THE DEGREES OF TOROIDAL REGULAR PROPER HYPERMAPS

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Abstract. Recently the classification of all possible faithful transitive permutation representations of the group of symmetries of a regular toroidal map was accomplished. In this paper we complete this investigation on a surface of genus 1 considering the group of a regular toroidal hypermap of type $(3,3,3)$.

Keywords: Regular Polytopes, Regular Toroidal Maps, Regular Toroidal Hypermaps, Permutation Groups.
2010 Math Subj. Class: 52B11, 05E18, 20B25.

1. Introduction

By Cayley's theorem, every group is isomorphic to some permutation group. A finite group G has a *faithful permutation representation of degree* n if there exists a monomorphism from G into the symmetric group S_n, or equivalently, if G acts faithfully on a set of n points. In this paper, only transitive actions will be considered. Faithful transitive permutation representations of a group G are in correspondence with core-free subgroups of G, that is, subgroups containing no nontrivial normal subgroups. The stabilizer of a point of a faithful transitive permutation representation is core-free and conversely, the action on the cosets of a core-free subgroup is faithful and transitive.

The minimal degree of a faithful permutation representation of G has been a subject of extensive study. In [7] it was shown that a faithful permutation representation of a simple group with minimal degree is primitive. The minimal degree of a faithful (transitive) permutation representation is known for all simple groups [8, Theorem 5.2.2].

We have particular interest on the study of the faithful transitive permutation representations of the automorphism groups of abstract regular polytopes, which are quotients of Coxeter groups with linear diagram [9], or more generally, of the groups of regular hypertopes [2]. The minimal faithful permutation representations of finite irreducible Coxeter groups, which include the automorphism groups of spherical polytopes, was recently determined in [10].

This paper is a sequel to [3] in which faithful transitive permutation representations of the groups of symmetries of toroidal regular maps were determined and rectified in [4]. In the present paper we complete the classification of toroidal regular hypermaps, answering a question made by Gareth Jones in the Bled Conference in Graph Theory 2019, where the results accomplished in [3] were presented.

The results can be summarized as follows. Let $s \geq 2$.

- for the hypermap $(3,3,3)_{(s,0)}$, the possible degrees are s^2, $2ds$, $3ds$ and $6ds$ where d is a divisor of s. Moreover, the degree $2ds$ exists if and only if all prime divisors of s/d are congruent to 1 modulo 6;

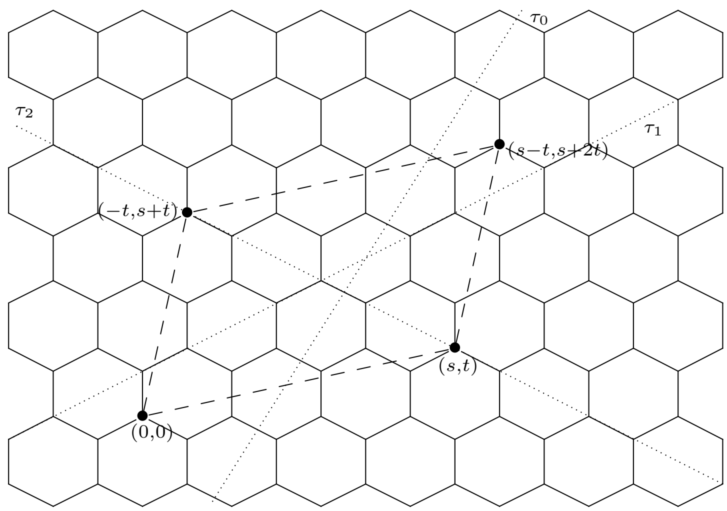


FIGURE 1. Toroidal map of type $\{6,3\}$ with $(s,t)=(4,1)$

- for the hypermap $(3,3,3)_{(s,s)}$, the possible degrees are those of the hypermap $(3,3,3)_{(s,0)}$ multiplied by 3.

Despite that $(3,3,3)_{\mathbf{s}}$ is an index two subgroup of $\{6,3\}_{\mathbf{s}}$ for $\mathbf{s}\in\{(s,0),(s,s)\}$, it is not true in general that if n is a degree of $\{6,3\}_{\mathbf{s}}$ then $n/2$ is the degree of a toroidal hypermap $(3,3,3)_{\mathbf{s}}$.

2. Toroidal hypermaps

Consider a regular tessellation of the plane by identical regular hexagons, whose full symmetry group is the Coxeter group $[6,3]$, generated by three reflections τ_0, τ_1 and τ_2, as shown in Figure 1.

By identifying opposite sides of a parallelogram with vertices $(0,0)$, (s,t), $(-t,s+t)$ and $(s-t,s+2t)$ of the tessellation, we obtain the toroidal map $\{6,3\}_{(s,t)}$, with $F=s^2+st+t^2$ faces, $3F$ edges and $2F$ vertices. This map is said to be regular when the group of symmetries acts regularly on the set of flags of the map (triples of mutually incident vertex, edge and face) [9], which is the case if and only if $st(s-t)=0$. Therefore, two families of toroidal regular maps of type $\{6,3\}$ arise: $\{6,3\}_{(s,0)}$ and $\{6,3\}_{(s,s)}$, which are factorizations of the infinite Coxeter group $[6,3]$ by $(\tau_0\tau_1\tau_2)^{2s}$ and $(\tau_0\tau_1\tau_0\tau_1\tau_2)^{2s}$, respectively. The number of flags of $\{6,3\}_{(s,0)}$ is $12s^2$ while the number of flags of $\{6,3\}_{(s,s)}$ is $36s^2$.

A hypermap can be defined as an embedding of a bipartite graph into a compact surface. The bipartition of vertices determines two types of vertices. We call hypervertices to the vertices of one type and hyperedges to the vertices of the other type (see [5] for more detail). A toroidal hypermap is obtained from a map of type $\{6,3\}$ by considering a bipartition on the set of its vertices (see Figure 2) and the translation subgroup of the map $\{6,3\}$ respects this bipartition. The toroidal hypermap constructed from $\{6,3\}_{(s,t)}$ is denoted by $(3,3,3)_{(s,t)}$, which is regular if

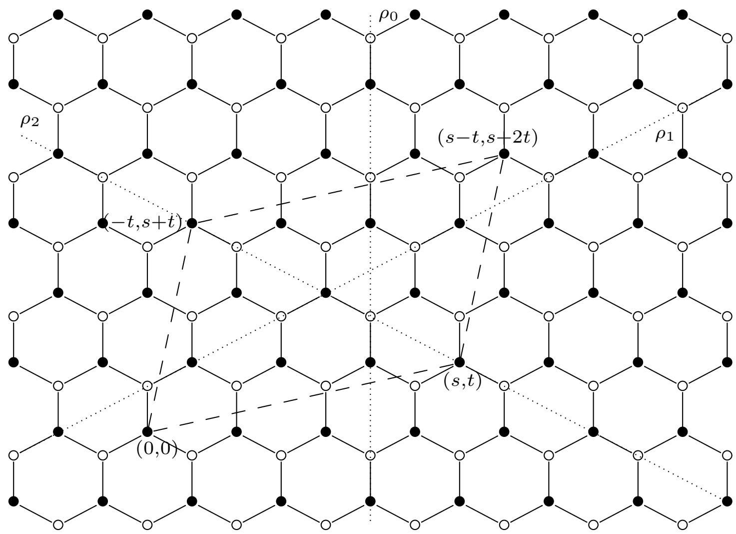


FIGURE 2. Toroidal map of type $(3,3,3)$

and only if $st(s-t)=0$. All the proper toroidal regular hypermaps arise in this way [1].

Analogously a bipartition on the set of vertices of a toroidal regular map of type $\{4,4\}$ results in another map of type $\{4,4\}$ (where a face-rotation, preserving the bipartition, has order 2).

The group G of symmetries of the hypermap $(3,3,3)_{(s,t)}$ is a subgroup of index 2 of the group of the map $\{6,3\}_{(s,t)}$,

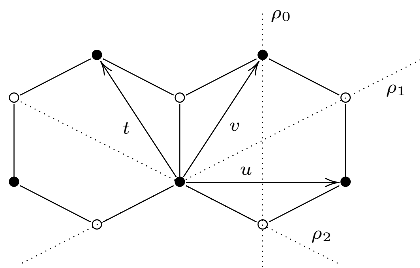
$$G := \langle \rho_0, \rho_1, \rho_2 \rangle, \text{ where } \rho_0 := \tau_0\tau_1\tau_0,\ \rho_1 := \tau_1 \text{ and } \rho_2 := \tau_2.$$

If the toroidal hypermap is regular, then G is the infinite Coxeter group $[3,3,3]$ factorized by either $(\rho_0\rho_1\rho_2\rho_1)^s$ or $(\rho_0\rho_1\rho_2)^{2s}$, depending on whether it is $(3,3,3)_{(s,0)}$ or $(3,3,3)_{(s,s)}$, respectively.

The automorphism group of the map $\{6,3\}_{(s,s)}$ (resp. $\{6,3\}_{(3s,0)}$) has a subgroup of index 3 isomorphic to the automorphism group of the map $\{6,3\}_{(s,0)}$ (resp. $\{6,3\}_{(s,s)}$). The same relations hold for the corresponding toroidal hypermaps. Particularly, the group of the $(3,3,3)_{(s,0)}$ is a quotient of the group of $(3,3,3)_{(s,s)}$ by $\langle(\rho_0\rho_1\rho_2\rho_1)^s\rangle$, and the latter is a quotient of the group of $(3,3,3)_{(3s,0)}$ by $\langle(\rho_0\rho_1\rho_2)^{2s}\rangle$.

Let u and v be two translations of order s forming an oblique basis for the group of translations of the hypermap $(3,3,3)_{(s,0)}$ (or $\{6,3\}_{(s,0)}$).

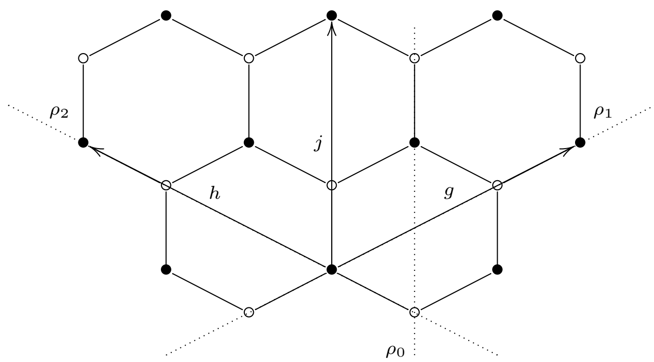
$$u := \rho_0\rho_1\rho_2\rho_1,\ v := u^{\rho_1} = \rho_1\rho_0\rho_1\rho_2 \text{ and } t := u^{-1}v.$$



We have the equalities

$$u^{\rho_0} = u^{-1},\ u^{\rho_2} = t^{-1},\ v^{\rho_2} = v^{-1},\ v^{\rho_0} = t \ \text{ and } \ t^{\rho_1} = t^{-1}. \tag{1}$$

For the hypermap $(3,3,3)_{(s,s)}$, consider the translations $g := uv = (\rho_0\rho_1\rho_2)^2$, $h := g^{\rho_0}$ and $j := gh$.



In this case we have the following equalities

$$g^{\rho_1} = g,\ g^{\rho_2} = j^{-1} \ \text{ and } \ h^{\rho_1} = j^{-1}. \tag{2}$$

3. Degrees of maps of type $\{6,3\}$ vs. degrees of toroidal hypermaps

The degrees of a faithful transitive permutation representation of the group of a regular map of type $\{3,6\}$ (or equivalently $\{6,3\}$) are given in [4] by the following two theorems.

Theorem 3.1. [4, Theorem 5.1] *Let $s \geq 2$. The degrees of a faithful transitive permutation representation of a toroidal regular map of type $\{3,6\}_{(s,0)}$ are*

(1) s^2,
(2) $3ds$, $6ds$ *or* $12ds$ *for any divisor d of s,*
(3) $2ds$ *and* $4ds$ *if and only if d is a divisor of s and all prime divisors of s/d are equal to* $1 \bmod 6$.

Theorem 3.2. [4, Theorem 5.1] *Let $s \geq 2$. The degrees of a faithful transitive permutation representation of a toroidal regular map of type $\{3,6\}_{(s,s)}$ are*

(1) $3s^2$,
(2) $9ds$, $18ds$ *or* $36ds$ *for any divisor d of s,*
(3) $6ds$ *and* $12ds$ *if and only if d is a divisor of s and all prime divisors of s/d are equal to* $1 \bmod 6$.

The basis for the proof of the above theorem is the following result that is a combination of both Lemma 3.4 of [3] and Lemma 2.1 of [4].

Lemma 3.3. *Let G be the group of a toroidal regular map. If $n \neq s^2$ then G is embedded into $S_k \wr S_m$ with $n = km$ $(m,\, k > 1)$ and*

(1) *$k = ds$ where d is a divisor of s and,*
(2) *m is a divisor of $\frac{|G|}{s^2}$.*

The above lemma assumes T is intransitive. Indeed we also have the following result.

Lemma 3.4. [3, Lemma 3.2] *If T is transitive, then $n = s^2$.*

The number m on Lemma 3.3 is the number of T-orbits, where T is the translation group with the translations as defined in Section 2 of this paper and of [3]. For the map $\{3, 6\}_{(s,0)}$ and $T = \langle u, v\rangle$ we have proved the following.

Proposition 3.5. [4, Proposition 3.3] *If $m = 4$, then $k = sd$ where d is a divisor of s and all prime divisors p of s/d are such that $p \equiv 1 \mod 6$.*

As seen in [3], there is a correspondence between core-free subgroups and faithful transitive actions. Moreover, if G has a faithful transitive permutation representation of degree n and is a subgroup of index α of U, then U has a faithful transitive permutation representation of degree αn. Similarly to Corollary 3.5 of [3], we have the following.

Corollary 3.6. *If n is a degree of $(3, 3, 3)_{(s,0)}$ (resp. $(3, 3, 3)_{(s,s)}$) then $3n$ is a degree of $(3, 3, 3)_{(s,s)}$ (resp. $(3, 3, 3)_{(3s,0)}$).*

Additionally, the group of symmetries of a toroidal hypermap $(3, 3, 3)_{(s,t)}$ is a subgroup of index 2 of the group of the toroidal map $\{6, 3\}_{(s,t)}$ and, hence, we have the following.

Corollary 3.7. *If n is a degree of $(3, 3, 3)_{(s,0)}$ (resp. $(3, 3, 3)_{(s,s)}$) then $2n$ is a degree of $\{6, 3\}_{(s,0)}$ (resp. $\{6, 3\}_{(s,s)}$).*

It must be pointed out that this property works only in one direction, meaning that a degree n of the group of a map $\{6, 3\}_{(s,t)}$ does not determine the degrees of $(3, 3, 3)_{(s,t)}$. However, by knowing the degrees of a map $\{6, 3\}_{(s,t)}$ we can restrict the set of possible degrees for $(3, 3, 3)_{(s,t)}$.

4. The degrees of $(3, 3, 3)_{(s,0)}$

In what follows let $U := \langle \tau_0, \tau_1, \tau_2\rangle$ be the group of $\{6, 3\}_{(s,0)}$, $G := \langle \rho_0, \rho_1, \rho_2\rangle$ be the group of $(3, 3, 3)_{(s,0)}$ and $T = \langle u, v\rangle$ be the translation group of order s^2 as defined in Section 2. We recall that the translation subgroups of $\{6, 3\}_{(s,0)}$ and $(3, 3, 3)_{(s,0)}$ are the same.

Lemma 4.1. *If n is a degree of $(3, 3, 3)_{(s,0)}$, then $n \in \left\{s^2,\, 2ds,\, 3ds,\, 6ds\right\}$ for some divisor d of s.*

Proof. By Corollary 3.7 the set of possible degrees of $(3, 3, 3)_{(s,0)}$ is a subset of

$$\left\{\frac{s^2}{2},\, \delta s,\, \frac{3\delta s}{2},\, 2\delta s,\, 3\delta s,\, 6\delta s\right\},$$

where δ is a divisor of s. Moreover, the degrees δs and $2\delta s$ of this list must be considered only if all prime factors of s/δ are equal to 1 modulo 6. To prove this lemma we need to prove that each of the degrees $n = \frac{s^2}{2}$, $n = \frac{3\delta s}{2}$ and $n = \delta s$, either belongs to the list given in this lemma, or cannot be a degree of $(3,3,3)_{(s,0)}$.

These degrees are attained when a faithful transitive permutation representation of G on n cosets corresponds to a faithful transitive permutation representation of U on $2n$ cosets; while G acts on a set X of cosets of a core-free subgroup, U acts on $X \cup X\tau_0$.

Note that, for $x \in X$, x and $x\tau_0$ must be in different T-orbits. Thus, the number of T-orbits for the action of U on $X \cup X\tau_0$ is $2m$ where m is the number of T-orbits on X. By Lemma 3.3, $2m \in \{2, 4, 6, 12\}$. The size of a T-orbit, denoted by k, is the same in both actions.

Let first $n = \frac{s^2}{2}$. By Lemma 3.4, T is not transitive on X, which imply that $m \neq 1$. Hence $2m \in \{4, 6, 12\}$. If $2m = 4$ then by Proposition 3.5, $2n = 4ds$ with d a divisor of s, where all prime factors of s/d are equal 1 modulo 6. But then one get $\frac{s^2}{2} = 4ds$, hence $\frac{s}{d} = 4$, which is not 1 modulo 6, a contradiction. If $2m \in \{6, 12\}$ then $2n \in \{6ds, 12ds\}$ for some divisor d of s. In any case n is one of the degrees given in the statement of this lemma.

Now let $n = \frac{3\delta s}{2}$. First if δ is even then $n = 3ds$ with d being a divisor of s which is one degrees given in the statement of this lemma. Suppose that δ is odd. As $2m \in \{4, 6, 12\}$, hence $2n = 3\delta s \in \{2ds, 4ds, 6ds, 12ds\}$ for some divisor d of s, which implies that δ is even, a contradiction.

Now suppose that $n = \delta s$ with all prime factors of s/δ equal to 1 mod 6. Particularly s/δ must be odd. If $2m \in \{4, 6, 12\}$ then the degrees are among the ones listed in this lemma. We may assume that $2m = 2$. Then $m = 1$, which implies that $n = s^2$. $\square$

The dihedral groups $\langle \rho_i, \rho_j \rangle$ of order 6 are core-free subgroups of G (for $i, j \in \{0, 1, 2\}$ and $i \neq j$), hence there are faithful transitive permutation representations of G of degree s^2. Similarly to Proposition 5.1 (1) of [3], $\langle u^a, v^b \rangle$ is a core-free subgroup of G. Hence G has a faithful transitive permutation representation of degree $n = 6ab$ for any integers a and b such that $s = lcm(a, b)$, or equivalently, of degree $n = 6ds$ for any divisor d of s. In what follows we give other core-free subgroups of G.

Proposition 4.2. *Let G be the group of $(3,3,3)_{(s,0)}$ with $s \geq 2$ and d be a divisor of s.*

(1) *The group $\langle u^d \rangle \rtimes \langle \rho_0 \rangle$ is a core-free subgroup of G of index $3ds$.*
(2) *Suppose that there exists α, coprime with s/d, such that $\alpha^2 - \alpha + 1 \equiv 0 \bmod (s/d)$. Then $\langle (v^{-\alpha}u)^d,\ \rho_1\rho_2 \rangle$ is a core-free subgroup of G with index $2sd$.*

Proof. (1) Let $H := \langle u^d \rangle \rtimes \langle \rho_0 \rangle$. Suppose that $x \in H \cap H^{\rho_1} = \langle u^d \rangle \rtimes \langle \rho_0 \rangle \cap \langle v^d \rangle \rtimes \langle \rho_0^{\rho_1} \rangle$. If $x \notin T$ then $\rho_0 \rho_0^{\rho_1} \in T$, a contradiction. Thus $x \in T$ and therefore as in (1) we conclude that x is trivial. The order of H is $\frac{2s}{d}$ thus $|G : H| = 3ds$.

(2) Let now $H := \langle (v^{-\alpha}u)^d,\ \rho_1\rho_2 \rangle$. First note that $\langle (v^{-\alpha}u)^d \rangle$ is a normal subgroup of H. Indeed we have $(v^{-\alpha}u)^{\rho_1\rho_2} = t^\alpha v^{-1} = u^\alpha v^{\alpha-1} = (v^{-\alpha}u)^\alpha$. Suppose that $x \in H \cap H^{\rho_1}$. Then $x = (v^{-\alpha}u)^{id}(\rho_1\rho_2)^j = (u^{-\alpha}v)^{i'd}(\rho_1\rho_2)^{j'}$. This implies that $j = j'$ and $i = i' = 0 \bmod (s/d)$. Thus $H \cap H^{\rho_1} = \langle \rho_1\rho_2 \rangle$. Now the intersection

of $\langle\rho_1\rho_2\rangle$ and H^{ρ_0} is trivial, otherwise we get that either $\rho_1\rho_2 \in T$, $(\rho_1\rho_2)^{\rho_0} \in T$, $\rho_0\rho_2 \in T$, $tu^{-1} \in \langle t^{-\alpha}u^{-1}\rangle$ or $uv \in \langle t^{-\alpha}u^{-1}\rangle$, which is never possible. $\square$

Let us also recall the following proposition.

Proposition 4.3. [4, Proposition 3.1] *Let q be an odd number. The modular equation*

$$x^2 - x + 1 \equiv 0 \bmod q$$

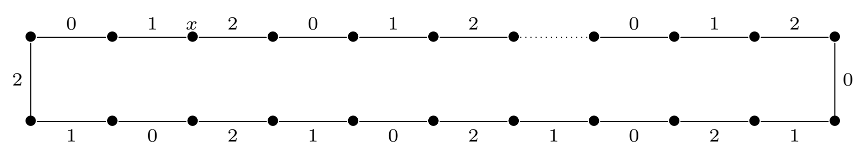
has a solution if and only if all prime divisors p of q are such that $p \equiv 1 \bmod 6$.

Theorem 4.4. *Let $s \geq 2$. A faithful transitive permutation representation of the group of symmetries of $(3,3,3)_{(s,0)}$ has degree n if and only if $n \in \{s^2, 3ds, 6ds\}$ where d is a divisor of s or; $n = 2ds$ where d is a divisor of s and all prime factors of s/d are equal $1 \bmod 6$.*

Proof. This is a consequence of Lemma 4.1 and the core-free subgroups indexes found in this section. $\square$

A *Schreier coset graph* of a group G is a graph $\mathcal{G}$ associated with a subgroup $H \leq G$ and a set of generators $\langle \rho_i \mid i \in \{0,\dots,r-1\}\rangle$ of G, where the vertices are the cosets G/H and there is an edge $\{Hx, Hy\}$ labeled i whenever $Hx\rho_i = Hy$ (for some $x, y \in G$). When H is core-free, a Schreier coset graph gives a faithful transitive permutation representation of the group G, of degree $n = |G : H|$.

Proposition 4.5. *Let $s \geq 2$. The following graph is a faithful transitive permutation representation graph of the automorphism group of $(3,3,3)_{(s,0)}$ with degree $3s$.*



Moreover, the stabilizer of a point is, up to conjugacy, $\langle u\rangle \rtimes \langle\rho_0\rangle$.

Proof. Let $G = \langle\rho_0, \rho_1, \rho_2\rangle$ be the group with the given permutation representation graph. It is clear from the graph that $\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^3 = (\rho_0\rho_2)^3 = (\rho_1\rho_2)^3 = (\rho_0\rho_1\rho_2\rho_1)^s = 1$. Hence G must be a subgroup of the automorphism group of the regular hypermap $(3,3,3)_{(s,0)}$ and $|G| \leq 6s^2$.

Consider the vertex x of the permutation representation. Its stabilizer G_x contains the subgroup $\langle\rho_0, u\rangle$ of order $2s$. Then, $|G_x| \geq 2s$ and, by the Orbit-Stabilizer theorem, $|G| \geq 6s^2$. Consequently, the graph is a faithful transitive permutation representation of the automorfism group of $(3,3,3)_{(s,0)}$. $\square$

Remark. The faithful transitive permutation representation given on Proposition 4.5 is of minimal degree whenever s is not a prime number congruent with $1 \bmod 6$.

Similarly to what was done in [3], it is possible to obtain permutation representation graphs for other degrees. As some of the graphs are very complicated to draw we decide to include only the simplest one.

5. The degrees of $(3,3,3)_{(s,s)}$

In this section we determine the degrees of $(3,3,3)_{(s,s)}$ using the degrees of $(3,3,3)_{(s,0)}$ and $(3,3,3)_{(3s,0)}$, given by Theorem 4.4. Let n be the degree of a faithful transitive permutation representation of $(3,3,3)_{(s,s)}$ and $T = \langle u, v\rangle$ the translation group of $(3,3,3)_{(3s,0)}$ of order $(3s)^2$.

Theorem 5.1. *Let $s \geq 2$. A faithful transitive permutation representation of the group of symmetries of $(3,3,3)_{(s,s)}$ has degree n if and only if $n \in \{3s^2,\ 9ds,\ 18ds\}$ where d is a divisor of s or; $n = 6ds$ where d is a divisor of s and all prime factors of s/d are equal* 1 mod 6.

Proof. Let G be the group of $(3,3,3)_{(s,s)}$. From Theorem 4.4 and Corollary 3.6 there are faithful transitive permutation representations with the degrees given in the statement of this theorem. By Theorem 4.4, a degree of $(3,3,3)_{(3s,0)}$ is either equal to $(3s)^2$, $3\delta(3s)$ and $6\delta(3s)$, with δ being a divisor of $3s$, or to $2\delta(3s)$, with δ being a divisor of $3s$ and all prime factors of $3s/\delta$ equal 1 mod 6.

Dividing the possible degrees of $(3,3,3)_{(3s,0)}$ by 3, we get that

$$n \in \{3s^2,\ 2\delta s,\ 3\delta s,\ 6\delta s\}$$

with δ dividing $3s$.

The degree $n = 3s^2$ is in set given in the statement of the theorem. If $n = 2\delta s$ then, as in this case δ is a divisor of $3s$ and all prime divisors of $3s/\delta$ must be equal 1 mod 6, $\delta = 3d$ for some divisor d of s. Hence this degree is already included in the set given in the statement of this theorem. Let us prove that also on the remaining cases $\delta = 3d$ for some divisor d of s.

The hypermap $(3,3,3)_{(3s,0)}$ contains three copies of the hypermap $(3,3,3)_{(s,s)}$. To be more precise the group of $(3,3,3)_{(s,s)}$ is the group of $(3,3,3)_{(3s,0)}$ factorized by the translation $(uv)^s$ of order 3. Hence, the points x, $x(uv)^s$ and $x(uv)^{2s}$ of any faithful transitive permutation representation of $(3,3,3)_{(3s,0)}$ are identified under this factorization. Any faithful transitive permutation representation of an action of $(3,3,3)_{(3s,0)}$ on a set X gives a permutation representation, of degree $|X|/3$, of $(3,3,3)_{(s,s)}$ on triples of points of X of the form

$$\left\{x, x(uv)^s, x(uv)^{2s}\right\}.$$

with $x \in X$. Note that these points are in the same T-orbit. Hence the number m of T-orbits is unchanged under this factorization.

To prove that the action on the triple of points is faithful only if $\delta = 3d$, for some divisor d, we can follow an identical proof as the one presented for Theorem 5.3 of [4]. We note that Lemma 2.1 of [4], that establishes the size of T-orbit, can be used here. □

6. Open Problems

The study of faithful transitive permutation representations can be extended to other regular polytopes, particularly to finite locally spherical regular polytopes, including the cubic tessellations and to the finite locally toroidal regular polytopes.

Problem 6.1. *Determine the degrees of faithful transitive permutation representations of the groups of spherical and euclidean type.*

Problem 6.2. *Determine the degrees of faithful transitive permutation representations of the groups of the finite toroidal regular polytopes.*

The problem of the classification locally toroidal regular polytopes dominated the theory of abstract polytopes for a while and it was originally posed by Grünbaum [6]. The meritoriously known as Grünbaum's Problem, is not yet totally solved [9].

7. Acknowledgements

The authors would like to thank an anonymous referee whose comments improved a preliminary version of this paper.

This work is supported by The Center for Research and Development in Mathematics and Applications (CIDMA) through the Portuguese Foundation for Science and Technology (FCT - Fundação para a Ciência e a Tecnologia), references UIDB/04106/2020 and UIDP/04106/2020.

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