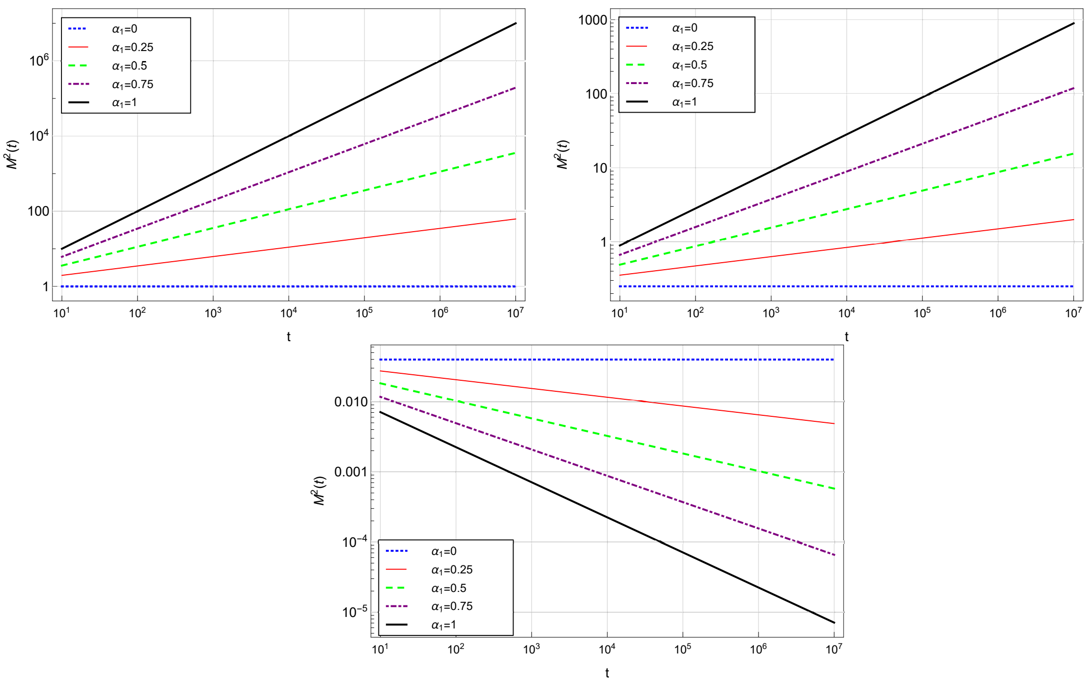

Figure 3: Representation of the *slow-diffusion* process for $n = 1, 2$ (first line from left), and $n = 4$ (second line), and different values of α_1.

The plots show the different behaviour of the power functions t^{α_1}, $t^{\frac{\alpha_1}{2}}$, $t^{-\frac{\alpha_1}{2}}$ for $n = 1, 2, 4$ respectively (see (99)). As the parameter α_1 tends to 1, the variance of the diffusion process tends to the variance of the *normal* *diffusion* process. The increase of the dimension induces a reduction of the range of the plots.

The wave case: In Figure 4 we have the graphical representation of (109), (112), (114), and (116) for $n = 1$, and using a logarithmic scale.

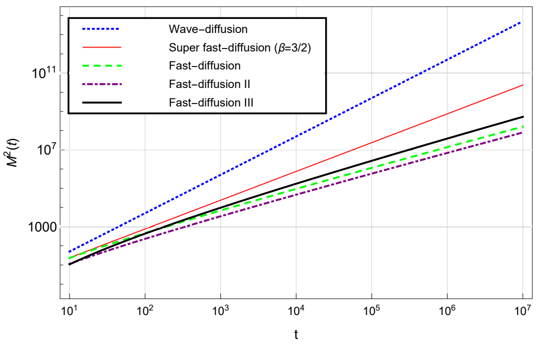

Figure 4: Representation for $n = 1$ of the *wave-diffusion, super fast-diffusion, and fast-diffusion* processes.

From the analysis of the previous figure, we see that, for large values of t, the transition from *fast-diffusion* to *super-fast diffusion* and then to normal *wave-diffusion* is characterized by a gradual increase of the variance. For the different super fast-diffusion processes that we exhibit, the behaviour of $\mathbf{M}^2(t)$ at initial times is not the same at large values of t, as we can see in the plot. Next, we present a graphical representation of (110) (on the left) and (109) (on the right) for $\beta_1 = 1.5, 1.75$, and different values of the dimension.

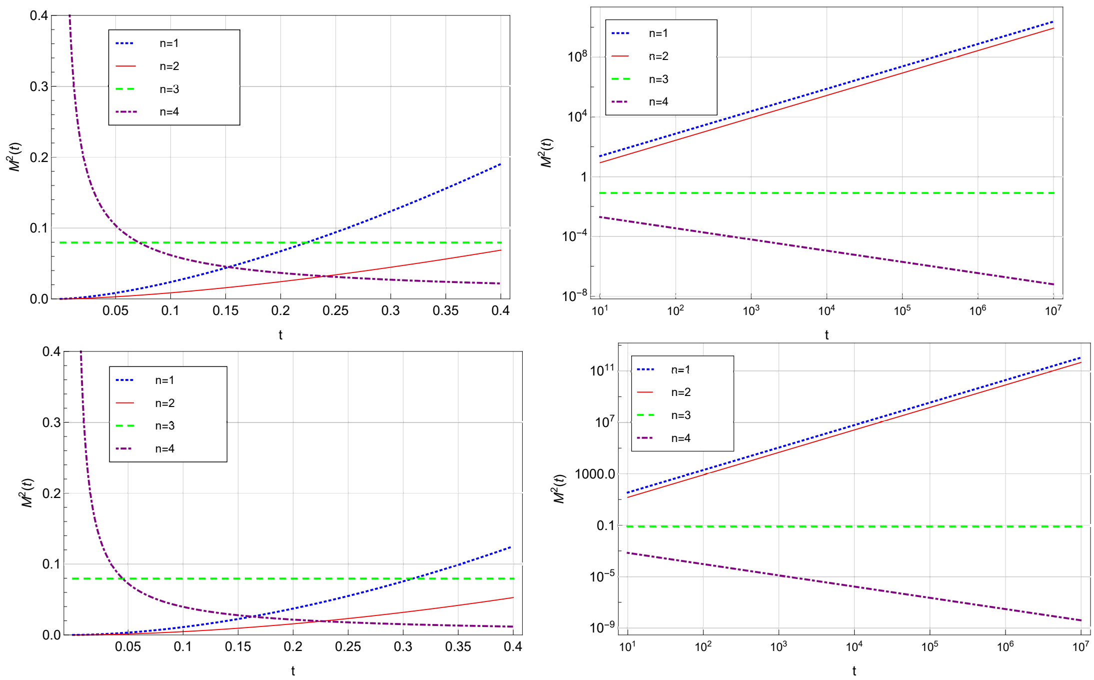

Figure 5: Representation of the *super fast-diffusion* process for $t \to 0^+$ (left) and for $t \to +\infty$ (right), for $\beta_1 = 1.5, 1.75$ (1st and 2nd lines respectively), and different values of n

The analysis of the previous figures leads to similar conclusions to those obtained from Figure 2. Again we see that there are different behaviours along the dimensions and occurs a transition at $n = 3$. In fact, for $n = 1$ and $n = 2$, the variance increases for large values of t with approximately the same slope, for $n = 3$ the variance is constant, while for $n = 4$ (and also $n = 4 + 2k$, $k \in \mathbb{N}$) the variance decreases for large values of t. This fact comes from the different behaviour of the variance for small values of t (see the plots on the left), where the diffusion is faster when the dimension increases. Finally, we present a graphical representation in logarithmic scales of (109) for $n = 1, 2, 4$, and different values of β_1.

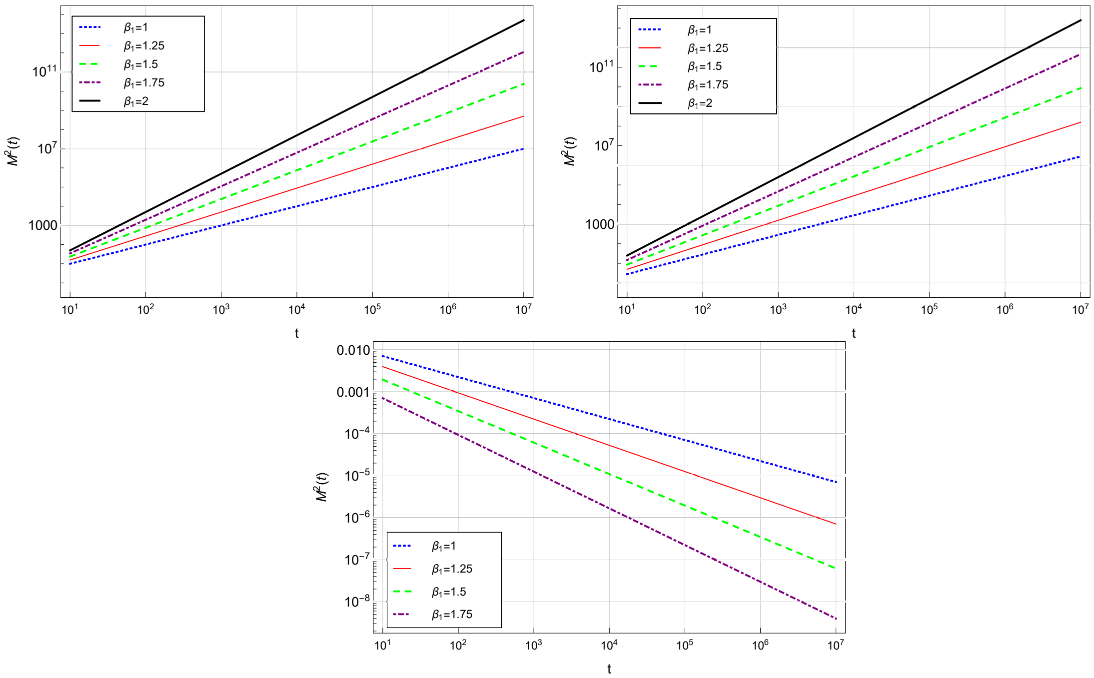

Figure 6: Representation of the *super fast-diffusion* process for $n = 1, 2$ (first line from left), and $n = 4$ (second line), and different values of β_1.

The plots reflect the different behaviour of the power functions t^{β_1}, $t^{\frac{\beta_1}{2}}$, and $t^{-\frac{\beta_1}{2}}$ for $n = 1, 2, 4$ respectively (see (109)). The variance of the diffusion process tends to the variance of the normal *wave-diffusion* process, when β_1 tends to 2.

6 Conclusions

The telegraph equation containing fractional derivatives in time and/or in space are usually adopted to describe both diffusive and wave-like anomalous phenomena, due to the simultaneous presence of the first and second order time derivatives, and therefore a detailed study of their solutions is required. Our attention in this work was focussed on the time-fractional telegraph equation in $\mathbb{R}^n \times \mathbb{R}^+$ of distributed order, which, for some particular choices of the density functions, can be related with sub/super-diffusive processes. Specifically, we were able to worked out how to express their fundamental solutions in terms of Fox H-functions by a combination of the Laplace, Fourier and Mellin transforms.

The presented approach corresponds to a generalization of the techniques used by several authors for time-fractional diffusion-wave equations of distributed order (see [15, 23], for example), however, the simultaneous presence of two density functions lead to more elaborate computations and a more complicated expression for the solution. We were able to obtain a representation of the fundamental solution in terms of a Laplace-type integral of a Fox H-function. Moreover, the general expression for the fractional moments of arbitrary order were deduced and the second-order moment (variance) was studied in detail via the Tauberian theorems, for specific choices of the density functions. We show during the paper that for particular choices of the density functions and/or some parameters in (1) we recover several results presented in the literature for the time-fractional diffusion-wave equations of single and distributed order, which reveals consistency of our results.

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