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Cauchy's formula on nonempty closed sets and a new notion of Riemann–Liouville fractional integral on time scales

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Abstract

We prove Cauchy's formula for repeated integration on time scales. The obtained relation gives rise to new notions of fractional integration and differentiation on arbitrary nonempty closed sets.

Keywords: fractional calculus; time-scale calculus; fractional integrals on time scales; Cauchy formula for repeated integration on time scales; Riemann–Liouville and Caputo operators.

MSC 2020: 26A33; 26E70.

1 Introduction

Fractional calculus (FC), the study of integration and differentiation of non-integer order, is an old subject of current interest [21]. On the set $\mathbb{T} = \mathbb{R}$ of real numbers, one can argue that FC, as a theory, has its origins in the 1823 work of Abel [1] on the tautochrone or isochrone problem, that is, the problem of finding the curve for which the time taken by an object sliding without friction in uniform gravity to its lowest point is independent of its starting point on the curve [13]. FC on $\mathbb{T} = \mathbb{R}$ has been extensively studied, with many books on the topic, the encyclopedic reference being [18]. Similarly, on the discrete setting $\mathbb{T} = \mathbb{Z}$ the pioneer work of FC appears in 1957 in connection with Kuttner's study of difference sequences [14]. A few years later, in 1966, the study of fractional quantum calculus on the set $\mathbb{T} = q^{\mathbb{Z}}$, $0 < q < 1$, of quantum numbers was initiated by Al-Salam [3]. Two decades later, with the introduction in 1988, by Aulbach and Hilger [4], of time-scale calculus on any nonempty closed set $\mathbb{T}$, the question as whether there exists a unified theory of FC on an arbitrary time scale $\mathbb{T}$ became a natural one. The first works, which considered the question of developing a FC on a generic time scale, are three 2011 papers by Bastos et al. [6, 7, 8]. The subject is nowadays under strong current development: see [16, 17, 19] and references therein.

One of the approaches to define fractional integration on time scales is related with the generalization of Euler's gamma function Γ to an arbitrary time scale $\mathbb{T}$. Such approach is followed in the quantum setting by Al-Salam [3] and Agarwal [2], where the q-gamma function Γ_q is used to define fractional q-integrals. Such path is problematic on arbitrary time scales, because of the difficulty to define a suitable gamma function on $\mathbb{T}$ [10]. A different approach is proposed in [9], where the fractional integral on time scales is introduced by integrating on the time scale but making use of the classical Euler's gamma function Γ. Although such notion is now being used, with success, in several contexts and by different authors, see, e.g., [5, 15, 20, 22, 23], here we show that the definition of [9] is not the most natural one on time scales.

The paper is organized as follows. In Section 2, we motivate the new definition of fractional integral on time scales, explaining its difference with respect to the one of [9]. Our central result, the Cauchy formula for repeated integration on time scales, in then proved in Section 3. We proceed with Section 4, giving the fundamental definition of any FC: a notion of fractional integration –

see Definition 3 of fractional integral on time scales in the sense of Riemann–Liouville. We end with Section 5 of conclusion, commenting on some possible future directions of research based on the results here presented.

2 Motivation

For an introduction to the calculus on time scales, we refer the reader to the comprehensible monograph [11]. Brief but sufficient preliminaries on the calculus on time scales are found in Section 2 of [9]. Here we motivate the current investigation on repeated integration on time scales, showing why the Riemann–Liouville fractional integral on time scales given by Definition 10 of [9] is not the best one. Let $\mathbb{T}$ be a time scale with forward operator σ and Δ derivative and integral. The product rule asserts that

$$(f \cdot g)^\Delta = f^\Delta \cdot g + f^\sigma \cdot g^\Delta. \tag{1}$$

For this reason, while the usual continuous calculus of $\mathbb{T} = \mathbb{R}$ asserts that $\left(t^2\right)' = 2t$, on a general time scale $\mathbb{T}$ one gets

$$\left(t^2\right)^\Delta = (t \cdot t)^\Delta = t + \sigma(t). \tag{2}$$

For $\mathbb{T} = \mathbb{R}$ one has $\sigma(t) = t$ and we get from (2) the expected equality $\left(t^2\right)' = 2t$; but for $\mathbb{T} = \mathbb{Z}$, for example, one has $\sigma(t) = t + 1$ and (2) is telling us that the forward difference of t^2 is $\Delta\left(t^2\right) = 2t + 1$. Relation (2) also means that while in $\mathbb{T} = \mathbb{R} \int_0^t 2s\,ds = t^2$, on a general time scale $\mathbb{T}$ the corresponding equality is

$$\int_0^t (s + \sigma(s))\,ds = t^2.$$

These simple examples should be enough for the reader to suspect that the definition of fractional integral on time scales proposed in [9], that is,

$$\left(_a^\mathbb{T} I^\alpha f\right)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} f(s)\,\Delta s, \tag{3}$$

is not the natural one on time scales. Indeed, here we claim that the correct definition should be

$$\left(_a^\mathbb{T} I^\alpha f\right)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t - \sigma(s))^{\alpha - 1} f(s)\,\Delta s$$

(note the appearance of σ). To show that, we generalize Cauchy's formula for repeated integration on time scales, proving that for $n \in \mathbb{N}$ one has

$$\left(_a^\mathbb{T} I^n f\right)(t) = \frac{1}{\Gamma(n)} \int_a^t (t - \sigma(s))^{n-1} f(s)\,\Delta s, \tag{4}$$

where $_a^\mathbb{T} I^n f$ denotes n-times integration on the time scale $\mathbb{T}$, that is,

$$\left(_a^\mathbb{T} I^n f\right)(t_n) := \int_a^{t_n} \cdots \int_a^{t_1} f(t_0)\,\Delta t_0 \cdots \Delta t_{n-1} \tag{5}$$

(cf. Theorem 2 in Section 3).

3 Cauchy's formula for repeated integration

There is nothing to prove when $n = 1$, because in this case (4) reduces to

$$\left(_a^\mathbb{T} I^1 f\right)(t) = \int_a^t f(s)\,\Delta s,$$

which is just definition (5) for $n = 1$. For simplicity and illustrative purposes, we begin by proving (4) when $n = 2$. As we shall see, the proof of (4), for a general n, is similar, just more technical.

2

Proposition 1. *Let $\mathbb{T}$ be a time scale with $a, t, \tau \in \mathbb{T}$, $t > a$ and $\tau > a$, and f an integrable function on $\mathbb{T}$. Then,*

$$\int_a^t \int_a^\tau f(s)\Delta s\Delta\tau = \int_a^t (t - \sigma(s))f(s)\Delta s. \tag{6}$$

Proof. Let $g(t)$ be the right hand side of (6):

$$g(t) := \int_a^t (t - \sigma(s))f(s)\Delta s. \tag{7}$$

Observe that $g(t)$ can be written as

$$g(t) = t\int_a^t f(s)\Delta s - \int_a^t \sigma(s)f(s)\Delta s. \tag{8}$$

Differentiating (8), we get from the product rule (1) and the fundamental theorem of the calculus on time scales that

$$g^\Delta(t) = \left[\int_a^t f(s)\Delta s + \sigma(t)f(t)\right] - \sigma(t)f(t) = \int_a^t f(s)\Delta s. \tag{9}$$

Since by definition (7) of $g(t)$ one has $g(a) = 0$, we know that

$$g(t) = g(t) - g(a) = \int_a^t g^\Delta(\tau)\Delta\tau. \tag{10}$$

Using (9) in (10), we arrive to

$$g(t) = \int_a^t \int_a^\tau f(s)\Delta s\Delta\tau,$$

which proves the intended relation. $\qquad\square$

We now do the proof of Cauchy's formula of repeated integration on time scales by expanding the term $(t - \sigma(s))^{n-1}$ of (4) with the help of the binomial theorem and then writing $g(t)$ in the manner of (8), with all the terms t^j outside the integral sign. Proposition 1 is just the particular case of Theorem 2 with $n = 2$.

Theorem 2 (Cauchy's result on time scales). *Let $n \in \mathbb{N}$, $\mathbb{T}$ be a time scale with $a, t_1, \ldots, t_n \in \mathbb{T}$, $t_i > a$, $i = 1, \ldots, n$, and f an integrable function on $\mathbb{T}$. Then,*

$$\int_a^{t_n} \cdots \int_a^{t_1} f(t_0)\Delta t_0 \cdots \Delta t_{n-1} = \frac{1}{(n-1)!}\int_a^{t_n} (t_n - \sigma(s))^{n-1}f(s)\Delta s. \tag{11}$$

Proof. Let $g(t_n)$ be the right hand side of (11):

$$g(t) := \frac{1}{(n-1)!}\int_a^t (t - \sigma(s))^{n-1}f(s)\Delta s. \tag{12}$$

By the binomial theorem, we observe that $g(t)$ can be written as

$$\begin{aligned}
g(t) &= \frac{1}{(n-1)!}\int_a^t \sum_{k=0}^{n-1}\left[\binom{n-1}{k}(-1)^k t^{n-1-k}\sigma^k(s)\right]f(s)\Delta s \\
&= \sum_{k=0}^{n-1}\frac{(-1)^k}{k!(n-1-k)!}t^{n-1-k}\int_a^t \sigma^k(s)f(s)\Delta s.
\end{aligned} \tag{13}$$

3

Differentiating (13), we get from the product rule and the fundamental theorem of the calculus on time scales that

$$g^{\Delta}(t) = \sum_{k=0}^{n-1} \left[\frac{(-1)^k (n-1-k)}{k!(n-1-k)!} t^{n-2-k} \int_a^t \sigma^k(s) f(s) \Delta s + \frac{(-1)^k}{k!(n-1-k)!} \sigma^{n-1}(t) f(t) \right]. \quad (14)$$

Since

$$\sum_{k=0}^{n-1} \frac{(-1)^k}{k!(n-1-k)!} = 0,$$

it follows from (14) that

$$g^{\Delta}(t) = \sum_{k=0}^{n-2} \frac{(-1)^k}{k!(n-2-k)!} t^{n-2-k} \int_a^t \sigma^k(s) f(s) \Delta s. \quad (15)$$

In general, differentiating (13) i times, we obtain that

$$g^{\Delta^i}(t) = \sum_{k=0}^{n-1-i} \frac{(-1)^k}{k!(n-1-i-k)!} t^{n-1-i-k} \int_a^t \sigma^k(s) f(s) \Delta s, \quad (16)$$

$i = 0, \dots, n-1$, where $g^{\Delta^0}(t) = g(t)$. In particular, for $i = n-1$ in (16), one has

$$g^{\Delta^{n-1}}(t) = \int_a^t f(s) \Delta s, \quad (17)$$

while $g^{\Delta^i}(a) = 0$, $i = 0, \dots, n-1$. Therefore, by the fundamental theorem of integral calculus, we have

$$g^{\Delta^j}(t_{n-j}) = g^{\Delta^j}(t_{n-j}) - g^{\Delta^j}(a) = \int_a^{t_{n-j}} g^{\Delta^{j+1}}(t_{n-j-1}) \Delta t_{n-j-1}, \quad (18)$$

$j = 0, \dots, n-2$. Relation (18) with $j = n-2$ asserts that

$$g^{\Delta^{n-2}}(t_2) = \int_a^{t_2} g^{\Delta^{n-1}}(t_1) \Delta t_1 \quad (19)$$

and using (17) in (19) allow us to write that

$$g^{\Delta^{n-2}}(t_2) = \int_a^{t_2} \int_a^{t_1} f(s) \Delta s \Delta t_1. \quad (20)$$

Similarly, relation (18) with $j = n-3$ asserts that

$$g^{\Delta^{n-3}}(t_3) = \int_a^{t_3} g^{\Delta^{n-2}}(t_2) \Delta t_2 \quad (21)$$

and using (20) in (21) implies that

$$g^{\Delta^{n-3}}(t_3) = \int_a^{t_3} \int_a^{t_2} \int_a^{t_1} f(s) \Delta s \Delta t_1 \Delta t_2.$$

By repeating this procedure till $j = 0$, we arrive at

$$g(t_n) = \int_a^{t_n} \int_a^{t_{n-1}} \cdots \int_a^{t_1} f(s) \Delta s \Delta t_1 \cdots \Delta t_{n-1}.$$

Since by definition (12) of $g(t)$ one has

$$g(t_n) = \frac{1}{(n-1)!} \int_a^{t_n} (t_n - \sigma(s))^{n-1} f(s) \Delta s,$$

the proof is complete. $\qquad \square$

Theorem 2 provides the foundation of a proper fractional calculus on time scales.

4 A proper Fractional Calculus on Time Scales

Although there is a myriad of different FC on the literature, all of them begin by defining a notion of fractional integral and then proceeding from there [18]. The most common definition of fractional integral is the one of Riemann–Liouville:

$$\left(_aI^\alpha f\right)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds. \tag{22}$$

The corresponding Riemann–Liouville fractional derivative of order α is then introduced by computing the nth order derivative over the fractional integral of order $n - \alpha$, where n is the smallest integer greater than α, that is, $n := \lceil \alpha \rceil$:

$$_aD^\alpha f = \frac{d^n}{dt^n} \left(_aI^{n-\alpha} f\right). \tag{23}$$

Another option for computing fractional derivatives was proposed by Caputo in 1967 [12], and consists in interchanging the order of the operators $\frac{d^n}{dt^n}$ and $_aI^{n-\alpha}$ in (23):

$$_aD_C^\alpha f = {_aI^{n-\alpha}} \left(f^{(n)}\right). \tag{24}$$

Having in mind that $\Gamma(n) = (n-1)!$, Theorem 2 shows that (4) holds for $n \in \mathbb{N}$. The question is what should be $_a^{\mathbb{T}}I^\alpha f$ when α is any positive number? The answer should be clear since (4) makes sense for any nonnegative $n \in \mathbb{R}$. This provides the proper notion of fractional integral on time scales in the sense of Riemann–Liouville.

Definition 3 (Riemann–Liouville fractional integral on time scales). *Suppose $\mathbb{T}$ is a time scale, $[a,b]$ is an interval of $\mathbb{T}$, and f is an integrable function on $[a,b]$. Let $\alpha > 0$. Then, the (left) fractional integral of order α of f is defined by*

$$\left(_a^{\mathbb{T}}I^\alpha f\right)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\sigma(s))^{\alpha-1} f(s)\Delta s, \tag{25}$$

where Γ is the gamma function.

In the case $\mathbb{T} = \mathbb{R}$, one gets from Definition 3 the standard notion (22). The generalization (25) is not trivial, in the sense that (3), used currently in the literature, also generalizes (22). However, only Definition 3 is coherent with Cauchy's formula (Theorem 2).

Using the new Definition 3, a fractional calculus on time scales can now be developed. As expected, the first steps of such fractional calculus are the notions of fractional differentiation on time scales in the sense of Riemann–Liouville,

$$_a^{\mathbb{T}}D^\alpha f = \left(_aI^{n-\alpha} f\right)^{\Delta^n}, \tag{26}$$

and Caputo,

$$_a^{\mathbb{T}}D_C^\alpha f = {_aI^{n-\alpha}} \left(f^{\Delta^n}\right), \tag{27}$$

where $n := \lceil \alpha \rceil$, which are the natural extensions of (23) and (24), respectively. Such fractional calculus on time scales is rich and technical and its development will be addressed elsewhere.

5 Conclusion

We have proposed a new definition of fractional integral on time scales, using Cauchy's formula for repeated integration on time scales. There is much further work to be done based on this new notion. Some next natural steps are to consider fractional differential equations on time scales, eigenvalue problems, fractional dynamic inequalities, which are examples of very active research areas in both time-scale and fractional communities. About applications, we claim that our fractional calculus on time scales has a big potential in mathematical modeling, for example in epidemiology and consensus problems.

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