On circulant like matrices properties involving Horadam, Fibonacci, Jacobsthal and Pell numbers

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Abstract

In this work a new type of matrix called circulant-like matrix is introduced. This type of matrix includes the classical k-circulant matrix, introduced in [5], in a natural sense. Its eigenvalues and its inverse and some other properties are studied, namely, it is shown that the inverse of a matrix of this type is also a matrix of this type and that its first row is the unique solution of a certain system of linear equations. Additionally, for some of these matrices whose entries are written as function of Horadam, Fibonacci, Jacobsthal and Pell numbers we study its eigenvalues and write it as function of those numbers. Moreover, the same study is done if the entries are arithmetic sequences.

Keywords: k-circulant matrix; symmetric matrix; eigenvalues; Horadam number; Fibonacci number; Jacobsthal number; Pell number.
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1. Introduction

We start this section introducing the definition of k-circulant matrix with $k \in \mathbb{C}$ and some of the state of the art related with this important concept that motivated us to introduce one of its generalizations.

Definition 1. *[5] Let $k \in \mathbb{C}$. A k-circulant matrix is a matrix of the form:*

$$A(k) = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{m-1} \\ ka_{m-1} & a_0 & a_1 & \cdots & a_{m-2} \\ ka_{m-2} & ka_{m-1} & a_0 & \cdots & a_{m-3} \\ \vdots & \vdots & \vdots & & \vdots \\ ka_1 & ka_2 & ka_3 & \cdots & a_0 \end{pmatrix}. \tag{1}$$

The matrix in (1) is completely determined by k and its first row. Therefore, we simply denote this matrices as $A(k) = k - circ(a_0, a_1, \ldots, a_{m-1})$. This matrix is also called weighted circulant matrix. Note that, for $k = 1$ and $k = -1$ a circulant matrix and skew-circulant matrix are obtained, respectively. In the case of circulant matrices we just write $circ(a_0, a_1, \ldots, a_{m-1})$. These matrices (especially, circulant and skew circulant matrices) are very useful in many areas for instance in probability, statistic, coding theory, signal and image processing, numerical analysis, etc. In fact, they have been widely studied. See for instance, [5], where the authors present necessary and sufficient conditions for a matrix $A \in \mathbb{C}^m$ to be a matrix in the form (1) presenting the structure of the multiplicative group of matrices in $\mathcal{C}_m$, the multiplicative semigroup of all $m \times m$ complex matrices, that commute with a given matrix having distinct eigenvalues. In [7] the authors construct circulant-like matrices, that were designated by generalized weighted circulant and are matrices that are generated by generalized permutation matrices (that is, square matrices having in each row and column only one nonzero element) corresponding to a subgroup of some permutation group. Therefore, it was defined that a generalized weighted circulant matrix is an element in the algebra generated by some generalized permutation matrix. The eigenvalues for these class of matrices were obtained. In [6] it is shown how $A(k)^q$ can be obtained where q is a positive integer greater than 1, by using the basis expression of k-circulant matrices in matrix spaces and the Multinomial Theorem.

Some authors focus on these matrices considering its entries as different types of number sequences. For instance, circulant matrices with geometric sequences were studied in [3]. It has been a tradition to use number sequences as entries for right circulant matrices C (matrices where each row is a right shift of the row above it) where C is determined by its first row $(c_0, \ldots, c_m)$ and $c_k = c_{i-j}$ whenever $i - j \equiv k (\mathrm{mod\, m})$. Therefore papers on these special matrices normally would concentrate on the investigation of their eigenvalues, determinants, Euclidean norms, spectral norms and inverses. In [4] the author studied these properties for matrices of this type whose first row is $(\frac{F_0}{g}, \frac{F_1}{gq}, \frac{F_2}{gq^2}, \ldots, \frac{F_{m-1}}{gq^{m-1}})$ where F_j is the j-th Fibonacci number, and $g \neq 0$ and $q \neq 0, 1$. Bahsi and Solak in [1] used arithmetic sequence as entries. They also provided explicit formulas for the determinant, eigenvalues, Euclidean and spectral norms, and inverses of the right circulant matrices. Moreover, circulant and skew circulant matrices with binomial coefficients were considered in [16]. The paper [17] is devoted to circulant and skew circulant matrices whose entries are binomial coefficients combined with either Fibonacci numbers or Lucas numbers. B. Radičić in [8] studied the eigenvalues of k-circulant matrices whose first row is $(J_1, \ldots, J_m)$ where J_i is the i-th Jacobsthal number, $i = 1, \ldots, m$. The author also studied k-circulant matrices whose first row is $(J_1^{-1}, \ldots, J_m^{-1})$. Moreover, B. Radičić in [9] studied the eigenvalues of k-circulant matrices whose first row is $(P_1, \ldots, P_m)$ where P_m is the m-th Pell number. In [13]-[15], the case with Horadam numbers is analysed.

Throghout the text, the identity matrix of order m is denoted by I_m and $diag\{a_{11}, \ldots, a_{mm}\}$ represent the diagonal matrix with diagonal elements $a_{11}, \ldots, a_{mm}$.

The paper is written as follows: motivated by all of previous results, at Section 2 a new type of matrix that generalizes the definition of k-circulant matrix, designated by (k, n)-*circulant like matrix,* is introduced ($(k, n)-circlike$ matrix for short). This matrix is denoted by $(k, n) - circlike(a_0, \ldots, a_{m-1})$ where $(a_0, \ldots, a_{m-1})$ is its first row. We study its properties, its eigenvalues and its inverse and some other properties, namely, it is shown that the product of two $(k, n) - circlike$ matrices its also $(k, n) - circlike$ matrix and the inverse of a $(k, n) - circlike$ matrix is also a $(k, n) - circlike$ matrix and its first row is the unique solution of a certain system of linear equations. Additionally, for a $(k, n) - circlike(g, qg, q^2g, \ldots, q^{m-1}g)$ we determine its inverse and write it as function of another (k, n)-circulant like matrix. At Section

3, for certain circulant like matrices whose entries are written as function of Horadam, Fibonacci, Jacobsthal and Pell numbers, we study its eigenvalues and write it as function of those numbers. Moreover, the same study is done if the entries are arithmetic sequences. At Section 4 the solution from polynomial equations through the information of the eigenvalues of (k,n)-circlike matrices are studied.

2. Circulant-like matrices and some properties

Throughout the text $\mathbb{K}$ denotes an algebraic closed field with characteristic zero and $k \in \mathbb{K}$. In this section some important properties for k-circulant matrices are recalled. Moreover, the concept of $(k,n)-$circulant like matrix, denoting it by $\mathbb{A}(k,n) = (k,n) - circlike(a_0, \ldots, a_{m-1})$ is introduced. Some of its properties are studied in particular, we prove that if A commutes with $\mathbb{Q} = \mathbb{Q}(k,n) = (k,n) - circlike(0,1,0,\ldots,0)$ then A is a $(k,n) - circlike$ matrix and $A = \sum_{r=0}^{m-1} a_r \mathbb{Q}^r$, with $a_r \in \mathbb{K}$. Moreover, the list of eigenvalues is written in an explicit way. It is also proved that the product and inverse of $(k,n) - circlike$ matrices is also a $(k,n) - circlike$ matrix. The first row of the $\mathbb{A}^{-1}(k,n)$ is the solution of a certain system of linear equations. Additionally, for a $(k,n) - circlike(g, qg, q^2 g, \ldots, q^{m-1} g)$ we determine its inverse and write it as function of another (k,n)-circulant like matrix.

The following known properties of k-circulant matrices that can be found in [5]. Let $\mathbb{R}^{m\times m}$ be the set of all $m \times m$ real matrices and $Q \in \mathbb{R}^{m\times m}$ be the following matrix

$$Q := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ k & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2}$$

We recall that Q has distinct eigenvalues, precisely, the m distinct m-th roots of k. The followings results are easily verified.

Lemma 2. *[5] Let $k \in \mathbb{K}\backslash\{0\}$. An $m \times m$ matrix A is a k-circulant matrix if only if $AQ = QA$. In this case A can be expressed as*

$$A = \sum_{i=0}^{m-1} a_i Q^i,$$

where $(a_0, a_1, \ldots, a_{m-1})$ is the first row of A, and Q is a matrix as in (2).

Let $F = (f_{ij})$ be the discrete Fourier matrix, with $f_{ij} = \frac{1}{\sqrt{m}}\omega^{ij},\ 0 \le i, j \le m-1$, where ω is any primitive m-th root of the unity,

$$\omega = \cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m}. \tag{3}$$

It is very well know that a matrix C is a circulant matrix if only if

$$F^*CF = diag\{\mu_0, \mu_1, \dots, \mu_{m-1}\},$$

where the numbers $\mu_i, i = 0, 1, \dots, m-1$ are the eigenvalues of C.
Let $k \in \mathbb{K}\backslash\{0\}$, and λ be any m-th root of k. Define $G(\lambda) = (g_{ij}(\lambda))$ as follows:

$$g_{ij}(\lambda) = \lambda^i f_{ij},\ 0 \le i, j \le m-1.$$

Lemma 3. *[5] Let $k \in \mathbb{K}\backslash\{0\}$. Then a matrix A is a k-circulant matrix if only if*

$$G(\lambda)^{-1}AG(\lambda) = diag\{\mu_0, \mu_1, \dots, \mu_{m-1}\},$$

where μ_i are the eigenvalues of A.

Lemma 4. *[5] Let $k \in \mathbb{K}\backslash\{0\}$ and $A(k)$ be a k-circulant matrix with first row $(a_0, a_1, \dots, a_{m-1})$ and suppose that λ is an m-th root of k. Then*

$$\mu_j = \sum_{i=0}^{m-1} a_i(\lambda\omega^j)^i,\ j = 0, 1, \dots, m-1$$

are the eigenvalues of $A(k)$. Moreover, in this case

$$a_i = \frac{1}{m}\sum_{j=0}^{m-1} \mu_j(\lambda\omega^j)^{-i},\ i = 0, 1, \dots, m-1.$$

In what follows, a new type of matrix that generalizes the definition of k-circulant matrix is introduced and called (k, n)-*circulant like matrix.*

Definition 5. *Let $k, n \in \mathbb{K}$. A (k, n)-circulant like matrix is a matrix in the form:*

$$\mathbb{A}(k,n) = \begin{pmatrix} a_0 & na_1 & na_2 & \cdots & na_{m-1} \\ ka_{m-1} & a_0 & a_1 & \ddots & \vdots \\ ka_{m-2} & kna_{m-1} & a_0 & \ddots & a_2 \\ \vdots & \ddots & \ddots & \ddots & a_1 \\ ka_1 & \cdots & kna_{m-2} & kna_{m-1} & a_0 \end{pmatrix}. \tag{4}$$

Then $\mathbb{A}(k,n)$ is simply written as $\mathbb{A}(k,n) = (k,n)-circlike(a_0, \dots, a_{m-1})$.

Remark 6. *If $\mathbb{A}(k,n)=(a_{i\ell})$ is or order m, then $a_{11}=a_0$, and*

$$a_{i\ell}=\begin{cases} na_{\ell-1}, & if \quad i=1,\ell\geq 2\\ a_{\ell-i}, & if \quad i>1,\ell\geq i\\ nka_{m-i+\ell} & if \quad i>\ell,\ell\geq 2\\ ka_{m-i+\ell} & if \quad i\geq 2,\ell=1.\end{cases}$$

Note that if $n=1$ then the matrix $\mathbb{A}(k,1)=A(k)$ for every $k\in\mathbb{K}$. Additionally, if $(A_\ell)_{\ell=0}^{m-1}$ is a family of $(k,n)-circlike(a_0^\ell,a_1^\ell,\dots,a_{m-1}^\ell)$ matrices and $\alpha_\ell\in\mathbb{K}$, $\ell=0,1,\dots,m-1$, then, it is easy to check that

$$\sum_{\ell=0}^{m-1}A_\ell=(k,n)-circlike(\sum_{\ell=0}^{m-1}\alpha_\ell a_0^\ell,\sum_{\ell=0}^{m-1}\alpha_\ell a_1^\ell,\dots,\sum_{\ell=0}^{m-1}\alpha_\ell a_{m-1}^\ell).$$

Consider now $\mathbb{Q}=\mathbb{Q}(k,n)=(k,n)-circlike(0,1,0,\dots,0)$ matrix:

$$\mathbb{Q}(k,n)=\begin{pmatrix} 0 & n & 0 & \cdots & 0\\ 0 & 0 & 1 & \ddots & \vdots\\ 0 & 0 & 0 & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & 1\\ k & \cdots & 0 & 0 & 0\end{pmatrix}.$$

It is easy to verify that $\mathbb{Q}^j(k,n)=(k,n)-circlike(0,\dots,0,1,0,\dots,0)$, where the element 1 is at the $(j+1)$-position, with $j=1,\dots,m$.

Lemma 7. *Let A be an $m\times m$ matrix. If $A\mathbb{Q}(k,n)=\mathbb{Q}(k,n)A$, then A is (k,n)-circulant.*

Proof. Let $A=(a_{ij}),i,j=1,\dots,m$. Consider

$$A\mathbb{Q}(k,n)-\mathbb{Q}(k,n)A=[R_1,R_2,\cdots,R_m]=0_m,$$

where R_i denotes the row $i=1,\dots,m$ of the matrix and

$$\begin{aligned} R_1 &= \begin{bmatrix} na_{21}-ka_{1m} & na_{22}-na_{11} & na_{23}-a_{12} & \cdots & na_{2m}-a_{1,m-1}\end{bmatrix},\\ R_p &= \begin{bmatrix} a_{p+1,1}-ka_{pm} & a_{p+1,2}-na_{p1} & a_{p+1,3}-a_{p,2} & \cdots & a_{p+1,m}-a_{p,m-1}\end{bmatrix},\\ R_m &= \begin{bmatrix} ka_{11}-ka_{m,m} & ka_{12}-na_{m1} & ka_{13}-na_{m2} & \cdots & ka_{1m}-a_{m,m-1}\end{bmatrix},\end{aligned}$$

for $p = 2, \ldots, m-1$. Then, from $R_i = 0, i = 1, \ldots, m$ we have:

$$R_1 = 0 \Rightarrow \begin{cases} a_{21} = \frac{k}{n} a_{1m} \\ a_{22} = a_{11} \\ a_{23} = \frac{1}{n} a_{12} \\ \vdots \\ a_{2m} = \frac{1}{n} a_{1,(m-1)} \end{cases} \qquad R_m = 0 \Rightarrow \begin{cases} a_{m1} = k a_{(m-1),m} = \frac{k}{n} a_{1,1} \\ a_{m2} = n a_{(m-1),1} = k a_{13} \\ a_{m3} = a_{(m-1),2} = k a_{14} \\ \vdots \\ a_{mm} = a_{(m-1),(m-1)} = a_{1,1} \end{cases}$$

and, for $p = 2, \ldots, m-1$ the following recursive formulas can be written:

$$R_p = 0 \Rightarrow \begin{cases} a_{(p+1),1} = k a_{pm} \\ a_{(p+1),2} = n a_{p1} \\ a_{(p+1),\ell} = \ a_{p,(\ell-1)}; \quad \ell = 3, 4, \ldots, m \end{cases}$$

Therefore, we can conclude that $A = (\frac{k}{n}, n) - circlike(\frac{1}{n} a_{11}, \frac{1}{n} a_{12}, \ldots, \frac{1}{n} a_{1m})$. $\square$
In the next lemma we get a similar decomposition for the matrices $\mathbb{A}(k,n)$ as in [5, Lemma 2].

Lemma 8. *An $m \times m$ matrix A is (k,n)-circulant like if and only if $A\mathbb{Q}(k,n) = \mathbb{Q}(k,n)A$. Moreover, if $A = \mathbb{A}(k,n)$ then it can be expressed as*

$$\mathbb{A}(k,n) = \sum_{i=0}^{m-1} a_i \mathbb{Q}^i(k,n),$$

where $a_i, i = 0, \ldots, m-1$ are the elements of the first row of $\mathbb{A}(k,n)$.

Proof. If $A\mathbb{Q}(k,n) = \mathbb{Q}(k,n)A$ then A is a $(k,n) - circlike$ matrix by Lemma 7.

Let $\mathbb{A}(k,n) = (a_{i\ell})$, and $\mathbb{Q}(k,n) = (q_{\ell j})$, then $\mathbb{A}(k,n)\mathbb{Q}(k,n) = (c_{ij})$, where $c_{i,j} = \sum_{l=1}^{m} a_{i\ell} q_{\ell j}$ with i, j $1 \leq i, j \leq m$. Note that $q_{\ell j} \in \{0, 1, k, n\}$, then

$$\begin{aligned} c_{i1} &= k a_{im}, c_{i2} = n a_{i1}, \forall i = 1, \ldots, m \\ c_{ij} &= a_{i,(j-1)}, \ j = 3, \ldots, m. \end{aligned}$$

Now considering $Q(k,n)A(k,n) = (d_{i,j})$, with $d_{i,j} = \sum_{l=1}^{m} q_{i\ell} a_{\ell j}$ with i, j $1 \leq i, j \leq m$. Then

$$d_{i1} = \begin{cases} n a_{21} & \text{if } i = 1; \\ a_{(i+1),1} & \text{if } i = 2, \ldots, m-1; \\ k a_{11} & \text{if } i = m. \end{cases}$$

But $ma_{21} = m(ka_{m-1}) = k(ma_{m-1}) = ka_{1m}$. For $i = 2, \dots, m-1$, $a_{i+1,1} = ka_{m-i} = ka_{im}$ and $ka_{11} = ka_0 = ka_{mm}$. Therefore

$$d_{i1} = \begin{cases} ka_{1m} & \text{if } i = 1ka_{im} \quad \text{if } i = 2, \dots, m-1; \\ ka_{mm} & \text{if } i = m. \end{cases}$$

In consequence,

$$c_{i1} = ka_{im} = d_{i1}, \forall i = 1, \dots, m.$$

In analogous way, we can prove that

$$c_{i2} = d_{i2}, \forall i = 1, \dots, m.$$

Let $i = 2, \dots, m$ and $j = 3, \dots, m$. Then

$$d_{ij} = \begin{cases} na_{2j} & \text{if } i = 1,\ j = 3, \dots, m; \\ a_{(i+1)(j)} & \text{if } i = 2, \dots, m-1,\ j = 3, \dots, m; \\ ka_{1j} & \text{if } i = m,\ j = 3, \dots, m. \end{cases}$$

Note that $a_{i(j-1)} = a_{(i+1)(j)}$ for all $i = 2, \dots, m$ and $j = 3, \dots, m$. Therefore $c_{i,j} = d_{ij}$ for all $i = 2, \dots, m-1$ and $j = 3, \dots, m$. Moreover as $a_{1(j-1)} = na_{j-2} = na_{2j}$ for all $j = 3, \dots, m$ then $c_{1j} = d_{1j}$ for all $j = 3, \dots, m$. Now, as $a_{m(j-1)} = kna_{j-1} = ka_{1j}$ then $c_{mj} = d_{mj}$ for all $j = 3, \dots, m$. Therefore $c_{ij} = d_{ij}$ for all $i, j = 1, \dots, m$.
Attending that if A, B are $(k,n)-$*circlike* matrices then $A + B$ is a $(k,n)-$*circlike* matrix, it is clear that,

$$\mathbb{A}(k,n) = a_0\mathbb{Q}^0(k,n) + a_1\mathbb{Q}(k,n) + \cdots + a_{m-1}\mathbb{Q}^{m-1}(k,n),$$

□

Let λ be any m-th root of (kn) and define the following diagonal matrix

$$\Lambda(k,n) = \begin{pmatrix} n & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \ddots & \vdots \\ 0 & 0 & \lambda^2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \lambda^{m-1} \end{pmatrix}.$$

Moreover, let ω be any primitive m-th root of unity, that is, ω is as in (3), and let F be the discrete Fourier matrix.

Then considering $T(k,n)=\Lambda(k,n)F$ the j-th column of $T(k,n)$ is given by

$$t(j)=\begin{pmatrix} t_{1j}\\ t_{2j}\\ t_{3j}\\ \vdots\\ t_{mj}\end{pmatrix}=\frac{1}{\sqrt{m}}\begin{pmatrix} n\\ \lambda\omega^{j-1}\\ (\lambda\omega^{j-1})^2\\ \vdots\\ (\lambda\omega^{j-1})^{m-1}\end{pmatrix}.$$

Moreover

$$\begin{aligned}\mathbb{Q}(k,n)t(j)=\begin{pmatrix} nt_{2j}\\ t_{3j}\\ t_{4j}\\ \vdots\\ kt_{1j}\end{pmatrix} &= \frac{1}{\sqrt{m}}\begin{pmatrix} n(\lambda\omega^{j-1})\\ (\lambda\omega^{j-1})^2\\ (\lambda\omega^{j-1})^3\\ \vdots\\ kn\end{pmatrix}\\ &= \lambda\omega^{j-1}\begin{pmatrix} t_{1j}\\ t_{2j}\\ t_{3j}\\ \vdots\\ t_{mj}\end{pmatrix}=\lambda\omega^{j-1}t(j),\ 1\le j\le m.\end{aligned}$$

Therefore

$$\mathbb{Q}(k,n)t(j)=\lambda\omega^{j-1}t(j),\ \forall 1\le j\le m.$$

And this implies that $T^{-1}(k,n)\mathbb{Q}(k,n)T(k,n) = diag\{\lambda,\lambda\omega,\ldots,\lambda\omega^{m-1}\}$, where $\lambda=(kn)^{1/m}$. Denote by $\mathcal{P}(\mathcal{D}_{\lambda,\omega})=\sum_{i=0}^{m-1}a_i\mathcal{D}^i_{\lambda,\omega}$, where $\mathcal{D}_{\lambda,\omega}= diag\{\lambda,\lambda\omega,\ldots,\lambda\omega^{m-1}\}$.

Lemma 9. *Let $\mathbb{A}(k,n)$ be a (k,n)-circulant like matrix as in (4). Then*

$$\mathbb{A}(k,n)=T\mathcal{P}(\mathcal{D}_{\lambda,\omega})T^{-1},$$

where $T=\Lambda(k,n)M$.

Proof. By Lemma 8, $\mathbb{A}(k,n)=a_0\mathbb{Q}^0(k,n)+a_1\mathbb{Q}(k,n)+\cdots+a_{n-1}\mathbb{Q}^{n-1}(k,n)$. Let $T=T(k,n)$ and $\mathbb{Q}=\mathbb{Q}(k,n)$. From previous observations, $\mathbb{Q}= T\mathcal{D}_{\lambda,\omega}T^{-1}$. Therefore $\mathbb{Q}^j=T\mathcal{D}^j_{\lambda,\omega}T^{-1}$ for all $j=1,\ldots,m-1$.

Then

$$\begin{aligned}\mathbb{A}(k,n) &= a_0 I_m + a_1 T^{-1}\mathcal{D}_{\lambda,\omega}T + \cdots + a_{n-1}T\mathcal{D}_{\lambda,\omega}^{m-1}T^{-1}\\ &= T(\sum_{i=0}^{m-1} a_i \mathcal{D}_{\lambda,\omega}^{i})T^{-1} = T\mathcal{P}(\mathcal{D}_{\lambda,\omega})T^{-1}.\end{aligned}$$

□

Lemma 10. *Let $\mathbb{A}(k,n)$ be a (k,n)-circlike$(a_0, a_1, \ldots, a_{m-1})$, and suppose that λ is an m-th root of kn Then*

$$\mu_j = \sum_{i=0}^{m-1} a_i(\lambda\omega^j)^i,\ j = 0, 1, \ldots, m-1 \tag{5}$$

are the eigenvalues of $\mathbb{A}(k,n)$ where $\lambda = \sqrt[m]{kn}$.

Proof. Denote by μ_j the eigenvalues of $\mathbb{A}(k,n)$ for $j = 0, \ldots, m-1$. By Lemma 9, the eigenvalues of $\mathbb{A}(k,n)$ are the same eigenvalues of $\mathcal{P}(\mathcal{D}_{\lambda,\omega})$. Moreover

$$\begin{aligned}\mathcal{P}(\mathcal{D}_{\lambda,\omega}) &= \sum_{i=0}^{m-1} a_i \mathcal{D}_{\lambda,\omega}^{i}\\ &= diag\Big\{\sum_{i=0}^{m-1} a_i\lambda^i, \sum_{i=0}^{m-1} a_i(\lambda\omega)^i, \sum_{i=0}^{m-1} a_i(\lambda\omega^2)^i, \ldots, \sum_{i=0}^{m-1} a_i(\lambda\omega^{m-1})^i\Big\}.\end{aligned}$$

Therefore $\mu_0 = \sum_{i=0}^{m-1} a_i\lambda^i, \mu_1 = \sum_{i=0}^{m-1} a_i(\lambda\omega)^i, \mu_2 = \sum_{i=0}^{m-1} a_i(\lambda\omega^2)^i, \ldots, \mu_{m-1} = \sum_{i=0}^{m-1} a_i(\lambda\omega^{m-1})^i$, and the proof follows. □

Remark 11. *In [7] it was defined that if a k-circulant matrix C is an element in the algebra generated by U where U is a generalized permutation matrix, that is $C = \sum_{r=0}^{m-1} c_r U^r$ for some $m \in \mathbb{N}$ where $c_r \in \mathbb{C}$, for all $r = 0, \ldots, m-1$ then it is designated by generalized weighted circulant matrices. The eigenvalues of C were studied and a closed formula for its expression was obtained. From Lemma 8, and attending to [7, Definition 1] the matrix $\mathbb{A}(k,n)$ is also a generalized weighted circulant matrix. That is $\mathbb{A}(k,n) = \sum_{r=0}^{m-1} a_r \mathbb{Q}(k,n)^r$. Then the spectrum of the matrices $\mathbb{A}(k,n)$ can also be shown as in [7, Theorem 1.17] however, we used at Lemma 10 a different approach that allowed us to study interesting properties of these matrices.*

Lemma 12. *A matrix A is a (k,n)-circulant like matrix if and only if*

$$T^{-1}AT = diag\{\mu_0, \mu_1, \ldots, \mu_{m-1}\}$$

where $T = \Lambda(k,n)M$ and $\mu_0, \ldots, \mu_{m-1}$ are the eigenvalues of A.

Proof. Firstly suppose that A is a (k,n)-circulant like matrix. Then, by Lemma 9 the necessary condition follows.
Reciprocally, suppose that $T^{-1}AT = diag\{\mu_0, \ldots, \mu_{m-1}\}$, with $T = \Lambda(k,n)M$. Let us denote by $D_\mu = diag\{\mu_0, \ldots, \mu_{m-1}\}$ and $\mathcal{D}_{\lambda,\omega} = diag\{\lambda, \lambda\omega, \ldots, \lambda\omega^{m-1}\}$. Then, $A = TD_\mu T^{-1}$ and $\mathbb{Q}(k,n) = TD_{\lambda,\omega}T^{-1}$. Moreover

$$\begin{aligned} A\mathbb{Q}(k,n) &= (TD_\mu T^{-1})(TD_{\lambda,\omega}T^{-1}) \\ &= TD_\mu D_{\lambda,\omega}T^{-1} = TD_{\lambda,\omega}D_\mu T^{-1} \end{aligned}$$

and

$$\mathbb{Q}(k,n)A = (TD_{\lambda,\omega}T^{-1})(TD_\mu T^{-1}) = TD_{\lambda,\omega}D_\mu T^{-1}$$

Therefore, A comutes with $\mathbb{Q}(k,n)$, by Lemma 8 we conclude that A is a $(k,n)-$ *circlike* matrix. ∎

Lemma 13. *Let λ be any m-th root of kn. Then a matrix A is $(k,n)-$ circlike matrix if and only if $A = \Lambda(k,n)C\Lambda(k,n)^{-1}$ for some circulant matrix C.*

Proof. By Lemma 12, a matrix A is $(k,n)-$ *circlike* matrix if and only if

$$T^{-1}AT = diag\{\mu_0, \mu_1, \ldots, \mu_{m-1}\},$$

where $T = \Lambda(k,n)M$ and $\mu_0, \ldots, \mu_{m-1}$ are the eigenvalues of A. Therefore the matrix A is $(k,n)-$ *circlike* if and only if

$$A = \Lambda(k,n)M diag\{\mu_0, \ldots, \mu_{m-1}\}M^*\Lambda(k,n)^{-1}.$$

Taking the circulant matrix $C = M diag\{\mu_0, \ldots, \mu_{m-1}\}M^*$ we get

$$A = \Lambda(k,n)C\Lambda(k,n)^{-1},$$

and the proof follows. ∎

Lemma 14. *If* $\mathbb{A}(k,n), \mathbb{B}(k,n)$ *are* $(k,n)-$*circlike matrices then* $\mathbb{A}(k,n)\mathbb{B}(k,n)$ *is also a* $(k,n)-$ *circlike matrix.*

Proof. Suppose that $\mathbb{A}(k,n)$ and $\mathbb{B}(k,n)$ are $(k,n)-$ *circlike* matrices. From Lemma 13, $\mathbb{A}(k,n) = \Lambda(k,n)C\Lambda(k,n)^{-1}$ and $\mathbb{B}(k,n) = \Lambda(k,n)\hat{C}\Lambda(k,n)^{-1}$ where C and $\hat{C}$ are some circulant matrices. Then

$$\begin{aligned}\mathbb{A}(k,n)\mathbb{B}(k,n) &= (\Lambda(k,n)C\Lambda(k,n)^{-1})(\Lambda(k,n)\hat{C}\Lambda(k,n)^{-1}) \\ &= \Lambda(k,n)C\hat{C}\Lambda(k,n)^{-1}\end{aligned}$$

As C and $\hat{C}$ are circulant matrices, then $C\hat{C}$ also is circulant matrix. Therefore by Lemma 13 we conclude that $\mathbb{A}(k,n)\mathbb{B}(k,n)$ is a $(k,n)-$ *circlike* matrix. $\square$

Lemma 15. *If* $\mathbb{A}(k,n)$ *is a* $(k,n)-$ *circlike matrix then its inverse is also a* $(k,n)-$ *circlike matrix.*

Proof. Suppose that $\mathbb{A}(k,n)$ is an invertible $(k,n)-$*circlike* matrix. From Lemma 13 we have

$$\mathbb{A}(k,n) = \Lambda(k,n)C\Lambda(k,n)^{-1}$$

for some C circulant matrix. Therefore $\mathbb{A}(k,n)^{-1} = \Lambda(k,n)C^{-1}\Lambda(k,n)^{-1}$. Note that A invertible if only if, none of its eigenvalues are zero, and in this case, C also is invertible. It follows from [5] that the inverse of a nonsingular circulant matrix is also a circulant matrix. Therefore by Lemma 13, $\mathbb{A}(k,n)^{-1}$ is a $(k,n)-$ *circlike* and the proof follow. $\square$

Lemma 16. *If* $kn \neq \gamma^m$ *then*

$$(\mathbb{Q}(k,n)-\gamma I_m)^{-1} = \frac{1}{kn-\gamma^m}\Big(\gamma^{m-1}I_m+\gamma^{m-2}\mathbb{Q}(k,n)+\gamma^{m-3}\mathbb{Q}^2(k,n)+\cdots+\mathbb{Q}^{m-1}(k,n)\Big).$$

Proof. As $\mathbb{Q}^m(k,n) = knI_m$, if

$$U = \frac{1}{kn-\gamma^m}\Big(\gamma^{m-1}I_m+\gamma^{m-2}\mathbb{Q}(k,n)+\gamma^{m-3}\mathbb{Q}^2(k,n)+\cdots+\mathbb{Q}^{m-1}(k,n)\Big).$$

It is easy to verify that

$$U(\mathbb{Q}(k,n)-\gamma I_m) = I_m \quad \textit{and} \quad (\mathbb{Q}(k,n)-\gamma I_m)U = I_m,$$

then $U = (\mathbb{Q}(k,n)-\gamma I_m)^{-1}$ $\square$

Lemma 17. *Let* $\mathbb{A}(k,n)$ *be an invertible* $(k,n)-circlike(a_0, a_1, a_2, \ldots, a_{m-1})$ *matrix as in (4). Then* $\mathbb{A}^{-1}(k,n)$ *is a* $(k,n) - circlike(a'_0, a'_1, a'_2, \ldots, a_{m-1})$ *matrix where* $(a'_0, a'_1, a'_2, \ldots, a'_{m-1})$ *is the unique solution of the following system of linear equations:*

$$\mathbb{A}(k,n) \begin{pmatrix} x_0 \\ kx_{m-1} \\ \vdots \\ kx_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{6}$$

Proof. Let $\mathbb{A}(k,n) = (k,n) - circlike(a_0, a_1, \ldots, a_{m-1})$ be an invertible matrix. From Lemma 15 the matrix $\mathbb{A}^{-1}(k,n)$ is also a $(k,n) - circlike$ matrix. Consider $\mathbb{A}^{-1}(k,n) = (k,n) - circlike(a'_0, a'_1, \ldots, a'_{m-1})$. Therefore, the identity $\mathbb{A}(k,n)\mathbb{A}^{-1}(k,n) = I_m$ is equivalent to

$$\begin{pmatrix} a_0 & na_1 & \cdots & na_{m-2} & na_{m-1} \\ ka_{m-1} & a_0 & \cdots & a_{m-3} & a_{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ ka_2 & kna_3 & \cdots & a_0 & a_1 \\ ka_1 & kna_2 & \cdots & kna_{m-1} & a_0 \end{pmatrix} \begin{pmatrix} a'_0 & na'_1 & \cdots & na'_{m-2} & na'_{m-1} \\ ka'_{m-1} & a'_0 & \cdots & a'_{m-3} & a'_{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ ka'_2 & kna'_3 & \cdots & a'_0 & a'_1 \\ ka'_1 & kna'_2 & & kna'_{m-1} & a'_0 \end{pmatrix} = I_m.$$

Thus, it is straightforward to conclude that $(a'_0, a'_1, a'_2, \ldots, a'_{m-1})$ is the solution of the system (6).

Moreover, since the matrix $\mathbb{A}^{-1}(k,n)$ is unique, it follows that $(a'_0, a'_1, a'_2, \ldots, a'_{m-1})$ is the unique solution of the system (6). $\square$

Remark 18. *From above we also can say that* $(a'_0, a'_1, a'_2, \ldots, a'_{m-1})$ *is the solution of* m *linear systems as*

$$\mathbb{A}(k,n) \begin{pmatrix} nx_{m-1} \\ x_{m-2} \\ \vdots \\ x_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Additionally, in this case this system does not depend on k*.*

Theorem 19. *Let* m *be a natural number greater than* 1 *and* $\mathbb{W}(k,n) = (k,n) - circlike(1, q, q^2, \ldots, q^{m-1})$*, where* $q \in \mathbb{R}$*. Then, if* $knq^m - 1 \neq 0$*,*

$$\mathbb{W}^{-1}(k,n) = \frac{1}{nkq^m - 1} \mathbb{W}^\star(k,n),$$

where $\mathbb{W}^\star(k,n) = (k,n) - circlike(-1, q, 0, \ldots, 0)$*.*

Proof. Consider the following system

$$\mathbb{W}(k,n)\begin{pmatrix} x_0 \\ kx_{m-1} \\ \vdots \\ kx_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{7}$$

In the system (7) let us consider $y_i = kx_i$ for $i = 1, \ldots, m-1$. Then, the system (7) can be rewritten as:

$$\mathbb{W}(k,n)\begin{pmatrix} x_0 \\ y_{m-1} \\ \vdots \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{8}$$

Let e_1 be the first canonical vector and $\overline{\mathbb{W}}(k,n) = (\mathbb{W}(k,n)|e_1)$, be the augmented matrix of (8). Applying elementary row operations in $\overline{\mathbb{W}}(k,n)$ we get $\overline{\mathbb{W}}(k,n) \sim \hat{\mathbb{W}}(k,n)$ where

$$\hat{\mathbb{W}}(k,n) = \begin{pmatrix} 1 & nq & nq^2 & \cdots & nq^{m-2} & nq^{m-1} & 1 \\ 0 & 1-nkq^m & q(1-nkq^m) & \cdots & q^{m-3}(1-nkq^m) & q^{m-2}(1-nkq^m) & -kq^{m-1} \\ 0 & 0 & 1-nkq^m & \cdots & q^{m-4}(1-nkq^m) & q^{m-3}(1-nkq^m) & -kq^{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-nkq^m & q(1-nkq^m) & -kq^2 \\ 0 & 0 & 0 & \cdots & 0 & 1-nkq^m & -kq \end{pmatrix}.$$

Therefore, from $\overline{\mathbb{W}}(k,n) = (\mathbb{W}(k,n)|e_1)$, the system of linear equations (8) is equivalent to the following system:

$$\begin{cases} x_0 + nq^{m-1}y_1 & = 1; \\ y_i & = 0 \quad \text{for } i = 2, \ldots, m-1 \\ (1-knq^m)y_1 & = -kq. \end{cases}$$

Then, the solution of the previous system is

$$\begin{cases} x_0 & = -\frac{1}{knq^m-1}; \\ y_1 & = \frac{kq}{knq^m-1}; \\ y_i & = 0,\ i = 2, \ldots, m-1. \end{cases}$$

As $y_i = kx_i$ for $i = 1, \ldots m-1$ the solution of the system (7) is given by

$$\begin{cases} x_0 & = -\frac{1}{knq^m-1}; \\ x_1 & = \frac{q}{knq^m-1}; \\ x_i & = 0, \quad \text{for } i = 2, \ldots, m-1. \end{cases}$$

Then, we proved that for e_1, the system of linear equations $W(k,n)x = e_1$ is consistent which implies that the matrix $W(k,n)$ is invertible. Moreover, from above the matrix $\mathbb{W}^{-1}(k,n)$ is a $(k,n)-cirlike(c_0, c_1, c_2, \ldots, c_{m-1})$ matrix and

$$\mathbb{W}^{-1}(k,n) \quad = \quad \frac{1}{knq^m - 1}\mathbb{W}^{\star}(k,n)$$

where $\mathbb{W}^{\star}(k,n) = (k,n)-circlike(-1, q, 0, \ldots, 0)$, and the theorem is proved. ∎

Corollary 20. *Let m be a natural number greater than 1 and suppost that $\mathbb{G}(k,n) = (k,n) - circlike(g, qg, q^2g, \ldots, q^{m-1}g)$ where $g \in \mathbb{C}$ with $g \neq 0$ and $q \in \mathbb{R}$. If $knq^m - 1 \neq 0$, then*

$$\mathbb{G}^{-1}(k,n) = \frac{1}{g(knq^m - 1)}\mathbb{G}^{\star}(k,n),$$

where $\mathbb{G}^{\star}(k,n) = (k,n) - circlike(-1, q, 0, \ldots, 0)$.

Proof. Since $\mathbb{G}(k,n) = g\mathbb{W}(k,n)$ then, by Theorem 19 we obtain

$$\mathbb{G}^{-1}(k,n) \quad = \quad \frac{1}{g}\mathbb{W}^{-1}(k,n) = \frac{1}{g(knq^m - 1)}\mathbb{W}^{\star}(k,n),$$

where $\mathbb{W}^{\star}(k,n) = (k,n)-circlike(-1, q, 0, \ldots, 0)$ and the proof follows. ∎

Remark 21. *If $k = 1$, then we get the result of Aldous Cesar F. Bueno (see [3, Theorem 3.8]). If $n = 1$ we get the result of Radičić (see [12, Corollary 2.1]).*

Example 22. *Let $\mathbb{A}(-2,3) = (-2,3)$-circlike$(a_0, a_1, \ldots, a_5)$ given by*

$$\begin{pmatrix}
\frac{1}{4}\left(1-\sqrt{3}i\right) & -\frac{3}{4}\left(1-\sqrt{3}i\right) & \frac{3}{4}\left(1-\sqrt{3}i\right) & -\frac{3}{4}\left(1-\sqrt{3}i\right) & \frac{3}{4}\left(1-\sqrt{3}i\right) & -\frac{3}{4}\left(1-\sqrt{3}i\right) \\
\frac{1}{2}\left(1-\sqrt{3}i\right) & \frac{1}{4}\left(1-\sqrt{3}i\right) & \frac{1}{4}\left(\sqrt{3}i-1\right) & \frac{1}{4}\left(1-\sqrt{3}i\right) & \frac{1}{4}\left(\sqrt{3}i-1\right) & \frac{1}{4}\left(1-\sqrt{3}i\right) \\
\frac{1}{2}\left(\sqrt{3}i-1\right) & \frac{3}{2}\left(1-\sqrt{3}i\right) & \frac{1}{4}\left(1-\sqrt{3}i\right) & \frac{1}{4}\left(\sqrt{3}i-1\right) & \frac{1}{4}\left(1-\sqrt{3}i\right) & \frac{1}{4}\left(\sqrt{3}i-1\right) \\
\frac{1}{2}\left(1-\sqrt{3}i\right) & -\frac{3}{2}\left(1-\sqrt{3}i\right) & \frac{3}{2}\left(1-\sqrt{3}i\right) & \frac{1}{4}\left(1-\sqrt{3}i\right) & \frac{1}{4}\left(\sqrt{3}i-1\right) & \frac{1}{4}\left(1-\sqrt{3}i\right) \\
\frac{1}{2}\left(\sqrt{3}i-1\right) & \frac{3}{2}\left(1-\sqrt{3}i\right) & -\frac{3}{2}\left(1-\sqrt{3}i\right) & \frac{3}{2}\left(1-\sqrt{3}i\right) & \frac{1}{4}\left(1-\sqrt{3}i\right) & \frac{1}{4}\left(\sqrt{3}i-1\right) \\
\frac{1}{2}\left(1-\sqrt{3}i\right) & -\frac{3}{2}\left(1-\sqrt{3}i\right) & \frac{3}{2}\left(1-\sqrt{3}i\right) & -\frac{3}{2}\left(1-\sqrt{3}i\right) & \frac{3}{2}\left(1-\sqrt{3}i\right) & \frac{1}{4}\left(1-\sqrt{3}i\right)
\end{pmatrix},$$

where $a_0 = \frac{1}{4}\left(1-\sqrt{3}i\right)$, $a_1 = -\frac{1}{4}\left(1-\sqrt{3}i\right)$, $a_2 = \frac{1}{4}\left(1-\sqrt{3}i\right)$, $a_3 = -\frac{1}{4}\left(1-\sqrt{3}i\right)$, $a_4 = \frac{1}{4}\left(1-\sqrt{3}i\right)$ *and* $a_5 = -\frac{1}{4}\left(1-\sqrt{3}i\right)$. *Then* $\mathbb{A}^{-1}(-2,3)$ *is given by*

$$\begin{pmatrix} \frac{1}{7}\left(\sqrt{3}i+1\right) & \frac{3}{7}\left(\sqrt{3}i+1\right) & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{7}\left(\sqrt{3}i+1\right) & \frac{1}{7}\left(\sqrt{3}i+1\right) & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{7}\left(\sqrt{3}i+1\right) & \frac{1}{7}\left(\sqrt{3}i+1\right) & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{7}\left(\sqrt{3}i+1\right) & \frac{1}{7}\left(\sqrt{3}i+1\right) & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{7}\left(\sqrt{3}i+1\right) & \frac{1}{7}\left(\sqrt{3}i+1\right) \\ -\frac{2}{7}\left(\sqrt{3}i+1\right) & 0 & 0 & 0 & 0 & \frac{1}{7}\left(\sqrt{3}i+1\right) \end{pmatrix}.$$

Therefore, if $\mathbb{A}(-2,3) = (-2,3)\text{–}circlike(\frac{1-\sqrt{3}i}{4}, \frac{\sqrt{3}i-1}{4}, \frac{1-\sqrt{3}i}{4}, \frac{\sqrt{3}i-1}{4}, \frac{1-\sqrt{3}i}{4}, \frac{\sqrt{3}i-1}{4})$, *then*

$$\begin{aligned} \mathbb{A}^{-1}(-2,3) &= (-2,3) - circlike(\frac{1+\sqrt{3}i}{7}, \frac{1+\sqrt{3}i}{7}, 0,0,0,0) \\ &= \left(\frac{1+\sqrt{3}i}{7}\right)((-2,3) - circlike(1,1,0,0,0,0)) \\ &= \left(\frac{1}{\frac{1-\sqrt{3}i}{4}(-7)}\right)((-2,3) - circlike(-1,-1,0,0,0,0)). \end{aligned}$$

3. Spectra of circulant like matrices whose entries sequence numbers are: Horadam, Fibonacci, Jacobsthal and Pell numbers

In this section we study the eigenvalues of circulant like matrices whose entries are as function of Horadam, Fibonacci, Jacobsthal and Pell numbers and arithmetic sequences.

3.1. Spectra of circulant like matrices whose entries are Horadam numbers

In [15] the generalized p-Horadam sequence $\{H_{p,m}\}_{m\in\mathbb{N}}$ was defined as:

$$H_{p,m+2} = f(p)H_{p,m+1} + g(p)H_{p,m} \tag{9}$$

with $H_{p,0} = a$ and $H_{p,1} = b$, $a, b \in \mathbb{K}$, $m \in \mathbb{N}$, $p \in \mathbb{R}^+$ and $f^2(p) + 4g(p) > 0$. If $f(p) = g(p) = 1$ and $a = 0$ and $b = 1$, the well known Fibonacci sequence is obtained. Moreover the Binet formula allows us to express the generalized p-Horadam number as function of the roots α and β of the characteristic equation $x^2 - f(p)x - g(p) = 0$. The Binet formula related to the sequence $\{H_{p,m}\}_{m\in\mathbb{N}}$ has the form

$$H_{p,m} = \frac{X\alpha^m - Y\beta^m}{\alpha - \beta}$$

where $X = b - a\beta$ and $Y = b - a\alpha$. Consider the following matrix $(k,n)-$ *circlike* matrix whose entries as written as function of Horadam numbers.

$$\mathbb{H}(k,n) = \begin{pmatrix} H_{p,0} & nH_{p,1} & nH_{p,2} & \cdots & nH_{p,m-1} \\ kH_{p,m-1} & H_{p,0} & H_{p,1} & \cdots & H_{p,m-2} \\ kH_{p,m-2} & kna_{m-1} & a_0 & \cdots & H_{p,m-3} \\ \vdots & \vdots & \vdots & & \vdots \\ kH_{p,1} & knH_{p,2} & knH_{p,3} & \cdots & H_{p,0} \end{pmatrix}. \tag{10}$$

Then, we can write $\mathbb{H}(k,n) = (k,n) - cirklike(H_{p,0},\ldots,H_{p,m-1})$. If $n = 1$ then the matrix (10) is analysed in [15]. When $n = 1$ we denote $\mathbb{H}(k,1) = H(k)$.
Consider a quadratic equation $Ax^2 + Bx + C = 0$ and let x_1 and x_2 be its roots, that is, $x_1 + x_2 = -\frac{B}{A}$, and $x_1x_2 = \frac{C}{A}$. Then as α and β are the roots of the characteristic equation $x^2 - f(p)x - g(p) = 0$ we get $\alpha + \beta = f(p)$ and $\alpha\beta = -g(p)$. The following result includes the case analyzed in [15].

Lemma 23. *Let $\mathbb{H}(k,n) = (k,n) - circlike(H_{p,0},\ldots,H_{p,m-1})$ where $H_{p,i}, i = 0,\ldots,m-1$ are the numbers as in (9). Then the eigenvalues of $\mathbb{H}(k,n)$ are given by:*

$$\mu_j = \frac{knH_{p,m} + \Big(nkg(p)H_{p,m-1} - b + af(p)\Big)\sqrt[m]{kn}\omega^j - H_{p,0}}{g(p)(\sqrt[m]{kn}\omega^j)^2 + f(p)\sqrt[m]{kn}\omega^j - 1}, \ \forall j = 0,1,\ldots,m-1$$

where $\omega = e^{\frac{2\pi i}{m}}$, $\sqrt{-1} = i$.

Proof. Let $\mathbb{H}(k,n) = (k,n) - circlike(H_{p,0},\ldots,H_{p,m-1})$. By Lemma 10, the eigenvalues of $\mathbb{H}(k,n)$ are given by

$$\mu_j = \sum_{i=0}^{m-1} H_{p,i}(\lambda\omega^j)^i, \ j = 0,1,\ldots,m-1, \text{ with } \lambda = \sqrt[m]{kn}.$$

Let $v = (kn)^{1/m}\omega^j$. Then using similar techniques as in [14] we have:

$$\begin{aligned}
\mu_j &= \sum_{i=0}^{m-1} H_{p,i}(\lambda\omega^j)^i = \frac{1}{\alpha-\beta}\Big(X\sum_{i=0}^{m-1}(\alpha v)^i - Y\sum_{i=0}^{m-1}(\beta v)^i\Big)\\
&= \frac{1}{\alpha-\beta}\Big(X\frac{(\alpha v)^m-1}{\alpha v-1} - Y\frac{(\beta v)^m-1}{\beta v-1}\Big)\\
&= \frac{1}{\alpha-\beta}\Big(\frac{-v^m(X\alpha^m - Y\beta^m) + \alpha\beta v^{m+1}(X\alpha^{m-1}-Y\beta^{m-1}) - (\beta X - \alpha Y)v + (X-Y)}{\alpha\beta v^2 - (\alpha+\beta)v + 1}\Big)\\
&= \frac{1}{\alpha-\beta}\Big(\frac{-v^m(X\alpha^m - Y\beta^m) - g(p)v^m(X\alpha^{m-1}-Y\beta^{m-1})v - (\beta X - \alpha Y)v + (X-Y)}{-g(p)v^2 - f(p)v + 1}\Big)\\
&= \frac{knH_{p,m} + \Big(nkg(p)H_{p,m-1} - b + af(p)\Big)\sqrt[m]{kn}\omega^j - H_{p,0}}{g(p)(\sqrt[m]{kn}\omega^j)^2 + f(p)\sqrt[m]{kn}\omega^j - 1}, \ \forall j = 0,1,\ldots,m-1.
\end{aligned}$$

∎

3.2. Spectra of circulant like matrices whose entries are Fibonacci numbers: We analyze here the spectra of circulant like matrices whose entries are Fibonacci numbers. We obtain a result on the eigenvalues for circulant like matrices $\mathbb{A}(k,n)$ which improves the result for the case when the matrices are k- circulant matrices whose elements are Fibonacci numbers [10].
The Fibonacci numbers F_j satisfy the following recursive relation:

$$F_j = F_{j-2} + F_{j-1}, \forall j \geq 2$$

with initial conditions $F_0 = 0$ and $F_1 = 1$. Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Note that $\alpha\beta = -1$, $\alpha+\beta = 1$ and $\alpha - \beta = \sqrt{5}$. Moreover, the Binet formula for the Fibonacci numbers is:

$$F_j = \frac{\alpha^j - \beta^j}{\alpha - \beta}, \ \forall j \geq 2. \tag{11}$$

The following result contains [10, Theorem 4].

Lemma 24. *Let $\mathbb{F}(k,n) = (k,n)-cirklike(F_1,\ldots,F_m)$, where $F_i, i = 1,\ldots,m$ are the numbers as in (11). Then the eigenvalues of $\mathbb{F}(k,n)$ are given by the following formulae:*

- *If $\sqrt[m]{kn}\omega^j = \frac{1}{\alpha}$, then $\mu_j = \frac{1}{\sqrt{5}}\Big(m\alpha + \frac{1-(-1)^m\beta^{2m}}{\sqrt{5}}\Big)$.*

- *If* $\sqrt[m]{kn}\omega^j = \frac{1}{\beta}$, *then* $\mu_j = \frac{1}{\sqrt{5}}\Big(\frac{1-(-1)^m\alpha^{2m}}{\sqrt{5}} - m\beta\Big)$.

- *If* $\sqrt[m]{kn}\omega^j \neq \frac{1}{\alpha}$ *and* $\sqrt[m]{kn}\omega^j \neq \frac{1}{\beta}$, *then* $\mu_j = \frac{knF_{m+1}+(nkF_m-1)\sqrt[m]{kn}\omega^j}{(\sqrt[m]{kn}\omega^j)^2+\sqrt[m]{kn}\omega^j-1}$.

Proof. Let $\mathbb{F}(k,n) = (k,n) - cirklike(F_1,\dots,F_m)$, where F_i are the numbers as in (11). By Lemma 10, the eigenvalues of $F(k,n)$ is given by

$$\mu_j = \sum_{i=0}^{m-1} F_{i+1}(\lambda\omega^j)^i, \;\; j = 0,1,\dots,m-1$$

where $\lambda = \sqrt[m]{nk}$. We divide the proof in two parts.

- If $\lambda\omega^j = \frac{1}{\alpha}$ or $\lambda\omega^j = \frac{1}{\beta}$, then the proof follows as in [10].

- If $\sqrt[m]{kn}\omega^j \neq \frac{1}{\alpha}$ and $\sqrt[m]{kn}\omega^j \neq \frac{1}{\beta}$. Then the proof follows from Lemma 23 taking $a = 0$, $b = 1$ and $f(p) = g(p) = 1$ and we get

$$\mu_j \;=\; \frac{knF_{m+1} + (knF_m - 1)\sqrt[m]{kn}\omega^j}{(\sqrt[m]{kn}\omega^j)^2 + \sqrt[m]{kn}\omega^j - 1}, \;\; \forall j = 0,1,\dots,m-1.$$

□

3.3. Spectra of circulant like matrices whose entries are Jacobsthal numbers

In this subsection we analyze the spectra of circulant like matrices whose entries are Jacobsthal numbers. Let us recall that the *Jacobsthal numbers* $\{J_m\}$ satisfy the following recurrence relation:

$$J_m = J_{m-1} + 2J_{m-2}, \forall m \geq 2, \tag{12}$$

with initial conditions $J_0 = 0$ and $J_1 = 1$. Let α and β be the roots of the equation $x^2 - x - 2 = 0$, that is $\alpha = 2$ and $\beta = -1$. Therefore, $\alpha\beta = -2$, $\alpha+\beta = 1$, and $\alpha-\beta = 3$. Moreover, the Binet formula for the Jacobsthal numbers is the following expression:

$$J_m = \frac{\alpha^m - \beta^m}{\alpha - \beta} = \frac{1}{3}\Big(2^m - (-1)^m\Big).$$

Lemma 25. *Let* $\mathbb{J}(k,n) = (k,n) - cirklike(J_1,\dots,J_m)$, *where* $J_i, i = 1,\dots,m$ *are the numbers as in (12). Then the eigenvalues of* $\mathbb{J}(k,n)$ *are given by:*

- *If* $\sqrt[m]{kn}\omega^j = \frac{1}{2}$, *then* $\mu_j = \frac{2}{9}\Big(3m - (-1)^m \frac{1}{2^m} + 1\Big)$, $j = 0, \ldots, m-1$.
- *If* $\sqrt[m]{kn}\omega^j = -1$, *then* $\mu_j = \frac{1}{9}\Big(3m + (-1)^{m+1}2^{m+1} + 2\Big)$, $j = 0, \ldots, m-1$.
- *If* $\sqrt[m]{kn}\omega^j \neq \frac{1}{2}$ *and* $\sqrt[m]{kn}\omega^j \neq -1$, *then* $\mu_j = \frac{knJ_{m+1} + (2knJ_m - 1)\sqrt[m]{kn}\omega^j}{2(\sqrt[m]{kn}\omega^j)^2 + \sqrt[m]{kn}\omega^j - 1}$, $j = 0, \ldots, m-1$.

Proof. Again, the proof is divided in two parts.

- if $\sqrt[m]{kn}\omega^j = \frac{1}{2}$ or $\sqrt[m]{kn}\omega^j = -1$, the proof follows as in [8].
- if $\sqrt[m]{kn}\omega^j \neq \frac{1}{2}$ and $\sqrt[m]{kn}\omega^j \neq -1$, then the proof follows from Lemma 23 taking $a = 0$, $b = 1$, $f(p) = 1$ and $g(p) = 2$ and we get

$$\mu_j = \frac{knJ_{m+1} + (2knJ_m - 1)\sqrt[m]{kn}\omega^j}{2(\sqrt[m]{kn}\omega^j)^2 + \sqrt[m]{kn}\omega^j - 1}, \ j = 0, \ldots, m-1.$$

∎

3.4. Spectra of circulant like matrices whose entries are Pell numbers

In this subsection we will focus on the eigenvalues of circulant like matrices whose entries are Pell numbers. Before we present our main results, let us recall that the Pell numbers $\{P_n\}$ satisfy the following recursive relation:

$$P_n = 2P_{n+1} + P_{n-2}, \ \forall n \geq 2, \tag{13}$$

with initial conditions $P_0 = 0$ and $P_1 = 1$. Let α and β be the roots of the equation $x^2 - 2x - 1 = 0$. Then $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Therefore, $\alpha\beta = -1$, $\alpha + \beta = 2$ and $\alpha - \beta = 2\sqrt{2}$. Moreover the Binet formula for the Pell numbers is given by

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{2\sqrt{2}}\Big((1+\sqrt{2})^n - (1-\sqrt{2})^n\Big).$$

Lemma 26. *Let* $\mathbb{P}(k,n) = (k,n) - cirklike(P_1, \ldots, P_m)$, *where* P_i *are the numbers as in (13). Then the eigenvalues of* $\mathbb{P}(k,n)$ *are given by:*

- *If* $\sqrt[m]{kn}\omega^j = \frac{1}{\alpha}$, *then* $\mu_j = \frac{1}{2\sqrt{2}}\Big(m\alpha + \frac{1-(-1)^m\beta^{2m}}{2\sqrt{2}}\Big)$.
- *If* $\sqrt[m]{kn}\omega^j = \frac{1}{\beta}$, *then* $\mu_j = \frac{1}{2\sqrt{2}}\Big(\frac{1-(-1)^m\alpha^{2m}}{2\sqrt{2}} - m\beta\Big)$.

- *If* $\sqrt[m]{kn}\omega^j \neq \frac{1}{\alpha}$ *and* $\sqrt[m]{kn}\omega^j \neq \frac{1}{\beta}$, *then* $\mu_j = \frac{knP_{m+1}+(nkP_m-1)\sqrt[m]{kn}\omega^j}{(\sqrt[m]{kn}\omega^j)^2+2\sqrt[m]{kn}\omega^j-1}$.

Proof. Let $\mathbb{P}(k,n) = (k,n) - circlike(P_1, \ldots, P_m)$. By Lemma 10, the eigenvalues of $\mathbb{P}(k,n)$ is given by $\mu_j = \sum_{i=0}^{m-1} F_{i+1}(\lambda\omega^j)^i$, $j = 0, 1, \ldots, m-1$,where $\lambda = \sqrt[m]{kn}$. As before, we divide the proof in two parts.

- if $\sqrt[m]{kn}\omega^j = \frac{1}{\alpha}$ or $\sqrt[m]{kn}\omega^j = \frac{1}{\beta}$, then the proof follows from [9].
- if $\sqrt[m]{kn}\omega^j \neq \frac{1}{\alpha}$ and $\sqrt[m]{kn}\omega^j \neq \frac{1}{\beta}$, the proof follows from Lemma 23 taking $a = 0$, $b = 1$, $f(p) = 2$ and $g(p) = 1$ and we get

$$\mu_j = \frac{knP_{m+1} + (nkP_m - 1)\sqrt[m]{kn}\omega^j}{(\sqrt[m]{kn}\omega^j)^2 + 2\sqrt[m]{kn}\omega^j - 1}.$$

∎

3.5. Spectra of circulant like matrices whose entries are arithmetic sequences
In this section we study the spectra of circulant like matrices whose entries are arithmetic sequences. In fact, the next result includes the result in [11, Theorem 2.1].

Lemma 27. *Let* $\mathbb{A}(k,n) = (k,n) - circlike(a, a+d, \ldots, a+(m-1)d)$. *Then*

- *If* $\sqrt[m]{kn}\omega^j = 1$, *then* $\mu_j = \frac{n}{2}(2a + (n-1)d)$.
- *If* $\sqrt[m]{kn}\omega^j \neq 1$, *then* $\mu_j = a\frac{kn-1}{\sqrt[m]{kn}\omega^j-1} + d\frac{\sqrt[m]{kn}\omega^j(1+mnk-nk)-mnk}{(1-\sqrt[m]{kn}\omega^j)^2}$.

Proof. *First we Suppose that* $\sqrt[m]{kn}\omega^j = 1$. *By Lemma 10, the eigenvalues of* $\mathbb{A}(k,n)$ *are given by* $\mu_j = \sum_{i=0}^{m-1} a_i(\lambda\omega^j)^i$, $j = 0, 1 \ldots, m-1$ *where* $\lambda = \sqrt[m]{kn}$. *Then* $\mu_j = \sum_{i=0}^{m-1} a_i(\lambda\omega^j)^i = \sum_{i=0}^{m-1} a_i(1) = \frac{m}{2}(2a + (m-1)d)$.
Consider now $\sqrt[m]{kn}\omega^j \neq 1$ *and let* $x = \sqrt[m]{kn}\omega^j$. *Using Lemma 10 we get*

$$\begin{aligned}
\mu_j &= \sum_{i=0}^{m-1} a_i(\lambda\omega^j)^i = \sum_{i=0}^{m-1}(a+id)x^i = a\sum_{i=0}^{m-1} x^i + d\sum_{i=0}^{m-1} ix^i \\
&= a\frac{x^m-1}{x-1} + dx(1 + 2x + 3x^2 + \cdots + (m-1)x^{m-2}) \\
&= a\frac{kn-1}{x-1} + dx\Big(\frac{x-x^m}{1-x}\Big)' = a\frac{kn-1}{x-1} + d\frac{x(1+mnk-nk)-mnk}{(1-x)^2} \\
&= a\frac{kn-1}{\sqrt[m]{kn}\omega^j - 1} + d\frac{\sqrt[m]{kn}\omega^j(1+mnk-nk)-mnk}{(1-\sqrt[m]{kn}\omega^j)^2},
\end{aligned}$$

and the proof follows. ∎

Remark 28. *We note that the results at Lemmas 24, 25, 26 and 27 can also be obtained using generalized permutation matrices. Additionally, we suspect that the properties proved at Lemmas 8, 14 and 15 can also be proven using generalized permutation matrices however, the proofs seem to the authors harder to obtain.*

4. Some examples and theoretical applications

In [7] the authors studied the solution from polynomial equations through the information of the eigenvalues of k-circulant matrices. Using same arguments, and using (k, n)-circulant matrices we perform a similar study using other polynomials equations.

Example 29. *For any numbers $a \neq 0$, the following polynomial*

$$P_5(x) = \frac{a^5}{3125} + \frac{a^4 x}{125} + \frac{2a^3x^2}{25} + \frac{2a^2x^3}{5} + ax^4 + x^5. \tag{14}$$

has to $x_0 = -\frac{a}{5}$ as root of multiplicity five. Indeed,

The polynomial P_5 can be written as:

$$P_5(x) \;=\; \Big(\frac{a}{5}\Big)^5 + 5\Big(\frac{a}{5}\Big)^4 x + 10\Big(\frac{a}{5}\Big)^3 x^2 + 10\Big(\frac{a}{5}\Big)^2 x^3 + 5\frac{a}{5}x^4 + x^5.$$

Let $\mathbb{A}(k, n) = (k, n)-circlike(a_0, a_1, a_3, a_4)$. By Lemma 10, its eigenvalues are given as in (5) with $m = 4$. Taking $a_1 = a_2 = a_3 = a_4 = 0$ then $\mathbb{A}(k, n) = a_0 I_5$. The characteristic polynomial of $\mathbb{A}(k, n)$ is given by

$$P_{\mathbb{A}(k,n)}(x) = a_0^5 - 5a_0^4 x + 10a_0^3 x^2 - 10a_0^2 x^3 + 5a_0 x^4 - x^5$$

Moreover, as the eigenvalues of $\mathbb{A}(k, n)$ are $\mu_0 = a_0$ with multiplicity five, comparing $P_{\mathbb{A}(k,n)}(x)$ and $P_5(x)$ we get $a_0 = -\frac{a}{5}$ and so $x_0 = -\frac{a}{5}$ is a root of $P_5(x)$ of multiplicity five.

Example 30. *For any non-zero real numbers a and b, the quintic equation*

$$-b + \frac{a^5}{3125} + \frac{a^4 x}{125} + \frac{2a^3x^2}{25} + \frac{2a^2x^3}{5} + ax^4 + x^5 = 0 \tag{15}$$

is solvable and its roots are $x_m = -\frac{a}{5} + \sqrt[5]{b}\omega^m$, $m = 0, 1, 2, 3, 4$. In fact, let us consider $\mathbb{A}(k, n) = (k, n) - circlike(a_0, a_1, a_3, a_4)$. By Lemma 10, its *eigenvalues are given as before as in (5) with* $m = 4$. *Taking* $a_2 = a_3 = a_4 = 0$ *we have* $\mathbb{A}(k,n) = a_0 I_5 + a_1 \mathbb{Q}(k,m)$. *The characteristic polynomial of* $\mathbb{A}(k,n)$ *is given by* $P_{\mathbb{A}(k,n)}(x) = nka_1^5 + a_0^5 - 5a_0^4 x + 10a_0^3 x^2 - 10a_0^2 x^3 + 5a_0 x^4 - x^5$, *and the eigenvalues are* $\mu_j = a_0 + a_1 \sqrt[5]{nk}\omega^j = a_0 + \sqrt[5]{a_1^5 nk}\omega^j$, $j = 0,1,2,3,4$. *As in Example 29 we get*

$$\left(x + \frac{a}{5}\right)^5 - b = 0.$$

Therefore the solutions are given by $x_m = -\frac{a}{5} + \sqrt[5]{b}\omega^m$, $m = 0,1,2,3,4$.

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