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Discrete Hardy Spaces for Bounded Domains in $\mathbb{R}^n$

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Abstract

Discrete function theory in higher-dimensional setting has been in active development since many years. However, available results focus on studying discrete setting for such canonical domains as half-space, while the case of bounded domains generally remained unconsidered. Therefore, this paper presents the extension of the higher-dimensional function theory to the case of arbitrary bounded domains in $\mathbb{R}^n$. On this way, discrete Stokes' formula, discrete Borel–Pompeiu formula, as well as discrete Hardy spaces for general bounded domains are constructed. Finally, several discrete Hilbert problems are considered.

Keywords Discrete Dirac operator · Discrete monogenic functions · Discrete function theory · Discrete Cauchy transform · Discrete boundary value problems

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1 Introduction

Construction of discrete analogues to the classical theory of monogenic functions has been an area of active research during last decades. The motivation for this construction has also evolved over the time: while initially the principal interest consisted in developing numerical methods based on integral representation formulae, see for example papers [9,11,12] and references therein, later, the growing importance of discrete modelling in various practical fields, e.g. [23], led to a genuine and native interest in discrete structures in the hypercomplex setting, see [2,7,10] and references therein.

The main advantage of discrete modelling lies in the fact that certain properties of a continuous problem are exactly replicated on the discrete level and they are not just an approximation as in conventional numerical schemes, e.g. factorisation of the discrete Laplace operator by pair of discrete Cauchy–Riemann operators. Therefore, it is not surprising that studies of discrete function theory and other related theories in different settings have been presented by many authors. Thus, to keep the presentation short and being completely aware of such classical topic as theory of discrete analytic functions [20], we will focus only on works relevant to the paper and related to the hypercomplex community, especially in higher dimensions.

As it has been mentioned, originally, discrete function theories have been related to numerical schemes for solving boundary value problems. For example, works [12, 13,16,17] presented the version of discrete function theory originating from the ideas of the discrete potential theory developed by Ryaben'kii [22]. This, although mostly two-dimensional, version of a discrete function theory has been later extended to the case of rectangular lattices in [14,15,18,19], allowing more flexibility in solving boundary value problems of mathematical physics in bounded domains. An alternative branch of research in discrete function theory is related to discrete Clifford analysis, which is a higher-dimensional version of discrete function theory. This line of research is generally associated with a more abstract algebraic point of view on the discrete function theory, typically based on Weyl calculus and its modifications, see for example [2,3,7,8]. One of the advantages of such more abstract approach is the possibility of constructing boundary value theory for discrete monogenic functions, which has been introduced in recent years, see e.g. [4,5].

The boundary value theory of discrete monogenic functions provides tools to define discrete counterparts of Hardy spaces, Plemelj-Sokhotzki formulae, as well as to study discrete Hilbert problems. In general, this theory is based on explicit calculations of discrete Fourier symbols of boundary operators. Original works [4,5] have been focusing on the idealised case, although being practically important, of the half-space. Naturally, the question of extension of this theory to the case of bounded domains and more complicated geometries appears. First ideas of extending the boundary value theory of discrete monogenic functions for bounded domain in $\mathbb{R}^3$ have been presented in [6]. The focus of this work was on introduction of discrete Stokes, Borel–Pompeiu, and Cauchy formulae for bounded domains, as well as first steps towards characterisation of boundary values of discrete monogenic functions via their boundary values have been presented. The results of [6] indicate that consideration of arbitrary bounded domains in higher dimensions requires a more "delicate"approach to the definition of the geometrical setting, as well as to the construction of boundary operators. Thus, in this paper, we present the necessary tools for extending the results in [6] to arbitrary bounded domains in $\mathbb{R}^n$. We consider constructions of discrete spaces and operators in interior and exterior settings. Moreover, explicit calculations of Fourier symbols along boundary layers are provided, which allow definitions of discrete Riesz kernels for arbitrary discrete bounded domains. Finally, to show the applicability of our theory several discrete Hilbert problems in bounded domains are considered.

2 Preliminaries and Notations

2.1 Geometrical Setting and Basic Operators

Let us consider n-dimensional Euclidean space $\mathbb{R}^n$ with the basis unit vectors e_k, $k = 1, 2, \ldots, n$ and points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. For $h > 0$ we introduce the unbounded uniform lattice $h\mathbb{Z}^n$ in the classical way, that is,

$$h\mathbb{Z}^n := \left\{\mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{x} = (m_1 h, m_2 h, \ldots, m_n h), m_j \in \mathbb{Z}, j = 1, 2, \ldots, n\right\}.$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded, simply connected domain with piecewise smooth boundary $\partial\Omega$. We introduce the discrete domain Ω_h associated to Ω as follows

$$\Omega_h := \Omega \cap h\mathbb{Z}^n, \Leftrightarrow \Omega_h := \left\{mh = (m_1 h, m_2 h, \ldots, m_n h) \,|\, mh \in \Omega \cap h\mathbb{Z}^n\right\}.$$

Now, for fixing notations, we introduce the following definition:

Definition 1 For a given discrete domain Ω_h, the following objects are introduced:

(i) a discrete complementary domain to Ω_h: $\Omega_h^c := h\mathbb{Z}^n \backslash \Omega_h$;
(ii) the discrete interior of Ω_h, denoted by $int(\Omega_h)$, is the set of all points $mh \in \Omega_h$ such that at least one of its immediate neighbour points $(m + e_j)h$, $(m - e_j)h$ for some $j = 1, \ldots, n$, also belongs to Ω_h, i.e.

$$int(\Omega_h) := \left\{mh \in \Omega_h \,|\, \exists\, j : (m + e_j)h \in \Omega_h \vee (m - e_j)h \in \Omega_h\right\};$$

(iii) the discrete exterior of Ω_h, denoted by Ω_h^{ext}, is defined symmetrically as the interior of the complementary domain Ω_h^c.

As the next step, the following definition provides a first classification of boundary points of a discrete domain Ω_h and of its exterior Ω_h^{ext} on the discrete lattice $h\mathbb{Z}^n$:

Definition 2 Let Ω_h be a discrete domain $h\mathbb{Z}^n$ with the lattice constant $h > 0$. We say that

(i) a point $mh \in \Omega_h$ is a point of the *interior boundary layer* γ_h^+, if at least one of its neighbour points does not belong to Ω_h, i.e.

$$\gamma_h^+ := \left\{mh \in \Omega_h \,|\, \exists\, j : (m + e_j)h \notin \Omega_h \vee (m - e_j)h \notin \Omega_h\right\};$$

(ii) a point mh is a point of the *middle boundary layer* γ_h^*, if its neighbourhood contains points belonging to Ω_h, as well as points belonging to Ω_h^{ext}, i.e.

$$\gamma_h^* := \left\{mh \mid \exists\, j\colon (m \pm e_j)h \in \Omega_h \wedge (m \mp e_j)h \in \Omega_h^{ext}\right\};$$

(iii) a point $mh \in \Omega_h^{ext}$ is a point of the *exterior boundary layer* γ_h^-, if at least one of its neighbour points does not belong to Ω_h^{ext}, i.e.

$$\gamma_h^- := \left\{mh \in \Omega_h^{ext} \mid \exists\, j\colon (m + e_j)h \notin \Omega_h^{ext} \vee (m - e_j)h \notin \Omega_h^{ext}\right\}.$$

Remark 1 We would like to remark, that alternatively, middle and exterior boundary layers can be defined solely by using the definition of interior boundary layer. In this case, the *middle boundary layer* γ_h^* of Ω_h is defined as the interior boundary of $\Omega_h^c = h\mathbb{Z}^n \backslash \Omega_h$, and the *exterior boundary layer* γ_h^- is the interior boundary of the $\Omega_h^{ext} = int(\Omega_h^c)$.

Notice that the middle layer lies "in between"the domain Ω_h and its associated exterior domain Ω_h^{ext}, and, in fact, is the "true"boundary of the domain. As an example, consider the classical case of $\Omega_h := h\mathbb{Z}_+^n = \{(m_1h, \ldots, m_nh) \in h\mathbb{Z}^n | m_n > 0\}$. Then the exterior domain is

$$\Omega_h^{ext} := h\mathbb{Z}_-^n = \{(m_1h, \ldots, m_nh) \in h\mathbb{Z}^n | m_n < 0\},$$

and we have

$$\begin{aligned}
\gamma_h^+ &= \left\{(m_1h, \ldots, m_nh) \in h\mathbb{Z}^n | m_n = 1\right\} \text{ (also, 1-layer)},\\
\gamma_h^* &= \left\{(m_1h, \ldots, m_nh) \in h\mathbb{Z}^n | m_n = 0\right\} \text{ (also, 0-layer)},\\
\gamma_h^- &= \left\{(m_1h, \ldots, m_nh) \in h\mathbb{Z}^n | m_n = -1\right\} \text{ (also, -1-layer)}.
\end{aligned}$$

As the next step, we define the classical forward and backward differences $\partial_h^{\pm j}$ as

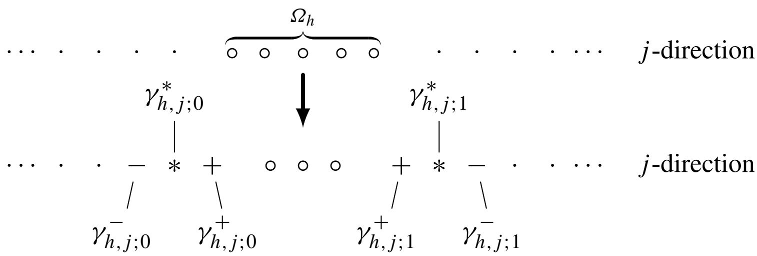
$$\begin{aligned}
\partial_h^{+j} f(mh) &:= h^{-1}(f(mh + e_jh) - f(mh)),\\
\partial_h^{-j} f(mh) &:= h^{-1}(f(mh) - f(mh - e_jh)),
\end{aligned}$$

for discrete functions $f(mh)$ with $mh \in h\mathbb{Z}^n$. Let us now introduce the characteristic functions for the discrete domains Ω_h and Ω_h^{ext} in the classical way

$$\chi_{\Omega_h}(mh) := \begin{cases} 1, & mh \in \Omega_h, \\ 0, & \text{overwise}, \end{cases} \qquad \chi_{\Omega_h^{ext}}(mh) := \begin{cases} 1, & mh \in \Omega_h^{ext}, \\ 0, & \text{overwise}. \end{cases}$$

Naturally, given a domain Ω_h the forward and backward derivatives of the characteristic function χ_{Ω_h} vanish everywhere except on the points of its interior and of its middle boundaries. Likewise, the characteristic function of the exterior domain $\chi_{\Omega_h^{ext}}$ has forward and backward derivatives which vanish everywhere except on the points of the middle and of the exterior boundaries.

For the upcoming discussion on boundary generators and discrete trace operators for bounded domains, we need further classify discrete boundary layers in order to address the direction in which the boundary is approached. Thus, we start with analysing the interior boundary γ_h^+ layer, and we say that $mh \in \gamma_h^+$ is relevant in the j-direction whenever $\partial_h^{\pm j} \chi_{\Omega_h}(mh) \neq 0$. The points for relevant directions are defined analogously for the middle and exterior boundary layers. This classification suggests splitting of the layers of the discrete boundaries into two parts $\gamma_{h,j;0}^{(\cdot)}$ and $\gamma_{h,j;1}^{(\cdot)}$ in each relevant direction j to the forward and backward difference operators, where $(\cdot) = \{+, *, -\}$. The splitting is illustrated as follows:



Now, the three-layer boundary introduced in Definition 2 can be characterised by help of backward and forward difference operators acting on the characteristic functions χ_{Ω_h} and $\chi_{\Omega_h^{ext}}$ as the following lemma states:

Lemma 1 *The parts of the three-layer boundary γ_h, namely γ_h^+, γ_h^* and γ_h^-, are given by:*

– *for the interior boundary layer γ_h^+ it holds:*

$$\partial_h^{-j} \chi_{\Omega_h}(mh) = \begin{cases} 1/h, & mh \in \gamma_{h,j;0}^+, \\ 0, & \textit{otherwise}, \end{cases}$$

$$\partial_h^{+j} \chi_{\Omega_h}(mh) = \begin{cases} -1/h, & mh \in \gamma_{h,j;1}^+, \\ 0, & \textit{otherwise}; \end{cases}$$

– *for the middle boundary layer γ_h^* it holds:*

$$\partial_h^{+j} \chi_{\Omega_h}(mh) = \begin{cases} 1/h, & mh \in \gamma_{h,j;0}^*, \\ 0, & \textit{otherwise}, \end{cases}$$

$$\partial_h^{-j} \chi_{\Omega_h}(mh) = \begin{cases} -1/h, & mh \in \gamma_{h,j;1}^*, \\ 0, & \textit{otherwise}; \end{cases}$$

– *for the exterior boundary layer γ_h^- it holds:*

$$\partial_h^{-j} \chi_{\Omega_h^{ext}}(mh) = \begin{cases} 1/h, & mh \in \gamma_{h,j;1}^-, \\ 0, & \textit{otherwise}, \end{cases}$$

$$\partial_h^{+j} \chi_{\Omega_h}(mh) = \begin{cases} -1/h, & mh \in \gamma_{h,j;0}^-, \\ 0, & \textit{otherwise.} \end{cases}$$

In the sequel we will work with functions defined on discrete lattices. Moreover, we need to introduce a convergence condition for the series appearing in the upcoming sections, which is for a discrete function f implies belonging to the space $l^p(\Omega_h, \mathbb{C}_n)$, $1 \leq p < \infty$.

Since our aim is to introduce a discrete Dirac operator factorising the star-Laplacian Δ_h, we follow the ideas from [2,10], and we split each basis element $e_k, k = 1, 2, \ldots, n$, into two basis elements e_k^+ and $e_k^-, k = 1, 2, \ldots, n$, i.e., $e_k = e_k^+ + e_k^-$, corresponding to the forward and backward directions. Among different possibilities to chose such a basis, see for example [3,9,12], we choose the one satisfying the following relations:

$$\begin{cases} e_j^- e_k^- + e_k^- e_j^- = 0, \\ e_j^+ e_k^+ + e_k^+ e_j^+ = 0, \\ e_j^+ e_k^- + e_k^- e_j^+ = -\delta_{jk}, \end{cases}$$

where δ_{jk} is the Kronecker delta. When allowing for complex coefficients, the basis elements $\{e_1, e_2, \ldots, e_n\}$ generate the complexified Clifford algebra $\mathbb{C}_n = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}_{0,n}$. In the sequel, we consider functions defined on $\Omega_h \subset h\mathbb{Z}^n$ and taking values in $\mathbb{C}_n$. As usual, all important properties such as, l^p-summability ($1 \leq p < \infty$), are defined component-wisely.

By help of the finite difference operator and the splitting of basis elements, the discrete Dirac operator $D^{+-}\colon l^p(\Omega_h, \mathbb{C}_n) \to l^p(\Omega_h, \mathbb{C}_n)$ and its adjoint operator $D^{-+}\colon l^p(\Omega_h, \mathbb{C}_n) \to l^p(\Omega_h, \mathbb{C}_n)$ are defined by

$$D_h^{+-} := \sum_{j=1}^{n} e_j^+ \partial_h^{+j} + e_j^- \partial_h^{-j}, \quad D_h^{-+} := \sum_{j=1}^{n} e_j^+ \partial_h^{-j} + e_j^- \partial_h^{+j}.$$

Therefore, the following factorisation of the star-Laplacian holds

$$(D_h^{+-})^2 = (D_h^{-+})^2 = -\Delta_h, \quad \text{with} \quad \Delta_h := \sum_{j=1}^{n} \partial_h^{+j} \partial_h^{-j}.$$

2.2 Discrete Fundamental Solution

In the sequel we will need the discrete fundamental solution of the discrete Dirac operator defined as follows:

Definition 3 The function $E_h^{-+} : h\mathbb{Z}^n \to \mathbb{C}_n$ is called a discrete fundamental solution of D_h^{-+} if it satisfies

$$D_h^{-+}E_h^{-+}(mh) = \delta_h(mh) = \begin{cases} h^{-n}, & \text{for } mh = 0, \\ 0, & \text{for } mh \neq 0, \end{cases}$$

for all grid points (mh) of $h\mathbb{Z}^n$.

While there are several ways to construct a discrete fundamental solution, in this paper, we follow the classical approach based on the discrete Fourier transform of $u \in l^p\left(h\mathbb{Z}^n, \mathbb{C}_n\right)$, $1 \leq p < +\infty$, defined as follows

$$\boldsymbol{\xi} \quad \mapsto \quad \mathcal{F}_h u(\xi) = \begin{cases} \sum\limits_{m \in \mathbb{Z}^n} e^{i\langle mh, \xi\rangle} u(mh)h^n, \; \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^n, \\ \\ 0, \qquad\qquad\qquad\qquad\quad \text{otherwise}, \end{cases}$$

where $\langle mh, \xi\rangle = h\sum_{j=1}^{n} m_j\xi_j$. The inverse transform is given by $\mathcal{F}_h^{-1} = R_h\mathcal{F}$, where $\mathcal{F}$ is the (standard) continuous Fourier transform

$$x \quad \mapsto \quad \mathcal{F}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(\xi)d\xi,$$

applied to a function f with $\text{supp}(f) \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^n$, and where R_h denotes its restriction to the lattice $h\mathbb{Z}^n$.

Additionally we recall the known symbols for the forward and backward differences $\partial_h^{\pm j}$, namely $\xi_{\pm j}^D = \mp h^{-1}\left(1 - e^{\mp ih\xi_j}\right)$, as well as the symbol for the star-Laplacian, i.e., $\mathcal{F}_h(\Delta_h u)(\xi) = d^2\mathcal{F}_h u(\xi)$, where

$$d^2 = \frac{4}{h^2}\sum_{j=1}^{n}\sin^2\left(\frac{\xi_j h}{2}\right).$$

Therefore, we have $\mathcal{F}_h(D_h^{-+}u)(\xi) = \left(\sum_{j=1}^{n} e_j^+\xi_{-j}^D + e_j^-\xi_{+j}^D\right)\mathcal{F}_h u(\xi)$ so that D^{-+} has symbol $\widetilde{\xi}_- = \sum_{j=1}^{n} e_j^+\xi_{-j}^D + e_j^-\xi_{+j}^D$. Thus, the fundamental solution E^{-+} is given by

$$E_h^{-+} = R_h\mathcal{F}\left(\frac{\widetilde{\xi}_-}{d^2}\right) = \sum_{j=1}^{n} e_j^+ R_h\mathcal{F}\left(\frac{\xi_{-j}^D}{d^2}\right) + e_j^- R_h\mathcal{F}\left(\frac{\xi_{+j}^D}{d^2}\right). \tag{1}$$

Remark 2 Considering that the symbol of the discrete operator D_h^{+-} is given by $\widetilde{\xi}_+ = \sum_{j=1}^n e_j^+ \xi_{+j}^D + e_j^- \xi_{-j}^D$, its fundamental solution E_h^{+-} can be calculated as follows

$$E_h^{+-} = R_h \mathcal{F}\left(\frac{\widetilde{\xi}_+}{d^2}\right) = \sum_{j=1}^n e_j^+ R_h \mathcal{F}\left(\frac{\xi_{+j}^D}{d^2}\right) + e_j^- R_h \mathcal{F}\left(\frac{\xi_{-j}^D}{d^2}\right).$$

Finally, for the results related to convergence analysis we will need the following fundamental lemma [4]:

Lemma 2 *Let E be the fundamental solution to the continuous Dirac operator in $\mathbb{R}^n$. For any point $mh \in h\mathbb{Z}^n$, with $m \neq 0$, there exists a constant C independent on h, such that*

$$|E_h^{-+}(mh) - E(mh)| \leq C \frac{h}{|mh|^n}.$$

We recall the following lemma without proof:

Lemma 3 [4] *The fundamental solution E^{-+} satisfies:*

(i) $D_h^{-+} E_h^{-+}(mh) = \delta_h(mh)$, *$mh \in h\mathbb{Z}^n$;*
(ii) $E_h^{-+} \in l^p(\mathbb{Z}^n, \mathbb{C}_n)$, *for $p > \frac{3}{2}$.*

3 Discrete Stokes, Borel–Pompeiu and Cauchy Formulae

3.1 Discrete Stokes' Formula for Bounded Domains

In this section we present the discrete Stokes' formula for bounded domains in $h\mathbb{Z}^n$. The discrete Stokes' formula will be used as a basis for introducing the discrete Borel–Pompeiu formula in Sect. 3.2. At first, we present a generic Stokes' formula in $h\mathbb{Z}^n$ and after that, we specify the generic formula by presenting formulae for a bounded domain and its exterior domain. To keep the presentation short, the proof of the discrete Stokes' formula is omitted.

Theorem 1 (See [4], Theorem 2.10) *We have*

$$\sum_{mh \in h\mathbb{Z}^n} \left[(g D_h^{-+})(mh) f(mh) + g(mh) \left(D_h^{+-} f(mh) \right) \right] h^n = 0, \quad \forall mh \in h\mathbb{Z}^n \quad (2)$$

for all functions f, g such that the series converge.

We wish to specify the generic Stokes' formulae for a bounded domain and its exterior domain (we recall, the interior of its complementary domain and, hence, an unbounded domain). For that effect, we consider the auxiliar characteristic functions, χ_{Ω_h} and $\chi_{\Omega_h^{ext}}$.

Theorem 2 (See also [6]) *The discrete Stokes' formula for an arbitrary domain $\Omega_h \subset h\mathbb{Z}^n$ is given by*

$$\sum_{mh\in h\mathbb{Z}^n} \left[(gD_h^{-+})(mh)f(mh) + g(mh)(D_h^{+-}f)(mh)\right]\chi_{\Omega_h}(mh)h^n$$
$$= -\sum_{mh\in h\mathbb{Z}^n}\sum_{j=1}^{n}\left[\left(\partial_h^{j,-}\chi_{\Omega_h}\right)((m+e_j)h)g(mh)e_j^+ f((m+e_j)h)\right.$$
$$\left. + \left(\partial_h^{j,+}\chi_{\Omega_h}\right)(mh)g((m+e_j)h)e_j^- f(mh)\right]h^n$$

for all discrete functions f and g such that the series converge, and where χ_{Ω_h} denotes the characteristic function of the domain.

Proof We present the general layout of the proof. By replacing g by $g\chi_{\Omega_h}$ in (2), and the use of the Leibniz rule (see [6])

$$\partial_h^{j,+}(g\chi_{\Omega_h})(mh) = \left(\partial_h^{j,+}g(mh)\right)\chi_{\Omega_h}(mh) + g((m+e_j)h)\left(\partial_h^{j,+}\chi_{\Omega_h}(mh)\right),$$
$$\partial_h^{j,-}(g\chi_{\Omega_h})(mh) = \left(\partial_h^{j,-}g(mh)\right)\chi_{\Omega_h}(mh) + g((m-e_j)h)\left(\partial_h^{j,-}\chi_{\Omega_h}(mh)\right),$$

we obtain directly that (recall that χ_{Ω_h} is a real-valued function)

$$0 = \sum_{mh\in h\mathbb{Z}^n}\left[((g\chi_{\Omega_h})D_h^{-+})(mh)f(mh) + g(mh)\chi_{\Omega_h}(mh)(D_h^{+-}f)(mh)\right]h^n$$
$$= \sum_{mh\in h\mathbb{Z}^n}\left[((gD_h^{-+})(mh)f(mh) + g(mh)(D_h^{+-}f)(mh)\right]\chi_{\Omega_h}(mh)h^n$$
$$+ \sum_{mh\in h\mathbb{Z}^n}\sum_{j=1}^{n}\left[\left(\partial_h^{j,-}\chi_{\Omega_h}\right)(mh)g((m-e_j)h)e_j^+ f(mh)\right.$$
$$\left. + \left(\partial_h^{j,+}\chi_{\Omega_h}\right)(mh)g((m+e_j)h)e_j^- f(mh)\right]h^n$$
$$= \sum_{mh\in h\mathbb{Z}^n}\left[((gD_h^{-+})(mh)f(mh) + g(mh)(D_h^{+-}f)(mh)\right]\chi_{\Omega_h}(mh)h^n$$
$$+ \sum_{mh\in h\mathbb{Z}^n}\sum_{j=1}^{n}\left[\left(\partial_h^{j,-}\chi_{\Omega_h}\right)((m+e_j)h)g(mh)e_j^+ f((m+e_j)h)\right.$$
$$\left. + \left(\partial_h^{j,+}\chi_{\Omega_h}\right)(mh)g((m+e_j)h)e_j^- f(mh)\right]h^n.$$

□

The backward and forward derivatives of χ_{Ω_h} are known at the points of the interior and middle boundaries, and vanish otherwise (see Lemma 1). Furthermore, the relations

$$mh - e_ih \in \gamma_{h,i;1}^{+} \Leftrightarrow mh \in \gamma_{h,i;1}^{*}, \qquad mh \in \gamma_{h,i;0}^{+} \Leftrightarrow mh - e_ih \in \gamma_{h,i;0}^{*},$$

allow us to express these sums in terms of points of the middle boundary γ_h^* alone.

Hence, we obtain the discrete Stokes' formula for an arbitrary domain Ω_h.

Theorem 3 *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary simply connected and bounded domain, and let $\Omega_h \subset h\mathbb{Z}^n$ be its discrete version with lattice constant h. Then it holds*

$$\sum_{mh\in h\mathbb{Z}^n} \left[(gD_h^{-+})(mh)f(mh) + g(mh)(D_h^{+-}f)(mh)\right]\chi_{\Omega_h}(mh)h^n$$
$$= \sum_{i=1}^{n}\left(-\sum_{mh\in\gamma^*_{h,i;0}}\left[g(mh)e_i^+ f(mh+e_ih) + g(mh+e_ih)e_i^- f(mh)\right]h^{n-1}\right.$$
$$\left. + \sum_{mh\in\gamma^*_{h,i;1}}\left[g(mh-e_ih)e_i^+ f(mh) + g(mh)e_i^- f(mh-e_ih)\right]h^{n-1}\right)$$

for all discrete functions f and g such that the series converge, and where χ_{Ω_h} is the characteristic function of the discrete domain.

Similar considerations with respect to the exterior domain $\Omega_h^{ext} = int(\Omega_h^c)$ lead to the following formula.

Theorem 4 *Let Ω_h^{ext} be the discrete exterior domain associated to Ω_h (as defined in Theorem 3). Then it holds*

$$\sum_{mh\in h\mathbb{Z}^n} \left[(gD_h^{-+})(mh)f(mh) + g(mh)(D_h^{+-}f)(mh)\right]\chi_{\Omega_h^{ext}}(mh)h^n$$
$$= \sum_{i=1}^{n}\left(\sum_{mh\in\gamma^*_{h,i;0}}\left[g(mh-e_ih)e_i^+ f(mh) + g(mh)e_i^- f(mh-e_ih)\right]h^{n-1}\right.$$
$$\left. - \sum_{mh\in\gamma^*_{h,i;1}}\left[g(mh)e_i^+ f(mh+e_ih) + g(mh+e_ih)e_i^- f(mh)\right]h^{n-1}\right)$$

for all discrete functions f and g such that the series converge, and where $\chi_{\Omega_h^{ext}}$ is the characteristic function of the discrete domain.

As in the previous case, both the interior boundary of Ω_h^{ext} (which corresponds to the exterior boundary γ_h^- of Ω_h) and its middle boundary γ_h^* are involved, because the following relations hold

$$mh \in \gamma^-_{h,i;0} \Leftrightarrow mh + e_ih \in \gamma^*_{h,i;0}, \qquad mh + e_ih \in \gamma^-_{h,i;1} \Leftrightarrow mh \in \gamma^*_{h,i;1}.$$

3.2 Borel–Pompeiu and Cauchy Formulae for Bounded Domains

By help of the discrete Stokes' formula introduced in Sect. 3.1 the discrete Borel–Pompeiu and Cauchy formulae can be established. We reinforce that, although all formulas are written in terms of the middle boundary γ_h^*, they, in fact, depend on the double boundary $\gamma_h = \gamma_h^+ \cup \gamma_h^*$ (or $\gamma_h = \gamma_h^* \cup \gamma_h^-$ in the case of the exterior domain).

We have the following theorem:

Theorem 5 *Given an arbitrary simply connected domain $\Omega \subset \mathbb{R}^n$, let Ω_h be its associated discrete domain with the lattice constant h, and let Ω_h^{ext} be its associated exterior domain. Then the discrete Borel–Pompeiu formula for Ω_h is given by*

$$\sum_{rh\in h\mathbb{Z}^n} E_h^{-+}\big((r-m)h\big)(D_h^{+-}f)(rh)\chi_{\Omega_h}(rh)h^n$$
$$+\sum_{i=1}^{n}\left(-\sum_{rh\in\gamma_{h,i;0}^*}\Big[E_h^{-+}\big((r-m)h\big)e_i^+ f\big((r+e_i)h\big)\right.$$
$$+E_h^{-+}\big((r+e_i-m)h\big)e_i^- f(rh)\Big]h^{n-1}$$
$$+\sum_{rh\in\gamma_{h,i;1}^*}\Big[E_h^{-+}\big((r-e_i-m)h\big)e_i^+ f(rh)$$
$$\left.+E_h^{-+}\big((r-m)h\big)e_i^- f(rh-e_ih)\Big]h^{n-1}\right) = \begin{cases} 0, & \text{if } mh\notin \Omega_h\cup\gamma_h^*, \\ -f(mh), & \text{if } mh\in\Omega_h\cup\gamma_h^*, \end{cases}$$

for any discrete function f such that the series converge, and where E_h^{-+} is the discrete fundamental solution to operator D_h^{-+} and χ_{Ω_h} is the characteristic function of the discrete domain.

Furthermore, the discrete Borel–Pompeiu formula for its associated exterior domain Ω_h^{ext} is given by

$$\sum_{rh\in h\mathbb{Z}^n} \Big[E_h^{-+}\left((r-m)h\right)(D_h^{+-}f)(rh)\Big]\chi_{\Omega_h^{ext}}(rh)h^n$$
$$+\sum_{i=1}^{n}\left(-\sum_{rh\in\gamma_{h,i;0}^*}\Big[E_h^{-+}\big((r-e_i-m)h\big)e_i^+ f(rh)\right.$$
$$+E_h^{-+}\big((r-m)h\big)e_i^- f\big((r-e_i)h\big)\Big]h^{n-1}$$
$$+\sum_{rh\in\gamma_{h,i;1}^*}\Big[E_h^{-+}\big((r-m)h\big)e_i^+ f\big((r+e_i)h\big)$$
$$\left.+E_h^{-+}\big((r+e_i-m)h\big)e_i^- f(rh)\Big]h^{n-1}\right) = \begin{cases} 0, & \text{if } mh\notin \Omega_h^{ext}\cup\gamma_h^*, \\ -f(mh), & \text{if } mh\in\Omega_h^{ext}\cup\gamma_h^*, \end{cases}$$

for any discrete function f such that the series converge, and where $\chi_{\Omega_h^{ext}}$ is the characteristic function of the exterior discrete domain.

Proof The proof of this theorem is essentially based on the use of the discrete Stokes's formula (2). We substitute g by $E_h^{-+}(\cdot - mh)$ in the discrete Stokes's. Considering that $mh \in \Omega_h$ and using the known properties of the discrete fundamental solution $\left[E_h^{-+}\big((\cdot - m)h\big)D_h^{-+}\right](rh) = 0, r \neq m$ and $\left[E_h^{-+}\big((\cdot - m)h\big)D_h^{-+}\right](rh) = h^{-n}, r = m$, the discrete Borel–Pompeiu formula follows immediately. □

Similar to the continuous case, the discrete Cauchy formula can be obtained immediately from the discrete Borel–Pompeiu formula if a function f is a discrete left-monogenic function. Thus, we have the following theorem:

Theorem 6 *Let f be a discrete left monogenic function with respect to operator D_h^{+-}, and let E_h^{-+} be the discrete fundamental solution to operator D_h^{-+}. Then the (interior and exterior) discrete Cauchy formulae for an arbitrary domain $\Omega_h \subset h\mathbb{Z}^n$ are given by*

$$\sum_{i=1}^{n}\left(-\sum_{rh\in\gamma_{h,i;0}^{*}}\left[E_h^{-+}\big((r-m)h\big)e_i^{+}f\big((r+e_i)h\big)\right.\right.$$
$$\left.+\,E_h^{-+}\big((r+e_i-m)h\big)e_i^{-}f(rh)\right]h^{n-1}$$
$$+\sum_{rh\in\gamma_{h,i;1}^{*}}\left[E_h^{-+}\big((r-e_i-m)h\big)e_i^{+}f(rh)\right.$$
$$\left.\left.+\,E_h^{-+}\big((r-m)h\big)e_i^{-}f(rh-e_ih)\right]h^{n-1}\right)=\begin{cases}0, & \text{if } mh\notin\Omega_h\cup\gamma_h^{*},\\ -f(mh), & \text{if } mh\in\Omega_h\cup\gamma_h^{*},\end{cases}$$

and

$$\sum_{i=1}^{n}\left(-\sum_{rh\in\gamma_{h,i;0}^{*}}\left[E_h^{-+}\big((r-e_i-m)h\big)e_i^{+}f(rh)\right.\right.$$
$$\left.+\,E_h^{-+}\big((r-m)h\big)e_i^{-}f\big((r-e_i)h\big)\right]h^{n-1}$$
$$+\sum_{rh\in\gamma_{h,i;1}^{*}}\left[E_h^{-+}\big((r-m)h\big)e_i^{+}f\big((r+e_i)h\big)\right.$$
$$\left.\left.+\,E_h^{-+}\big((r+e_i-m)h\big)e_i^{-}f(rh)\right]h^{n-1}\right)=\begin{cases}0, & \text{if } mh\notin\Omega_h^{ext}\cup\gamma_h^{*},\\ -f(mh), & \text{if } mh\in\Omega_h^{ext}\cup\gamma_h^{*},\end{cases}$$

which hold for any discrete function f such that the series converge.

The discrete (interior and exterior) Cauchy transforms are a direct consequence of Theorem 6, and they are introduced by the following definition:

Definition 4 Let us consider discrete bounded domain Ω_h, with the lattice constant $h > 0$. Then for a discrete l^p-function f, $1 \leq p < +\infty$, defined on the boundary layers γ_h^+ and γ_h^* the discrete interior Cauchy transform is defined by

$$\begin{aligned} C^+[f](mh) = \sum_{i=1}^{n} \Bigg(& \sum_{mh \in \gamma_{h,i;0}^*} \big[E_h^{-+}\big((r-m)h\big)e_i^+ f\big((r+e_i)h\big) \\ & + E_h^{-+}\big((r+e_i-m)h\big)e_i^- f(rh)\big]\, h^{n-1} \\ & - \sum_{rh \in \gamma_{h,i;1}^*} \big[E_h^{-+}\big((r-e_i-m)h\big)e_i^+ f(rh) \\ & + E_h^{-+}\big((r-m)h\big)e_i^- f(rh-e_ih)\big]\, h^{n-1} \Bigg). \end{aligned} \tag{3}$$

Likewise, if the discrete l^p-function f, $1 \leq p < +\infty$, is defined on the boundary layers γ_h^* and γ_h^- then its discrete exterior Cauchy transform is defined by

$$\begin{aligned} C^-[f](mh) = \sum_{i=1}^{n} \Bigg(& \sum_{rh \in \gamma_{h,i;0}^*} \big[E_h^{-+}\big((r-e_i-m)h\big)e_i^+ f(rh) \\ & + E_h^{-+}\big((r-m)h\big)e_i^- f\big((r-e_i)h\big)\big]\, h^{n-1} \\ & - \sum_{rh \in \gamma_{h,i;1}^*} \big[E_h^{-+}\big((r-m)h\big)e_i^+ f\big((r+e_i)h\big) \\ & + E_h^{-+}\big((r+e_i-m)h\big)e_i^- f(rh)\big]\, h^{n-1} \Bigg). \end{aligned} \tag{4}$$

These formulae hold for any discrete function f such that the series converge.

Both formulae (3) and (4) can be written as

$$\begin{aligned} C^\pm[f](mh) := & \sum_{rh \in \mathbb{R}_h^n} \sum_{j=1}^{n} \left(e_{\mp j} \partial_h^{\pm j} \chi_{(\cdot)}(rh) \right) f\left((r \pm e_j)h\right) \\ & \times E_h^{-+}\left((r-m)h\right) h^n, \end{aligned} \tag{5}$$

whereas $\chi_{(\cdot)}$ denotes the correspondent characteristic function at the domain Ω or its associated exterior domain Ω^{ext}.

The discrete Cauchy formula, similar to the continuous case, states clearly the dependence of a discrete left-monogenic function on its boundary values. We finish this section by the theorem stating important properties of the discrete Cauchy transform:

Theorem 7 *Let us consider a discrete bounded domain Ω_h and its associated discrete exterior domain Ω_h^{ext}, together with the three boundary layers γ_h^+, γ_h^*, and γ_h^-. Moreover, let us introduce the sets $\Gamma_h^+ := \gamma_h^+ \cup \gamma_h^*$ and $\Gamma_h^- := \gamma_h^* \cup \gamma_h^-$. Then the discrete Cauchy transforms (5) satisfy the following properties:*

(i) *The interior and exterior Cauchy transforms have the following mapping properties:*

$$C^+ : l^p(\Gamma_h^+, \mathbb{C}_{n-1}) \to l^q(\Omega_h, \mathbb{C}_n),\ 1 \leq p, q \leq \infty,$$
$$C^- : l^p(\Gamma_h^-, \mathbb{C}_{n-1}) \to l^q(\Omega_h^{ext}, \mathbb{C}_n),\ 1 \leq p < \infty, \frac{3}{2} < q < \infty.$$

(ii) $D_h^{+-} C^+[f](mh) = 0,\ \forall\, mh \in \Omega_h \backslash \gamma_h^+$.
(iii) $D_h^{+-} C^-[f](mh) = 0,\ \forall\, mh \in \Omega_h^{ext} \backslash \gamma_h^-$.

Proof The proof of this theorem is straightforward. The first property is proved by a direct application of Hölder's inequality and using the properties of the discrete fundamental solution. Applying the discrete operator D_h^{+-} to the discrete Cauchy transform, and considering the properties:

$$\partial_h^{+j} \left(E_h^{-+}((\cdot - m)h) \right) = -(\partial_h^{-j} E^{-+})(\cdot - m),$$
$$\partial_h^{-j} \left(E_h^{-+}((\cdot - m)h) \right) = -(\partial_h^{+j} E^{-+})(\cdot - m),$$

the second property is proved. The proof of the third property is analogue to the second. $\square$

4 Boundary Values of Discrete Monogenic Functions and Relations to Discrete Riemann–Hilbert Problems

To keep the presentation short, we will present only the operator form of equations in all upcoming discussions. The discrete Cauchy formula presented in Theorem 6 provides an additional condition specifying if a function given on the discrete boundary $\Gamma_h^+ = \gamma_h^+ \cup \gamma_h^*$ or $\Gamma_h^- = \gamma_h^- \cup \gamma_h^*$ represents boundary values of a discrete left-monogenic function in Ω_h or Ω_h^{ext}, correspondingly. Therefore, we have two sets of conditions:

$$C^+ f(mh) = \begin{cases} f(mh), & \text{for all } mh \in \Omega_h, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

for the interior domain, and

$$C^- f(mh) = \begin{cases} f(mh), & \text{for all } mh \in \Omega_h^{ext}, \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

for the exterior domain.

In the sequel, we will sometimes need explicitly the n-th component of elements in $\mathbb{R}^n$ or in $\mathbb{Z}^n$, or, more generally, n-th component of an arbitrary element of n-dimensional space. In these cases we will use such notations as $\xi = (\underline{\xi}, \xi_n) \in \mathbb{R}^n$, $m = (\underline{m}, m_n) \in \mathbb{Z}^n$ etc. By help of these notations, n-dimensional elements can be represented as a sum of $(n-1)$-dimensional part $\underline{\xi}$ and n-th component ξ_n.

Next, we want to introduce discrete Plemelj (or Hardy) projections, which require at first definition of discrete Riesz kernels (convolution kernels). To defining properly the discrete Riesz kernels, behaviour of the discrete fundamental solution E_h^{-+} on boundary layers needs to be studied, as it has been done in [4] for the case of a half-space. Recalling that the discrete fundamental solution is given by (1),

$$E_h^{-+}(mh) = R_h \mathcal{F}\left(\frac{\widetilde{\xi_-}}{d^2}\right) = \frac{1}{(2\pi)^n} \int_{\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^n} e^{-ih\langle m, \xi\rangle} \frac{\widetilde{\xi_-}}{d^2} d\xi,$$

where $\widetilde{\xi_-} := \sum_{j=1}^n \left(e_j^+ \frac{1-e^{ih\xi_j}}{h} + e_j^- \frac{e^{-ih\xi_j}-1}{h}\right)$, we need to study the Fourier symbols on the boundary layer. Thus we apply the $(n-1)$-dimensional discrete Fourier transform to the discrete fundamental solution:

$$\begin{aligned}
&\mathcal{F}_h^{(n-1)} E_h^{-+}(\underline{\eta}, m_n h)\\
&= \sum_{\underline{m}h \in h\mathbb{Z}^{n-1}} e^{-ih\langle \underline{m}, \underline{\eta}\rangle} \left[\frac{1}{(2\pi)^n} \int_{\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^n} e^{-ih\langle m, \xi\rangle} \frac{\widetilde{\xi_-}}{d^2} d\xi\right]\\
&= \frac{1}{(2\pi)^{n-1}} \int_{\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^{n-1}} \sum_{\underline{m}h \in h\mathbb{Z}^{n-1}} e^{-ih\langle \underline{m}, \underline{\eta}-\underline{\xi}\rangle} \underbrace{\left[\frac{1}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-ihm_n\xi_n} \frac{\widetilde{\xi_-}}{d^2} d\xi_n\right]}_{(I)} d\underline{\xi}.
\end{aligned}$$

Under the above introduced convention, we can represent the integral (I) as follows

$$\begin{aligned}
(I) &= \frac{1}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-ihm_n\xi_n} \frac{\widetilde{\xi_-}}{d^2} d\xi_n = \frac{1}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-ihm_n\xi_n} \frac{\underline{\widetilde{\xi}}_- + \underline{\widetilde{\xi}}_{-,n}}{\underline{d}^2 + \frac{4}{h^2}\sin^2\left(\frac{h\xi_n}{2}\right)} d\xi_n\\
&= \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i2tm_n} \frac{\underline{\widetilde{\xi}}_- + e_j^+ \frac{1-e^{2it}}{h} + e_j^- \frac{e^{-2it}-1}{h}}{\underline{d}^2 + \frac{4}{h^2}\sin^2(t)} \frac{h}{2} dt\\
&= \frac{1}{4\pi} h\underline{\widetilde{\xi}}_- \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-i2tm_n}}{\underline{d}^2 + \frac{4}{h^2}\sin^2(t)} dt\\
&\quad + \frac{1}{4\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i2tm_n} \frac{e_j^+(1-e^{2it}) + e_j^-(e^{-2it}-1)}{\underline{d}^2 + \frac{4}{h^2}\sin^2(t)} dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4\pi} h\underline{\widetilde{\xi}}_{-} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-i2tm_n}}{\underline{d}^2 + \frac{4}{h^2}\sin^2(t)} dt + \frac{1}{4\pi}(e_j^+ - e_j^-) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-i2tm_n}}{\underline{d}^2 + \frac{4}{h^2}\sin^2(t)} dt \\
&\quad - \frac{1}{4\pi} e_j^+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-i2t(m_n-1)}}{\underline{d}^2 + \frac{4}{h^2}\sin^2(t)} dt + \frac{1}{4\pi} e_j^- \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-i2t(m_n+1)}}{\underline{d}^2 + \frac{4}{h^2}\sin^2(t)} dt,
\end{aligned}
$$

where the change of variable $t = \frac{h\xi_n}{2}$ has been used. We concentrate our attention on the first of these integrals. Since we consider the general case of bounded domains in $\Omega_h \subset h\mathbb{Z}^n$, the exact position of boundaries depends on a specific discrete domain Ω_h, and therefore, we have to keep construction general to cover all possible situations. Thus, we need to distinguish several cases for m_n, which will be denoted as k for shortening:

– Case of $k = 0$ has been considered in [4], therefore we do not discuss it here.
– For $|k| \geq 1$ we obtain

$$
\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i2tk} \frac{1}{\underline{d}^2 + \frac{4}{h^2}\sin^2(t)} dt &= h^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-i2tk}}{h^2\underline{d}^2 + 4\sin^2(t)} dt \\
&= \frac{h^2}{2} \int_{|z|=1} \frac{z^{-2k}}{h^2\underline{d}^2 - (z - \overline{z})^2} \frac{dz}{iz} \\
&= \frac{h^2}{2i} \int_{|z|=1} \frac{dz}{z^{2k+1}\left[h^2\underline{d}^2 - \left(z - \frac{1}{z}\right)^2\right]} \\
&= \frac{h^2}{2i} \int_{|z|=1} \frac{dz}{z^{2k+1}\left[h^2\underline{d}^2 - \left(z - \frac{1}{z}\right)^2\right]} \\
&= -\frac{h^2}{2i} \int_{|z|=1} \frac{dz}{z^{2k-1}\left[\left(z^2 - 1\right)^2 - h^2\underline{d}^2 z^2\right]}.
\end{aligned}
$$

Further, if $k \geq 1$, then the polynomial in the denominator has 5 distinct zeros, namely

$$
\begin{cases}
z_0 = 0 & \text{of multiplicity } 2k-1; \\
z_{\pm,\pm} = \pm\frac{h\underline{d}}{2} \pm \frac{1}{2}\sqrt{h^2\underline{d}^2 + 4} & \text{each of multiplicity } 1.
\end{cases}
$$

Of these, $z_{+,+}, z_{-,-}$ lie outside the circle $|z| = 1$ while $z_{+,-}, z_{-,+}$ lie inside the circle $|z| = 1$. Therefore, we have

$$
\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i2tk} \frac{1}{\underline{d}^2 + \frac{4}{h^2}\sin^2(t)} dt &= -\frac{h^2}{2i} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dz}{z^{2k-1}\left[\left(z^2 - 1\right)^2 - h^2\underline{d}^2 z^2\right]} \\
&= -\pi h^2 \left[Res(z_0) + Res(z_{+,-}) + Res(z_{-,+})\right],
\end{aligned}
$$

where the first term is computed by the Faà di Bruno's formula

$$\begin{aligned}
Res(z_0) &= \frac{\partial_z^{2k-2}\left[\frac{1}{z^4-(2+h^2\underline{d}^2)z^2+1}\right]\Big|_{z_0=0}}{(2k-2)!} \\
&= \sum_{m_2+2m_4=k-1} (-1)^{m_4}\binom{m_2+m_4}{m_2}(2+h^2\underline{d}^2)^{m_2},
\end{aligned}$$

and

$$Res(z_{+,-}) = \left(\frac{h\underline{d}}{2}-\frac{1}{2}\sqrt{h^2\underline{d}^2+4}\right)^{1-2k}\left[\frac{1}{h\underline{d}\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)}\right],$$

$$\begin{aligned}
Res(z_{-,+}) &= \left(-\frac{h\underline{d}}{2}+\frac{1}{2}\sqrt{h^2\underline{d}^2+4}\right)^{1-2k}\left[\frac{1}{h\underline{d}\sqrt{h^2\underline{d}^2+4}\left(-h\underline{d}+\sqrt{h^2\underline{d}^2+4}\right)}\right] \\
&= (-1)^{1-2k+1}\left(\frac{h\underline{d}}{2}-\frac{1}{2}\sqrt{h^2\underline{d}^2+4}\right)^{1-2k} \\
&\quad \left[\frac{1}{h\underline{d}\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)}\right] \\
&= Res(z_{+,-}).
\end{aligned}$$

This leads to

$$\begin{aligned}
&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i2tk}\,\frac{1}{\underline{d}^2+\frac{4}{h^2}\sin^2(t)}dt \\
&\quad = -\pi h^2 \sum_{m_2+2m_4=k-1} (-1)^{m_4}\binom{m_2+m_4}{m_2}(2+h^2\underline{d}^2)^{m_2} \\
&\qquad -\frac{2^{2k}\pi h^2}{h\underline{d}\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)^{2k}}.
\end{aligned}$$

Finally, if $k \leq -1$, then $z_{+,-}$, $z_{-,+}$ are the only poles (of order 1), and, therefore, we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i2tk} \frac{1}{\underline{d}^2 + \frac{4}{h^2}\sin^2(t)} dt = -2\pi h^2 Res(z_{+,-})$$
$$= -\frac{2^{2k}\pi h^2}{h\underline{d}\sqrt{h^2\underline{d}^2+4}\left(h\underline{d} - \sqrt{h^2\underline{d}^2+4}\right)^{2k}}.$$

To finalise our computations, we denote by

$$I(k) = \frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-i2tk}}{\underline{d}^2 + \frac{4}{h^2}\sin^2(t)} dt,$$

and we remark that the coefficients of e_j^+ and of e_j^- are given, respectively, by

$$I(m_n) - I(m_n - 1), \qquad I(m_n + 1) - I(m_n).$$

Again, we remark that the case when $m_n = 0$ is done already in [4]. Hence, we now look into this difference when $k < 0$,

$$\begin{aligned}
&I(k) - I(k-1)\\
&= -\frac{2^{2k-2}h^2}{h\underline{d}\sqrt{h^2\underline{d}^2+4}\left(h\underline{d} - \sqrt{h^2\underline{d}^2+4}\right)^{2k}}\\
&\quad + \frac{2^{2k-4}h^2}{h\underline{d}\sqrt{h^2\underline{d}^2+4}\left(h\underline{d} - \sqrt{h^2\underline{d}^2+4}\right)^{2k-2}}\\
&= \frac{2^{2k-4}h^2}{h\underline{d}\sqrt{h^2\underline{d}^2+4}\left(h\underline{d} - \sqrt{h^2\underline{d}^2+4}\right)^{2k-2}}\left[1 - \frac{2^2}{h\underline{d}\sqrt{h^2\underline{d}^2+4}\left(h\underline{d} - \sqrt{h^2\underline{d}^2+4}\right)^{2}}\right]\\
&= \frac{2^{2k-3}h^2}{\sqrt{h^2\underline{d}^2+4}\left(h\underline{d} - \sqrt{h^2\underline{d}^2+4}\right)^{2k-1}}.
\end{aligned}$$

For $k > 1$ this term must be added to difference coming from the pole $z_0 = 0$. Hereby, we have to distinguish between the cases where k is even and k is odd. In the case of k odd we get

$$\begin{aligned}
&I(k) - I(k-1)\\
&= \sum_{m_2+2m_4=k-1} (-1)^{m_4}\binom{m_2+m_4}{m_2}(2+h^2\underline{d}^2)^{m_2}\\
&\quad - \sum_{m_2+2m_4=k-2} (-1)^{m_4}\binom{m_2+m_4}{m_2}(2+h^2\underline{d}^2)^{m_2}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2^{2k-3}h^2}{\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)^{2k-1}}\\
&=\sum_{m_2+2m_4=k-2}(-1)^{m_4}\binom{m_2+m_4+1}{m_2+1}(2+h^2\underline{d}^2)^{m_2+1}\\
&\quad-\sum_{m_2+2m_4=k-2}(-1)^{m_4}\binom{m_2+m_4}{m_2}(2+h^2\underline{d}^2)^{m_2}+(-1)^{(k-1)/2}\\
&\quad+\frac{2^{2k-3}h^2}{\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)^{2k-1}}\\
&=\sum_{m_2+2m_4=k-2}(-1)^{m_4}(2+h^2\underline{d}^2)^{m_2}\binom{m_2+m_4}{m_2}\left[\frac{(m_2+m_4+1)h^2\underline{d}^2)+k-1}{m_2+1}\right]\\
&\quad+(-1)^{(k-1)/2}+\frac{2^{2k-3}h^2}{\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)^{2k-1}},
\end{aligned}
$$

while in the case of k even, the term $(-1)^{(k-1)/2}$ will be missing, i.e.

$$
\begin{aligned}
&I(k)-I(k-1)\\
&=\sum_{m_2+2m_4=k-2}(-1)^{m_4}(2+h^2\underline{d}^2)^{m_2}\binom{m_2+m_4}{m_2}\left[\frac{m_2+m_4+1)h^2\underline{d}^2)+k-1}{m_2+1}\right]\\
&\quad+\frac{2^{2k-3}h^2}{\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)^{2k-1}}.
\end{aligned}
$$

Consequently we get for the Fourier multiplier (I) in the case of $m_n<0$:

$$
\begin{aligned}
(I)=&-\frac{1}{4\pi}h\widetilde{\underline{\xi}}_{-}\frac{2^{2m_n}\pi h^2}{h\underline{d}\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)^{2m_n}}\\
&+e_j^{+}\frac{1}{4\pi}\frac{2^{2m_n-3}h^2}{\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)^{2m_n-1}}\\
&+e_j^{-}\frac{1}{4\pi}\frac{2^{2m_n-1}h^2}{\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)^{2m_n+1}},
\end{aligned}
$$

while in the case of $m_n>0$, m_n even, we have

$$
\begin{aligned}
(I) = -\frac{1}{4\pi} h\underline{\widetilde{\xi}} & \left(\pi h^2 \sum_{m_2+2m_4=m_n-1} (-1)^{m_4} \binom{m_2+m_4}{m_2} (2+h^2\underline{d}^2)^{m_2} \right. \\
& \left. - \frac{2^{2m_n} \pi h^2}{h\underline{d}\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)^{2m_n}} \right) \\
& + e_j^+ \left(\sum_{m_2+2m_4=m_n-2} (-1)^{m_4} (2+h^2\underline{d}^2)^{m_2} \binom{m_2+m_4}{m_2} \left[\frac{(m_2+m_4+1)h^2\underline{d}^2)+m_n-1}{m_2+1} \right] \right. \\
& \left. + \frac{2^{2m_n-3}h^2}{\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)^{2m_n-1}} + \frac{2^{2m_n-3}h^2}{\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)^{2m_n-1}} \right) \\
& + e_j^- \frac{1}{4\pi} \left(\frac{2^{2m_n-1}h^2}{\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)^{2m_n+1}} \right. \\
& + \sum_{m_2+2m_4=m_n-1} (-1)^{m_4} (2+h^2\underline{d}^2)^{m_2} \binom{m_2+m_4}{m_2} \left[\frac{((m_2+m_4+1)h^2\underline{d}^2)+m_n-1}{m_2+1} \right] \\
& \left. + \frac{2^{2m_n-1}h^2}{\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)^{2m_n+1}} + (-1)^{(k-1)/2} \right),
\end{aligned}
$$

while in the case of $m_n > 0$, m_n odd, we have

$$
\begin{aligned}
(I) = -\frac{1}{4\pi} h\underline{\widetilde{\xi}} & \left(\pi h^2 \sum_{m_2+2m_4=m_n-1} (-1)^{m_4} \binom{m_2+m_4}{m_2} (2+h^2\underline{d}^2)^{m_2} \right. \\
& \left. - \frac{2^{2m_n} \pi h^2}{h\underline{d}\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)^{2m_n}} \right) \\
& + e_j^+ \frac{1}{4\pi} \left(\sum_{m_2+2m_4=m_n-2} (-1)^{m_4} (2+h^2\underline{d}^2)^{m_2} \binom{m_2+m_4}{m_2} \right. \\
& \left[\frac{(m_2+m_4+1)h^2\underline{d}^2)+m_n-1}{m_2+1} \right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2^{2m_n-3}h^2}{\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)^{2m_n-1}} \\
&\left.+\frac{2^{2m_n-3}h^2}{\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)^{2m_n-1}}+(-1)^{(k-1)/2}\right) \\
&+e_j^-\frac{1}{4\pi}\left(\frac{2^{2m_n-1}h^2}{\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)^{2m_n+1}}\right. \\
&+\sum_{m_2+2m_4=m_n-1}(-1)^{m_4}(2+h^2\underline{d}^2)^{m_2}\binom{m_2+m_4}{m_2}\left[\frac{(m_2+m_4+1)h^2\underline{d}^2)+m_n-1}{m_2+1}\right] \\
&\left.+\frac{2^{2m_n-1}h^2}{\sqrt{h^2\underline{d}^2+4}\left(h\underline{d}-\sqrt{h^2\underline{d}^2+4}\right)^{2m_n+1}}\right).
\end{aligned}
$$

Now we are ready to introduce discrete version of the classical Plemelj-Sokhotski formulae. For this we have to establish the corresponding Riesz kernels. Here in the discrete case we will use the possibility to defined it via the corresponding Fourier symbols on three boundary layers. For the general case of bounded domains in $\mathbb{R}^n$ which is considered in this paper we can do it directly via the above calculated symbols over the boundary layers γ_h^-, γ_h^*, and γ_h^+. This approach is similar to the continuous case of calculating Fourier transform along arbitrary curves, see for example [1] and references therein. But while it is extremely useful for concrete calculations and implementations for the theoretical part a more generic approach makes it easier to present. Hereby, we will follow the standard one in the continuous case where the convolution kernels of the Riesz operators are determined from their Fourier symbols over the real line and then mapped to the corresponding curve. This results in the discrete Riesz kernels given by

$$
\begin{aligned}
G_i^+ &:= \boldsymbol{\Phi}^{-1}\mathcal{F}_h^{(n-1)}\left[\frac{\widetilde{\underline{\xi}}_{-,i}}{\underline{d}}\frac{h\underline{d}-\sqrt{4+h^2\underline{d}^2}}{2}\right], \\
H_i^+ &:= \boldsymbol{\Phi}^{-1}\mathcal{F}_h^{(n-1)}\left[\frac{\widetilde{\underline{\xi}}_{-,i}}{\underline{d}}\left(e_i^+\frac{h\underline{d}-\sqrt{4+h^2\underline{d}^2}}{2}+e_i^-\frac{2}{h\underline{d}-\sqrt{4+h^2\underline{d}^2}}\right)\right], \\
H_i^- &:= \boldsymbol{\Phi}^{-1}\mathcal{F}_h^{(n-1)}\left[\frac{\widetilde{\underline{\xi}}_{-,i}}{\underline{d}}\left(e_i^+\frac{2}{h\underline{d}-\sqrt{4+h^2\underline{d}^2}}+e_i^-\frac{h\underline{d}-\sqrt{4-h^2\underline{d}^2}}{2}\right)\right],
\end{aligned}
$$

where $\mathcal{F}_h^{(n-1)}$ denotes the $(n-1)$-dimensional Fourier transform which treats the coefficient $m_i h$, associated with the discretisation in e_i direction, as constant, while the term $\widetilde{\underline{\xi}}_{-,i}$ omits the basis elements e_i^+, e_i^-, i.e. $\widetilde{\underline{\xi}}_- = \sum_{j=1, j\neq i}^n e_j^+ \xi_{-j}^D + e_j^- \xi_{+j}^D$, and $\boldsymbol{\Phi}$ denotes a mapping of individual boundary parts to the real line. Considering that in the discrete setting any geometry is a composition of hypercubes, the mappings $\boldsymbol{\Phi}$ is, in fact, a composition of rotations and translations of the uniform lattice.

Using the convolution kernels introduced above, a pair of operators can be defined:

$$H_+ f(mh) := \sum_{i=1}^n \left[\sum_{rh \in \gamma_i^+} H_i^+(rh - mh) f(rh) \right] h^{n-1},$$

for $mh \in \gamma_h^+$, and

$$H_- f(mh) := \sum_{i=1}^n \left[\sum_{rh \in \gamma_i^-} H_i^-(rh - mh) f(rh) \right] h^{n-1},$$

for $mh \in \gamma_h^-$. It can be easily checked, that both operators fulfil the condition $(H_+)^2 = (H_-)^2 = I$. Furthermore, conditions (6)–(7) can be reformulated using the operators H_+ and H_- as follows

$$f(mh) = H_+ f(mh), \text{ for } mh \in \gamma_h^+,$$
$$f(mh) = H_- f(mh), \text{ for } mh \in \gamma_h^-.$$

Thus, we can now introduce discrete Hardy space as follows:

Definition 5 The space of discrete functions $f \in l^p(\gamma_h^+, \mathbb{C}_n)$ whose discrete $(n-1)$-dimensional Fourier transform fulfils $f = H_+ f$ on γ_h^+ is called the *interior discrete Hardy space*, and it is denoted by $h_p^+(\gamma_h^+)$. Analogously, the space of discrete functions $f \in l^p(\gamma_h^-, \mathbb{C}_n)$ whose discrete $(n-1)$-dimensional Fourier transform fulfils $f = H_- f$ on γ_h^- is called the *exterior discrete Hardy space*, and it is denoted by $h_p^-(\gamma_h^-)$.

Finally, by using the operators H_+ and H_-, we can introduce the Plemelj (or Hardy) projectors

$$P_+ := \frac{1}{2}(I + H_+) \quad \text{and} \quad P_- := \frac{1}{2}(I + H_-).$$

Moreover, based on the last definition, it is clear that $f \in h_p^+(\gamma_h^+)$ is equivalent to $P_+ f = f$, while $f \in h_p^-(\gamma_h^-)$ means $P_- f = f$. Likewise we define the complementary projectors $Q_\pm := \frac{1}{2}(I - H_\pm)$ and we say that $f \in h_p^\pm(\gamma_h^\mp)$ iff $Q_\pm f = f$.

Remark 3 We remark that the relation

$$h_p^{\pm}(\gamma_h^+) = h_p^{\mp}(\gamma_h^-)$$

holds for the dual discrete Hardy spaces w.r.t. the associated domain Ω_h^{ext}.

The next step is to introduce the so-called discrete *extension* and *trace* operators, which have been introduced in [5] in the context of discrete Riemann–Hilbert problems over the half space. The role of an extension operator is to recover the function values on the discrete boundary layer γ_h^* from its values on γ_h^+, in the case of an interior discrete Riemann–Hilbert problem, and on γ_h^- in the case of the exterior problem, respectively. We formally introduce the extension operators as follows:

Definition 6 The interior extension operator, denoted as $\mathcal{A}_+$, is an operator extending a function given on the interior boundary layer γ_h^+ to the middle boundary layer γ_h^*, i.e. it is a mapping $\mathcal{A}_+ : l^p(\gamma_h^+) \to l^p(\gamma_h^*)$ given by

$$\mathcal{A}_+[f](mh) := \sum_{i=1}^{n} \left[\sum_{rh \in \gamma_i^+} A_i^+(rh - mh) f(rh) \right] h^{n-1},$$

$$A_i^+ := \boldsymbol{\Phi}^{-1} \mathcal{F}_h^{(n-1)} \left[\frac{\underline{\xi}^D}{\underline{d}} \left(\frac{2}{\sqrt{4 + h^2 \underline{d}^2} - h\underline{d}} \right) \right]$$

with $mh \in \gamma^*$. Similarly, the exterior extension operator, denoted as $\mathcal{A}_-$, is an operator extending a function given on the exterior boundary layer γ_h^- to the middle boundary layer γ_h^*, i.e. it is a mapping $\mathcal{A}_- : l^p(\gamma_h^-) \to l^p(\gamma_h^*)$, which is given by

$$\mathcal{A}_-[f](mh) := \sum_{i=1}^{n} \left[\sum_{rh \in \gamma_i^-} A_i^-(rh - mh) f(rh) \right] h^{n-1},$$

$$A_i^- := \boldsymbol{\Phi}^{-1} \mathcal{F}_h^{(n-1)} \left[\left(\frac{\sqrt{4 + h^2 \underline{d}^2} + h\underline{d}}{\sqrt{4 + h^2 \underline{d}^2} - h\underline{d}} \right) \right]$$

with $mh \in \gamma^*$.

As for the interior and exterior trace operators (see [5] in the context of the half space), they play the role of recovering the values of a discrete left-monogenic function on the boundary layer γ_h^* from its values on the boundary layers γ_h^+ and γ_h^-, and this by means of $\mathcal{A}_+$ and $\mathcal{A}_-$, respectively. The definition of the extension operators is kept identical to the one presented in [5], because the extension procedure is identical and the use of the extension operators is controlled only by normal vectors given on different parts of a boundary of the discrete domain Ω_h. Therefore, the principal difference to the case of a half-space will only become evident in the upcoming constructions.

Now, we can introduce the interior and exterior trace operators. For this let us recall that $mh \in \gamma^{\pm}_{h,i;0}$ or $mh \in \gamma^{\pm}_{h,i;1}$ if $\partial_h^{\pm j}\chi_{\Omega_h}(mh) \neq 0$ in the corresponding points (c.f. Lemma 1). The collection of these points will be characterized by the characteristic functions $\chi_{\gamma^{\pm}_{h,i}}$.

Definition 7 Let $\gamma^{(\cdot)}_{h,i}$, $i = 1, \ldots, n$ denote the components of the three-layer discrete boundary with $(\cdot) = \{+, *, -\}$ corresponding to the i-direction, then the trace operators are introduced as follows:

(i) *the interior trace operator acting on the i-component of the boundary in the i-direction* $\left(\chi_{\gamma_h^+}\mathrm{tr}^+\right)_i : l^p(\Omega_h) \to l^p(\gamma^+_{h,i}) \times l^p(\gamma^*_{h,i})$:

$$\left(\chi_{\gamma_h^+}\mathrm{tr}^+\right)_i [f] := (e_i^- \mathcal{A}_+[-e_i^+ f_i^+], e_i^+ f_i^+),$$

for functions $f \in l^p(\Omega_h)$ where $f_i^+ := f|_{\gamma^+_{h,i}}$. Based on this, we define the trace operator $\mathrm{tr}^+ : l^p(\Omega_h) \to l^p(\gamma_h^+) \times l^p(\gamma_h^*)$ as:

$$\mathrm{tr}^+[f] := \sum_{i=1}^{n} \left(\chi_{\gamma_h^+}\mathrm{tr}^+\right)_i [f].$$

(ii) *the exterior trace operator acting on the i-component of the boundary in the i-direction* $\left(\chi_{\gamma_h^-}\mathrm{tr}^-\right)_i : l^p(\Omega_h^{ext}) \to l^p(\gamma^-_{h,i}) \times l^p(\gamma^*_{h,i})$ is defined by

$$\left(\chi_{\gamma_h^-}\mathrm{tr}^-\right)_i [f] := (e_i^+ \mathcal{A}_-[f_i^-], e_i^- f_i^-),$$

for functions $f \in l^p(\Omega_h^{ext})$ with $f_i^- := f|_{\gamma^-_{h,i}}$. In a similar way, we define the trace operator $\mathrm{tr}^- : l^p(\Omega_h) \to l^p(\gamma_h^-) \times l^p(\gamma_h^*)$ as:

$$\mathrm{tr}^-[f] := \sum_{i=1}^{n} \left(\chi_{\gamma_h^-}\mathrm{tr}^-\right)_i [f].$$

The interior and exterior trace operators allow us to generate boundary data, which then can be monogenically extended by the Cauchy transform into interior or exterior of the discrete domain Ω_h. Moreover, as it is expected, a discrete version of the projection properties of the trace operator of the discrete Cauchy transform is obtained. Thus, we have the following corollary:

Corollary 1 *The following two properties hold:*

(i) *if $f \in l^p(\Omega_h)$, then $C^+\mathrm{tr}^+\left[C^+\mathrm{tr}^+(f)\right] = C^+\mathrm{tr}^+(f)$;*
(ii) *if $f \in l^p(\Omega_h^{ext})$, then $C^-\mathrm{tr}^-\left[C^-\mathrm{tr}^-(f)\right] = C^-\mathrm{tr}^-(f)$.*

As the next step, we introduce *interior and exterior boundary generators*, which are a speciality of the discrete setting. The interior and exterior boundary generators act similar to the trace operators, but they act on functions which are given either on the interior boundary layer γ_h^+ or on the exterior boundary layer γ_h^-, respectively. Thus, we have the following definition:

Definition 8 The boundary generators in the i-direction ($i = 1, \ldots, n$) are introduced as follows:

(1) *the interior boundary generator in the i-direction* $\left(\chi_{\gamma_h^+}\mathcal{G}^+\right)_i : l^p(\gamma_{h,i}^+) \to l^p(\gamma_{h,i}^+) \times l^p(\gamma_{h,i}^*)$:

$$\left(\chi_{\gamma_h^+}\mathcal{G}^+\right)_i [g] := (e_i^- \mathcal{A}_+[-e_i^+ g_i], e_i^+ g_i)$$

for functions $g \in l^p(\gamma_h^+)$ with $g_i := g|_{\gamma_{h,i}^+}$. Then, the interior generator operator $\mathcal{G}^+ : l^p(\gamma_h^+) \to l^p(\gamma_h^+) \times l^p(\gamma_h^*)$ is defined by:

$$\mathcal{G}^+[g] := \sum_{i=1}^{n} \left(\chi_{\gamma_h^+}\mathcal{G}^+\right)_i [g].$$

(2) *the exterior boundary generator in the i-direction* $\left(\chi_{\gamma_h^-}\mathcal{G}^-\right)_i : l^p(\gamma_{h,i}^-) \to l^p(\gamma_{h,i}^-) \times l^p(\gamma_{h,i}^*)$:

$$\left(\chi_{\gamma_h^-}\mathcal{G}^-\right)_i [g] := (e_i^+ \mathcal{A}_-[g_i], e_i^- g_i)$$

for a discrete function $g \in l^p(\gamma_h^-)$ with $g_i := g|_{\gamma_{h,i}^-}$. Then, the exterior generator operator $\mathcal{G}^- : l^p(\gamma_h^-) \to l^p(\gamma_h^-) \times l^p(\gamma_h^*)$ is given by:

$$\mathcal{G}^-[g] := \sum_{i=1}^{n} \left(\chi_{\gamma_h^-}\mathcal{G}^-\right)_i [g].$$

By the help of the above construction, the discrete Hardy projections can be characterised now as follows:

Lemma 4 *The discrete Hardy projections for the interior of a bounded discrete domain Ω_h and its exterior can be characterised by the following relations:*

(i) $P_+ f(mh) = C^+\mathcal{G}^+[f|_{\gamma_h^+}](mh)$, *for all* $mh \in \gamma_h^+$;
(ii) $P_- f(mh) = C^-\mathcal{G}^-[f|_{\gamma_h^-}](mh)$, *for all* $mh \in \gamma_h^-$.

Proof At first, we construct the proof for the interior case, while the exterior case can be proved analogously via a correct interchanging of interior and exterior setting-related objects. Let us consider a function $f := C^+\mathrm{tr}^+[g]$, where $g \in l^p(\Omega_h)$. The function $f^+ = f|_{\gamma_h^+}$ (the restriction of f to the interior boundary layer) satisfies the relation $e_i^+ g_i^+ = e_i^+ f_i^+$, where $g_i^+ = g|_{\gamma_{h,i}^+}$ in all directions i. Applying the boundary generator $\left(\chi_{\gamma_h^+}\mathcal{G}^+\right)_i$ to f_i^+ we obtain

$$\left(\chi_{\gamma_h^+}\mathcal{G}^+\right)_i [f_i^+] = (e_i^- \mathcal{A}_+[-e_i^+ f_i^+], e_i^+ f_i^+) = \left(\chi_{\gamma_h^+}\mathcal{G}^+\right)_i [g_i^+] = \left(\chi_{\gamma_h^+}\mathrm{tr}^+\right)_i [g],$$

for $i = 1, \ldots, n$. Therefore,

$$C^+\mathcal{G}^+[f|_{\gamma_h^+}] = C^+\mathrm{tr}^+[g] = C^+\mathrm{tr}^+\left[C^+\mathrm{tr}^+[g]\right] = \left[C^+\mathrm{tr}^+\right]^2[g],$$

implying that $C^+\mathcal{G}^+$ is a projector. Thus, we can identify

$$C^+\mathrm{tr}^+[g] = C^+\mathcal{G}^+[f|_{\gamma_h^+}] = P_+[f].$$

The same proof holds for the exterior case. □

4.1 Classic Hilbert Problems for Discrete Monogenic Functions

In this section, we consider the classic Hilbert problems of reconstructing a discrete monogenic function in the interior of domain Ω_h from its boundary data:
Problem I. Given $g \in l^p(\gamma_h^*)$, find $f : \Omega_h \to \mathbb{C}_n$ such that

$$\begin{cases} D_h^{+-} f(mh) = 0, & \text{for } mh \in \Omega_h, \\ f(rh) = g(rh), & \text{for } rh \in \gamma_h^+. \end{cases} \tag{8}$$

The solution of this problem is given by the following theorem:

Theorem 8 *The discrete boundary value problem (8) has a unique solution iff the boundary data g is in $h_p^+(\gamma_h^+)$, and the solution is given by*

$$f(mh) = C^+\mathcal{G}^+[g](mh), \quad mh \in \Omega_h.$$

Proof The condition $g \in h_p^+(\gamma_h^+)$ comes naturally, since if a function g does not belong to $h_p^+(\gamma_h^+)$, then no discrete monogenic function f exists satisfying the given boundary condition on γ_h^+. Thus, $g \in h_p^+(\gamma_h^+)$ is a necessary condition for the existence of a solution to problem (8).

Next step of the proof is to show that a solution exists in the case of $g \in h_p^+(\gamma_h^+)$. Applying boundary generator to g and by using properties of the discrete Cauchy transform, we get that $f = C^+\mathcal{G}^+[g]$ is a discrete monogenic function in Ω_h satisfying boundary conditions $f(rh) = g(rh)$ for $rh \in \gamma_h^+$. The uniqueness of solution is guaranteed by the discrete maximum principle, see [4] for the details. □

Analogously we get similar result for the exterior Hilbert problem:

Corollary 2 *The discrete exterior Hilbert boundary value problem has a unique solution iff the boundary data g is in $h_p^-(\gamma_h^-)$, and the solution is given by*

$$f(mh) = C^- \mathcal{G}^-[g](mh), \quad mh \in \Omega_h^{ext}.$$

Discrete Hilbert boundary value problems with jump relations, similar to the one presented in [5], can also be formulated for discrete bounded domains. Formulation of jump problems requires a notion of normal vectors for each part of a boundary layer γ_h^*, and the direction of these normal vectors depends on if the boundary layer is passed from interior to exterior, or the opposite. Taking into account that in the discrete setting we have as normal vectors $-e_i^\pm$ in the case of $\gamma_{h,i;0}$ and $e_i^\pm$ in the case of $\gamma_{h,i;1}$ we can use the usual decomposition in terms of the coordinate directions. We start with the classical jump problem for discrete monogenic functions:
Problem II. Given $g \in l^p(\gamma_h^*)$, find $f\colon h\mathbb{Z}^n \to \mathbb{C}_n$ such that

$$\begin{cases} D_h^{+-} f(mh) = 0, & \text{for } mh \in h\mathbb{Z}^n \backslash \gamma_h^*, \\ \sum\limits_{i=1}^{n} \big(\chi_{\gamma_{h,i;0}}(rh)(e_i^+ f_+(rh) - e_i^- f_-(rh)) & \\ + \chi_{\gamma_{h,i;1}}(rh)(-e_i^+ f_+(rh) + e_i^- f_-(rh)) \big) & \\ = \sum\limits_{i=1}^{n} \big(\chi_{\gamma_{h,i;0}}(rh) e_i g(rh) - \chi_{\gamma_{h,i;1}}(rh) e_i g(rh) \big), \text{ for } rh \in \gamma_h^*. & \end{cases} \tag{9}$$

Solvability of this problem is stated by the following theorem:

Theorem 9 *The discrete boundary value problem (9) has a unique solution for arbitrary boundary data $g \in l^p(\gamma_h^*$ with $1 \leq p < n$, and the solution is given by*

$$f(mh) = \begin{cases} C^+ \mathcal{G}^+[g^+](mh), \; mh \in \Omega_h, \\ C^- \mathcal{G}^-[g^-](mh), \; mh \in \Omega_h^{ext}. \end{cases}$$

where

$$g^+ = \sum_{i=1}^{n} (\chi_{\gamma_{h,i;0}} - \chi_{\gamma_{h,i;1}}) e_i^- (g_i^1 + e_i^- g_i^3),$$

$$g^- = \sum_{i=1}^{n} (\chi_{\gamma_{h,i;0}} - \chi_{\gamma_{h,i;1}}) e_i^+ (g_i^1 - g_i^4 + e_i^+ g_i^2),$$

with $g_i^1, g_i^2, g_i^3, g_i^4$ being the component functions with respect to the decomposition $\mathbb{C}_n = \mathbb{C}_{n-1} + e_i^+ \mathbb{C}_{n-1} + e_i^- \mathbb{C}_{n-1} + e_i^+ e_i^- \mathbb{C}_{n-1}$.

Proof First of all, let us remark that we can always decompose a Clifford-valued function $g = g_i^1 + e_i^+ g_i^2 + e_i^- g_i^3 + e_i^+ e_i^- g_i^4$ with respect to any basis elements e_i^+ and e_i^-. Obviously, the component functions $g_i^1, g_i^2, g_i^3, g_i^4$ depend on the particular choice of i. Furthermore, since all e_i are invertible our boundary data can be written as

$$
\begin{aligned}
&\sum_{i=1}^{n}\left(\chi_{\gamma_{h,i;0}} e_i g - \chi_{\gamma_{h,i;1}} e_i g\right) \\
&= \sum_{i=1}^{n}\Big(\chi_{\gamma_{h,i;0}}(e_i^+(g_i^1 + e_i^- g_i^3) + e_i^-(g_i^1 + e_i^+ g_i^2 - g_i^4)) \\
&\quad -\chi_{\gamma_{h,i;1}}(e_i^+(g_i^1 + e_i^- g_i^3) + e_i^-(g_i^1 + e_i^+ g_i^2 - g_i^4))\Big) \\
&=: \sum_{i=1}^{n}\left((\chi_{\gamma_{h,i;0}} - \chi_{\gamma_{h,i;1}}) e_i^- g_i^+ + (\chi_{\gamma_{h,i;0}} - \chi_{\gamma_{h,i;1}}) e_i^+ g_i^-\right).
\end{aligned}
$$

Let us remark while in the last line it looks like we only have a difference but due to the property of the characteristic functions it is in fact only a fixing of the signal over the corresponding points.

From the above relation we get for the first component of the upper and lower trace of f in the i-direction the decomposition

$$
\chi_{\gamma_{h,i;j}} e_i^- f^+ = \chi_{\gamma_{h,i;j}} e_i^- g_i^+, \qquad \chi_{\gamma_{h,i;j}} e_i^+ f^- = -\chi_{\gamma_{h,i;j}} e_i^+ g_i^-
$$

for $j = 0, 1$. This means that in the direction i the traces of f^+, f^- coincide with $\left(\chi_{\gamma_{h;j}} G^+\right)_i (g_i^+)$ and $\left(\chi_{\gamma_{h;j}} G^-\right)_i (g_i^-)$ for the parts of the boundary with $j = 0, 1$, respectively. Summing up over i and j this allows us to consider the function:

$$
f(mh) = \begin{cases} C^+\mathcal{G}^+[g^+](mh), & mh \in \Omega_h, \\ C^-\mathcal{G}^-[-g^-](mh), & mh \in \Omega_h^{ext}. \end{cases}
$$

which satisfies the discrete boundary value problem under consideration. □

Next we consider the boundary value problem relating values on boundary layers γ_h^+ and γ_h^-, which is unique for the discrete setting due to three-layer structure of the discrete boundary:

Problem III. Given $g \in l^p(\gamma_h^*)$, find $f : \mathbb{R}_h^n \to \mathbb{C}_n$ such that

$$
\begin{cases} D_h^{+-} f(mh) = 0, \text{ for } mh \in h\mathbb{Z}^n \backslash \gamma_h^*, \\ \displaystyle\sum_{i=1}^{n}\left(\chi_{\gamma_{h,i;0}^*} e_i^- A_i^+ f - \chi_{\gamma_{h,i;0}^*} e_i^+ A_i^- f - \chi_{\gamma_{h,i;1}^*} e_i^- A_i^+ f + \chi_{\gamma_{h,i;1}^*} e_i^+ A_i^- f\right) \\ \displaystyle= \sum_{i=1}^{n} \chi_{\gamma_{h,i;0}^*} g - \chi_{\gamma_{h,i;1}^*} g, \ \text{ on } \gamma_h^*. \end{cases} \tag{10}
$$

We have the following theorem:

Theorem 10 *The discrete boundary value problem* (10) *has a unique solution for arbitrary boundary data $g \in l^p(\gamma_h^*)$ with $1 \leq p < n$, and the solution is given by*

$$f(mh) = \begin{cases} C^+ \left[g^+, g^{++}\right](mh), \; mh \in \Omega_h, \\ C^- \left[g^-, g^{--}\right](mh), \; mh \in \Omega_h^{ext}, \end{cases}$$

with

$$\begin{aligned}
g^+ &= \sum_{i=1}^{n} \left(\chi_{\gamma^*_{h,i;0}} e_i^+ e_i g - \chi_{\gamma^*_{h,i;1}} e_i^+ e_i g \right), \\
g^{++} &= \sum_{i=1}^{n} \left(\chi_{\gamma^+_{h,i;0}} e_i^- (A_i^+)^{-1} (e_i g) - \chi_{\gamma^+_{h,i;1}} e_i^- (A_i^+)^{-1} (e_i g) \right), \\
g^- &= \sum_{i=1}^{n} \left(\chi_{\gamma^*_{h,i;0}} e_i^- e_i g - \chi_{\gamma^*_{h,i;1}} e_i^- e_i g \right), \\
g^{--} &= \sum_{i=1}^{n} \left(\chi_{\gamma^-_{h,i;0}} e_i^+ (A_i^-)^{-1} (e_i g) - \chi_{\gamma^-_{h,i;1}} e_i^+ (A_i^-)^{-1} (e_i g) \right).
\end{aligned}$$

Proof Let us take a closer look at the boundary condition

$$\chi_{\gamma^*_{h,i;0}} e_i^- A_i^+ f - \chi_{\gamma^*_{h,i;0}} e_i^+ A_i^- f = \chi_{\gamma^*_{h,i;0}} g.$$

Because of $e_i^2 = -1$, this boundary condition can be rewritten as follows

$$\chi_{\gamma^*_{h,i;0}} e_i^- A_i^+ f - \chi_{\gamma^*_{h,i;0}} e_i^+ A_i^- f = \chi_{\gamma_{h,i;0}} (-e_i^2 g) = \chi_{\gamma_{h,i;0}} (e_i^+ (-e_i g) + e_i^- (-e_i g)).$$

Since the operators $A_i^{\pm}$ are invertible, we immediately get

$$\chi_{\gamma^-_{h,i;0}} e_i^+ f = \chi_{\gamma^-_{h,i;0}} e_i^+ (A_i^-)^{-1} (e_i g) \text{ and } \chi_{\gamma^+_{h,i;0}} e_i^- f = \chi_{\gamma^+_{h,i;0}} e_i^- (A_i^+)^{-1} (e_i g).$$

Finally, application of the Cauchy transform leads to the claim of the theorem. □

It is important to remark, that formulation of Problem III is possible only for a fixed h (although it can be arbitrary small), and for $h \to 0$ it reduces to Problem II, as expected.

Finally, we consider a more general problem:

Problem IV. Given $g \in l^p(\gamma_h^*)$ and a constant $\kappa \in \mathbb{C}_n$ with a right inverse κ_r^{-1}, find $f : h\mathbb{Z}^n \to \mathbb{C}_n$ such that

$$\begin{cases} D_h^{+-} f(mh) = 0, \text{ for } mh \in h\mathbb{Z}^n \backslash \gamma_h^*, \\ \sum_{i=1}^{n} \left(\chi_{\gamma_{h,i;0}^*} e_i^- A_i^+ f - \chi_{\gamma_{h,i;0}^*} e_i^+ A_i^- f\kappa - \chi_{\gamma_{h,i;1}^*} e_i^- A_i^+ f + \chi_{\gamma_{h,i;1}^*} e_i^+ A_i^- f\kappa \right) \\ = \sum_{i=1}^{n} \chi_{\gamma_{h,i;0}^*} g - \chi_{\gamma_{h,i;1}^*} g, \text{ on } \gamma_h^*. \end{cases} \tag{11}$$

Problem (11) is a particular case of a general Riemann–Hilbert problem, see for example [21] for details. Considering Theorem 10, the solvability of Problem IV can be easily obtained, and it is provided by the following theorem:

Theorem 11 *The discrete boundary value problem* (11) *has a unique solution for arbitrary boundary data* $g \in l^p(\gamma_h^*)$ *with* $1 \leq p < n$, *and a constant* $\kappa \in \mathbb{C}_n$ *with a right inverse* κ_r^{-1}, *and the solution is given by*

$$f(mh) = \begin{cases} C^+\left[g^+, g^{++}\right](mh), & mh \in \Omega_h, \\ C^-\left[g^-, g^{--}\kappa\right](mh), & mh \in \Omega_h^{ext}, \end{cases}$$

with

$$g^+ = \sum_{i=1}^{n} \left(\chi_{\gamma_{h,i;0}^*} e_i^+ e_i g - \chi_{\gamma_{h,i;1}^*} e_i^+ e_i g \right),$$

$$g^{++} = \sum_{i=1}^{n} \left(\chi_{\gamma_{h,i;0}^+} e_i^- (A_i^+)^{-1}(e_i g) - \chi_{\gamma_{h,i;1}^+} e_i^- (A_i^+)^{-1}(e_i g) \right),$$

$$g^- = \sum_{i=1}^{n} \left(\chi_{\gamma_{h,i;0}^*} e_i^- e_i g - \chi_{\gamma_{h,i;1}^*} e_i^- e_i g \right),$$

$$g^{--} = \sum_{i=1}^{n} \left(\chi_{\gamma_{h,i;0}^-} e_i^+ (A_i^-)^{-1}(e_i g) - \chi_{\gamma_{h,i;1}^-} e_i^+ (A_i^-)^{-1}(e_i g) \right).$$

5 Summary

In this paper we have presented the extension of the boundary value theory of discrete monogenic functions to arbitrary bounded domains in $\mathbb{R}^n$. Especially, all the constructions on the discrete level are provided for more general types of domain, than considered in previous works, i.e. cuboids. Moreover, a general characterisation of discrete geometry in higher-dimensional case has been shown allowing compact presentations of discrete Stokes', Borel–Pompeiu, and Cauchy formulae. By help of explicit calculations of the discrete Fourier transform on boundary layers, general formulae for discrete Riesz kernels could be obtained. Finally, the discrete operators introduced in this paper have been used to discuss solvability of several discrete Hilbert problems.

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