The H-join of arbitrary families of graphs

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Abstract

The H-join of a family of graphs $\mathcal{G} = \{G_1, \ldots, G_p\}$, also called the generalized composition, $H[G_1, \ldots, G_p]$, where all graphs are undirected, simple and finite, is the graph obtained from the graph H replacing each vertex i of H by G_i and adding to the edges of all graphs in $\mathcal{G}$ the edges of the join $G_i \vee G_j$, for every edge ij of H. Some well known graph operations are particular cases of the H-join of a family of graphs $\mathcal{G}$ as it is the case of the lexicographic product (also called composition) of two graphs H and G, $H[G]$, which coincides with the H-join of family of graphs $\mathcal{G}$ where all the graphs in $\mathcal{G}$ are isomorphic to a fixed graph G.
So far, the known expressions for the determination of the entire spectrum of the H-join in terms of the spectra of its components and an associated matrix are limited to families of regular graphs. In this paper, we extend such a determination to families of arbitrary graphs.

Keywords: H-join, lexicographic product, graph spectra.

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1 Introduction

Nearly five decades since the publication in 1974 of Allen Shweenk's article [17], the determination of the spectrum of the generalized composition $H[G_1, \ldots, G_p]$ (recently designated H-join of $\mathcal{G} = \{G_1, \ldots, G_p\}$ [2]), in terms of the spectra of the graphs in $\mathcal{G}$ and an associated matrix, where all graphs are undirected, simple and finite, was limited to families $\mathcal{G}$ of regular graphs. In this work, the determination of this spectrum is extended to families of arbitrary graphs (which should be undirected, simple and finite).

The generalized composition $H[G_1, \ldots, G_p]$, introduced in [17, p. 167] was rediscovered in [2] under the designation of H-join of a family of graphs $\mathcal{G} = \{G_1, \ldots, G_p\}$, where H is a graph of order p. In [17, Th. 7], assuming that $G_1, \ldots, G_p$ are all regular graphs and taking into account that $V(G_1) \cup \cdots \cup V(G_p)$ is an equitable partition π, the characteristic polynomial of $H[G_1, \ldots, G_p]$ is determined in terms of the characteristic polynomials of the graphs $G_1, \ldots, G_p$ and the matrix associated to π.

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Using a generalization of a Fiedler's result [7, Lem. 2.2] obtained in [2, Th. 3], the spectrum of the H-join of a family of regular graphs (not necessarily connected) is determined in [2, Th. 5]. When the graphs of the family $\mathcal{G}$ are all isomorphic to a fixed graph G, the H-join of $\mathcal{G}$ is the same as the lexicographic product (also called the composition) of the graphs H and G which is denoted as $H[G]$ (or $H \circ G$). The lexicographic product of two graphs was introduced by Harary in [11] and Sabidussi in [16] (see also [12, 10]). From the definition, it is immediate that this graph operation is associative but not commutative.

In [1], as an application of the H-join spectral properties, the lexicographic powers of a graph H were considered and their spectra determined, when H is regular. The k-th lexicographic power of H, H^k, is the lexicographic product of H by itself k times (then $H^2 = H[H], H^3 = H[H^2] = H^2[H], \dots$). As an example, in [1], the spectrum of the 100-th lexicographic power of the Petersen graph, which has a gogool number (that is, 10^{100}) of vertices, was determined. With these powers, H^k, in [3] the lexicographic polynomials were introduced and their spectra determined, for connected regular graphs H, in terms of the spectrum of H and the coefficients of the polynomial.

Other particular H-joins appear in the literature under different designations, as it is the case of the mixed extension of a graph H studied in [8], where special attention is given to the mixed extensions of P_3. The mixed extension of a graph H, with vertex set $V(H) = \{1, \dots, p\}$, is the H-join of a family of graphs $\mathcal{G} = \{G_1, \dots, G_p\}$, where each graph $G_i \in \mathcal{G}$ is a complete graph or its complement. From the H-join spectral properties, we may conclude that the mixed extensions of a graph H of order p has at most p eigenvalues unequal to 0 and -1.

The remaining part of the paper is organized as follows. The focus of Section 2 is the preliminaries. Namely, the notation and basic definitions, the main spectral results of the H-join graph operation and the more relevant properties, in the context of this work, of the main characteristic polynomial and walk matrix of a graph. In section 3, the main result of this artice, the determination of the spectrum of the H-join of a family of arbitrary graphs is deduced.

2 Preliminaries

2.1 Notation and basic definitions

Throughout the text we consider undirected, simple and finite graphs, which are just called graphs. The vertex set and the edge set of a graph G is denoted by $V(G)$ and $E(G)$, respectively. The order of G is the cardinality of its vertex set and when it is n we consider that $V(G) = \{1, \dots, n\}$. The eigenvalues of adjacency matrix of a graph G, $A(G)$, of order n are also called the eigenvalues of G. For each distinct eigenvalue μ of G, $\mathcal{E}_G(\mu)$ denotes the eigenspace of μ whose dimension is equal to the algebraic multiplicity of μ, $m(\mu)$. The spectrum of G is denoted $\sigma(G) = \{\mu_1^{[m_1]}, \dots, \mu_s^{[m_s]}, \mu_{s+1}^{[m_{s+1}]}, \dots, \mu_t^{[m_t]}\}$, where $t \le n$ and $\mu_i^{[m_i]}$ means that $m(\mu_i) = m_i$. When we say that μ is an eigenvalue of G with zero multiplicity (that is, $m(\mu) = 0$) it means that $\mu \notin \sigma(G)$. The distinct eigenvalues of G are indexed in such way that the eigenspaces $\mathcal{E}_G(\mu_i)$, for $1 \le i \le s$, are not orthogonal to $\mathbf{j}_n$, the all-1 vector with n entries. The eigenvalues μ_i, with $1 \le i \le s$ are called main eigenvalues of G and the remaining distinct eigenvalues non-main. The concept of main (non-main) eigenvalue was introduced in [4] and further investigated in several publications. As it is well known, the largest eigenvalue of a connected graph is main and its remaining distinct eigenvalues are non-main [5]. A survey on main eigenvalues was published in [15].

2.2 The H-join operation

Now we recall the definition of the H-join of a family of graphs [2].

Definition 2.1. *Consider a graph H with vertex subset $V(H) = \{1, \ldots, p\}$ and a family of graphs $\mathcal{G} = \{G_1, \ldots, G_p\}$ such that $|V(G_1)| = n_1, \ldots, |V(G_p)| = n_p$. The H-join of $\mathcal{G}$ is the graph*

$$G = \bigvee_H \mathcal{G}$$

in which $V(G) = \bigcup_{j=1}^{p} V(G_j)$ and $E(G) = \left(\bigcup_{j=1}^{p} E(G_j)\right) \cup \left(\bigcup_{rs \in E(H)} E(G_r \vee G_s)\right)$, where $G_r \vee G_s$ denotes the join.

Theorem 2.2. *[2] Let G be the H-join as in Definition 2.1, where $\mathcal{G}$ is a family of regular graphs such that G_1 is d_1-regular, G_2 is d_2-regular, … and G_p is d_p-regular. Then*

$$\sigma(G) = \left(\bigcup_{j=1}^{p} \left(\sigma(G_j) \setminus \{d_j\} \right) \right) \cup \sigma(\widetilde{C}), \tag{1}$$

where the matrix $\widetilde{C}$ has order p and is such that

$$\left(\widetilde{C}\right)_{rs} = \begin{cases} d_r & \text{if } r = s, \\ \sqrt{n_r n_s} & \text{if } rs \in E(H), \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

and the set operations in (1) are done considering possible repetitions of elements of the multisets.

From the above theorem, if there is $G_i \in \mathcal{G}$ which is disconnected, with q components, then its regularity d_i appears q times in the multiset $\sigma(G_i)$. Therefore, according to (1), remains as an eigenvalue of G with multiplicity $q - 1$.

From now on, given a graph H, we consider the following notation:

$$\delta_{i,j}(H) = \begin{cases} 1 & \text{if } ij \in E(H), \\ 0 & \text{otherwise.} \end{cases}$$

Before the next result, it is worth observe the following. Considering a graph G, it is always possible to extend a basis of the eigensubspace associated to a main eigenvalue μ_j, $\mathcal{E}_G(\mu_j) \cap \mathbf{j}^\top$, to one of $\mathcal{E}_G(\mu_j)$ by adding an eigenvector $\hat{\mathbf{u}}_{\mu_j}$ which is uniquely determined without considering its multiplication by a nonzero scalar. The eigenvector $\hat{\mathbf{u}}_{\mu_j}$ is called the main eigenvector of μ_j. The subspace with basis $\{\hat{\mathbf{u}}_{\mu_1}, \ldots, \hat{\mathbf{u}}_{\mu_s}\}$ is the main subspace of G and is denoted as $Main(G)$. Note that for each main eigenvector $\hat{\mathbf{u}}_{\mu_j}$ of the basis of $Main(G)$, $\hat{\mathbf{u}}_{\mu_j}^T \mathbf{j} \neq 0$.

Lemma 2.3. *Let G be the H-join as in Definition 2.1 and $\mu_{i,j} \in \sigma(G_i)$. Then $\mu_{i,j} \in \sigma(G)$ with multiplicity*

$$\begin{cases} m(\mu_{i,j}) & \text{whether } \mu_{i,j} \text{ is a non-main eigenvalue of } G_i, \\ m(\mu_{i,j}) - 1 & \text{whether } \mu_{i,j} \text{ is a main eigenvalue of } G_i. \end{cases}$$

Proof. Denoting $\delta_{i,j} = \delta_{i,j}(H)$, then $\delta_{i,j}\mathbf{j}_{n_i}\mathbf{j}_{n_j}^T$ is an $n_i \times n_j$ matrix whose entries are 1 if $ij \in E(H)$ and 0 otherwise. Then the adjacency matrix of G has the form

$$A(G) = \begin{pmatrix} A(G_1) & \delta_{1,2}\mathbf{j}_{n_1}\mathbf{j}_{n_2}^T & \cdots & \delta_{1,p-1}\mathbf{j}_{n_1}\mathbf{j}_{n_{p-1}}^T & \delta_{1,p}\mathbf{j}_{n_1}\mathbf{j}_{n_p}^T \\ \delta_{2,1}\mathbf{j}_{n_2}\mathbf{j}_{n_1}^T & A(G_2) & \cdots & \delta_{2,p-1}\mathbf{j}_{n_2}\mathbf{j}_{n_{p-1}}^T & \delta_{2,p}\mathbf{j}_{n_2}\mathbf{j}_{n_p}^T \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta_{p-1,1}\mathbf{j}_{n_{p-1}}\mathbf{j}_{n_1}^T & \delta_{p-1,2}\mathbf{j}_{n_{p-1}}\mathbf{j}_{n_2}^T & \cdots & A(G_{p-1}) & \delta_{p-1,p}\mathbf{j}_{n_{p-1}}\mathbf{j}_{n_p}^T \\ \delta_{p,1}\mathbf{j}_{n_p}\mathbf{j}_{n_1}^T & \delta_{p,2}\mathbf{j}_{n_p}\mathbf{j}_{n_2}^T & \cdots & \delta_{p,p-1}\mathbf{j}_{n_p}\mathbf{j}_{n_{p-1}}^T & A(G_p) \end{pmatrix}.$$

Let $\mathbf{u}_{i,j}$ be an eigenvector of $A(G_i)$ associated to an eigenvalue $\mu_{i,j}$ whose sum of its components is zero (then, $\mu_{i,j}$ is non-main or it is main with multiplicity greater than one). Then,

$$A(G)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{u}_{i,j} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \delta_{1,i}\left(\mathbf{j}_{n_i}^T\mathbf{u}_{i,j}\right)\mathbf{j}_{n_1} \\ \vdots \\ \delta_{i-1,i}\left(\mathbf{j}_{n_i}^T\mathbf{u}_{i,j}\right)\mathbf{j}_{n_{i-1}} \\ A(G_i)\mathbf{u}_{i,j} \\ \delta_{i+1,i}\left(\mathbf{j}_{n_i}^T\mathbf{u}_{i,j}\right)\mathbf{j}_{n_{i+1}} \\ \vdots \\ \delta_{p,i}\left(\mathbf{j}_{n_i}^T\mathbf{u}_{i,j}\right)\mathbf{j}_{n_p} \end{pmatrix}. \tag{3}$$

It should be noted that when $\mu_{i,j}$ is main, there are $m(\mu_{i,j}) - 1$ linear independent eigenvectors belonging to $\mathcal{E}_G(\mu_{i,j}) \cap \mathbf{j}^\top$. □

2.3 The main characteristic polynomial and the walk matrix

If G has s distinct main eigenvalues $\mu_1, \ldots, \mu_s$, then the main characteristic polynomial of G is the polynomial of degree s [15]

$$\begin{aligned} m_G(x) &= \Pi_{i=1}^s (x - \mu_i) \\ &= x^s - c_0 - c_1 x - \cdots - c_{s-2}x^{s-2} - c_{s-1}x^{s-1}. \end{aligned} \tag{4}$$

Note that if μ is a main eigenvalue of G so is its algebraic conjugate μ^*. Therefore, the coefficients of $m_G(x)$ are integers as referred in [15] (see also [6]).

Let G be a graph. From [15, Prop. 2.1] it is immediate that $m_G(A(G))\mathbf{j} = \mathbf{0}$. Therefore,

$$A^s(G)\mathbf{j} = c_0\mathbf{j} + c_1 A(G)\mathbf{j} + \cdots + c_{s-2}A^{s-2}(G)\mathbf{j} + c_{s-1}A^{s-1}(G)\mathbf{j}. \tag{5}$$

Given a graph G of order n, let us consider the $n \times k$ matrix [13, 14]

$$\mathbf{W}_{G;k} = \left(\mathbf{j}, A(G)\mathbf{j}, A^2(G)\mathbf{j}, \ldots, A^{k-1}(G)\mathbf{j}\right).$$

The vector space spanned by the columns of $\mathbf{W}_{G;k}$ is denoted by $ColSp\mathbf{W}_{G;k}$.

Theorem 2.4. [9] *Let G be a graph of order n with s distinct main eigenvalues. If $k \geq s$, then $\mathbf{W}_{G;k}$ has rank s.*

As an immediate consequence of Theorem 2.4, the number of distinct main eigenvalues is $s = \min\{k : \{\mathbf{j}, A(G)\mathbf{j}, A^2(G)\mathbf{j}, \dots, A^k(G)\mathbf{j}\}$ is linearly dependent$\}$.

For a graph G of order n with s distinct main eigenvalues, the $n \times s$ matrix $\mathbf{W}_{G;s} = \big(\mathbf{j}, A(G)\mathbf{j}, A^2(G)\mathbf{j}, \dots, A^{s-1}(G)\mathbf{j}\big)$ is referred to be the walk matrix of G and is just denoted as $\mathbf{W}_G$.

From (5) we have the following corollary.

Corollary 2.5. *The s-th column of $A(G)\mathbf{W}_G$ is $A^s(G)\mathbf{j} = \mathbf{W}_G \begin{pmatrix} c_0 \\ \vdots \\ c_{s-2} \\ c_{s-1} \end{pmatrix}$, where c_j, for $j = 0, \dots, s-1$, are the coefficients of the main characteristic polynomial of $m_G(x)$, given in* (4).

From this corollary we may conclude that the coefficients of the main characteristic polynomial of G can be determined from its walk matrix $\mathbf{W}_G$, solving the linear system $\mathbf{W_G x} = A^s(G)\mathbf{j}$.

Theorem 2.6. *[15, Th. 2.4] Let G be a graph with s distinct main eigenvalues. Then the column space $ColSp\mathbf{W_G}$ coincides with $Main(G)$.*

Moreover $Main(G)$ and the vector space spanned by the vectors orthogonal to $Main(G)$, $(Main(G))^\perp$, are both $\mathbf{A}$–invariant [15, Th. 2.4].

From the above definitions, if G is a r-regular graph of order n, since its largest eigenvalue, r, is the unique main eigenvalue, then $m_G(x) = x - r$ and $W_G = (\mathbf{j}_n)$.

3 The spectrum of the H-join of a family of arbitrary graphs

Before the main result of this paper we need to define a special matrix $\widetilde{\mathbf{W}}$ which will be called the H-join associated matrix.

Definition 3.1. *Let G be the H-join as in Definition 2.1. The main eigenvalues of each $G_i \in \mathcal{G}$ are $\mu_{i,1}, \dots, \mu_{i,s_i}$ and the corresponding main characteristic polynomial* (4) *is $m_{G_i}(x) = x^s - c_{i,0} - c_{i,1}x - \dots - c_{i,s_i-1}x^{s-1}$. For $1 \leq i \leq p$, let W_{G_i} be the walk matrix of G_i and consider the matrix*

$$\widetilde{\mathbf{W}}_i = \begin{pmatrix} \overbrace{\delta_{i,1}\mathbf{j}_{n_1}^T W_{G_1}}^{s_1 \ columns} & \cdots & \overbrace{\delta_{i,i-1}\mathbf{j}_{n_{i-1}}^T W_{G_{i-1}}}^{s_{i-1} \ columns} & 0 & 0 & \cdots & 0 & c_{i,0} & \overbrace{\delta_{i,i+1}\mathbf{j}_{n_{i+1}}^T W_{G_{i+1}}}^{s_{i+1} \ columns} & \cdots & \overbrace{\delta_{i,p}\mathbf{j}_{n_p}^T W_{G_p}}^{s_p \ columns} \\ \mathbf{0} & \cdots & \mathbf{0} & 1 & 0 & \cdots & 0 & c_{i,1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & 0 & 1 & \cdots & 0 & c_{i,2} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 1 & c_{i,s_i-1} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}.$$

The H-join associated matrix is the $s \times s$ matrix, where $s = \sum_{i=1}^{p} s_i$,

$$\widetilde{\mathbf{W}} = \begin{pmatrix} \widetilde{\mathbf{W}}_1 \\ \widetilde{\mathbf{W}}_2 \\ \vdots \\ \widetilde{\mathbf{W}}_p \end{pmatrix}.$$

Observe that the submatrix in $\widetilde{\mathbf{W}}_i$, $\mathbf{C}(m_{G_i}) = \begin{pmatrix} 0 & 0 & \dots & 0 & c_{i,0} \\ 1 & 0 & \dots & 0 & c_{i,1} \\ 0 & 1 & \dots & 0 & c_{i,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & c_{i,s_i-1} \end{pmatrix}$, is the Frobenius companion matrix of the main characteristic polynomial

$$m_{G_i}(x) = x^{s_i} - c_{i,0} - c_{i,1}x - \cdots - c_{i,s_i-1}x^{s_i-1},$$

whose roots (that is, eigenvalues of $\mathbf{C}(m_{G_i})$) are the main eingenvalues of G_i.

Defining $\mathbf{M}_i = \begin{pmatrix} \mathbf{j}_n^T \mathbf{W}_{G_i} \\ 0 \dots 0 \\ \vdots \ddots \vdots \\ 0 \dots 0 \end{pmatrix}$, a $s_i \times s_i$ submatrix of the $s_i \times s$ matrix $\widetilde{W}_i$, then

$$\widetilde{W}_i = \begin{pmatrix} \delta_{i,1}M_1 & \dots & \delta_{i,i-1}M_{i-1} & \mathbf{C}(m_{G_i}) & \delta_{i,i+1}M_{i+1} & \dots & \delta_{i,p}M_p \end{pmatrix}.$$

Using this notation,

$$\widetilde{W} = \begin{pmatrix} \mathbf{C}(m_{G_1}) & \delta_{1,2}M_2 & \dots & \delta_{1,p-1}M_{p-1} & \delta_{1,p}M_p \\ \delta_{2,1}M_1 & \mathbf{C}(m_{G_2}) & \dots & \delta_{2,p-1}M_{p-1} & \delta_{2,p}M_p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta_{p,1}M_1 & \delta_{p,2}M_2 & \dots & \delta_{p,p-1}M_{p-1} & \mathbf{C}(m_{G_p}) \end{pmatrix}.$$

Theorem 3.2. *Let G be the H-join as in Definition 2.1, where $\mathcal{G}$ is a family of arbitrary graphs. If for each of the graphs G_i, with $1 \le i \le p$,*

$$\sigma(G_i) = \{\mu_{i,1}^{[m_{i,1}]}, \dots, \mu_{i,s_i}^{[m_{i,s_i}]}, \mu_{i,s_i+1}^{[m_{i,s_i+1}]}, \dots, \mu_{i,t_i}^{[m_{i,t_i}]}\},$$

where $t_i \le n_i$, $m_{i,j} = m(\mu_{i,j})$ and $\mu_{i,1}, \dots, \mu_{i,s_i}$ are the main eigenvalues of G_i, then

$$\sigma(G) = \bigcup_{i=1}^{p} \{\mu_{i,1}^{[m_{i,1}-1]}, \dots, \mu_{i,s_i}^{[m_{i,s_i}-1]}\} \cup \bigcup_{i=1}^{p} \{\mu_{i,s_i+1}^{[m_{i,s_i+1}]}, \dots, \mu_{i,t_i}^{[m_{i,t_i}]}\} \cup \sigma(\widetilde{\mathbf{W}}),$$

where the union of multisets is considered with possible repetitions.

Proof. From Lemma 2.3 it is immediate that

$$\bigcup_{i=1}^{p} \{\mu_{i,1}^{[m_{i,1}-1]}, \dots, \mu_{i,s_i}^{[m_{i,s_i}-1]}\} \cup \bigcup_{i=1}^{p} \{\mu_{i,s_i+1}^{[m_{i,s_i+1}]}, \dots, \mu_{i,t_i}^{[m_{i,t_i}]}\} \subseteq \sigma(G).$$

So it just remains to prove that $\sigma(\widetilde{\mathbf{W}}) \subseteq \sigma(G)$.

Let us define the vector $\hat{\mathbf{v}} = \begin{pmatrix} \hat{\mathbf{v}}_1 \\ \vdots \\ \hat{\mathbf{v}}_p \end{pmatrix}$ such that

$$\hat{\mathbf{v}}_i = \sum_{k=0}^{s_i-1} \alpha_{i,k} A^k(G_i)\mathbf{j}_{n_i} = \mathbf{W}_{G_i} \begin{pmatrix} \alpha_{i,0} \\ \alpha_{i,1} \\ \vdots \\ \alpha_{i,s_i-1} \end{pmatrix} = \mathbf{W}_{G_i}\hat{\alpha}_i, \tag{6}$$

where $\hat{\alpha}_i = \begin{pmatrix} \alpha_{i,0} \\ \alpha_{i,1} \\ \vdots \\ \alpha_{i,s_i-1} \end{pmatrix}$, for $1 \leq i \leq p$. Then each $\hat{\mathbf{v}}_i \in Main(G_i)$ and

$$A(G_i)\hat{\mathbf{v}}_i = A(G_i)\mathbf{W}_{G_i}\hat{\alpha}_i = \sum_{k=0}^{s_i-1} \alpha_{i,k} A^{k+1}(G_i)\mathbf{j}_{n_i}, \text{ for } 1 \leq i \leq p. \tag{7}$$

Therefore,

$$\begin{aligned}
A(G)\hat{\mathbf{v}} &= \begin{pmatrix} A(G_1) & \delta_{1,2}\mathbf{j}_{n_1}\mathbf{j}_{n_2}^T & \cdots & \delta_{1,p}\mathbf{j}_{n_1}\mathbf{j}_{n_p}^T \\ \delta_{2,1}\mathbf{j}_{n_2}\mathbf{j}_{n_1}^T & A(G_2) & \cdots & \delta_{2,p}\mathbf{j}_{n_2}\mathbf{j}_{n_p}^T \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{p,1}\mathbf{j}_{n_p}\mathbf{j}_{n_1}^T & \delta_{p,2}\mathbf{j}_{n_p}\mathbf{j}_{n_2}^T & \cdots & A(G_p) \end{pmatrix} \begin{pmatrix} \hat{\mathbf{v}}_1 \\ \hat{\mathbf{v}}_2 \\ \vdots \\ \hat{\mathbf{v}}_p \end{pmatrix} \\
&= \begin{pmatrix} A(G_1)\hat{\mathbf{v}}_1 + \left(\sum_{q\in[p]\setminus\{1\}} \delta_{1,q}\mathbf{j}_{n_q}^T\hat{\mathbf{v}}_q\right)\mathbf{j}_{n_1} \\ A(G_2)\hat{\mathbf{v}}_2 + \left(\sum_{q\in[p]\setminus\{2\}} \delta_{2,q}\mathbf{j}_{n_q}^T\hat{\mathbf{v}}_q\right)\mathbf{j}_{n_2} \\ \vdots \\ A(G_p)\hat{\mathbf{v}}_p + \left(\sum_{q\in[p]\setminus\{p\}} \delta_{p,q}\mathbf{j}_{n_q}^T\hat{\mathbf{v}}_q\right)\mathbf{j}_{n_p} \end{pmatrix} &(8) \\
&= \begin{pmatrix} A(G_1)\hat{\mathbf{v}}_1 + \left(\sum_{q\in[p]\setminus\{1\}} \delta_{1,q}\mathbf{j}_{n_q}^T\mathbf{W}_{G_q}\hat{\alpha}_q\right)\mathbf{j}_{n_1} \\ A(G_2)\hat{\mathbf{v}}_2 + \left(\sum_{q\in[p]\setminus\{2\}} \delta_{2,q}\mathbf{j}_{n_q}^T\mathbf{W}_{G_q}\hat{\alpha}_q\right)\mathbf{j}_{n_2} \\ \vdots \\ A(G_p)\hat{\mathbf{v}}_p + \left(\sum_{q\in[p]\setminus\{p\}} \delta_{p,q}\mathbf{j}_{n_q}^T\mathbf{W}_{G_q}\hat{\alpha}_q\right)\mathbf{j}_{n_p} \end{pmatrix}, &(9)
\end{aligned}$$

where (9) is obtained applying (6) in (8). Defining

$$\beta_{i,0} = \sum_{q\in[p]\setminus\{i\}} \delta_{i,q}\mathbf{j}_{n_q}^T\hat{\mathbf{v}}_q = \sum_{q\in[p]\setminus\{i\}} \delta_{i,q}\mathbf{j}_{n_q}^T\mathbf{W}_{G_q}\hat{\alpha}_q, \text{ for } 1 \leq i \leq p,$$

the i-th row of (9) can be written as

$$\begin{aligned}
\beta_{i,0}\mathbf{j}_{n_i} + A(G_i)\hat{\mathbf{v}}_i &= \left(\underbrace{\sum_{k\in[p]\setminus\{i\}} \delta_{i,k}\mathbf{j}_{n_k}^T W_{G_k}\hat{\alpha}_k}_{\beta_{i,0}}\right)\mathbf{j}_{n_i} + \sum_{k=0}^{s_i-1}\alpha_{i,k}A^{k+1}(G_i)\mathbf{j}_{n_i} \\
&= \beta_{i,0}\mathbf{j}_{n_i} + \sum_{k=1}^{s_i-1}\alpha_{i,k-1}A^{k}(G_i)\mathbf{j}_{n_i} + \alpha_{i,s_i-1}A^{s_i}(G_i)\mathbf{j}_{n_i} \qquad (10) \\
&= \beta_{i,0}\mathbf{j}_{n_i} + \sum_{k=1}^{s_i-1}\alpha_{i,k-1}A^{k}(G_i)\mathbf{j}_{n_i} + \alpha_{i,s_i-1}\mathbf{W}_{G_i}\begin{pmatrix} c_{i,0} \\ c_{i,1} \\ \vdots \\ c_{i,s_i} \end{pmatrix} \qquad (11)
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{W}_{G_i}\begin{pmatrix} \beta_{i,0} + \alpha_{i,s_i-1}c_{i,0} \\ \alpha_{i,0} + \alpha_{i,s_i-1}c_{i,1} \\ \vdots \\ \alpha_{i,s_i-2} + \alpha_{i,s_i-1}c_{i,s_i-1} \end{pmatrix} \qquad (12) \\
&= \mathbf{W}_{G_i}\underbrace{\begin{pmatrix} \delta_{i,1}M_1 & \ldots & \mathbf{C}(m_{G_i}) & \ldots & \delta_{i,p}M_p \end{pmatrix}}_{\widetilde{\mathbf{W}}_i}\begin{pmatrix} \hat{\alpha}_1 \\ \vdots \\ \hat{\alpha}_i \\ \vdots \\ \hat{\alpha}_p \end{pmatrix}.
\end{aligned}$$

Observe that (11) is obtained applying Corollary 2.5 to (10).

Finally, if $A(G)\hat{\mathbf{v}} = \rho\hat{\mathbf{v}}$, then $\hat{\alpha}_1, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_p$ can be determined as follows.

$$\begin{aligned}
A(G)\hat{\mathbf{v}} &= \begin{pmatrix} \mathbf{W}_{G_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{G_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{W}_{G_p} \end{pmatrix}\begin{pmatrix} \widetilde{\mathbf{W}}_1 \\ \widetilde{\mathbf{W}}_2 \\ \vdots \\ \widetilde{\mathbf{W}}_p \end{pmatrix}\begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \vdots \\ \hat{\alpha}_p \end{pmatrix} \\
&= \rho\hat{\mathbf{v}} \\
&= \rho\begin{pmatrix} \mathbf{W}_{G_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{G_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{W}_{G_p} \end{pmatrix}\begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \vdots \\ \hat{\alpha}_p \end{pmatrix}, \text{ taking into account (6).}
\end{aligned}$$

Then we obtain

$$\underbrace{\begin{pmatrix} \mathbf{W}_{G_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{G_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{W}_{G_p} \end{pmatrix}}_{(*)} \left(\begin{pmatrix} \widetilde{\mathbf{W}}_1 \\ \widetilde{\mathbf{W}}_2 \\ \vdots \\ \widetilde{\mathbf{W}}_p \end{pmatrix} - \rho I_s \right) \begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \vdots \\ \hat{\alpha}_p \end{pmatrix} = \mathbf{0}. \tag{13}$$

Since the columns of each matrix $\mathbf{W}_{G_i}$ are linear independent, the columns of the matrix $(*)$ are also linear independent and, consequently, (13) is equivalent to $\left(\widetilde{\mathbf{W}} - \rho I_s\right) \begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \vdots \\ \hat{\alpha}_p \end{pmatrix} = \mathbf{0}$, where $\widetilde{\mathbf{W}} = \begin{pmatrix} \widetilde{\mathbf{W}}_1 \\ \widetilde{\mathbf{W}}_2 \\ \vdots \\ \widetilde{\mathbf{W}}_p \end{pmatrix}$. Therefore, the eigenvalue ρ is a root of the characteristic polynomial of the matrix $\widetilde{\mathbf{W}}$. □

Example 3.3. *Consider the graph $H = P_3$, the path with three vertices, and the graphs $K_{1,3}$, K_2 and P_3 depicted in the Figure 1.*

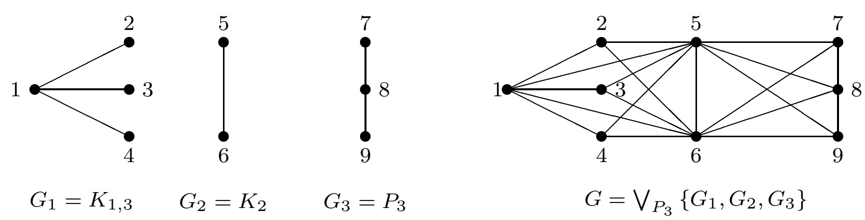

Figure 1: The P_3-join of the family of graphs G_1, G_2 and G_3.

The spectra of the graphs G_1, G_2 and G_3, depicted in Figure 1, are

$$\begin{aligned} \sigma(K_{1,3}) &= \{\sqrt{3}, -\sqrt{3}, 0^{[2]}\}, \\ \sigma(K_2) &= \{1, -1\}, \\ \sigma(P_3) &= \{\sqrt{2}, -\sqrt{2}, 0\}, \end{aligned}$$

and their main characteristic polynomials are $m_{G_1}(x) = x^2 - 3$, $m_{G_2}(x) = x - 1$ and $m_{G_3}(x) = x^2 - 2$, respectively. Since

$$\begin{aligned} \widetilde{\mathbf{W}}_1 &= \begin{pmatrix} 0 & c_{1,0} & \delta_{1,2}2 & \delta_{1,3}3 & \delta_{1,3}4 \\ 1 & c_{1,1} & 0 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 3 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \widetilde{\mathbf{W}}_2 &= \begin{pmatrix} \delta_{2,1}4 & \delta_{2,1}6 & c_{2,0} & \delta_{2,3}3 & \delta_{2,3}4 \end{pmatrix} &= \begin{pmatrix} 4 & 6 & 1 & 3 & 4 \end{pmatrix} \text{ and} \\ \widetilde{\mathbf{W}}_3 &= \begin{pmatrix} \delta_{3,1}4 & \delta_{3,1}6 & \delta_{3,2}2 & 0 & c_{3,0} \\ 0 & 0 & 0 & 1 & c_{3,1} \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \end{aligned}$$

it follows that

$$\widetilde{\mathbf{W}} = \begin{pmatrix} \widetilde{\mathbf{W}}_1 \\ \widetilde{\mathbf{W}}_2 \\ \widetilde{\mathbf{W}}_3 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 4 & 6 & 1 & 3 & 4 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and thus the characteristic polynomial of $\widetilde{\mathbf{W}}$ *is the polynomial*

$$p_{\widetilde{\mathbf{W}}}(x) = -42 - 40x + 15x^2 + 19x^3 + x^4 - x^5.$$

Therefore, applying Theorem 3.2, the characteristic polynomial of G *is*

$$p_G(x) = x^3(x+1)p_{\widetilde{\mathbf{W}}}(x) = x^3(x+1)(-42 - 40x + 15x^2 + 19x^3 + x^4 - x^5).$$

When all graphs of the family $\mathcal{G}$ are regular, that is, G_1 is d_1-regular, G_2 is d_2-regular, $\ldots$, G_p is d_p-regular, the walk matrices are $W_{G_1} = (\mathbf{j}_{n_1})$, $W_{G_2} = (\mathbf{j}_{n_2})$, $\ldots$, $W_{G_p} = (\mathbf{j}_{n_p})$, respectively. Consequently, the main polynomials are $m_{G_1}(x) = x - d_1$, $m_{G_2}(x) = x - d_2$, $\ldots$, $m_{G_p}(x) = x - d_p$. As direct consequence, for this particular case, the H-join associated matrix is

$$\widetilde{\mathbf{W}} = \begin{pmatrix} d_1 & \delta_{1,2}\mathbf{j}_{n_2}^T W_{G_2} & \cdots & \delta_{1,p}\mathbf{j}_{n_p}^T W_{G_p} \\ \delta_{2,1}\mathbf{j}_{n_1}^T W_{G_1} & d_2 & \cdots & \delta_{2,p}\mathbf{j}_{n_p}^T W_{G_p} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{p,1}\mathbf{j}_{n_1}^T W_{G_1} & \delta_{p,2}\mathbf{j}_{n_p}^T W_{G_2} & \cdots & d_p \end{pmatrix} = \begin{pmatrix} d_1 & \delta_{1,2}n_2 & \cdots & \delta_{1,p}n_p \\ \delta_{2,1}n_1 & d_2 & \cdots & \delta_{2,p}n_p \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{p,1}n_1 & \delta_{p,2}n_2 & \cdots & d_p \end{pmatrix}.$$

Therefore, it is immediate that when all the graphs of the family $\mathcal{G}$ are regular, the matrix $\widetilde{\mathbf{W}}$ and the matrix $\widetilde{C}$ in (2) are similar matrices. Note that $\widetilde{C} = D\widetilde{\mathbf{W}}D^{-1}$, where $D = \mathrm{diag}\left(\sqrt{n_1}, \sqrt{n_2}, \ldots, \sqrt{n_p}\right)$ and thus $\widetilde{\mathbf{W}}$ and $\widetilde{C}$ are cospectral matrices as it should be.

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