A fractional analysis in higher dimensions for the Sturm-Liouville problem*

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Abstract

In this work, we consider the n-dimensional fractional Sturm-Liouville eigenvalue problem, by using fractional versions of the gradient operator involving left and right Riemann-Liouville fractional derivatives. We study the main properties of the eigenfunctions and the eigenvalues of the associated fractional boundary problem. More precisely, we show that the eigenfunctions are orthogonal and the eigenvalues are real and simple. Moreover, using techniques from fractional variational calculus, we prove in the main result that the eigenvalues are separated and form an infinite sequence, where the eigenvalues can be ordered according to increasing magnitude. Finally, a connection with Clifford analysis is established.

Keywords: Fractional derivatives; Fractional Sturm-Liouville problem; Fractional variational calculus; Eigenvalue problem; Eigenfunctions; Fractional Clifford analysis.

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1 Introduction

The Sturm-Liouville theory deals with the following general linear, homogeneous second-order differential equation

$$\frac{d}{dx}\left(\mu(x)\,\frac{df}{dx}(x)\right)+\nu(x)\,f(x)\,=\lambda\,r(x)f(x), \tag{1}$$

where $x\in[a,b]$ and in some cases the functions $\mu(x)$, $\nu(x)$, and $r(x)$ are known. The problem consists in finding the values of $\lambda\in\mathbb{C}$ (called eigenvalues) for which there exits a non-trivial solution $f(x)$ (called eigenfunction) satisfying (1) and some additional boundary conditions at a and b. The previous equation was initially studied by Jacques Sturm and Joseph Liouville in 1837, more precisely they studied problems involving the previous equation and the properties of the solutions. The so-called Sturm-Liouville equation (1) arises directly as a mathematical model of motion according to second Newton's law, but more often as a result of using the method of separation of variables to solve the classical partial differential equations of physics, such as Laplace's equation, the heat equation, and the wave equation (see $[1,7,9]$).

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In recent years, many mathematicians directed their attention to some generalizations of the Sturm-Liouville problem in connections with other fields of Mathematics. One of the most important reasons for this emerging interest is the fact that the orthogonal eigenfunctions' system of the fractional Sturm-Liouville problem can be used to solve fractional partial differential equations that are related with anomalous diffusion processes (see [12]). For example, in the one dimensional case we can find in [12, 14–16, 18] fractional approaches for the Sturm-Liouville problem in the sense that the integer derivatives are replaced by derivatives of arbitrary order. In [15, 16] the authors defined some fractional Sturm-Liouville operators containing left and right Riemann-Liouville and left and right Caputo fractional derivatives. Here, the eigenvalue and eigenfunction properties of the fractional Sturm-Liouville problem were theoretically investigated in detail. These properties were also proved in [18] for other fractional operators. The use of fractional variational calculus in [14], allowed the authors to improve the work in [18] proving the existence of a countable set of orthogonal solutions and corresponding eigenvalues. Moreover, it was shown that the lowest eigenvalue is the minimum value for a certain variational functional. More recently, in [12] the authors studied exact and numerical solutions for the regular Sturm-Liouville problem in a bounded domain subject to homogeneous mixed boundary conditions. Their approach consisted on transformation of the differential fractional Sturm-Liouville problem into an integral problem and the use of operator theory. Nowadays, there are several definitions for fractional derivatives. Its purpose is to represent the physical reality more accurately by introducing a kind of memory mechanism in the process to obtain models for anomalous physical processes. For more details about fractional calculus and its applications we refer to [6, 11, 12, 17, 19, 20].

This paper aims to study the fractional Sturm-Liouville problem in higher dimension combining the ideas and some techniques presented in [14,18] for the one-dimensional case. We point out that some of the techniques used in [14, 18] can only be applied in the one dimensional case. For example, the Ritz method used in [14] to approximate the solutions is replaced by the Rayleigh-Ritz method when we deal with problems in $\mathbb{R}^n$. The fractional Sturm-Liouville operator introduced in this work includes the generalization of the eigenvalue problem (1) to n-dimensions, i.e.

$$\nabla \cdot (\mu(x)\, \nabla f(x)) + \nu(x)\, f(x) \;= \lambda\, r(x) f(x), \tag{2}$$

where $x \in \Omega \subset \mathbb{R}^n$ and ∇ is the gradient operator in $\mathbb{R}^n$ (for more details see [9]). In our work, the gradient operator is replaced by fractional gradients involving left and right Riemann-Liouville fractional derivatives giving rise to the new eigenvalue problem:

$$-\left({}^{RL}\nabla^{\alpha}_{b^-} \cdot \left(\mu\, {}^{RL}\nabla^{\alpha}_{a^+} f\right)\right)(x) + \nu(x)\, f(x) = \lambda\, r(x)\, f(x). \tag{3}$$

The use of simultaneous left and right fractional gradients allows the fractional Sturm-Liouville operator to be self-adjoint and the use of a fractional variational approach (see Section 4.2). Moreover, a formulation of (3) in terms of one-sided fractional derivatives does not lead to orthogonal eigenfunctions (see [13]).

The structure of the paper reads as follows. In the Preliminaries section, we recall some basic facts about fractional calculus which are necessary for the development of this work. In Section 3 we establish a new generalization of the fractional Sturm-Liouville problem in $\mathbb{R}^n$ and we study the properties of the eigenfunctions and eigenvalues. In Section 4 we prove that the eigenvalues are separated and form an infinite sequence, where the eigenvalues can be ordered according to their increasing magnitude, and we prove also the existence of orthogonal solutions for the fractional Sturm-Liouville problem. In the following section, we indicate how the obtained results can be extended to the context of Clifford analysis and in the last section we present the conclusions of this work.

2 Preliminaries

Let $a, b \in \mathbb{R}$ with $a < b$ and $\alpha > 0$. The left and right Riemann-Liouville fractional integrals $I^{\alpha}_{a^+}$ and $I^{\alpha}_{b^-}$ of order α are given by (see [11])

$$\left(I^{\alpha}_{a^+} f\right)(x) \;= \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \qquad x > a \tag{4}$$

$$\left(I^{\alpha}_{b^-} f\right)(x) \;= \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}\, dt, \qquad x < b. \tag{5}$$

By ${}^{RL}D^{\alpha}_{a+}$ and ${}^{RL}D^{\alpha}_{b-}$ we denote the left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$ on $[a,b] \subset \mathbb{R}$, which are defined by (see [11])

$$\left({}^{RL}D^{\alpha}_{a+} f\right)(x) = \left(D^m I^{m-\alpha}_{a+} f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x \frac{f(t)}{(x-t)^{\alpha-m+1}}\, dt, \qquad x > a \tag{6}$$

$$\left({}^{RL}D^{\alpha}_{b-} f\right)(x) = (-1)^m \left(D^m I^{m-\alpha}_{b-} f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^b \frac{f(t)}{(t-x)^{\alpha-m+1}}\, dt, \qquad x < b. \tag{7}$$

Here, $m = [\alpha] + 1$ and $[\alpha]$ means the integer part of α. Let ${}^{C}D^{\alpha}_{a+}$ and ${}^{C}D^{\alpha}_{b-}$ denote, respectively, the left and right Caputo fractional derivative of order $\alpha > 0$ on $[a,b] \subset \mathbb{R}$, which are defined by (see [11])

$$\left({}^{C}D^{\alpha}_{a+} f\right)(x) = \left(I^{m-\alpha}_{a+} D^m f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}}\, dt, \qquad x > a \tag{8}$$

$$\left({}^{C}D^{\alpha}_{b-} f\right)(x) = (-1)^m \left(I^{m-\alpha}_{b-} D^m f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b \frac{f^{(m)}(t)}{(t-x)^{\alpha-m+1}}\, dt, \qquad x < b. \tag{9}$$

The left and right Caputo fractional derivatives can be defined in terms of left and right Riemann-Liouville fractional derivatives by (see [11])

$$\left({}^{C}D^{\alpha}_{a+} f\right)(x) := \left({}^{RL}D^{\alpha}_{a+} \left[f(t) - \sum_{k=0}^{m} \frac{f^{(k)}(a)}{k!}(t-a)^k\right]\right)(x), \tag{10}$$

$$\left({}^{C}D^{\alpha}_{b-} f\right)(x) := \left({}^{RL}D^{\alpha}_{b-} \left[f(t) - \sum_{k=0}^{m} \frac{f^{(k)}(b)}{k!}(b-t)^k\right]\right)(x). \tag{11}$$

Remark 2.1 *If $f(a) = 0$ then from (10) we conclude that $\left({}^{C}D^{\alpha}_{a+} f\right)(x) = \left({}^{RL}D^{\alpha}_{a+} f\right)(x)$, and if $f(b) = 0$ then from (11) we conclude that $\left({}^{C}D^{\alpha}_{b-} f\right)(x) = \left({}^{RL}D^{\alpha}_{b-} f\right)(x)$.*

We denote by $I^{\alpha}_{a+}(L_p)$, $p \geq 1$ the class of functions f that are represented by the fractional integral (4) of a summable function, that is $f = I^{\alpha}_{a+}\varphi$, with $\varphi \in L_p(a,b)$. A description of the space $I^{\alpha}_{a+}(L_1)$ is given in [19].

Theorem 2.2 *(cf. [19]) A function f belongs to $I^{\alpha}_{a+}(L_1)$, with $\alpha > 0$, if and only if $I^{m-\alpha}_{a+} f$ belongs to $AC^m([a,b])$, $m = [\alpha]+1$ and $(I^{m-\alpha}_{a+} f)^{(k)}(a) = 0$, $k = 0, \ldots, m-1$.*

In Theorem 2.2, $AC^m([a,b])$ denotes the class of functions f which are continuously differentiable on the segment $[a,b]$ up to the order $m-1$ and $f^{(m-1)}$ is absolutely continuous on $[a,b]$. We note that the conditions $(I^{m-\alpha}_{a+} f)^{(k)}(a) = 0$, $k = 0, \ldots, m-1$, imply that $f^{(k)}(a) = 0$, $k = 0, \ldots, m-1$ (see [17,19]). Removing the last condition in Theorem 2.2 we obtain the class of functions that admit a summable fractional derivative.

Definition 2.3 *(see [19]) A function $f \in L_1(a,b)$ has a summable fractional derivative $\left(D^{\alpha}_{a+} f\right)(x)$ if $\left(I^{m-\alpha}_{a+} f\right)(x)$ belongs to $AC^m([a,b])$, where $m = [\alpha]+1$.*

If a function f admits a summable fractional derivative, then we have the following composition rules (see [17,19])

$$\left(I^{\alpha}_{a+}\, {}^{RL}D^{\alpha}_{a+} f\right)(x) = f(x) - \sum_{k=0}^{m-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \left(I^{m-\alpha}_{a+} f\right)^{(m-k-1)}(a), \qquad m = [\alpha]+1 \tag{12}$$

$$\left(I^{\alpha}_{b-}\, {}^{RL}D^{\alpha}_{b-} f\right)(x) = f(x) - \sum_{k=0}^{m-1} \frac{(b-x)^{\alpha-k-1}}{\Gamma(\alpha-k)} \left(I^{m-\alpha}_{a+} f\right)^{(m-k-1)}(b), \qquad m = [\alpha]+1. \tag{13}$$

We remark that if $f \in I^{\alpha}_{a+}(L_1)$ then (12) and (13) reduce to $\left(I^{\alpha}_{a+}\, {}^{RL}D^{\alpha}_{a+} f\right)(x) = \left(I^{\alpha}_{b-}\, {}^{RL}D^{\alpha}_{b-} f\right)(x) = f(x)$. Nevertheless we note that ${}^{RL}D^{\alpha}_{a+}\, I^{\alpha}_{a+} f = {}^{RL}D^{\alpha}_{b-}\, I^{\alpha}_{b-} f = f$ in both cases. This is a particular case of a more general property (cf. [17, (2.114)])

$$D^{\alpha}_{a+}\left(I^{\gamma}_{a+} f\right) = D^{\alpha-\gamma}_{a+} f, \qquad \alpha \geq \gamma > 0. \tag{14}$$

It is important to remark that the semigroup property for the composition of fractional derivatives does not hold in general (see [17, Sect. 2.3.6]). In fact, the property

$$D^{\alpha}_{a^+}\left(D^{\beta}_{a^+}f\right) = D^{\alpha+\beta}_{a^+}f \tag{15}$$

holds whenever $f \in AC^{m-1}([a,b])$, $f^{(m)} \in L_1(a,b)$ with $m = [\beta]+1$, and

$$f^{(j)}(a^+) = 0, \qquad j = 0,1,\ldots,m-1. \tag{16}$$

Moreover, for $m-1 < \alpha < m$ with $m \in \mathbb{N}$ and $\beta > 0$ we have

$$^{RL}D^{\alpha}_{a^+}(x-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\,(x-a)^{\beta-\alpha-1}, \tag{17}$$

$$I^{\alpha}_{b^-}(b-x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\,(b-x)^{\beta+\alpha-1}. \tag{18}$$

Now we present the rules for fractional integration by parts.

Lemma 2.4 *(cf. [11, 18]) Let $\alpha > 0$, $p \geq 1$, $q \geq 1$, and $\frac{1}{p}+\frac{1}{q} \leq 1+\alpha$ ($p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p}+\frac{1}{q} = 1+\alpha$).*

(a) If $g(x) \in L_p(a,b)$ and $f(x) \in L_q(a,b)$ then

$$\int_a^b g(x)\,\left(I^{\alpha}_{a^+}f\right)(x)\,dx = \int_a^b f(x)\,\left(I^{\alpha}_{b^-}g\right)(x)\,dx. \tag{19}$$

(b) If $f(x) \in I^{\alpha}_{b^-}(L_p)$ and $g(x) \in I^{\alpha}_{a^+}(L_q)$ then

$$\int_a^b g(x)\,\left({}^{C}D^{\alpha}_{a^+}f\right)(x)\,dx = \int_a^b f(x)\,\left({}^{RL}D^{\alpha}_{b^-}g\right)(x)\,dx. \tag{20}$$

(c) If $f(x) \in I^{\alpha}_{a^+}(L_p)$ and $g(x) \in I^{\alpha}_{b^-}(L_q)$ then

$$\int_a^b g(x)\,\left({}^{C}D^{\alpha}_{b^-}f\right)(x)\,dx = \int_a^b f(x)\,\left({}^{RL}D^{\alpha}_{a^+}g\right)(x)\,dx. \tag{21}$$

Now, we recall the following result regarding the boundedness of the left Riemann-Liouville fractional integrals in the space $L_p(a,b)$ space (see [11, Lem. 2.1]).

Theorem 2.5 *Let $\beta \in \mathbb{R}^+$ and $p \geq 1$. The fractional integral operator $I^{\beta}_{a^+}$ is bounded in $L_p(a,b)$, i.e.,*

$$\left\|I^{\beta}_{a^+}f\right\|_{L_p} \leq K_\beta\,\|f\|_{L_p}, \qquad \text{with} \;\; K_\beta = \frac{(b-a)^\beta}{\Gamma(\beta+1)}.$$

The previous definitions of fractional integrals and derivatives can be naturally extended to $\mathbb{R}^n$ considering partial fractional integrals and derivatives (see Chapter 5 in [19]). Finally, in what follows when vectorial functions are considered all theirs properties should be understood componentwise. For example, if $f(x) = (f_1(x),\ldots,f_n(x))$ is said to be $C(\Omega)$ where $\Omega \subset \mathbb{R}^n$, then $f_i(x) \in C(\Omega)$ for all $i = 1,\ldots,n$.

3 Fractional Sturm-Liouville problem in higher dimensions - definition and basis properties

In this work we consider the following Riemann-Liouville fractional Sturm-Liouville problem in n-dimensions with mixed Dirichlet and Neuman boundary conditions in the form

$$\begin{cases} -\left({}^{RL}\nabla^{\alpha}_{b^-}\cdot\left(\mu(x)\,{}^{RL}\nabla^{\alpha}_{a^+}f\right)\right)(x) + \nu(x)\,f(x) = \lambda\,r(x)\,f(x) \\ \beta_1^{[j]}\,f(x)\Big|_{x_j=a_j} + \beta_2^{[j]}\,I^{1-\alpha_j}_{b_j^-}\left(\mu\,{}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j}f\right)(x)\Big|_{x_j=a_j} = 0, \quad j=1,\ldots,n \\ \beta_3^{[j]}\,f(x)\Big|_{x_j=b_j} + \beta_4^{[j]}\,I^{1-\alpha_j}_{b_j^-}\left(\mu\,{}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j}f\right)(x)\Big|_{x_j=b_j} = 0, \quad j=1,\ldots,n \end{cases} \tag{22}$$

where:

- $x \in \Omega = \prod_{i=1}^{n}]a_i, b_i[\subset \mathbb{R}^n$ and "$\cdot$" is the usual scalar product between two vectors in $\mathbb{R}^n$;
- ${}^{RL}\nabla^{\alpha}_{b^-}$ and ${}^{RL}\nabla^{\alpha}_{a^+}$ are, respectively, the right and left Riemann-Liouville fractional gradient operators of order $\alpha = (\alpha_1, \ldots, \alpha_n)$ given by

$$
{}^{RL}\nabla^{\alpha}_{a^+} = \sum_{i=1}^{n} e_i \, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} \qquad \text{and} \qquad {}^{RL}\nabla^{\alpha}_{b^-} = \sum_{i=1}^{n} e_i \, {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}, \tag{23}
$$

  where for $i = 1, \ldots, n$, e_i denotes the standard unit vector in the direction of x_i, the partial derivatives ${}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}$, ${}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}$, are the left and right Riemann-Liouville fractional derivatives of order $\alpha_i \in \left]\frac{1}{2}, 1\right]$ with respect to the variable $x_i \in]a_i, b_i[$;
- $I^{1-\alpha_j}_{b_j^-}$ denotes the right Riemann-Liouville fractional integral of order $1 - \alpha_j$ with respect to the variable $x_j \in]a_j, b_j[$, where $\alpha_j \in \left]\frac{1}{2}, 1\right]$ and $j = 1, \ldots, n$;
- μ, ν, and r are continuous scalar functions defined on Ω. The function r is called the weight or density function. Moreover, $\mu(x) > 0$ and $r(x) > 0$ for all $x \in \Omega$;
- the values of $\lambda \in \mathbb{C}$ for which there exists non-trivial solutions $f(x) \in I^{\alpha_j}_{a_j^+}(L_p(\Omega))$, $p > 1$ and $j = 1, \ldots, n$, are called the eigenvalues of the problem.

We remark that $L_p(\Omega) \subset L_1(\Omega)$, for $p > 1$, then since $f(x) \in I^{\alpha_j}_{a_j^+}(L_p(\Omega))$ we have that $f(x) \in I^{\alpha_j}_{a_j^+}(L_1(\Omega))$, for every $j = 1, \ldots, n$. Therefore, from Theorem 2.2 we conclude that $f(x)|_{x_j = a_j} = 0$. Let us define the fractional Sturm-Liouville operator ${}^{RL}L^{\alpha}$ associated to problem (22):

$$
{}^{RL}L^{\alpha} := -{}^{RL}\nabla^{\alpha}_{b^-} \cdot \left(\mu \, {}^{RL}\nabla^{\alpha}_{a^+}\right) + \nu. \tag{24}
$$

This operator can be seen as a fractional differential operator of second order since $\alpha_i \in \left]\frac{1}{2}, 1\right]$, for every $i = 1, \ldots, n$. Moreover, in the special case of $\alpha = (1, \ldots, 1)$ and $\mu(x) = 1$ we recover the Euclidean Laplace operator. We prove now that ${}^{RL}L^{\alpha}$ is a self-adjoint operator with respect to the inner product in $\mathbb{R}^n$ from $C^2(\Omega)$ to $C^2(\Omega)$.

Theorem 3.1 *Let $\alpha^* = \min_{1 \le i \le n}\{\alpha_i\}$, $p \ge 1$, $q \ge 1$ and $\frac{1}{q} + \frac{1}{p} \le 1 + \alpha^*$ ($p \ne 1$ and $q \ne 1$ in the case $\frac{1}{p} + \frac{1}{q} = 1 + \alpha^*$). If $h \in I^{\alpha}_{b^-}(L_p)$ and $\mu(x) \, {}^{RL}\nabla^{\alpha}_{a^+} g(x) \in I^{\alpha}_{a^+}(L_p)$, then*

$$
\int_{\Omega} h(x) \, {}^{RL}L^{\alpha} g(x) \, dx = \int_{\Omega} g(x) \, {}^{RL}L^{\alpha} f(x) \, dx.
$$

Proof: From the definition of ${}^{RL}L^{\alpha}$ (see (24)) we have

$$
\begin{aligned}
I &= \int_{\Omega} h(x) \, {}^{RL}L^{\alpha} g(x) \, dx = -\int_{\Omega} h(x) \left({}^{RL}\nabla^{\alpha}_{b^-} \cdot \left(\mu(x) \, {}^{RL}\nabla^{\alpha}_{a^+} g(x)\right)\right) dx + \int_{\Omega} \nu(x) \, h(x) \, g(x) \, dx \\
&= -\sum_{i=1}^{n} \int_{\Omega} h(x) \left({}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \left(\mu(x) \, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} g(x)\right)\right) dx + \int_{\Omega} \nu(x) \, h(x) \, g(x) \, dx.
\end{aligned}
$$

Now, using the rule of integration by parts in Theorem 2.4 and taking into account Remark 2.1, we get

$$
\begin{aligned}
I &= -\sum_{i=1}^{n} \int_{\Omega} \left(\mu(x) \, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} g(x)\right) {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h(x) \, dx + \int_{\Omega} \nu(x) \, h(x) \, g(x) \, dx \\
&= -\sum_{i=1}^{n} \int_{\Omega} {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} g(x) \left(\mu(x) \, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h(x)\right) dx + \int_{\Omega} \nu(x) \, h(x) \, g(x) \, dx \\
&= -\sum_{i=1}^{n} \int_{\Omega} g(x) \left({}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \left(\mu(x) \, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h(x)\right)\right) dx + \int_{\Omega} \nu(x) \, h(x) \, g(x) \, dx \\
&= -\int_{\Omega} g(x) \left({}^{RL}\nabla^{\alpha}_{b^-} \cdot \left(\mu \, {}^{RL}\nabla^{\alpha}_{a^+} h(x)\right)\right) dx + \int_{\Omega} \nu(x) \, h(x) \, g(x) \, dx \\
&= \int_{\Omega} g(x) \, {}^{RL}L^{\alpha} h(x) \, dx.
\end{aligned}
$$

■

For the fractional Sturm-Liouville problem (22) we prove now some classical results.

Theorem 3.2 *All the eigenvalues of the fractional Sturm-Liouville problem (22) are real.*

Proof: Suppose that λ is a complex eigenvalue of the fractional problem (22), with the corresponding eigenfunction f, possibly complex-valued. Then the eigenvalue and its eigenfunction verify

$$ {}^{RL}L^{\alpha} f(x) = \lambda\, r(x)\, f(x). \tag{25} $$

Moreover, taking the complex conjugate we get

$$ {}^{RL}L^{\alpha} \overline{f(x)} = \overline{\lambda}\, r(x)\, \overline{f(x)}. \tag{26} $$

From Theorem 3.1 we obtain

$$ \begin{aligned} \int_{\Omega} f(x)\, {}^{RL}L^{\alpha} \overline{f}(x)\, dx = \int_{\Omega} \overline{f}(x)\, {}^{RL}L^{\alpha} f(x)\, dx \ &\Leftrightarrow \int_{\Omega} f(x)\, \overline{\lambda}\, r(x)\, \overline{f}(x)\, dx - \int_{\Omega} \overline{f}(x)\, \lambda\, r(x)\, f(x)\, dx \ = 0 \\ &\Leftrightarrow (\overline{\lambda} - \lambda) \int_{\Omega} r(x)\, f(x)\, \overline{f}(x)\, dx \ = 0 \\ &\Leftrightarrow (\overline{\lambda} - \lambda) \int_{\Omega} r(x)\, \|f(x)\|^2\, dx \ = 0. \end{aligned} $$

Since the integral is always positive we get that $\lambda = \overline{\lambda}$, i.e., λ is a real number.

■

Theorem 3.3 *If f and g are two eigenfunctions of the fractional Sturm-Liouville problem (22) corresponding to the eigenvalues λ_1 and λ_2, respectively, with $\lambda_1 \neq \lambda_2$, then the eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weight function r, i.e.,*

$$ \int_{\Omega} r(x)\, f(x)\, g(x)\, dx = 0. $$

Proof: Let λ_1 and λ_2 be eigenvalues with corresponding eigenfunctions f and g, and such that $\lambda_1 \neq \lambda_2$. These eigenvalues and eigenfunctions verify

$$ {}^{RL}L^{\alpha} f(x) = \lambda_1\, r(x)\, f(x) \qquad \text{and} \qquad {}^{RL}L^{\alpha} g(x) = \lambda_2\, r(x)\, g(x). \tag{27} $$

Multiplying the first and second equalities in (27) by g and f, respectively, leads to

$$ g(x)\, {}^{RL}L^{\alpha} f(x) = \lambda_1\, r(x)\, f(x)\, g(x) \qquad \text{and} \qquad f(x)\, {}^{RL}L^{\alpha} g(x) = \lambda_2\, r(x)\, g(x)\, f(x). \tag{28} $$

Subtracting the equalities in (28) and integrating over Ω, we get

$$ \int_{\Omega} \left(g(x)\, {}^{RL}L^{\alpha} f(x) - f(x)\, {}^{RL}L^{\alpha} g(x) \right) dx \ = (\lambda_1 - \lambda_2) \int_{\Omega} r(x)\, f(x)\, g(x)\, dx, \tag{29} $$

where due to Theorem 3.1 the left hand side of (29) is equal to zero. Since $\lambda_1 \neq \lambda_2$, the orthogonality of f and g with respect to the weight function r is verified.

■

Now, we prove under which conditions the eigenfunctions of the fractional Sturm-Liouville problem (22) are simple, i.e., under which conditions we have that for each eigenvalue corresponds only one linearly independent eigenfunction, up to a constant. Let λ be an eigenvalue of the fractional Sturm-Liouville problem (22) and f the eigenfunction associated to it. For the equation in (22) we have

$$ \begin{aligned} -\left({}^{RL}\nabla^{\alpha}_{b^-} \cdot \left(\mu\, {}^{RL}\nabla^{\alpha}_{a^+} f \right) \right)(x) + \nu(x)\, f(x) \ = \lambda\, r(x)\, f(x) \Leftrightarrow \left({}^{RL}\nabla^{\alpha}_{b^-} \cdot \left(\mu\, {}^{RL}\nabla^{\alpha}_{a^+} f \right) \right)(x) \ &= \underbrace{\nu(x)\, f(x) - \lambda\, r(x)\, f(x)}_{F_{\lambda}(f)} \\ \Leftrightarrow \sum_{i=1}^{n} {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \left(\mu(x)\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} f(x) \right) \ &= F_{\lambda}(f). \end{aligned} \tag{30} $$

In order to incorporate (30) and the boundary conditions defined in (22) in a single equation, we need to apply fractional integral operators to (30). Applying first $I^{\alpha_j}_{b_j^-}$ to both sides of (30), taking into account (13) and making straightforward calculations, we get that (30) is equivalent to

$$ {}^{RL}_{a_j^+}\partial^{\alpha_i}_{x_i} f(x) = \frac{(b_j - x_j)^{\alpha_j - 1}}{\mu(x)\,\Gamma(\alpha_j)} \; \xi_2^{[j]}\Big|_{x_j = b_j} - \sum_{\substack{i=1\\ i\neq j}}^{n} \frac{1}{\mu(x)}\, I^{\alpha_j}_{b_j^-}\, {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \left(\mu(x)\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} f(x)\right) + \frac{1}{\mu(x)}\, I^{\alpha_j}_{b_j^-} F_\lambda(f), \tag{31} $$

where the constant $\xi_2^{[j]}\Big|_{x_j = b_j}$ with respect to the variable x_j is given by

$$ \xi_2^{[j]}\Big|_{x_j = b_j} = I^{1-\alpha_j}_{b_j^-} \left(\mu(x)\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} f(x)\right)\Big|_{x_j = b_j}. $$

Applying now $I^{\alpha_j}_{a_j^+}$ to both sides of (31), taking into account (12) and making straightforward calculations, we get that (31) is equivalent to

$$ \begin{aligned} f(x) = {}& \frac{(x_j - a_j)^{\alpha_j - 1}}{\Gamma(\alpha_j)} \; \xi_1^{[j]}\Big|_{x_j = a_j} + \xi_2^{[j]}\Big|_{x_j = b_j} \; I^{\alpha_j}_{a_j^+} \left(\frac{(b_j - x_j)^{\alpha_j - 1}}{\mu(x)\,\Gamma(\alpha_j)}\right) \\ & + \sum_{\substack{i=1\\ i\neq j}}^{n} I^{\alpha_j}_{a_j^+} \left(\frac{1}{\mu(x)}\, I^{\alpha_j}_{b_j^-}\, {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \left(\mu(x)\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} f(x)\right)\right) + I^{\alpha_j}_{a_j^+} \left(\frac{1}{\mu(x)}\, I^{\alpha_j}_{b_j^-} F_\lambda(f)\right), \end{aligned} \tag{32} $$

where the constant $\xi_1^{[j]}\Big|_{x_j = a_j}$ with respect to the variable x_j is given by

$$ \xi_1^{[j]}\Big|_{x_j = a_j} = I^{1-\alpha_j}_{a_j^+} f(x)\Big|_{x_j = a_j}. $$

Finally, applying the operator $I^{1-\alpha_j}_{b_j^-} \left(\mu(x)\, {}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j}\right)$ to both sides of (32) and taking into account the relations (17) and (18), we obtain that (32) is equivalent to

$$ \xi_2^{[j]}\Big|_{x_j = b_j} = I^{1-\alpha_j}_{b_j^-} \left(\mu(x)\, {}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j} f(x)\right) - \sum_{\substack{i=1\\ i\neq j}}^{n} I^{1}_{b_j^-}\, {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \left(\mu(x)\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} f(x)\right) - I^{1}_{b_j^-} F_\lambda(f). \tag{33} $$

Considering now the boundary condition in (22) and the fact that $f(x)|_{x_j = a_j} = 0$, we have

$$ I^{1-\alpha_j}_{b_j^-} \left(\mu(x)\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} f(x)\right)\Big|_{x_j = a_j} = 0. \tag{34} $$

Combining (34) and (33) with $x_j = a_j$, we get

$$ \xi_2^{[j]}\Big|_{x_j = b_j} = -I^{1}_{b_j^-} F_\lambda(f) - \sum_{\substack{i=1\\ i\neq j}}^{n} I^{1}_{b_j^-}\, {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \left(\mu(x)\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} f(x)\right). \tag{35} $$

Taking into account (35) and (32) with $x_j = b_j$, the second boundary condition in (22) can be written in terms of $\xi_1^{[j]}\Big|_{x_j = a_j}$ as

$$ \begin{aligned} & \beta_3^{[j]} f(x)\Big|_{x_j = b_j} + \beta_4^{[j]} I^{1-\alpha_j}_{b_j^-} \left(\mu(x)\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} f\right)(x)\Big|_{x_j = b_j} = 0 \\ & \Leftrightarrow \xi_1^{[j]}\Big|_{x_j = a_j} = \frac{\Gamma(\alpha_j)}{(b_j - a_j)^{\alpha_j - 1}} \left[\left(I^{1}_{b_j^-} F_\lambda(f) + \sum_{\substack{i=1\\ i\neq j}}^{n} I^{1}_{b_j^-}\, {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \left(\mu(x)\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} f(x)\right)\right) I^{\alpha_j}_{a_j^+} \left(\frac{(b_j - x_j)^{\alpha_j - 1}}{\mu(x)\,\Gamma(\alpha_j)}\right)\Big|_{x_j = b_j} \right. \\ & \left. \qquad - \sum_{\substack{i=1\\ i\neq j}}^{n} I^{\alpha_j}_{a_j^+} \left(\frac{1}{\mu(x)}\, I^{\alpha_j}_{b_j^-}\, {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \left(\mu(x)\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} f(x)\right)\right)\Big|_{x_j = b_j} - I^{\alpha_j}_{a_j^+} \left(\frac{1}{\mu(x)}\, I^{\alpha_j}_{b_j^-} F_\lambda(f)\right)\Big|_{x_j = b_j} \right]. \end{aligned} \tag{36} $$

Let us observe that when $n=1$ expressions (35) and (36) become, respectively,

$$\xi_2 = -I^1_{b^-}\,F_\lambda(f) \tag{37}$$

$$\xi_1 = \frac{1}{(b-a)^{\alpha-1}}\,I^\alpha_{a^+}\,\frac{(b-x)^{\alpha-1}}{\mu(x)}\Big|_{x=b}\,I^1_{b^-}\,F_\lambda(f) - \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\,I^\alpha_{a^+}\left(\frac{1}{\mu(x)}\,I^\alpha_{b^-}\,F_\lambda(f)\right)\Big|_{x=b}. \tag{38}$$

Expression (37) corresponds to expression (59) in [18], while (38) corresponds to expression (58) where some misprints are corrected. Summing up each member of (32) from $j=1,\ldots,n$ we obtain

$$f(x) = \sum_{j=1}^{n}\Bigg\{\frac{(x_j-a_j)^{\alpha_j-1}}{\Gamma(\alpha_j)}\;\xi_1^{[j]}\Big|_{x_j=a_j} + \xi_2^{[j]}\Big|_{x_j=b_j}\;I^{\alpha_j}_{a_j^+}\left(\frac{(b_j-x_j)^{\alpha_j-1}}{\mu(x)\,\Gamma(\alpha_j)}\right)$$
$$+\sum_{\substack{i=1\\ i\neq j}}^{n} I^{\alpha_j}_{a_j^+}\left(\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\,{}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\left(\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}\,f(x)\right)\right) + I^{\alpha_j}_{a_j^+}\left(\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\,F_\lambda(f)\right)\Bigg\}, \tag{39}$$

where $\xi_1^{[j]}\Big|_{x_j=a_j}$ and $\xi_2^{[j]}\Big|_{x_j=b_j}$ are given by (36) and (35), respectively. We can consider (39) as a fixed point condition on the function space $C(\Omega)$ of the form $f=Tf$, where Tf is the right-hand side of (39). Now we calculate the norm of the difference between Tf and Tg for $f,g\in C(\Omega)$.

$$\|Tf-Tg\| \le \frac{1}{n}\sum_{j=1}^{n}\Big(\|T_1f-T_1g\| + \|T_2f-T_2g\| + \|T_3f-T_3g\| + \|T_4f-T_4g\|\Big), \tag{40}$$

where each term inside the sum in (40) is associated with a term in (39). Let us now estimate the norms of the four terms inside the sum in (40). For the first term, taking into account that

$$\|F_\lambda(f)-F_\lambda(g)\| \le \frac{1}{n^2}\,\|\nu-\lambda\,r\|\,\|fg\|, \tag{41}$$

and the expression for $\xi_1^{[j]}\Big|_{x_j=a_j}$ given in (36) we have after straightforward calculations

$$\|T_1f-T_1g\| \le \|\phi_1\|\,\Bigg[\Bigg(\left\|I^1_{b_j^-}\,(F_\lambda(f)-F_\lambda(g))\right\| + \left\|\sum_{\substack{i=1\\ i\neq j}}^{n} I^1_{b_j^-}\,{}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\left(\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}\,(f-g)\right)\right\|\Bigg)\,\left\|\phi_2|_{x_j=b_j}\right\|$$
$$+\left\|\sum_{\substack{i=1\\ i\neq j}}^{n} I^{\alpha_j}_{a_j^+}\left(\frac{1}{\mu}\,I^{\alpha_j}_{b_j^-}\,{}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\left(\mu\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}\,(f-g)\right)\right)\Bigg|_{x_j=a_j}\right\| + \left\|I^{\alpha_j}_{a_j^+}\left(\frac{1}{\mu}\,I^{\alpha_j}_{b_j^-}\,(F_\lambda(f)-F_\lambda(g))\right)\right\|\Bigg]$$
$$\le \|\phi_1\|\left[\left(\frac{\|\nu-\lambda\,r\|\,\|\phi_3\|}{n^2} + \|\phi_4\|\,M_\mu\right)\left\|\phi_2|_{x_j=b_j}\right\| + \frac{M_\mu}{m_\mu}\,\left\|\phi_5|_{x_j=a_j}\right\| + \frac{\|\nu-\lambda\,r\|\,\|\phi_6\|}{n^2\,m_\mu}\right]\|f-g\| \tag{42}$$

where

$$\phi_1(x_j) = \left(\frac{x_j-a_j}{b_j-a_j}\right)^{\alpha_j-1}, \qquad \phi_2(x_j) = I^{\alpha_j}_{a_j^+}\,\frac{(b_j-x_j)^{\alpha_j-1}}{\mu(x)\,\Gamma(\alpha_j)}, \qquad \phi_3(x) = I^1_{b_j^-}1,$$
$$\phi_4(x) = \sum_{\substack{i=1\\ i\neq j}}^{n} I^1_{b_j^-}\,{}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}1, \quad \phi_5(x) = I^{\alpha_j}_{a_j^+}\,I^{\alpha_j}_{b_j^-}\sum_{\substack{i=1\\ i\neq j}}^{n}{}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}1, \quad \phi_6(x) = I^{\alpha_i}_{a_i^+}\,I^{\alpha_i}_{b_i^-}1, \tag{43}$$
$$M_\mu = \max_{x\in\Omega}|\mu(x)|, \qquad m_\mu = \min_{x\in\Omega}|\mu(x)|.$$

Concerning the 2nd term on the right-hand side of (40), taking into account (41) and the expression for $\left.\xi_2^{[j]}\right|_{x_j=b_j}$ given in (35), we have that

$$\|T_2 f - T_2 g\| \leq \|\phi_2\| \left[\left\| I^1_{b_j^-} \left(F_\lambda(f) - F_\lambda(g)\right) \right\| + \left\| \sum_{\substack{i=1\\ i\neq j}}^{n} I^1_{b_j^-} \, {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \left(\mu \, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} (f-g) \right) \right\| \right]$$

$$\leq \|\phi_2\| \left[\frac{\|\nu - \lambda\, r\| \; \|\phi_3\|}{n^2} + \|\phi_4\| \; M_\mu \right] \|f-g\| \,. \tag{44}$$

We pass now to the 3rd term on the right-hand side of (40). We have that

$$\|T_3 f - T_3 g\| \leq \left\| \sum_{\substack{i=1\\ i\neq j}}^{n} I^{\alpha_j}_{a_j^+} \left(\frac{1}{\mu} \, I^{\alpha_j}_{b_j^-} \, {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \left(\mu \, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} (f-g) \right) \right) \right\|$$

$$\leq \frac{\|\phi_5\| \; M_\mu}{m_\mu} \; \|f-g\| \,. \tag{45}$$

Finally we study the last term in the right-hand side of (40). Taking into account (41) we have

$$\|T_4 f - T_4 g\| \; = \left\| I^{\alpha_i}_{a_i^+} \left(\frac{1}{\mu(x)} \, I^{\alpha_j}_{b_j^-} \left(F_\lambda(f) - F_\lambda(g)\right) \right) \right\| \; \leq \frac{\|\nu - \lambda\, r\| \; \|\phi_6\|}{n^2 \, m_\mu} \; \|f-g\| \,. \tag{46}$$

From (42), (44), (45), and (46), expression (40) becomes

$$\|Tf - Tg\| \; \leq \phi_8 \; \|f-g\| \,,$$

where

$$\phi_8 = \frac{1}{n} \sum_{j=1}^{n} \left[\|\phi_1\| \left[\left(\frac{\|\nu - \lambda\, r\| \; \|\phi_3\|}{n^2} + \|\phi_4\| \; M_\mu \right) \left\| \phi_2|_{x_j=b_j} \right\| + \frac{M_\mu}{m_\mu} \; \left\| \phi_5|_{x_j=a_j} \right\| + \frac{\|\nu - \lambda\, r\| \; \|\phi_2\|}{n^2 \, m_\mu} \right] \right.$$

$$\left. + \|\phi_2\| \left[\frac{\|\nu - \lambda\, r\| \; \|\phi_3\|}{n^2} + \|\phi_4\| \; M_\mu \right] + \frac{\|\phi_5\| \; M_\mu}{m_\mu} + \frac{\|\nu - \lambda\, r\| \; \|\phi_6\|}{n^2 \, m_\mu} \right],$$

and ϕ_i, $i = 1, \ldots, 6$, M_μ and m_μ are given in (43). Under the assumption that $\phi_8 < 1$ we have that T is a contraction on the space $C(\Omega)$ for a chosen norm. Therefore, the unique fixed point f exists, up to constant, and solve (22).

4 Main result - increasing sequence of eigenvalues

4.1 Auxiliar results

In this section, we present and prove some auxiliar results that are necessary to the proof of the main result of the paper (see Section 4.2).

Lemma 4.1 *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \left]\frac{1}{2}, 1\right]$, and a vectorial function $\gamma(x) = (\gamma_1(x), \ldots, \gamma_n(x)) \in C(\Omega)$ such that*

$${}^{RL}_{a_i^+}\partial^{1+\alpha_i}_{x_i} \sum_{\substack{j=1\\ j\neq i}}^{n} I^{1+\alpha_i}_{a_i^+} \gamma_j(x) = 0, \quad i = 1, \ldots, n. \tag{47}$$

If

$$\int_\Omega \gamma(x) \, \cdot \, {}^{RL}\nabla^{1+\alpha}_{a^+} h(x) \, dx = 0 \tag{48}$$

for $h \in C^1(\Omega)$ such that ${}^{RL}\nabla^{1+\alpha}_{a^+} h \in C(\Omega)$ and h satisfies the boundary conditions

$$h(x)\Big|_{x_j=a_j} = 0, \qquad I^{1-\alpha_i}_{a_i^+} h(x)\Big|_{x_i=b_i} = 0,$$
$$ {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h(x)\Big|_{x_i=a_i} = 0, \qquad {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h(x)\Big|_{x_i=b_i} = 0, \qquad i,j = 1,\dots,n, \quad i \neq j, \tag{49}$$

then for each $i = 1,\dots,n$ we have $\gamma_i(x) = c_0^{[i]} + c_1^{[i]} x_i$, where $c_0^{[i]}$ and $c_1^{[i]}$ are real constants.

Proof: Let us define a function h as follows

$$h(x) = I^{1+\alpha}_{a^+} \cdot \left(\sum_{j=1}^{n} e_j \left(\gamma_j(x) - c_0^{[j]} - c_1^{[j]} x_j\right)\right) = \sum_{j=1}^{n} I^{1+\alpha_j}_{a_j^+} \left(\gamma_j(x) - c_0^{[j]} - c_1^{[j]} x_j\right),$$

with constants $c_0^{[j]}$ and $c_1^{[j]}$ fixed by the $2n$ conditions (for each $i = 1,\dots,n$)

$$I^{2}_{a_i^+} \left(\gamma_i(x) - c_0^{[i]} - c_1^{[i]} x_i\right)\Big|_{x_i=b_i} = 0 \tag{50}$$

$$I^{1}_{a_i^+} \left(\gamma_i(x) - c_0^{[i]} - c_1^{[i]} x_i\right)\Big|_{x_i=b_i} = 0 \tag{51}$$

$$I^{1-\alpha_i}_{a_i^+} I^{1+\alpha_j}_{a_j^+} \left(\gamma_j(x) - c_0^{[j]} - c_1^{[j]} x_j\right)\Big|_{x_i=b_i} = 0, \qquad j = 1,\dots,n \quad \text{and} \quad j \neq i, \tag{52}$$

$$ {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} I^{1+\alpha_j}_{a_j^+} \left(\gamma_j(x) - c_0^{[j]} - c_1^{[j]} x_j\right)\Big|_{x_i=b_i} = 0, \qquad j = 1,\dots,n \quad \text{and} \quad j \neq i. \tag{53}$$

It is direct to see that function h is continuous and fulfils the boundary conditions (49). Using (4), (6), and the fact that the integration over a point is equal to zero, we have for $j, i = 1,\dots,n$ that

$$h(x)\Big|_{x_j=a_j} = \sum_{j=1}^{n} I^{1+\alpha_j}_{a_j^+} \left(\gamma_j(x) - c_0^{[j]} - c_1^{[j]} x_i\right)\Big|_{x_j=a_j} = 0,$$

$$ {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h(x)\Big|_{x_i=a_i} = \sum_{j=1}^{n} {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} I^{1+\alpha_j}_{a_j^+} \left(\gamma_j(x) - c_0^{[j]} - c_1^{[j]} x_j\right)\Big|_{x_i=a_i} = 0.$$

Taking into account (52), (50), (53), and (51) we have that

$$I^{1-\alpha_i}_{a_i^+} h(x)\Big|_{x_i=b_i} = \sum_{\substack{j=1\\ j\neq i}}^{n} I^{1-\alpha_i}_{a_i^+} I^{1+\alpha_j}_{a_j^+} \left(\gamma_j(x) - c_0^{[j]} - c_1^{[j]} x_j\right)\Big|_{x_i=b_i} + I^{2}_{a_i^+} \left(\gamma_i(x) - c_0^{[i]} - c_1^{[i]} x_i\right)\Big|_{x_i=b_i} = 0,$$

$$ {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h(x)\Big|_{x_i=b_i} = \sum_{\substack{j=1\\ j\neq i}}^{n} {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} I^{1+\alpha_j}_{a_j^+} \left(\gamma_j(x) - c_0^{[j]} - c_1^{[j]} x_j\right)\Big|_{x_i=b_i} + I^{1}_{a_i^+} \left(\gamma_i(x) - c_0^{[i]} - c_1^{[i]} x_i\right)\Big|_{x_i=b_i} = 0.$$

In addition, for $i = 1,\dots,n$

$$\partial_{x_i} h(x) = \sum_{\substack{j=1\\ j\neq i}}^{n} \partial_{x_i} I^{1+\alpha_j}_{a_j^+} \left(\gamma_j(x) - c_0^{[j]} - c_1^{[j]} x_j\right) + I^{\alpha_i}_{a_i^+} \left(\gamma_i(x) - c_0^{[i]} - c_1^{[i]} x_i\right)$$

$$= \sum_{\substack{j=1\\ j\neq i}}^{n} I^{1+\alpha_j}_{a_j^+} \partial_{x_i} \gamma_j(x) + I^{\alpha_i}_{a_i^+} \left(\gamma_i(x) - c_0^{[i]} - c_1^{[i]} x_i\right)$$

is a function in $C(\Omega)$. Moreover, taking into account Remark 2.1 and the assumption (47), we have

$$
\begin{aligned}
{}^{RL}\nabla^{1+\alpha}_{a^+} h(x) &= \sum_{i=1}^{n} e_i \, {}^{RL}_{a_i^+}\partial^{1+\alpha_i}_{x_i} \left[\sum_{j=1}^{n} I^{1+\alpha_j}_{a_j^+} \left(\gamma_j(x) - c_0^{[j]} - c_1^{[j]} x_j \right) \right] \\
&= \sum_{i=1}^{n} e_i \sum_{j=1 \neq i}^{n} {}^{RL}_{a_i^+}\partial^{1+\alpha_i}_{x_i} \, I^{1+\alpha_j}_{a_j^+} \left(\gamma_j(x) - c_0^{[j]} - c_1^{[j]} x_j \right) + \sum_{i=1}^{n} e_i \left(\gamma_i(x) - c_0^{[i]} - c_1^{[i]} x_i \right) \\
&= \sum_{i=1}^{n} e_i \sum_{j=1 \neq i}^{n} {}^{C}_{a_i^+}\partial^{1+\alpha_i}_{x_i} \, I^{1+\alpha_j}_{a_j^+} \left(\gamma_j(x) - c_0^{[j]} - c_1^{[j]} x_j \right) + \sum_{i=1}^{n} e_i \left(\gamma_i(x) - c_0^{[i]} - c_1^{[i]} x_i \right) \\
&= \sum_{i=1}^{n} e_i \, {}^{RL}_{a_i^+}\partial^{1+\alpha_i}_{x_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} I^{1+\alpha_i}_{a_i^+} \gamma_j(x) + \sum_{i=1}^{n} e_i \left(\gamma_i(x) - c_0^{[i]} - c_1^{[i]} x_i \right) \\
&= \sum_{i=1}^{n} e_i \left(\gamma_i(x) - c_0^{[i]} - c_1^{[i]} x_i \right)
\end{aligned} \tag{54}
$$

is also a function in $C(\Omega)$. From (48) and (49) we get for $\widehat{\Omega} = \prod_{\substack{j=1 \\ j\neq i}}^{n}]a_i, b_i[$

$$
\begin{aligned}
&\int_\Omega \left(\sum_{i=1}^{n} e_i \left(\gamma_i(x) - c_0^{[i]} - c_1^{[i]} x_i \right) \right) \cdot {}^{RL}\nabla^{1+\alpha}_{a^+} h(x) \, dx \\
&= \int_\Omega \gamma(x) \cdot {}^{RL}\nabla^{1+\alpha}_{a^+} h(x) \, dx - \int_\Omega \left(\sum_{i=1}^{n} e_i \left(c_0^{[i]} + c_1^{[i]} x_i \right) \right) \cdot {}^{RL}\nabla^{1+\alpha}_{a^+} h(x) \, dx \\
&= 0 - \sum_{i=1}^{n} \int_{\widehat{\Omega}} \int_{a_j}^{b_j} \left(c_0^{[i]} + c_1^{[i]} x_i \right) \partial_{x_i} \, {}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j} h(x) \, dx_i \, d\widehat{x} \\
&= - \sum_{i=1}^{n} \int_{\widehat{\Omega}} \left[\left(c_0^{[i]} + c_1^{[i]} x_i \right) {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h(x) \Big|_{x_i = a_i}^{x_i = b_i} - c_1^{[i]} \int_{a_i}^{b_i} {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h(x) \, dx_i \right] d\widehat{x} \\
&= 0.
\end{aligned} \tag{55}
$$

Finally, from (55) and (54) we obtain

$$
\begin{aligned}
0 &= \int_\Omega \left(\sum_{i=1}^{n} e_i \left(\gamma_i(x) - c_0^{[i]} - c_1^{[i]} x_i \right) \right) \cdot {}^{RL}\nabla^{1+\alpha}_{a^+} h(x) \, dx \\
&= \int_\Omega \sum_{i=1}^{n} \left(\gamma_i(x) - c_0^{[i]} - c_1^{[i]} x_i \right)^2 dx
\end{aligned}
$$

which implies that for each $i = 1, \ldots, n$ we have

$$\gamma_i(x) = c_0^{[i]} + c_1^{[i]} x_i.$$

■

Lemma 4.2 *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \left]\frac{1}{2}, 1\right]$. Let also $\gamma(x) = (\gamma_1(x), \ldots, \gamma_n(x)) \in C(\Omega)$ with ${}^{RL}\nabla^{1-\alpha}_{a^+} \cdot \gamma \in L_2(\Omega)$. If*

$$\int_\Omega \gamma(x) \cdot {}^{RL}\nabla^{1+\alpha}_{a^+} h(x) \, dx = 0$$

for $h \in C^1(\Omega)$ such that $\partial^2_{x_i} h \in L(\Omega)$, $i = 1, \ldots, n$, and ${}^{RL}\nabla^{1+\alpha}_{a^+} h \in C(\Omega)$ fulfilling the boundary conditions (49). Then for each $i = 1, \ldots, n$ we have that $\gamma_i(x) = c_0^{[i]} + c_1^{[i]} x_i$, where $c_0^{[i]}$ and $c_1^{[i]}$ are real constants.

Proof: We define a function h as in the proof of Lemma 4.1

$$h(x) = \sum_{j=1}^{n} I_{a_j^+}^{1+\alpha_j} \left(\gamma_j(x) - c_0^{[j]} - c_1^{[j]} x_j \right),$$

with constants $c_0^{[j]}$ and $c_1^{[j]}$ fixed by the conditions (50), (51), (52), and (53). The proof of this lemma is analogous to the proof of Lemma 4.1. Additionally, for the second order derivative with respect to x_i we have for $i = 1, \ldots, n$

$$\begin{aligned}
\partial_{x_i}^2 h(x) &= \sum_{\substack{j=1 \\ j\neq i}}^{n} \partial_{x_i}^2 \, I_{a_j^+}^{1+\alpha_j} \left(\gamma_j(x) - c_0^{[j]} - c_1^{[j]} x_j \right) + {}^{RL}_{a_i^+}\partial_{x_i}^{1-\alpha_i} \left(\gamma_i(x) - c_0^{[i]} - c_1^{[i]} x_i \right) \\
&= \sum_{\substack{j=1 \\ j\neq i}}^{n} I_{a_j^+}^{1+\alpha_j} \, \partial_{x_i}^2 \gamma_j(x) + {}^{RL}_{a_i^+}\partial_{x_i}^{1-\alpha_i} \left(\gamma_i(x) - c_0^{[i]} - c_1^{[i]} x_i \right),
\end{aligned}$$

which is a function in $L_2(\Omega)$. Moreover, function h satisfies the conditions of Lemma 4.2. The remaining part of the proof is analogous to that in Lemma 4.1.

■

Lemma 4.3 *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \in \left]\frac{1}{2}, 1\right]$, and $\gamma_1 \in C(\Omega)$ a scalar function. Let $\gamma_2(x) = \left(\gamma_i^{[2]}, \ldots, \gamma_n^{[2]}\right)$, where for each $i = 1, \ldots, n$ and $p > 1$, $\gamma_i^{[2]} \in I_{b_i^-}^2(L_p)$, i.e., $\gamma_i^{[2]} = I_{b_i^-}^2 \varphi_i$ with $\varphi_i \in L_p(\Omega)$. If*

$$\int_\Omega \left(\gamma_2(x) \cdot {}^{RL}\nabla_{a^+}^{\alpha} h(x) + \gamma_1(x) \, h(x) \right) dx = 0$$

for each $h \in C^1(\Omega)$ such that $\partial_{x_i}^2 h \in L_2(\Omega)$, $i = 1, \ldots, n$, and ${}^{RL}\nabla_{a^+}^{1+\alpha} h \in C(\Omega)$ fulfilling the boundary conditions in Lemma 4.1, then

$${}^{RL}\nabla_{b^-}^{\alpha} \cdot \varphi(x) + \gamma_1(x) = 0.$$

Proof: In the one hand, since $\gamma_i^{[2]} \in I_{b_i^-}^2(L_p)$, with $i = 1, \ldots, n$ and $p > 1$, we have

$$\begin{aligned}
\int_\Omega \gamma_2(x) \cdot {}^{RL}\nabla_{a^+}^{\alpha} h(x) \, dx &= \int_\Omega \sum_{i=1}^{n} \gamma_i^{[2]}(x) \, {}^{RL}_{a_i^+}\partial_{x_i}^{\alpha_i} h(x) \, dx \\
&= \sum_{i=1}^{n} \int_{\widehat{\Omega}} \int_{a_i}^{b_i} I_{b_i^-}^2 \varphi_i(x) \, {}^{RL}_{a_i^+}\partial_{x_i}^{\alpha_i} h(x) \, dx_i \, d\widehat{x}.
\end{aligned}$$

Applying the formula of integration by parts for fractional integrals (19) we get

$$\begin{aligned}
\int_\Omega \gamma_2(x) \cdot {}^{RL}\nabla_{a^+}^{\alpha} h(x) \, dx &= \sum_{i=1}^{n} \int_{\widehat{\Omega}} \int_{a_i}^{b_i} I_{b_i^-}^1 \varphi_i(x) \, I_{a_i^+}^1 \, {}^{RL}_{a_i^+}\partial_{x_i}^{\alpha_i} h(x) \, dx_i \, d\widehat{x} \\
&= \sum_{i=1}^{n} \int_{\widehat{\Omega}} \int_{a_i}^{b_i} I_{b_i^-}^1 \varphi_i(x) \, {}^{RL}_{a_i^+}\partial_{x_i}^{1+\alpha_i} h(x) \, dx_i \, d\widehat{x} \\
&= \int_\Omega I_{b^-}^1 \varphi(x) \cdot {}^{RL}\nabla_{a^+}^{1+\alpha} h(x) \, dx,
\end{aligned} \tag{56}$$

where

$$I_{b^-}^1 \varphi(x) = \sum_{i=1}^{n} e_i \, I_{b_i^-}^1 \varphi_i(x).$$

On the other hand, we have that

$$\int_\Omega \gamma_1(x) \, h(x) \, dx = \frac{1}{n} \int_\Omega \gamma_1(x) \sum_{i=1}^{n} I_{a_i^+}^{\alpha_i} \, {}^{RL}_{a_i^+}\partial_{x_i}^{\alpha_i} \, h(x) \, dx.$$

Applying the formula of integration by parts for fractional integrals (19) we get

$$\int_\Omega \gamma_1(x)\, h(x)\, dx = \frac{1}{n} \sum_{i=1}^{n} \int_{\widehat{\Omega}} \int_{a_i}^{b_i} I^{\alpha_i}_{b_i^-}\gamma_1(x)\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h(x)\, dx_i\, d\widehat{x}.$$

Taking into account the boundary conditions in Lemma 4.1 and making use of the classical formula for integration by parts, we arrive to

$$\int_\Omega \gamma_1(x)\, h(x)\, dx = -\frac{1}{n} \sum_{i=1}^{n} \int_{\widehat{\Omega}} \int_{a_i}^{b_i} I^{1}_{a_i^+}\, I^{\alpha_i}_{b_i^-}\gamma_1(x)\, \partial_{x_i}\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h(x)\, dx_i\, d\widehat{x}.$$

Since $I^1_{a_i^+} = I^1_{b_i^-}$ we have the following simplification of the previous expression

$$\begin{aligned}\int_\Omega \gamma_1(x)\, h(x)\, dx &= -\frac{1}{n} \sum_{i=1}^{n} \int_{\widehat{\Omega}} \int_{a_i}^{b_i} I^{1+\alpha_i}_{b_i^-}\gamma_1(x)\, {}^{RL}_{a_i^+}\partial^{1+\alpha_i}_{x_i} h(x)\, dx_i\, d\widehat{x} \\ &= -\frac{1}{n} \int_\Omega I^{1+\alpha}_{b^-}\gamma_1(x) \cdot {}^{RL}\nabla^{1+\alpha}_{a^+} h(x)\, dx. \qquad (57)\end{aligned}$$

Since γ_1 and φ are continuous in Ω and

$$I^1_{b^-}\varphi(x) - \frac{1}{n}\, I^{1+\alpha}_{b^-}\gamma_1(x) \in C(\Omega) \subset L_2(\Omega),$$

we conclude from Lemma 4.2 that there exist reals constants $c_0^{[i]}$ and $c_1^{[i]}$, with $i = 1, \ldots, n$, such that

$$I^1_{b^-}\varphi(x) - \frac{1}{n}\, I^{1+\alpha}_{b^-}\gamma_1(x) = \sum_{i=1}^{n} e_i \left(c_0^{[i]} + c_1^{[i]} x_i \right).$$

Applying the operator ${}^{RL}\nabla^{\alpha}_{b^-}$ to the previous equality, and taking into account Remark 2.1 and the definition of the right Caputo fractional derivative (9) leads to our result

$$\begin{aligned} & {}^{RL}\nabla^{\alpha}_{b^-} \cdot \left(I^1_{b^-}\varphi(x) - \frac{1}{n}\, I^{1+\alpha}_{b^-}\gamma_1(x) \right) = {}^{RL}\nabla^{\alpha}_{b^-} \cdot \left(\sum_{i=1}^{n} e_i \left(c_0^{[i]} + c_1^{[i]} x_i \right) \right) \\ \Leftrightarrow\ & \sum_{i=1}^{n} {}^{RL}_{b_i^-}\partial^{1+\alpha_i}_{x_i}\, I^1_{b_i^-}\varphi_i(x) = \sum_{i=1}^{n} {}^{C}_{b_i^-}\partial^{1+\alpha_i}_{x_i} \left(c_0^{[i]} + c_1^{[i]} x_i \right) \\ \Leftrightarrow\ & {}^{RL}\nabla^{\alpha}_{b^-} \cdot \varphi(x) - \gamma_1(x) = 0. \end{aligned}$$

■

Remark 4.4 *When $n = 1$, Lemma 4.1, Lemma 4.2 and Lemma 4.3 reduce, respectively, to Lemma 1, Lemma 2 and Lemma 3-Part (2) in [14].*

4.2 Main result

Now we prove the main result of this work, namely that the eigenvalues are separated and form an infinite sequence, where the eigenvalues can be ordered according to their increasing magnitude so that

$$\lambda^{(1)} < \lambda^{(2)} < \ldots < \lambda^{(n)} < \ldots; \qquad \lambda^{(n)} \to +\infty \quad \text{as} \quad n \to +\infty.$$

Theorem 4.5 *The fractional Sturm-Liouville problem (22) has an infinite increasing sequence of eigenvalues $\lambda^{(1)}, \lambda^{(2)}, \ldots$, and to each eigenvalue $\lambda^{(n)}$ there is a correspondent eigenfunction $f^{(n)}$ which is unique up to a constant factor. Furthermore, the eigenfunctions $f^{(n)}$ form an orthogonal set of solutions.*

Proof: The proof of this theorem is based on the proof presented in [7, 14], but here we consider the n-dimensional case and fractional derivatives in the Riemann-Liouville sense. The proof is divided into six parts and aims to derive a method for approximating eigenvalues and eigenfunctions at the same time.

Part 1 - we solve the minimizing problem associated to (22)

Let us consider the minimization problem of the functional

$$J(f) = \int_\Omega \left[\mu(x) \left({}^{RL}\nabla^\alpha_{a^+} f(x)\right) \cdot \left({}^{RL}\nabla^\alpha_{a^+} f(x)\right) + \nu(x)\, f^2(x)\right] dx \tag{58}$$

subject to the boundary condition in (22) and the additional condition

$$I(f) = \int_\Omega r(x)\, f^2(x)\, dx = 1. \tag{59}$$

First we observe that the functional (58) is bounded from below. In fact, since $r(x) > 0$ and $\mu(x) \left({}^{RL}\nabla^\alpha_{a^+} f(x)\right) \cdot \left({}^{RL}\nabla^\alpha_{a^+} f(x)\right) > 0$ because $\mu(x) > 0$, we have

$$\begin{aligned} J(f) &= \int_\Omega \left[\mu(x) \left({}^{RL}\nabla^\alpha_{a^+} f(x)\right) \cdot \left({}^{RL}\nabla^\alpha_{a^+} f(x)\right) + \nu(x)\, f^2(x)\right] dx \\ &\geq \int_\Omega \nu(x)\, f^2(x)\, dx \\ &\geq \underbrace{\min_{x\in\Omega} \frac{\mu(x)}{r(x)}}_{M} \underbrace{\int_\Omega r(x)\, f^2(x)\, dx}_{1} \\ &= M. \end{aligned}$$

Let $\{\phi_i\}_{i\in\mathbb{N}}$ be an orthonormal basis in $L_2(\Omega)$ vanishing on $\partial\Omega$. Taking into account the Rayleigh-Ritz method (see [21]), we approximate the solution of the variational model by the truncated sum

$$f_m(x) = \frac{1}{\sqrt{r(x)}} \sum_{j=1}^m \beta_j\, \phi_j(x), \tag{60}$$

where for each $j = 1, \ldots, n$, we have $\beta_j \in \mathbb{R}$. Due to the choice of ϕ_j it is immediate that f_m vanishes on the boundary of Ω. Substituting (60) into (58) and (59) we obtain the problem of minimizing

$$\widetilde{J}_m(\beta_1, \ldots, \beta_m) = \int_\Omega \left[\mu(x) \sum_{i=1}^n \left(\sum_{j=1}^m \beta_j\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} \left(\frac{\phi_j(x)}{\sqrt{r(x)}}\right)\right)^2 + \frac{\nu(x)}{r(x)} \left(\sum_{j=1}^m \beta_j\, \phi_j(x)\right)^2\right] dx \tag{61}$$

subject to the condition

$$\widetilde{I}_m(\beta_1, \ldots, \beta_m) = \int_\Omega \Big(\sum_{j=1}^m \beta_j\, \phi_j(x)\Big)^2 dx = \sum_{j=1}^m \beta_j^2 \int_\Omega \phi_j^2(x)\, dx = \sum_{j=1}^m \beta_j^2 = 1. \tag{62}$$

We have that (61) is a function of $\underline{\beta} = (\beta_1, \ldots, \beta_m)$, then our problem is to minimize $\widetilde{J}_m$ on the surface σ_m of the m-dimensional unit sphere with equation (62). Since σ_m is a compact set and $\widetilde{J}_m$ is continuous on σ_m, then $\widetilde{J}_m$ has a minimum denoted by $\lambda_m^{(1)}$ at some point $\underline{\beta}^{(1)} = (\beta_1^{(1)}, \ldots, \beta_m^{(1)})$. If this procedure is carried out for each $m = 1, 2, 3, \ldots$ we obtain a sequence of numbers $\lambda_1^{(1)}$, $\lambda_2^{(1)}$, $\lambda_3^{(1)}, \ldots$ and corresponding sequence of functions $f_1^{(1)}$, $f_2^{(1)}$, $f_3^{(1)}, \ldots$. Taking into account that σ_m is a subset of σ_{m+1} obtained by setting $\beta_{m+1} = 0$, while $\widetilde{J}_m(\beta_1, \ldots, \beta_m) = \widetilde{J}_m(\beta_1, \ldots, \beta_m, 0)$ we set $\lambda_{m+1}^{(1)} \leq \lambda_m^{(1)}$, since increasing the domain of definition of a function can only decrease its minimum. It follows from the relation $\lambda_{m+1}^{(1)} \leq \lambda_m^{(1)}$ and the fact that $J(f)$ is bounded from below that exits the limit

$$\lim_{m\to+\infty} \lambda_m^{(1)} = \widetilde{\lambda}^{(1)}, \quad \widetilde{\lambda}^{(1)} \in \mathbb{R}. \tag{63}$$

Part 2: we prove that $\left(f_m^{(1)}\right)_{m\in\mathbb{N}}$ *has a uniformly convergent subsequence* $\left(f_{m_p}^{(1)}\right)_{p\in\mathbb{N}}$
Let

$$f_m^{(1)}(x) = \frac{1}{\sqrt{r(x)}} \sum_{j=1}^{m} \beta_j^{(1)} \, \phi_j(x) \tag{64}$$

denote the linear combination (60) achieving the minimum $f_m^{(1)}$. We prove now that the sequence $\left(f_m^{(1)}\right)_{m\in\mathbb{N}}$ has a uniformly convergent subsequence. Recall that

$$\lambda_m^{(1)} = \int_\Omega \left[\mu(x) \, \left({}^{RL}\nabla_{a^+}^{\alpha} f_m^{(1)}(x)\right) \cdot \left({}^{RL}\nabla_{a^+}^{\alpha} f_m^{(1)}(x)\right) + \nu(x) \, \left(f_m^{(1)}\right)^2(x)\right] dx$$

is convergent, which implies that it is also bounded, i.e., there exists a constant $M_0 > 0$ such that for all $m \in \mathbb{N}$

$$\int_\Omega \left[\mu(x) \, \left({}^{RL}\nabla_{a^+}^{\alpha} f_m^{(1)}(x)\right) \cdot \left({}^{RL}\nabla_{a^+}^{\alpha} f_m^{(1)}(x)\right) + \nu(x) \, \left(f_m^{(1)}(x)\right)^2\right] dx \leq M_0.$$

Hence, for all $m \in \mathbb{N}$ and taking into account (59) it holds

$$\begin{aligned} 0 &\leq \int_\Omega \mu(x) \, \left({}^{RL}\nabla_{a^+}^{\alpha} f_m^{(1)}(x)\right) \cdot \left({}^{RL}\nabla_{a^+}^{\alpha} f_m^{(1)}(x)\right) \, dx \\ &\leq M_0 + \left|\int_\Omega \nu(x) \, \left(f_m^{(1)}(x)\right)^2 \, dx\right| \\ &\leq \underbrace{M_0 + \max_{x\in\Omega}\left|\frac{\nu(x)}{r(x)}\right| \underbrace{\int_\Omega r(x) \, \left(f_m^{(1)}(x)\right)^2 \, dx}_{1}}_{M_1}, \end{aligned}$$

and since $\mu(x) > 0$,

$$\min_{x\in\Omega} \mu(x) \int_\Omega \left({}^{RL}\nabla_{a^+}^{\alpha} f_m^{(1)}(x)\right) \cdot \left({}^{RL}\nabla_{a^+}^{\alpha} f_m^{(1)}(x)\right) \, dx \;\leq \int_\Omega \mu(x) \, \left({}^{RL}\nabla_{a^+}^{\alpha} f_m^{(1)}(x)\right) \cdot \left({}^{RL}\nabla_{a^+}^{\alpha} f_m^{(1)}(x)\right) \, dx \;\leq M_1,$$

which leads to

$$\int_\Omega \left({}^{RL}\nabla_{a^+}^{\alpha} f_m^{(1)}(x)\right) \cdot \left({}^{RL}\nabla_{a^+}^{\alpha} f_m^{(1)}(x)\right) \, dx \;\leq \frac{M_1}{\min_{x\in\Omega} \mu(x)} \;=: M_2. \tag{65}$$

Using (65) and taking into account that $I_{a^+}^{\alpha} \cdot {}^{RL}\nabla_{a^+}^{\alpha} f_m^{(1)}(x) = n \, f_m^{(1)}(x)$ and using the Schwartz inequality, we obtain the following estimate (where $\widehat{x} = (x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$)

$$\begin{aligned} \left|n \, f_m^{(1)}\right|^2 &= \left|I_{a^+}^{\alpha} \cdot {}^{RL}\nabla_{a^+}^{\alpha} f_m^{(1)}(x)\right|^2 \\ &\leq \sum_{i=1}^{n} \frac{1}{\Gamma^2(\alpha_i)} \left|\int_{a_i}^{x_i} (x_i - t)^{\alpha_i - 1} \, \left({}^{RL}_{a_i^+}\partial_{x_i}^{\alpha_i} f_m^{(1)}\right)(\widehat{x}) \, dt\right|^2 \\ &\leq \sum_{i=1}^{n} \frac{1}{\Gamma^2(\alpha_i)} \left(\int_{a_i}^{b_i} \left|\left({}^{RL}_{a_i^+}\partial_{x_i}^{\alpha_i} f_m^{(1)}\right)(\widehat{x})\right|^2 \, dt\right) \, \left(\int_{a_i}^{x_i} (x_i - t)^{2(\alpha_i - 1)} \, dt\right) \\ &\leq \sum_{i=1}^{n} \frac{1}{\Gamma^2(\alpha_i)} \, M_2 \, \left(\int_{a_i}^{x_i} (x_i - t)^{2(\alpha_i - 1)} \, dt\right) \\ &\leq \sum_{i=1}^{n} \frac{M_2}{\Gamma^2(\alpha_i)} \, \frac{b_i^{2\alpha_i - 1} - a_i^{2\alpha_i - 1}}{2\alpha_i - 1}, \end{aligned}$$

and therefore $\left(f_m^{(1)}\right)_{m\in\mathbb{N}}$ is uniformly bounded. Using the Schwartz inequality and (65) we have for any

$a_i < x_1^{[i]} < x_2^{[i]} \leq b_i$, with $i = 1, \dots, n$

$$\begin{aligned}
\left| n\, f_m^{(1)}(x_2) - n\, f_m^{(1)}(x_1) \right| &= \left| I_{a^+}^{\alpha} \cdot {}^{RL}\nabla_{a^+}^{\alpha} f_m^{(1)}(x_2) - I_{a^+}^{\alpha} \cdot {}^{RL}\nabla_{a^+}^{\alpha} f_m^{(1)}(x_1) \right| \\
\leq \sum_{i=1}^{n} \frac{1}{\Gamma(\alpha_i)} & \left| \int_{a_i}^{x_2^{[i]}} \left(x_2^{[i]} - t \right)^{\alpha_i - 1} \left({}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_2^{[i]}} f_m^{(1)} \right) (\widehat{x}_2)\, dt - \int_{a_i}^{x_1^{[i]}} \left(x_1^{[i]} - t \right)^{\alpha_i - 1} \left({}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_1^{[i]}} f_m^{(1)} \right) (\widehat{x}_1)\, dt \right| \\
\leq \sum_{i=1}^{n} \frac{1}{\Gamma(\alpha_i)} & \left[\left(\int_{a_i}^{x_2^{[i]}} \left({}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_2^{[i]}} f_m^{(1)} \right) (\widehat{x}_2)\, dt \right)^{\frac{1}{2}} \left(\int_{a_i}^{x_2^{[i]}} \left(x_2^{[i]} - t \right)^{2(\alpha_i - 1)} dt \right)^{\frac{1}{2}} \right. \\
& \left. + \left(\int_{a_i}^{x_1^{[i]}} \left({}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_1^{[i]}} f_m^{(1)} \right) (\widehat{x}_1)\, dt \right)^{\frac{1}{2}} \left(\int_{a_i}^{x_1^{[i]}} \left(x_1^{[i]} - t \right)^{2(\alpha_i - 1)} dt \right)^{\frac{1}{2}} \right] \\
\leq \sum_{i=1}^{n} \frac{\sqrt{M_2}}{\Gamma(\alpha_i)} & \left(\sqrt{\frac{\left(x_2^{[i]} - a_i \right)^{2\alpha_i - 1}}{2\alpha_i - 1}} + \sqrt{\frac{\left(x_1^{[i]} - a_i \right)^{2\alpha_i - 1}}{2\alpha_i - 1}} \right) \\
& \leq 2\sqrt{M_2} \sum_{i=1}^{n} \frac{\sqrt{b_i^{2\alpha_i - 1} - a_i^{2\alpha_i - 1}}}{\sqrt{2\alpha_i - 1}\, \Gamma(\alpha_i)},
\end{aligned} \tag{66}$$

where the right-hand side of (66) is finite. Hence $\left(f_m^{(1)} \right)_{m \in \mathbb{N}}$ is equicontinuous. Therefore, by the Arzalá-Ascoli's Theorem, we can select a uniformly convergent subsequence $\left(f_{m_p}^{(1)} \right)_{p \in \mathbb{N}}$, i.e., we can find $f^{(1)} \in C(\Omega)$ such that

$$\lim_{p \to +\infty} f_{m_p}^{(1)} = f^{(1)}. \tag{67}$$

Part 3: we prove that $f^{(1)}$ is solution of the minimizing problem
By the Lagrange multiplier rule at $\underline{\beta}$ we have that

$$\frac{\partial}{\partial \beta_k} \left[\widetilde{J}_m(\underline{\beta}) - \lambda_m^{(1)} (\widetilde{I}_m(\underline{\beta}) - 1) \right]_{\underline{\beta} = \underline{\beta}^{(1)}} = 0, \qquad k = 1, \dots, m,$$

i.e., for each $k = 1, \dots, m$, we have

$$\int_{\Omega} \left[\mu(x) \sum_{i=1}^{n} \Big(\sum_{j=1}^{m} \beta_j \, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} \Big(\frac{\phi_j(x)}{\sqrt{r(x)}} \Big) \Big) \, {}^{RL}_{a_k^+}\partial^{\alpha_k}_{x_k} \Big(\frac{\phi_j(x)}{\sqrt{r(x)}} \Big) + \frac{\nu(x)}{r(x)} \Big(\sum_{j=1}^{m} \beta_j \, \phi_j(x) \Big) \phi_k(x) \right] dx - \lambda_m^{(1)} \, \beta_k \; = 0, \tag{68}$$

where we used (61), (62), and the fact that $\{\phi_i\}_{i \in \mathbb{N}}$ is an orthonormal basis in $L_2(\Omega)$. Multiplying each equation by an arbitrary constante C_k, summing from 1 to m, recalling (64) and considering

$$h_m^{(1)}(x) = \frac{1}{\sqrt{r(x)}} \sum_{k=1}^{m} C_k \, \phi_k(x),$$

we can rewrite (68) in the form

$$\int_{\Omega} \left[\mu(x) \left({}^{RL}\nabla_{a^+}^{\alpha} f_m^{(1)}(x) \right) \cdot \left({}^{RL}\nabla_{a^+}^{\alpha} h_m^{(1)}(x) \right) + \nu(x) \, f_m^{(1)}(x) \, h_m^{(1)}(x) \right] dx - \lambda_m^{(1)} \sum_{k=1}^{m} \beta_k \, C_k = 0. \tag{69}$$

Integrating by parts the 1st term of the left-hand side of (69) and considering $\Omega^* = \prod_{\substack{j=1 \\ j \neq i}}^{n}]a_i, b_i[$ and $dx^* =$

$dx_1\,\ldots\,dx_{i-1}\,dx_{i+1}\,\ldots\,dx_n$,we get

$$\int_\Omega \mu(x)\,\left({}^{RL}\nabla^{\alpha}_{a^+} f^{(1)}_m(x)\right)\cdot\left({}^{RL}\nabla^{\alpha}_{a^+} h^{(1)}_m(x)\right)\,dx$$

$$= \sum_{i=1}^{n}\int_{\Omega^*}\int_{a_i}^{b_i}\mu(x)\,\partial_{x_i} I^{1-\alpha_i}_{a_i^+} f^{(1)}_m(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}_m(x)\,dx_i\,dx^*$$

$$= \sum_{i=1}^{n}\left\{\left[\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}_m(x)\,I^{1-\alpha_i}_{a_i^+} f^{(1)}_m(x)\right]_{x=a_i}^{x=b_i} dx^* - \int_{\Omega^*}\int_{a_i}^{b_i}\partial_{x_i}\left[\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}_m(x)\right]\,I^{1-\alpha_i}_{a_i^+} f^{(1)}_m(x)\,dx_i\,dx^*\right\}$$

$$= \sum_{i=1}^{n}\left\{\int_{\Omega^*}\left[\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}_m(x)\,I^{1-\alpha_i}_{a_i^+} f^{(1)}_m(x)\right]_{x=a_i}^{x=b_i} dx^*\right.$$

$$\left. - \int_\Omega\int_{a_i}^{b_i}\left[\partial_{x_i}\left[\mu(x)\right]\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}_m(x) + \mu(x)\,\partial_{x_i}\left[{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}_m(x)\right]\right]\,I^{1-\alpha_i}_{a_i^+} f^{(1)}_m(x)\,dx\right\}. \tag{70}$$

From (70) we can rewrite (69) in the following way

$$\begin{aligned}
0 = &-\sum_{i=1}^{n}\int_\Omega\int_{a_i}^{b_i}\left[\partial_{x_i}\left[\mu(x)\right]\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}_m(x) + \mu(x)\,\partial_{x_i}\left[{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}_m(x)\right]\right]\,I^{1-\alpha_i}_{a_i^+} f^{(1)}_m(x)\,dx\\
&+\sum_{i=1}^{n}\int_{\Omega^*}\left[\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}_m(x)\,I^{1-\alpha_i}_{a_i^+} f^{(1)}_m(x)\right]_{x=a_i}^{x=b_i} dx^*\\
&+\int_\Omega \nu(x)\,f^{(1)}_m(x)\,h^{(1)}_m(x)\,dx - \lambda^{(1)}_m\sum_{k=1}^{m}\beta_k\,C_k\\
&=: I_m.
\end{aligned} \tag{71}$$

At least for a subsequence $f^{(1)}_{m_p}$, with $p\in\mathbb{N}$, we shall obtain in the limit the relation

$$\begin{aligned}
0 = &-\sum_{i=1}^{n}\int_\Omega\int_{a_i}^{b_i}\left[\partial_{x_i}\left[\mu(x)\right]\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}(x) + \mu(x)\,\partial_{x_i}\left[{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}(x)\right]\right]\,I^{1-\alpha_i}_{a_i^+} f^{(1)}(x)\,dx\\
&+\sum_{i=1}^{n}\int_{\Omega^*}\left[\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}(x)\,I^{1-\alpha_i}_{a_i^+} f^{(1)}(x)\right]_{x=a_i}^{x=b_i} dx^*\\
&+\int_\Omega \nu(x)\,f^{(1)}(x)\,h^{(1)}(x)\,dx - \widetilde{\lambda}^{(1)}\sum_{k=1}^{+\infty}\beta_k\,C_k\\
&=: I.
\end{aligned} \tag{72}$$

Let us check the convergence of (71) to (72) explicitly

$$\begin{aligned}
|I_m - I| \le &\sum_{i=1}^{n}\int_\Omega\left|-\partial_{x_i}\left[\mu(x)\right]\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}_m(x)\,I^{1-\alpha_i}_{a_i^+} f^{(1)}_m(x) + \partial_{x_i}\left[\mu(x)\right]\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}(x)\,I^{1-\alpha_i}_{a_i^+} f^{(1)}(x)\right|\,dx\\
&+\sum_{i=1}^{n}\int_\Omega\left|-\mu(x)\,\partial_{x_i}\left[{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}_m(x)\right]\,I^{1-\alpha_i}_{a_i^+} f^{(1)}_m(x) + \mu(x)\,\partial_{x_i}\left[{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}(x)\right]\,I^{1-\alpha_i}_{a_i^+} f^{(1)}(x)\right|\,dx\\
&+\sum_{i=1}^{n}\left|\left[\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}_m(x)\,I^{1-\alpha_i}_{a_i^+} f^{(1)}_m(x)\right]_{x=b_i} - \left[\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}(x)\,I^{1-\alpha_i}_{a_i^+} f^{(1)}(x)\right]_{x=b_i}\right|\\
&+\sum_{i=1}^{n}\left|\left[-\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}_m(x)\,I^{1-\alpha_i}_{a_i^+} f^{(1)}_m(x)\right]_{x=a_i} + \left[\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} h^{(1)}(x)\,I^{1-\alpha_i}_{a_i^+} f^{(1)}(x)\right]_{x=a_i}\right|\\
&+\int_\Omega\left|\nu(x)\left[f^{(1)}_m(x)\,h^{(1)}_m(x) - f^{(1)}(x)\,h^{(1)}(x)\right]\right|\,dx + \left|\lambda^{(1)}_m\sum_{k=1}^{m}\beta_k\,C_k - \widetilde{\lambda}^{(1)}\sum_{k=1}^{+\infty}\beta_k\,C_k\right|.
\end{aligned} \tag{73}$$

For the first term in (73) we get

$$\sum_{i=1}^{n}\int_{\Omega}\left|-\partial_{x_i}\left[\mu(x)\right]\ {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}h^{(1)}_m(x)\, I^{1-\alpha_i}_{a_i^+}f^{(1)}_m(x)+\partial_{x_i}\left[\mu(x)\right]\ {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}h^{(1)}(x)\, I^{1-\alpha_i}_{a_i^+}f^{(1)}(x)\right| dx$$

$$=\sum_{i=1}^{n}\int_{\Omega}\Big|-\partial_{x_i}\left[\mu(x)\right]\ {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}h^{(1)}_m(x)\, I^{1-\alpha_i}_{a_i^+}f^{(1)}_m(x)-\partial_{x_i}\left[\mu(x)\right]\ {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}h^{(1)}(x)\, I^{1-\alpha_i}_{a_i^+}f^{(1)}_m(x)$$

$$+\partial_{x_i}\left[\mu(x)\right]\ {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}h^{(1)}(x)\, I^{1-\alpha_i}_{a_i^+}f^{(1)}_m(x)+\partial_{x_i}\left[\mu(x)\right]\ {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}h^{(1)}(x)\, I^{1-\alpha_i}_{a_i^+}f^{(1)}(x)\Big| dx$$

$$\leq\sum_{i=1}^{n}\|\partial_{x_i}\mu\|\left[\left\|{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}h^{(1)}\right\|_{L_2}\left\|I^{1-\alpha_i}_{a_i^+}\left(f^{(1)}_m-f^{(1)}\right)\right\|_{L_2}+\left\|I^{1-\alpha_i}_{a_i^+}f^{(1)}_m\right\|_{L_2}\left\|{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}\left(h^{(1)}_m-h^{(1)}\right)\right\|_{L_2}\right]$$

$$\leq\sum_{i=1}^{n}\|\partial_{x_i}\mu\|\left[\left\|{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}h^{(1)}\right\|_{L_2}\left\|I^{1-\alpha_i}_{a_i^+}\left(f^{(1)}_m-f^{(1)}\right)\right\|_{L_2}+K_{1-\alpha}\,M_3\left\|{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}\left(h^{(1)}_m-h^{(1)}\right)\right\|_{L_2}\right],$$

where

$$M_3=\sup_{x\in\Omega}\left\|f^{(1)}_m\right\|,\qquad K_{1-\alpha}=\left\|\frac{\pi^{1-\alpha}}{\Gamma(-\alpha)}\right\|=\left\|\left(\frac{\pi^{1-\alpha_1}}{\Gamma(-\alpha_1)},\dots,\frac{\pi^{1-\alpha_n}}{\Gamma(-\alpha_n)}\right)\right\|.$$

Proceeding in a very similar way for the second term in (73) we get

$$\sum_{i=1}^{n}\int_{\Omega}\left|\mu(x)\,\partial_{x_i}\left[{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}h^{(1)}_m(x)\right]\, I^{1-\alpha_i}_{a_i^+}f^{(1)}_m(x)-\mu(x)\,\partial_{x_i}\left[{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}h^{(1)}(x)\right]\, I^{1-\alpha_i}_{a_i^+}f^{(1)}(x)\right| dx$$

$$\leq\sum_{i=1}^{n}\|\mu\|\left[\left\|\partial_{x_i}\left[{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}h^{(1)}\right]\right\|_{L_2}\left\|I^{1-\alpha_i}_{a_i^+}\left(f^{(1)}_m-f^{(1)}\right)\right\|_{L_2}+K_{1-\alpha}\,M_3\left\|\partial_{x_i}\left[{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}\left(h^{(1)}_m-h^{(1)}\right)\right]\right\|_{L_2}\right].$$

For the 3rd and 4th terms we have, for $i=1,\dots,n$,

$$I^{1-\alpha_i}_{a_i^+}f^{(1)}_m(a)\longrightarrow I^{1-\alpha_i}_{a_i^+}f^{(1)}(a),\qquad I^{1-\alpha_i}_{a_i^+}f^{(1)}_m(b)\longrightarrow I^{1-\alpha_i}_{a_i^+}f^{(1)}(b),\qquad I^{1-\alpha_i}_{a_i^+}h^{(1)}_m(x)\longrightarrow I^{1-\alpha_i}_{a_i^+}h^{(1)}(a),$$

due to the convergences $\|f^{(1)}_m-f^{(1)}\|\to 0$ and $\|h^{(1)}_m-h^{(1)}\|\to 0$ as $m\to+\infty$. Moreover, we have

$$I^{1-\alpha_i}_{a_i^+}h^{(1)}_m(x)\longrightarrow I^{1-\alpha_i}_{a_i^+}h^{(1)}(x)\Longrightarrow\lim_{m\to+\infty}\left\|{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}\left(h^{(1)}_m-h^{(1)}\right)\right\|_{L_2}=0$$

$$\Longrightarrow\lim_{m\to+\infty}\left\|\partial_{x_i}\left[{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}\left(h^{(1)}_m-h^{(1)}\right)\right]\right\|_{L_2}=0.$$

Hence

$$\partial_{x_i}\left[{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}h^{(1)}_m\right]_{x_i=a_i}\longrightarrow\partial_{x_i}\left[{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}h^{(1)}\right]_{x_i=a_i},$$

$$\partial_{x_i}\left[{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}h^{(1)}_m\right]_{x_i=b_i}\longrightarrow\partial_{x_i}\left[{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}h^{(1)}\right]_{x_i=b_i}.$$

The above pointwise convergence implies that the 3rd and 4th terms tend to zero as $m\to+\infty$. For the 5th term in (73) we have

$$\int_{\Omega}\left|\nu(x)\left[f^{(1)}_m(x)\,h^{(1)}_m(x)-f^{(1)}(x)\,h^{(1)}(x)\right]\right| dx$$

$$=\int_{\Omega}\left|\nu(x)\left(f^{(1)}_m(x)\,h^{(1)}_m(x)+f^{(1)}_m(x)\,h^{(1)}(x)-f^{(1)}_m(x)\,h^{(1)}(x)-f^{(1)}\,h^{(1)}(x)\right)\right| dx$$

$$\leq\|\nu\|\left[\sup_{m\in\mathbb{N}}\left\|f^{(1)}_m\right\|_{L_2}\left\|h^{(1)}_m-h^{(1)}\right\|_{L_2}+\left\|h^{(1)}\right\|_{L_2}\left\|f^{(1)}_m-f^{(1)}\right\|_{L_2}\right],$$

which tends to zero as m goes to infinity. For the last term it is immediate by (63) and (67) that

$$\lambda^{(1)}_m\sum_{k=1}^{m}\beta_k\,C_k\longrightarrow\widetilde{\lambda}^{(1)}\sum_{k=1}^{+\infty}\beta_k\,C_k,\tag{74}$$

as $m \to +\infty$. Finally, we conclude that I_m tends to I as $m \to +\infty$ and (72) is valid for a function $f^{(1)}$ being the limit of the subsequence $\left(f^{(1)}_{m_p}\right)_{p\in\mathbb{N}}$ of the sequence $\left(f^{(1)}_m\right)_{m\in\mathbb{N}}$.

Part 4: we prove that $f^{(1)}$ is solution of the fractional Sturm-Liouville problem (22)

Taking into account the calculations of *Part 3*, we can rewrite integral (72) as

$$\int_\Omega \left[\mu(x)\,\left({}^{RL}\nabla^\alpha_{a+} f^{(1)}(x)\right)\cdot\left({}^{RL}\nabla^\alpha_{a+} h^{(1)}(x)\right) + \left[\nu(x) - \widetilde{\lambda}^{(1)}\, r(x)\right]\, f^{(1)}(x)\, h^{(1)}(x)\right] dx \,=0.$$

Now, we apply Lemma 4.3. Defining

$$\gamma_1(x) \,= \left(\nu(x) - \widetilde{\lambda}^{(1)}\, r(x)\right) f^{(1)}(x) \quad \text{ and } \quad \gamma_2(x) \,= \mu(x)\, {}^{RL}\nabla^\alpha_{a+} f^{(1)}(x),$$

we have that $\gamma_1, \gamma_2 \in C(\Omega)$ and therefore

$$\left[{}^{RL}\nabla^\alpha_{b-}\cdot\left(\mu(x)\,{}^{RL}\nabla^\alpha_{a+}\right) + \nu(x)\right]\, f^{(1)}(x) \,= \widetilde{\lambda}^{(1)}\, r(x)\, f^{(1)}(x).$$

By our construction the $f^{(1)}$ solution satisfies the boundary condition in (22), i.e., $f^{(1)}$ vanishes on $\partial\Omega$, and it is nontrivial due to

$$I\left(f^{(1)}\right) \,= \int_\Omega r(x)\,\left(f^{(1)}(x)\right)^2\, dx \,=1.$$

Moreover, for the solution $f^{(1)}$, we have that ${}^{RL}\nabla^\alpha_{a+} f^{(1)} ={}^C\,\nabla^\alpha_{a+} f^{(1)}$ is a function in $C(\Omega)$. Therefore we have that $f^{(1)}$ solves (22) and $\widetilde{\lambda}^{(1)} = \lambda^{(1)}$ is its correspondent eigenvalue.

Part 5: we prove that $f^{(1)}$ is also the limit of $\left(f^{(1)}_m\right)_{m\in\mathbb{N}}$

For a given λ, the solution of

$$\left[{}^{RL}\nabla^\alpha_{b-}\cdot\left(\nu(x)\,{}^{RL}\nabla^\alpha_{a+}\right) + \nu(x)\right]\, f(x) \,= \lambda\, r(x)\, f(x) \tag{75}$$

such that f vanishes on $\partial\Omega$ and satisfies (59) is unique except for a sign. Next, let us assume that $f^{(1)}$ solves (75) and that the corresponding eigenvalue is $\lambda^{(1)}$. Additionally, suppose that $f^{(1)}$ is nontrivial, i.e., there exists $x_0 \in \Omega$ such that $f^{(1)}(x_0) \neq 0$ and choose the sign so that $f^{(1)}(x_0) > 0$. Similarly, for all $m \in \mathbb{N}$, let $f^{(1)}_m$ solves (75) with the corresponding eigenvalues $\lambda^{(1)}_m$ and let us choose the signs such that $f^{(1)}_m(x_0) \geq 0$. Suppose that $\left(f^{(1)}_m\right)_{m\in\mathbb{N}}$ does not converge to $f^{(1)}$, then we can find another subsequence of $\left(f^{(1)}_m\right)_{m\in\mathbb{N}}$ such it converges to another solution $\overline{f}^{(1)}$ of (75) with the same eigenvalue $\lambda^{(1)}$. We know that for $\lambda^{(1)}$ the solution of (75) must be unique except for a sign, hence $\overline{f}^{(1)} = -f^{(1)}$, i.e., $\overline{f}^{(1)}(x_0) < 0$, which is impossible because $f^{(1)}_m(x_0) \geq 0$ for all $m \in \mathbb{N}$. Therefore $f^{(1)}_m \to f^{(1)}$, provided we choose each $f^{(1)}_m$ with the proper sign.

Part 6: we construct the remaining eigenvalues and correspondent eigenfunctions

In order to find the eigenfunction $f^{(2)}$ and the corresponding eigenvalue $\lambda^{(2)}$, we again minimize the functional (58) such that (59) is verified and $f^{(2)}$ vanishes on $\partial\Omega$ and satisfies an extra orthogonality condition

$$\int_\Omega r(x)\, f(x)\, f^{(1)}(x)\, dx = 0. \tag{76}$$

If we approximate the solution by

$$f_m(x) = \frac{1}{\sqrt{r(x)}}\sum_{i=1}^m \beta_i\, \phi_i(x),$$

such that $f_m(x) = 0$ for $x \in \partial\Omega$, then we get again (61). However, in this case, admissible solutions are points satisfying (62) and the extra condition

$$\sum_{i=1}^m \beta_i\, \beta_i^{(1)} = 0, \tag{77}$$

i.e., they lay on a $(m-1)-$dimensional sphere resulting from the intersection of the m-dimensional unit sphere defined by (62) and the hyperplane defined by (77). By the same argument as before, we conclude that $\widetilde{J}_m(\underline{\beta})$ has a minimum $\lambda^{(2)}_m$ and there exits $\lambda^{(2)}$ such that $\lim_{m\to+\infty}\lambda^{(2)}_m \,= \lambda^{(2)}$. Moreover, we have that

$\lambda^{(1)} \leq \lambda^{(2)}$. Now let $f_m^{(2)}(x) = \sum_{i=1}^m \beta_i^{(2)}\, \phi_i(x)$ be the linear combination achieving the minimum $\lambda_m^{(2)}$, where $\underline{\beta}^{(2)} = (\beta_1^{(2)}, \ldots, \beta_m^{(2)})$ is the point satisfying (62) and (77). By the same arguments as before, we can prove that the sequence $\left(f_m^{(2)}\right)_{m\in\mathbb{N}}$ converges uniformly to a limit function $f^{(2)}$, which satisfies (22) with eigenvalue $\lambda^{(2)}$, vanishes on $\partial\Omega$, and satisfies (59) and (76). Furthermore, because orthogonal functions cannot be linearly dependent, and since each eigenfunction corresponds to one eigenvalue (except for a constant factor) we have the strictly inequality $\lambda^{(1)} < \lambda^{(2)}$. If we repeat the above procedure, with the necessary modifications, we can obtain the eigenvalues $\lambda^{(3)}, \lambda^{(4)}, \ldots$ and the corresponding eigenfunctions $f^{(3)}, f^{(4)}, \ldots$. Now the proof is complete.

■

As a consequence of the previous results, we could use the normalized eigenfunctions $f^{(1)}, f^{(2)}, f^{(3)}, \ldots$ of the fractional Sturm-Liouville problem (22) to obtain a series expansion for a function g

$$g(x) = \sum_{i=1}^{+\infty} c_i\, f^{(i)}(x)$$

where the coefficients c_i are given by

$$c_i \,=\, \int_\Omega r(x)\, g(x)\, f^{(i)}(x)\, dx \,=\, \langle g,\, r\, f^{(i)}\rangle, \qquad i = 1, 2, \ldots.$$

We observe that when $n = 1$ we recover the results presented in [14, 18]. Moreover, if $\alpha = (1, \ldots, 1)$ we obtain the classical Sturm-Liouville problem in higher dimensions (see [1, 7, 9]).

5 Clifford analysis

The aim of this section is to indicated how the previous results can be presented in the context of Clifford analysis. Clifford analysis offers a higher dimensional generalization of the classical theory of complex holomorphic functions. Its tools can be applied to several different areas, for instance to quantum mechanics, quantum field theory [8], projective geometry, computer graphics [22], neural network theory [2] and to many other areas of physics and engineering [10]. The corresponding analogy of the class of complex holomorphic functions is that of monogenic functions, which are the null solutions of the Dirac operator.

Now, we recall some basic facts about this geometric algebras: let $\{e_1, \cdots, e_n\}$ be the standard basis of the Euclidean vector space in $\mathbb{R}^n$. The associated Clifford algebra $\mathbb{R}_{0,n}$ is the free algebra generated by $\mathbb{R}^n$ modulo $x^2 = -||x||^2\, e_0$, where $x \in \mathbb{R}^n$ and e_0 is the neutral element with respect to the multiplication operation in the Clifford algebra $\mathbb{R}_{0,n}$. The defining relation induces the multiplication rules

$$e_i e_j + e_j e_i = -2\delta_{ij},$$

where δ_{ij} denotes the Kronecker's delta. In particular, $e_i^2 = -1$ for all $i = 1, \ldots, n$. The standard basis vectors thus operate as imaginary units. A vector space basis for $\mathbb{R}_{0,n}$ is given by the set $\{e_A : \ A \subseteq \{1, \ldots, n\}\}$ with $e_A = e_{l_1} e_{l_2} \ldots e_{l_r}$, where $1 \leq l_1 < \ldots < l_r \leq n$, $0 \leq r \leq n$, $e_\emptyset := e_0 := 1$. Each $a \in \mathbb{R}_{0,n}$ can be written in the form $a = \sum_A a_A\, e_A$, with $a_A \in \mathbb{R}$. The conjugation in the Clifford algebra $\mathbb{R}_{0,n}$ is defined by $\overline{a} = \sum_A a_A\, \overline{e}_A$, where $\overline{e}_A = \overline{e}_{l_r}\, \overline{e}_{l_{r-1}} \ldots \overline{e}_{l_1}$, and $\overline{e}_j = -e_j$ for $j = 1, \ldots, n$, $\overline{e}_0 = e_0 = 1$.

A Clifford valued function, i.e, an $\mathbb{R}_{0,n}-$valued function f over $\Omega \subseteq \mathbb{R}^n$ has the representation $f = \sum_A e_A f_A$ with components $f_A : \Omega \to \mathbb{R}_{0,n}$. Properties such as continuity or differentiability have to be understood componentwise. Next, we recall the Euclidean Dirac operator $\mathcal{D} = \sum_{j=1}^n e_j \partial_{x_j}$. This operator satisfies $\mathcal{D}^2 = -\Delta$, where Δ is the n-dimensional Euclidean Laplacian, and provides a first order generalization of the well-known Cauchy-Riemann operator in complex analysis. An $\mathbb{R}_{0,n}$-valued function f is called *left-monogenic* if it satisfies $\mathcal{D}u = 0$ on Ω (resp. *right-monogenic* if it satisfies $u\mathcal{D} = 0$ on Ω). For more details about Clifford algebras and basic concepts of its associated function theory we refer the interested reader for example to [3, 4].

From the above short description of the Clifford analysis setting, it is possible to adapt the obtained results and the correspondent calculations to the context of Clifford analysis by the following considerations:

- Vectors in $\mathbb{R}^n$ are identified with 1-vectors in $\mathbb{R}_{0,n}$, i.e. $x = \sum_{i=1}^{n} e_i\, x_i$. Moreover, for two 1-vectors x and y in $\mathbb{R}_{0,n}$ we have

$$x \cdot y = -\, [xy]_0 \,,$$

  where $[xy]_0$ means the scalar part of the Clifford product xy.

- The vectorial functions are replaced by Clifford-valued functions;

- The fractional gradient operators that appear in the previous sections can be identified with the so-called fractional Dirac operators introduced in [5]:

$$^{RL}\nabla^{\alpha}_{a^+} = {}^{RL}\mathcal{D}^{\alpha}_{a^+} = \sum_{i=1}^{n} e_i \, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} \qquad \text{and} \qquad ^{RL}\nabla^{\alpha}_{b^-} = {}^{RL}\mathcal{D}^{\alpha}_{b^-} = \sum_{i=1}^{n} e_i \, {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}. \tag{78}$$

  In [5] the authors used these operators to develop a fractional integrodifferential operator calculus for Clifford-algebra valued functions. As it happens in the formulation of the fractional Sturm-Liouville problem (22), the Clifford analysis setting also requires the simultaneous use of left and right fractional derivatives.

6 Conclusions

In this work, we use a composition of fractional gradient operators defined in terms of left and right partial fractional derivatives in the Riemann-Liouville sense to propose a fractional approach to the Sturm-Liouville problem in $\mathbb{R}^n$. We studied the main properties of the eigenfunctions and the associated eigenvalues if this fractional boundary problem, more precisely we proved that the eigenfunctions are orthogonal and the eigenvalues are real and simple. Moreover, we proved in the main result that the eigenvalues are separated and form an infinite sequence, where the eigenvalues can be ordered according to increasing magnitude. The proved results allow us to use these eigenfunctions and eigenvalues to define a novel kind of series expansion that depends on $\alpha = (\alpha_1, \ldots, \alpha_n)$. The connections of this work with Clifford analysis is presented, more precisely, it was indicated the necessary adaptations to rewrite our results for Clifford-values functions. Finally, we point out that similar results can be obtained using fractional gradients defined in terms of Caputo and Riemann-Liouville fractional derivatives, i.e., for the fractional Sturm-Liouville operator $L^{\alpha} = -^{RL}\nabla^{\alpha}_{b^-} \cdot \left(\mu \, ^{C}\nabla^{\alpha}_{a^+}\right) + \nu$, since this operator is self-adjoint.

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