Static black holes without spatial isometries: from AdS multipoles to electrovacuum scalarization

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May 2020

Abstract

We review recent results on the existence of static black holes without spatial isometries in four spacetime dimensions and propose a general framework for their study. These configurations are regular on and outside a horizon of spherical topology. Two different mechanisms allowing for their existence are identified. The first one relies on the presence of a solitonic limit of the black holes; when the solitons have no spatial isometries, the black holes, being a non-linear bound state between the solitons and a horizon, inherit this property. The second one is related to black hole scalarization, and the existence of zero modes of the scalar field without isometries around a spherical horizon. When the zero modes have no spatial isometries, the backreaction of their non-linear continuation makes the scalarized black holes inherit the absence of spatial continuous symmetries. A number of general features of the solutions are discussed together with possible generalizations.

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1 Introduction

An interesting property of some non-linear field theory models in a flat spacetime background is the existence of solitonic configurations¹ which are static but not spherically symmetric [1]. Moreover, in some cases, the configurations minimizing the action in a given sector of the theory possess discrete symmetries only. Examples include magnetic monopoles in Yang-Mills-Higgs models [2], Hopfions [3,4] and Skyrmions [5].

Rather surprisingly, the situation in Einstein's theory of gravity is different². The natural counterparts of the aforementioned field theory solitons, as non-perturbative elementary solutions, are black holes $(BHs)^3$. However, as implied by a number of classic results [6], the spectrum of BHs is very limited; at least in the (electro)vacuum case the solutions possess a high degree of symmetry. For example, Israel's theorem [9, 10], has established that a static BH solution in Einstein-Maxwell theory is spherically symmetric and described by only two parameters, its ADM mass and electric/magnetic charge.

The extrapolation of these results for a more general matter content and/or different spacetime asymptotics is, however, unjustified. In particular, in analogy with the situation in soliton physics, in certain models one finds static BH solutions *without* any continuous spatial symmetry.

To the best of our knowledge, two different mechanisms allowing for such configurations have been identified in the literature. The first one can easily be understood from the discussion above and relies on three different ingredients:

- i) the existence of (non-gravitating) solitons in certain field theory models, which are static and nonspherically symmetric;
- ii) these configurations possess generalizations as test fields on a Schwarzschild BH background⁴, *i.e.* one can add a (small) horizon at the center of a soliton [12];
- iii) the solutions survive in the fully non-linear regime, when including the soliton's backreaction⁵, without any enhancement of symmetry.

Restricting to the asymptotically flat case, a number of partial results in this direction can be found in Refs. [14]- [17]. However, all these configurations still possess a Killing vector associated with rotational symmetry. Although BHs without isometries should also exist in those cases no explicit non-perturbative construction has been reported, presumably due to the highly non-linear considered matter models (see, however, the partial results in [18]).

The work reported in [19] has provided an explicit realization of this mechanism, the technical obstacle of dealing with non-linear matter fields being avoided by considering an electromagnetic field and AdS asymptotics. Then, steps i)-iii) above are easily fulfilled, which results in a tower of solutions without isometries, naturally interpreted as Schwarzschild-AdS BHs with electric (or magnetic) multipole hair.

The second mechanism allowing for static BHs with discrete spatial symmetries only is rather different, and relies on the phenomenon of *spontaneous scalarization*⁶. Two different steps can be identified in this case:

i) the static, spherically symmetric BHs S_0 of a given model are unstable against scalar perturbations, with the existence of *scalar clouds*. That is, the model possesses zero modes, with an infinitesimally small scalar field, which, however, exist for a certain set of S_0 -parameters.

 2 In this work we shall restrict to the case of a four dimensional spacetime.

¹By solitons we mean non-singular, finite energy localized solutions, without any assumption about their stability.

 $^{^{3}}$ We recall that, for a Minkowski spacetime background, no smooth horizonless configurations exist in Einstein's theory [7] - see *e.g.* the discussion in [8].

 $^{^{4}}$ This step is nontrivial: not all solitons survive when adding a BH horizon at their center. For instance, spherically symmetric boson stars admit no BH generalization [11].

 $^{^{5}}$ This is not automatically guaranteed. For example, the BPST Yang-Mills solution is destroyed when including gravity effects, for an asymptotically flat, five-dimensional spacetime [13].

 $^{^{6}}$ Similar solutions should also exist for higher spin fields (*e.g.* spontaneous vectorization), in which case, however, the numerical problem becomes more complicated.

ii) The scalar clouds survive when including backreaction on the spacetime geometry, with a bifurcation to a new family S_e of BH solutions. In particular, there are zero modes resulting in S_e -BHs with no rotational symmetry at all.

An explicit realization of these setup has been investigated in the recent work [20], which has studied the spontaneous scalarization of Reissner-Nordström (RN) BH. As discussed there, unlike the case of electrovacuum, the Einstein-Maxwell-scalar model admits static, asymptotically flat, regular on and outside the horizon BHs without any spatial isometries.

This paper is structured as follows. In the next Section we exhibit a general framework for the study of static BHs with no rotational symmetries. Sections 3 and 4 discuss explicit realizations of the two mechanisms discuss above, for Einstein-Maxwell-AdS and Einstein-Maxwell-scalar theories, respectively. Finally, in Section 5 we present some conclusions and further remarks. The Appendix contains technical details on the far field asymptotics of Einstein-Maxwell-AdS BHs

2 Static black holes with no isometries: a general framework

2.1 The metric

In the absence of closed form solutions, static BHs without spatial isometries are constructed numerically. This task, however, is a numerical challenge, since the geometry, and the matter functions, depend on all three space coordinates, making the expression of the Einstein tensor very complicated. Also, some techniques which work well for axially symmetric General Relativity problems (e.g. the metric gauge choice) do not have a straightforward extension in this case.

These difficulties can be (at least partially) circumvented by employing the Einstein-De Turck (EDT) approach, proposed in [21, 22]. This approach has become, in recent years, a standard tool in the numerical treatment of stationary problems in general relativity, and has the advantage of not fixing *a priori* a metric gauge, yielding at the same time elliptic equations - see [23, 24] for reviews.

In this approach one solves the so called EDT equations

$$R_{\mu\nu} - \nabla_{(\mu}\xi_{\nu)} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu} \right) .$$
 (1)

Here, ξ^{μ} is a vector defined as $\xi^{\mu} \equiv g^{\nu\rho}(\Gamma^{\mu}_{\nu\rho} - \bar{\Gamma}^{\mu}_{\nu\rho})$, where $\Gamma^{\mu}_{\nu\rho}$ is the Levi-Civita connection associated to the spacetime metric g that one wants to determine, and a reference metric \bar{g} is introduced, $\bar{\Gamma}^{\mu}_{\nu\rho}$ being the corresponding Levi-Civita connection; $T_{\mu\nu}$ is the energy-momentum tensor of the matter field(s) and G is Newton's constant.

Solutions to (1) solve the Einstein equations iff $\xi^{\mu} \equiv 0$ everywhere. To achieve this, we impose boundary conditions which are compatible with $\xi^{\mu} = 0$ on the boundary of the domain of integration. Then, this should imply $\xi^{\mu} \equiv 0$ everywhere, a condition which is verified from the numerical output.

The BHs with no isometries are more easily constructed by employing spherical coordinates (r, θ, φ) , such that the horizon is located at some constant value of r. As such, one introduces a general metric ansatz with seven functions, $F_1, F_2, F_3, F_0, S_1, S_2, S_3$:

$$ds^{2} = -F_{0}(r,\theta,\varphi)N(r)dt^{2} + F_{1}(r,\theta,\varphi)\frac{dr^{2}}{N(r)} + F_{2}(r,\theta,\varphi)\left(rd\theta + S_{1}(r,\theta,\varphi)dr\right)^{2}$$

$$+F_{3}(r,\theta,\varphi)\left(r\sin\theta d\varphi + S_{2}(r,\theta,\varphi)dr + S_{3}(r,\theta,\varphi)rd\theta\right)^{2}.$$

$$(2)$$

N(r) is an input 'background' function whose explicit form depends on the considered problem. A general enough expression for N(r) which covers both models in this paper is

$$N(r) = \left(1 - \frac{r_H}{r}\right) \left(1 + \frac{r^2 + r_H^2}{L^2} + \frac{rr_H}{L^2} - \frac{q^2}{rr_H}\right),\tag{3}$$

with $r_H \ge 0$ denoting the event horizon radius and q, L two input constants which will be defined below. The reference metric \bar{g} is found by taking $F_1 = F_2 = F_3 = F_0 = 1$, $S_1 = S_2 = S_3 = 0$ in the ansatz (2).

We remark that the static axially symmetric (or spherically symmetric) BHs can also be studied within framework. For example, the axially symmetric metric has

$$S_2 = S_3 = 0,$$
 (4)

all remaining functions depending on (r, θ) only.

2.2 The boundary conditions

In this approach, the problem reduces to solving a set of seven partial differential equations (PDEs) resulting from (1)-(2), for the metric functions, together with a set of PDEs for the matter functions. These equations are solved with suitable boundary conditions (BCs) which are found by constructing an approximate form of the solutions on the boundary of the domain of integration compatible with the requirement $\xi^{\mu} = 0$ plus regularity and AdS/Minkowski asymptotics of the solutions, as appropriate for the problem at hand.

The configurations reported in this work possess a reflection symmetry along the equatorial plane ($\theta = \pi/2$) and two \mathbb{Z}_2 -symmetries w.r.t. the φ -coordinate. Then the domain of integration for the (θ, φ)-coordinates is $[0, \pi/2] \times [0, \pi/2]$.

The metric functions satisfy the following BCs at infinity:

$$F_1 = F_2 = F_3 = F_0 = 1, \ S_1 = S_2 = S_3 = 0, \tag{5}$$

such that the background geometry is approached. The BCs at $\theta = 0$ are

$$\partial_{\theta}F_1 = \partial_{\theta}F_2 = \partial_{\theta}F_3 = \partial_{\theta}F_0 = 0, \ S_1 = S_2 = \partial_{\theta}S_3 = 0, \tag{6}$$

while at $\theta = \pi/2$ we impose

$$\partial_{\theta}F_1 = \partial_{\theta}F_2 = \partial_{\theta}F_3 = \partial_{\theta}F_0 = 0, \ S_1 = \partial_{\theta}S_2 = S_3 = 0.$$

$$\tag{7}$$

The BCs at $\varphi = 0$ are

$$\partial_{\varphi}F_1 = \partial_{\varphi}F_2 = \partial_{\varphi}F_3 = \partial_{\varphi}F_0 = 0, \ \partial_{\varphi}S_1 = S_2 = S_3 = 0, \tag{8}$$

while at $\varphi = \pi/2$ we impose

$$\partial_{\varphi}F_1 = \partial_{\varphi}F_2 = \partial_{\varphi}F_3 = \partial_{\varphi}F_0 = 0, \ \partial_{\varphi}S_1 = S_2 = S_3 = 0.$$
(9)

In order to obtain simple boundary conditions at the event horizon, it proves useful to introduce a new (compact) radial coordinate x, with⁷

$$r = \frac{r_H}{1 - x^2} \tag{10}$$

with range $0 \le x < 1$, such that the horizon is located at x = 0. This results in the following boundary conditions at the horizon:

$$\partial_x F_1 = \partial_x F_2 = \partial_x F_3 = \partial_x F_0 = 0, \ S_1 = S_2 = \partial_x S_3 = 0.$$

$$(11)$$

⁷Numerics are performed by employing the compactified coordinate x.

2.3 Horizon quantities

The induced metric on the spatial sections of the horizon is

$$d\sigma^2 = r_H^2 \left(F_2(r_H, \theta, \varphi) d\theta^2 + F_3(r_H, \theta, \varphi) (\sin \theta d\varphi + S_3(r_H, \theta, \varphi) d\theta)^2 \right).$$
(12)

The horizon area and the Hawking temperature are

$$A_{H} = r_{H}^{2} \int_{0}^{\pi} d\theta \int_{0}^{2\pi} \sin \theta \sqrt{F_{2}(r_{H}, \theta, \varphi) F_{3}(r_{H}, \theta, \varphi)}, \qquad T_{H} = \frac{1}{4\pi r_{H}} \left(1 + \frac{3r_{H}^{2}}{L^{2}} - q^{2} \right).$$
(13)

The horizon has a spherical topology, as one can see by evaluating its Euler characteristic⁸ χ . However, the deviation from sphericity can be significant, in particular for the EM-AdS BHs in the next Section. A measure of this deviation can be found *e.g.* by evaluating the circumference of the horizon along the equator

$$L_{e} = r_{H} \int_{0}^{2\pi} d\varphi \sqrt{F_{3}(r_{H}, \pi/2, \varphi)},$$
(14)

together with the circumference of the horizon along the poles (which is φ -dependent):

$$L_p(\varphi) = 2r_H \int_0^\pi d\theta \sqrt{F_2(r_H, \theta, \varphi) + F_3(r_H, \theta, \varphi)S_3^2(r_H, \theta, \varphi)}.$$
(15)

Also, in principle, the horizon geometry (12) can be visualised by considering its isometric embedding in a three-dimensional Euclidean space, with

$$ds_E^2 = dX^2 + dY^2 + dZ^2.$$
 (16)

The embedding functions $X(r,\theta)$, $Y(r,\theta)$, $Z(r,\theta)$ are found by integrating the following system of non-linear PDEs

$$X_{,\theta}^{2} + Y_{,\theta}^{2} + Z_{,\theta}^{2} = r_{H}^{2} \left(F_{2}(r_{H},\theta,\varphi) + F_{3}(r_{H},\theta,\varphi) S_{3}^{2}(r_{H},\theta,\varphi) \right)$$

$$X_{,\varphi}^{2} + Y_{,\varphi}^{2} + Z_{,\varphi}^{2} = r_{H}^{2} F_{3}(r_{H},\theta,\varphi) \sin^{2}\theta,$$

$$X_{,\theta}X_{,\varphi} + Y_{,\theta}Y_{,\varphi} + Z_{,\theta}Z_{,\varphi} = r_{H}^{2} \sin\theta F_{3}(r_{H},\theta,\varphi) S_{3}(r_{H},\theta,\varphi).$$
(17)

One remarks, however, that in general, these global isometric embeddings could only be obtained up to some threshold configuration, beyond which an obstruction, similar to the one found for fast spinning Kerr BH [25,26], arises.

2.4 The numerical approach

The EDT equations (1) together with the matter field(s) equations can be solved by using various approaches. For all cases discussed in this work, the numerical methods we have chosen to use can be summarized as follows. First, the equations are discretized on a (r, θ, φ) -grid with $N_r \times N_\theta \times N_\varphi$ points⁹ The resulting system is then solved iteratively until convergence is achieved. All numerical calculations for axially symmetric configurations are performed by using a professional software based on the iterative Newton-Raphson method [27]. This code requests the system of non-linear PDEs to be written in the form $F(r, \theta, \varphi; u; u_r, u_\theta, u_\varphi; u_{rr}, u_{\theta\theta}, u_{\varphi\varphi}, u_{r\theta}, u_{r\varphi}, u_{\theta\varphi}) = 0$, (where u denotes the set of unknown functions) subject to a set of boundary conditions on a rectangular domain. The user must deliver the equations, the boundary conditions, and the Jacobian matrices for the equations and the boundary conditions. Starting with a guess solution, small corrections are computed until a desired accuracy is reached. The code automatically provides also an error estimate for each unknown function, which is the maximum of the discretization error divided by the maximum of the function. For most of the solutions reported in this work, the typical numerical error for the functions is estimated to be lower than 10^{-3} .

⁸This provides a further test of the accuracy of solutions, as the (numerical) integral of the horizon Ricci scalar over the horizon should equal χ .

⁹ Typical grids have sizes around $100 \times 30 \times 30$ points. The grid spacing in the *r*-direction is non-uniform; but the angular grids are uniform.

3 The first mechanism: Einstein–Maxwell–AdS BHs

3.1 Maxwell 'solitons' in AdS spacetime

For a Minkowski spacetime background, the existence of solitons in a given model is supported by non-linearities of the field(s). However, this is not necessarily the case for different spacetime asymptotics. Perhaps the simplest example in this direction is provided by a Maxwell field in a globally AdS fixed geometry, with the usual action

$$I_A = -\frac{1}{4} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu}, \qquad (18)$$

where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is the U(1) field strength. The 4-potential A_{μ} satisfies the Maxwell equations

$$\nabla_{\mu}F^{\mu\nu} = 0, \tag{19}$$

with an energy-momentum tensor

$$T_{\mu\nu} = F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta} - \frac{1}{4}g_{\mu\nu}F^2, \qquad (20)$$

where $\rho = -T_t^t$ is taken as the energy density.

For the geometry, we consider a static spherically symmetric background, with a line element

$$ds^{2} = -N(r)dt^{2} + \frac{dr^{2}}{N(r)} + r^{2}(d\theta^{2} + \sin^{2}\theta d\varphi^{2}), \qquad (21)$$

where r, θ, φ are spherical coordinates. Globally AdS spacetime corresponds to taking

$$N(r) = 1 + \frac{r^2}{L^2},\tag{22}$$

in (21), where L is the AdS length scale, the cosmological constant being $\Lambda = -3/L^2$.

Given a static solution of the Maxwell equations, one can define a total mass-energy E and an electric charge Q_e as

$$E = -\int d^3x \sqrt{-g} T_t^t, \quad Q_e = \oint_{\infty} dS_r F^{rt} .$$
⁽²³⁾

Restricting to a purely electric U(1) potential with

$$A = V(r, \theta, \varphi) dt, \tag{24}$$

and assuming separation of variables for the electric potential, a generic configuration has the expression

$$V(r,\theta,\varphi) = \sum_{\ell \ge 0} \sum_{m=-\ell}^{m=\ell} c_{\ell m} R_{\ell}(r) Y_{\ell m}(\theta,\varphi), \qquad (25)$$

where $c_{\ell m}$ are arbitrary constant and $Y_{\ell m}(\theta, \varphi)$ are the *real* spherical harmonics. For each ℓ, m , the radial function R_{ℓ} is a solution of the equation

$$(r^2 R'_{\ell})' = \frac{1}{N(r)} \ell(\ell+1) R_{\ell},$$
(26)

where a prime denotes the derivative w.r.t. the radial coordinate. Note that the integer m does not enter the above equation.

For a Minkowski spacetime background N(r) = 1, the general solution of the equation (26) reads

$$R_{\ell}(r) = c_1 r^{\ell} + \frac{c_2}{r^{\ell+1}},\tag{27}$$

with $c_{1,2}$ arbitrary constants. Thus, any non-trivial solution diverges either at the origin or at infinity; also, the total energy associated to any such multipolar field is infinite. However, "boxing" Minkowski spacetime allows for everywhere regular electric multipoles, with a finite total energy inside the box. That is, physically reasonable configurations can be obtained by taking $c_1 = 0$ in (27), and confining the electric field to be inside a box – say, spherical and of radius r_B .

In gravitational physics, AdS is a natural "box", in view of its conformal timelike boundary. This suggest the existence of solutions of Maxwell equations in a (globally) AdS background which are everywhere regular, with a finite total energy. Indeed, this conjecture has been confirmed in [28]. For any $\ell \ge 1$ the equation (26) possesses a solution¹⁰ which is regular everywhere, in particular at r = 0:

$$R_{\ell}(r) = \frac{\Gamma(\frac{\ell+\ell}{2})\Gamma(\frac{3+\ell}{2})}{\sqrt{\pi}\Gamma(\frac{3}{2}+\ell)} \frac{r^{\ell}}{L^{\ell}} {}_{2}F_{1}\left(\frac{1+\ell}{2}, \frac{\ell}{2}, \frac{3}{2}+\ell, -\frac{r^{2}}{L^{2}}\right),$$
(28)

where $_{2}F_{1}$ is the hypergeometric function. The simplified forms of the first two such functions are

$$R_1(r) = -\frac{2}{\pi} \left[\frac{L}{r} \left(-1 + \frac{L}{r} \right) \arctan\left(\frac{r}{L}\right) \right], \qquad R_2(r) = 1 + \frac{3L^2}{2r^2} - \frac{3L(r^2 + L^2)}{2r^3} \arctan\left(\frac{r}{L}\right).$$

The asymptotic form of the general solution reads

$$R_{\ell}(r) = c_0 \left(\frac{r}{L}\right)^{\ell} + \dots \text{ as } r \to 0, \qquad \text{and} \qquad R_{\ell}(r) = 1 - c_1 \frac{L}{r} + \dots \text{ as } r \to \infty, \tag{29}$$

where c_0 , c_1 are ℓ -dependent constants [28].

One can easily see that the energy density ρ of the solutions is finite everywhere, being strongly localized in a finite region of space, and depending on both θ, φ in a complicated way. In particular, for $m \neq 0$, a surface of constant energy density possesses discrete symmetries only [19]. Also, ρ is nonzero at $\theta = 0$; and it vanishes at the origin unless $\ell = 1$. At infinity, ρ decays as $1/r^4$, such that the total energy of these solutions is finite

$$E_{\ell} = L \frac{\Gamma(\frac{1+\ell}{2})\Gamma(\frac{3+\ell}{2})}{\Gamma(1+\frac{\ell}{2})\Gamma(\frac{\ell}{2})} .$$
(30)

By using a simple scaling argument, one can show that these static Maxwell configurations in a fixed AdS spacetime satisfy the virial identity

$$\int_{0}^{\infty} r^{2} dr \int_{0}^{\pi} \sin \theta \left(V_{,r}^{2} + \frac{1 - \frac{r^{2}}{L^{2}}}{N^{2}(r)r^{2}} \left(V_{,\theta}^{2} + \frac{1}{\sin^{2}\theta} V_{,\varphi}^{2} \right) \right) = 0.$$
(31)

This makes it clear that the AdS geometry supplies the attractive force needed to balance the repulsive force of the gauge interactions. Also, it is clear that the configurations are supported by the nontrivial angular dependence of V, *i.e.* they should possess a multipolar structure.

Finally, let us also mention that, due to the electric-magnetic duality of Maxwell's theory in four dimensions, these configurations possess also an equivalent purely magnetic picture [29].

3.2 Adding a BH horizon. Electrostatics in Schwarzschild-AdS background

A rather similar picture is found when taking instead a Schwarzschild-AdS (SAdS) BH backgound, with a line element still given by (21), where this time

$$N(r) = \left(1 - \frac{r_H}{r}\right) \left(1 + \frac{r^2}{L^2} \left(1 + \frac{r_H}{r} + \frac{r_H^2}{r^2}\right)\right).$$
(32)

¹⁰The solution is normalized such that $R_{\ell}(r) \to 1$ asymptotically.



Figure 1: Left panel: The radial function R_{ℓ} (with $\ell = 1, 2, 3$) is shown for Maxwell field solutions on a fixed Schwarzschild-AdS background. Right panel: The total mass-energy for families of $\ell = 1, 2, 3, m = 0$ solutions is shown as a function of the event horizon radius.

Unfortunately, equation (26) cannot be solved in closed form in this case¹¹, except for $\ell = 0$, in which case $R_0(r) = c_0 - c_1/r$. However, an approximate expression of the solution can be found both near the horizon and in the far field. Assuming the existence of a power series in $(r - r_H)$, one can easily see from the eq. (26) that, for $\ell > 0$, the radial function necessarily vanishes on the horizon¹², the first terms in the solution therein being

$$R_{\ell}(r) = r_1(r - r_H) + \frac{r_1\left((\ell - 1)(\ell + 2) - \frac{6r_H^2}{L^2}\right)}{2r_H(1 + \frac{3r_H^2}{L^2})}(r - r_H)^2 + \mathcal{O}(r - r_H)^3,$$
(33)

where r_1 is a parameter which results from the numerics. As $r \to \infty$, the solution reads

$$R_{\ell}(r) = 1 - c_1 \frac{L}{r} + \frac{1}{2}\ell(\ell+1)\frac{L^2}{r^2} + \dots,$$
(34)

where we normalized it such that $R_{\ell}(r) \to 1$ asymptotically.

The $\ell \ge 1$ solutions interpolating smoothly between the asymptotics (33), (34) are constructed numerically. In Figure 1 (left panel) we exhibit the radial function R_{ℓ} for a SAdS background with a fixed horizon radius $r_H = 1$ and $\ell = 1, 2, 3$. The dependence of the total mass-energy of the $\ell = 1, 2, 3$; m = 0 solutions as a function of the event horizon radius is shown in Figure 1 (right panel). Note that in both plots we take an AdS length scale L = 1.

3.3 Including the back-reaction: Einstein-Maxwell-AdS BHs

The existence of these everywhere regular, finite energy Maxwell fields, as described above, suggests that fully non-linear Einstein-Maxwell-AdS solitons, as well as deformed BHs, exist, as the backreacting non-linear version of the test field solutions. In the absence of analytic methods¹³ to tackle the fully non-linear Einstein-Maxwell-AdS solutions, the problem is approached by using numerical methods, as described in Section 2. Thus the

¹¹However, an exact solution can be found for a Schwarzschild BH background, (*i.e.* in the limit $L \to \infty$), $R_{\ell}(r)$ being the sum of two modes, one of them diverging at the horizon and the other one at infinity. Thus, again, the "boxing" feature of AdS spacetime regularizes the far field asymptotics.

¹²This results from the presence of the N(r)-factor in the denominator of the *r.h.s.* in eq. (26). Also, note that the condition $R_{\ell}(r_H) = 0$, together with the expansion (33), implies that the components of the energy-momentum tensor (20) are finite at the horizon.

¹³Closed form perturbative solitons can be found, as a power series in the parameter c_e - see [28–30] for work in the axially symmetric case. However, the lowest order solutions are already extremely complicated.

EDT equations (1) are solved together with the Maxwell equations by employing an electric U(1) ansatz with

$$A = V(r, \theta, \varphi) dt. \tag{35}$$



Figure 2: The isometric embeddings in Euclidean 3-space for the spatial sections of the horizon of typical AdS-electrovacuum BHs with $(\ell = 2; m = 0)$ (left) and $(\ell = 3; m = 1)$ (right).



Figure 3: Equatorial slices for the horizon isometric embeddings of AdS-electrovacuum BHs with different boundary data. The different lines correspond to BHs that have the same temperature, but increasing the values of the parameter c_e . The right panel is extracted from [19].

While the metric ansatz and the corresponding BCs are those displayed in Section 2, the boundary conditions satisfied by the electrostatic potential are

$$V\big|_{r=0} = 0, \ V\big|_{r\to\infty} = c_e Y_{\ell m}(\theta,\varphi), \ V\big|_{\theta=0} = 0, \ V\big|_{\theta=\pi/2} = 0, \ \partial_{\varphi} V\big|_{\varphi=0} = 0, \ V\big|_{\varphi=\pi/2} = 0.$$
(36)

with c_e a constant. Note that if $\ell + m$ an even number we impose instead $\partial_{\theta} V|_{\theta=\pi/2} = 0$; also, for even m we shall require $\partial_{\varphi} V|_{\varphi=\pi/2} = 0$, as implied by the symmetries of the problem. No upper bound on c_e seems to exist, although the numerical accuracy decreases for large values of this parameter. For example, for the $(\ell = 2, m = 2)$ case, the maximal considered value of c_e was 15, while for $(\ell = 3, m = 3)$ we have constructed solutions up to $c_e = 12$.

The numerical solutions were reported in Ref. [19]; here we review their basic properties. First, the configurations which were found in the probe limit for a SAdS geometry survive when including backreaction on the spacetime geometry. The case ($\ell = m = 0$) is special, corresponding to the RN-AdS BHs. Non-linear continuation exist for all (ℓ, m)-modes. While the solutions found starting with a m = 0 mode are axially symmetric, static BHs without isometries exist as well, being the nonlinear continuation of the probe solutions with $m \neq 0$. A systematic study of the axially symmetric ($\ell = 1, m = 0$) case was reported in Ref. [30], where the solutions were dubbed '*polarised BHs*', as justified by the local distribution of the electric charge.

These BHs possesses a single global charge, corresponding to their total mass, which is computed *e.g.* by employing either the prescriptions in Ref. [31], as described in Appendix A, while the net electric charge vanishes. As such, most of the basic thermodynamical features of these configurations are similar to the (vacuum) SAdS case. For example, the BH temperature is always bounded from below. At low temperatures we have a single solution, which corresponds to the thermal globally regular solution. At high temperatures there exist two additional solutions that correspond to the small and large BHs [19]. Also, the minimal temperature of the BHs decreases with the increase of c_e , although, at least for the explored solutions, it never reaches zero.

Despite these SAdS-like features, the horizon geometry of these configurations can strongly depart from spherical symmetry, as shown in Figures 2, 3.

4 The second mechanism: scalarized RN BHs

4.1 The general setting and the zero mode

The situation is rather different in this case. There is no solitonic limit whose basic properties (in particular the absence of isometries) are inherited by the BH generalizations [8] (but see [32]). Instead, the mechanism at work here has a different origin, residing on the phenomenon of 'spontaneous scalarization' and the existence of (ℓ, m) -scalar clouds for a given (electro-)vacuum configuration S_0 . The non-linear continuation of these clouds results in a set S_e of static BHs without isometries. These solutions violates the well-known no-hair theorems due to the non-standard scalar field action. In a nutshell, the scalar field action to be considered can be expressed as

$$I_{\phi} = \int d^4x \sqrt{-g} \left[\frac{1}{2} (\nabla \phi)^2 + f(\phi) \mathcal{I}(\psi; g) \right], \qquad (37)$$

with $f(\phi)$ some coupling function and \mathcal{I} a source term which generically depends on some extra-matter field(s) ψ and metric tensor $g_{\mu\nu}$. Then the corresponding equation of motion for the scalar field ϕ reads

$$\nabla^2 \phi = \frac{\partial f}{\partial \phi} \mathcal{I}.$$
(38)

Next, one assumes the existence of a ground state for the scalar field with $\phi = 0$, which is the fundamental solution of the equation (38); then the coupling function should satisfy the condition¹⁴

$$\left. \frac{\partial f}{\partial \phi} \right|_{\phi=0} = 0. \tag{39}$$

Thus, the usual electrovacuum solutions solve the considered model, (37) supplemented with the Einstein-Hilbert action and, possibly, the action of other matter field(s) ψ . This provides the *fundamental solutions* of the model. Apart from that, the model possesses a second set of solutions, with a nontrivial scalar field – *the scalarized BHs.* These solutions are usually entropically preferred over the fundamental ones (*i.e.* they maximize the entropy for given global charges). Moreover, they are smoothly connected with the fundamental set, approaching it for $\phi = 0$.

At the linear level, spontaneous scalarization manifests itself as a tachyonic instability triggered by a negative effective mass squared of the scalar field. This can be seen by considering the linearized form of the equation

 $^{^{14}}$ This condition excludes the case of a dilaton coupling, $f(\phi)=e^{-\alpha\phi}.$

(38) (*i.e.* with a small- ϕ):

$$(\nabla^2 - \mu_{eff}^2)\phi = 0, \quad \text{where} \quad \frac{1}{2}\mu_{eff}^2 = \frac{\partial^2 f}{\partial\phi^2}\Big|_{\phi=0}\mathcal{I}.$$
(40)

Assuming a spherically symmetric background as given by (21) and a decomposition of the scalar field in spherical harmonics similar to (25), equation (40) implies that the amplitude $R_{\ell}(r)$ is a solution of the equation

$$\frac{1}{r^2} (r^2 N R'_{\ell})' = \left(\frac{\ell(\ell+1)}{r^2} + \mu_{eff}^2\right) R_{\ell}.$$
(41)

The solutions of the above equation describe scalar clouds. In fact, solving (41) can be viewed as an eigenvalue problem: for a given ℓ , the condition for a smooth scalar field which vanishes asymptotically selects a discrete set of S_0 configurations.



Figure 4: Existence lines for scalar clouds on a RN background are shown as a function of the parameter α , for $\ell = 0, 1, 2$. For each (α, ℓ) , a branch of scalarised solutions bifurcates from a RN BH with a particular charge to mass ratio q = Q/M.

In this setting, one can distinguish two types of scalarisation, depending on the 'source' term \mathcal{I} . Geometric scalarisation started being considered in Refs. [33–35], using the Gauss-Bonnet invariant as the source term

$$\mathcal{I} = L_{GB},\tag{42}$$

with S_0 the Schwarzschild BH. Matter-induced scalarisation is illustrated by the work [20], which studied the spontaneous scalarisation of electrovacuum BHs, and where the source term was

$$\mathcal{I} = F_{\mu\nu}F^{\mu\nu},\tag{43}$$

with S_0 the RN BH. In both cases, the explicit form of the coupling function does not appear for be important as long as the condition (39) is satisfied. For concreteness, we shall present results for

$$f(\phi) = e^{-\alpha\phi^2},\tag{44}$$

with the coupling constant $\alpha < 0$ being an input parameter.

The main advantage of the matter-induced scalarisation model (43) consists in its simplicity. For example, the equations are simple enough to allow a non-perturbative study of the *general* non-spherical solutions, which

is not the case for model (42). Moreover, the zero-mode equation (41) has $\mu_{eff}^2 = \alpha Q^2/r^4$ and can be solved in closed form for $\ell = 0$, with

$$R_0(r) = P_u \left[1 + \frac{2Q^2(r - r_H)}{r(r_H^2 - Q^2)} \right], \tag{45}$$

where $u \equiv (\sqrt{4\alpha + 1} - 1)/2$, $r_H \equiv M + \sqrt{M^2 - Q^2}$ and P_u is a Legendre function ((M, Q) being the mass and charge parameter of the RN BH). Thus, for generic parameters (α, Q, r_H) , finding the $\ell = 0$ bifurcation points from RN reduces to studying the zeros of this function as $r \to \infty$, such that the condition $R_0 \to 0$ is satisfied. Although the general $\ell \ge 1$ solution of the Eq. (41) is not known in closed form, the results in Figure 4 indicated that the $\ell = 0$ results are generic, although the minimal value of $|\alpha|$ increases with ℓ . Some further analytic studies of these scalar clouds can be found in [36].



Figure 5: Reduced area, $a_H \equiv \frac{A_H}{16\pi M^2}$, of scalarized BHs with $\alpha = -71.8 \ vs.$ reduced charge q. BHs with $m \neq 0$ have no spatial isometries. Extracted from [20].

4.2 Scalarized BHs with no isometries

The non-linear realisation of a general (ℓ, m) -mode results in a set S_e of scalarized RN BHs. These solution were reported in Ref. [20] (see also Ref. [37] for a more detailed study of the spherical sector). They possess a variety of interesting properties, the most important one being that some spherically symmetric scalarized BHs emerge dynamically as end points of numerical evolutions starting with small scalar perturbations of a RN BH. Also, differently from the AdS BHs in the previous Section, these configurations do not possess a solitonic limit.

In the context of this work, of main interest are the configurations originating from scalar clouds with $m \neq 0$, which result in branches of static BH solutions with no isometries. In particular, the horizon, despite being a topological two-sphere, possesses discrete symmetries only.

Since no exact solution seems accessible (not even at the perturbative level), these configurations are found numerically, by using the framework in Section 1. The background metric is provided by the RN BH, with N(r)given by the $L \to \infty$ limit in eq. (3) (with q = Q). The ansatz for the Maxwell field is still given by (35), while for the scalar field the anstaz has the most general expression compatible with a static spacetime,

$$\phi = \phi(r, \theta, \varphi). \tag{46}$$

In numerics, we impose the following BCs for the matter fields

$$V\big|_{x=0} = 0, \ \partial_x \phi\big|_{x=0} = 0, \ \partial_x V\big|_{x=1} = Q, \ \phi\big|_{x=1} = 0, \ \partial_\theta V\big|_{\theta=0} = \phi\big|_{\theta=0} = 0, \ \partial_\theta V\big|_{\theta=\pi/2} = \phi\big|_{\theta=\pi/2} = 0, \ \partial_\varphi V\big|_{\varphi=0} = 0, \ \phi\big|_{\varphi=\pi/2} = 0, \ \phi\big|_{\varphi=\pi/2} = 0, \ (47)$$

while for even *m* we impose instead $\partial_{\varphi} \phi |_{\varphi=0,\pi/2} = 0$; *Q* is an input parameter fixing the electric charge, and we recall that $r = r_H/(1-x^2)$.

The only global charges are the mass and the electric charge, which are read off from the far field asymptotics

$$-g_{tt} = F_0 N = 1 - \frac{2M}{r} + \dots, \quad V = \Phi - \frac{Q}{r} + \dots$$
(48)

with Φ the electrostatic potential. A general large-*r* expression of the solutions can be constructed as a series in 1/r, with e.g. $F_0 = 1 + c_t/r + \ldots$, where c_t is a constant.

The main properties of the solutions can be summarized as follows. First, all configurations are regular on and outside the horizon and approach asymptotically the Minkowski spacetime background. Although the deformation from sphericity is much less pronounced than in the AdS case discussed in the previous Section, the horizon geometry differs from that of the RN BH, possessing no isometries. Perhaps the most interesting result is that, for given m, the entropy is maximized by the solutions emerging from the $\ell = m$ zero mode, see Figure 5. However, in a full diagram, the entropy of the solutions with a given mass and electric charge is maximized by the spherically symmetric (l = m = 0) scalarized BHs [20].

5 Further remarks

The main purpose of this work was to propose a general framework for the investigation of static BHs without isometries, together with two different physical mechanisms allowing for such configurations. The first mechanism is based on the existence in some models of solitons without isometries, while the second one relies on BH scalarization. Explicit realizations were considered in both cases, corresponding to Einstein-Maxwell-AdS and Einstein-Maxwell-scalar models, respectively. These BHs have a smooth, topologically spherical horizon, but without isometries, and approach, asymptotically, the AdS or the Minkowski spacetime backgrounds.

We expect similar solutions to exist in a variety of other models. As such, the simple picture found in the electrovacuum case [9,10], cannot be taken as a rule; in more general models, staticity does not guarantee the existence of any continuous spatial symmetry, for physically acceptable BH solutions.

For example, concerning the first mechanism, we expect the known field theory solutions without isometries in Refs. [2–5] to possess BH generalizations. In principle, these solutions can be constructed within the framework proposed here, the only obvious obstacle being the complexity of the (highly nonlinear) matter field equations¹⁵. In this context, we mention the existence of asymptotically flat, static BHs without isometries in a simple model consisting in Einstein gravity coupled with a self-interacting scalar field, a much simpler case than the matter field models in Refs. [2–5]. Static and axially symmetric BHs solutions of this model were constructed in Ref. [17]. They evade the no-hair theorems by having a scalar potential which is not strictly positive, and possess a solitonic limit. We have found that the same mechanism allows for the existence of solitons and BHs without spatial isometries, which will be discussed elsewhere.

Concerning the second mechanism, we would like to comment on the results in the recent work [38], which deals with BH scalarization in Einstein-Gauss-Bonnet-scalar theory, *i.e.* a source term (42). The results there indicate that, at least for the ($\ell \neq 0$, m = 0) case, some basic features found for scalarized RN BHs are generic, with the existence of static, axially symmetric solutions. These configurations can be viewed as non-linear continuations of the corresponding zero-mode solutions of the equation (41). As remarked in [38], more general static configurations without rotational symmetry should also exist, the only obstacle in their study being the tremendous complexity of the gravity equations in the presence of a Gauss-Bonnet term.

Still on the same subject, we mention the existence in some models of a similar instability of Schwarzschild/RN BHs with respect to higher spin fields. An interesting case here are the pure Einstein-Weyl gravity solutions in [40]. The configurations there are spherically symmetric, bifurcating from a critical Schwarzschild BH, *i.e.* with an ($\ell = m = 0$) zero mode. Similar configurations are likely to exist for the higher (ℓ, m) case, which would correspond to static BHs without isometries in a pure gravity model.

¹⁵For example, the issue of gauge fixing for non-Abelian fields with a nontrivial dependence of all space coordinates for a numerical approach is an open problem. The same problem for the axially symmetric case is non-trivial, possessing a number of subtleties [39].

Acknowlegements

This work is supported by the Center for Research and Development in Mathematics and Applications (CIDMA) through the Portuguese Foundation for Science and Technology (FCT - Fundacao para a Ciência e a Tecnologia), references UIDB/04106/2020 and UIDP/04106/2020 and by national funds (OE), through FCT, I.P., in the scope of the framework contract foreseen in the numbers 4, 5 and 6 of the article 23, of the Decree-Law 57/2016, of August 29, changed by Law 57/2017, of July 19. We acknowledge support from the projects PTDC/FIS-OUT/28407/2017 and CERN/FIS-PAR/0027/2019. This work has further been supported by the European Union Horizon 2020 research and innovation (RISE) programme H2020-MSCA-RISE-2017 Grant No. FunFiCO-777740. The authors would like to acknowledge networking support by the COST Action CA16104.

A Einstein-Maxwell-AdS BHs: far field asymptotics and boundary stress tensor

The general Einstein-Maxwell-AdS solutions discussed in Section 3 possess a far field expansion with the following leading order terms (with $\alpha^2 = 4\pi G$):

$$\begin{aligned} V(r,\theta,\varphi) &= v_0(\theta,\varphi) + v_1(\theta,\varphi) \frac{L}{r} - \frac{1}{2} \left(v_{0,\theta}(\theta,\varphi) + \cot\theta v_{0,\theta}(\theta,\varphi) + \frac{1}{\sin^2 \theta} v_{0,\varphi\varphi}(\theta,\varphi) \right) \left(\frac{L}{r} \right)^2 + \dots, \\ F_1(r,\theta,\varphi) &= 1 + \left(\frac{q^2}{L^2} + \alpha^2 \left(v_{0,\theta}^2(\theta,\varphi) + \frac{1}{\sin^2 \theta} v_{0,\varphi}(\theta,\varphi)^2 \right) \right) \left(\frac{L}{r} \right)^4 + \dots \\ F_2(r,\theta,\varphi) &= 1 + f_{23}(\theta,\varphi) \left(\frac{L}{r} \right)^3 - \alpha^2 \frac{1}{\sin^2 \theta} v_{0,\varphi}(\theta,\varphi)^2 \left(\frac{L}{r} \right)^4 + \dots \\ F_3(r,\theta,\varphi) &= 1 + f_{33}(\theta,\varphi) \left(\frac{L}{r} \right)^3 - \alpha^2 v_{0,\theta}^2(\theta,\varphi) \left(\frac{L}{r} \right)^4 + \dots \\ F_0(r,\theta,\varphi) &= 1 + f_{03}(\theta,\varphi) \left(\frac{L}{r} \right)^3 + \left(-\frac{q^2}{L^2} + \alpha^2 v_1^2(\theta,\varphi) \right) \left(\frac{L}{r} \right)^4 + \dots \\ S_1(r,\theta,\varphi) &= \frac{1}{\sin \theta} \left(\sin \theta f_{23,\theta}(\theta,\varphi) + \cos \theta (f_{23}(\theta,\varphi) - f_{33}(\theta,\varphi)) + s_{33,\varphi}(\theta,\varphi) \right) \log(\frac{L}{r}) \left(\frac{L}{r} \right)^5 + \dots \\ S_2(r,\theta,\varphi) &= \frac{1}{\sin \theta} \left(f_{33,\varphi}(\theta,\varphi) + \sin \theta s_{33,\theta}(\theta,\varphi) + 2 \cos \theta s_{33}(\theta,\varphi) \right) \log(\frac{L}{r}) \left(\frac{L}{r} \right)^5 + \dots \\ S_3(r,\theta,\varphi) &= s_{33}(\theta,\varphi) \left(\frac{L}{r} \right)^3 + \alpha^2 \frac{1}{\sin \theta} v_{0,\theta}(\theta,\varphi) v_{0,\varphi}(\theta,\varphi) \left(\frac{L}{r} \right)^4 + \dots . \end{aligned}$$

In our approach, $v_0(\theta, \varphi)$ is imposed as a BC, while $v_1(\theta, \varphi)$ results from numerics. The metric functions contain $f_{03}(\theta, \varphi), f_{23}(\theta, \varphi), f_{33}(\theta, \varphi), s_{33}(\theta, \varphi)$ which are also fixed by the numerics, subject to the constraints (which follow from the field eqs.):

$$f_{00} + f_{23} + f_{33} = 0,$$

$$\cos \theta (f_{23} - f_{33}) + s_{33,\varphi} + \sin \theta f_{23,\theta} + \frac{4}{3} \alpha^2 \sin \theta v_1 v_{0,\theta} = 0,$$

$$2 \cos \theta s_{23} - f_{33,\varphi} + \sin \theta s_{33,\theta} + \frac{4}{3} \alpha^2 \sin \theta v_1 v_{0,\varphi} = 0.$$
(50)

The mass of the solutions is computed by employing the boundary counterterm approach in [31], as the conserved charge associated with Killing symmetry ∂_t of the induced boundary metric, found for a large value r = constant. A straightforward computation leads to the following expression:

$$M = M^{(b)} + \frac{1}{8\pi G} \frac{3L}{2} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin \theta \left(f_{23}(\theta, \varphi) + f_{33}(\theta, \varphi) \right) , \qquad (51)$$

with

$$M^{(b)} = \frac{r_H}{2G} \left(1 + \frac{r_H^2}{L^2} + \frac{q^2}{r_H^2} \right)$$
(52)

the contribution from the background metric. Note that the same result can be derived by using the Ashtekar-Magnon-Das conformal mass definition [41].

We expect these solutions to be relevant in the context of AdS/CFT and more generally in the context of gauge/gravity dualities. Thus it is of interest to evaluate the holographic stress tensor. In order to extract it, we first transform the (asymptotic metric) into Fefferman–Graham coordinates, by using a new radial coordinate z, with

$$r = \frac{L^2}{z} - \frac{2r_H^2 + L^2}{4L^2}z + \frac{r_H^4 + (q^2 + r_H^2)L^2}{6r_H L^4}z^2.$$
(53)

In these coordinates, the line element can be expanded around z = 0 (*i.e.* as $r \to \infty$) in the standard form

$$ds^{2} = \frac{L^{2}}{z^{2}} \bigg[dz^{2} + \big(g_{(0)} + z^{2}g_{(2)} + z^{3}g_{(3)} + O(z^{4})\big)_{ij} dx^{i} dx^{j} \bigg],$$
(54)

where $x^i = (\theta, \varphi, t)$ and

$$\left(g_{(0)} + z^2 g_{(2)}\right)_{ij} dx^i dx^j = \left(L^2 - \frac{z^2}{2}\right) \left(d\theta^2 + \sin^2\theta d\varphi^2\right) - \left(1 + \frac{z^2}{2L^2}\right) dt^2.$$
(55)

Then the background metric upon which the dual field theory resides is $d\sigma^2 = g_{(0)ij}dx^i dx^j = -dt^2 + L^2(d\theta^2 + \sin^2\theta d\varphi^2)$, which corresponds to a static Einstein universe in (2+1) dimensions.

From (54) one can read the *v.e.v.* of the holographic stress tensor [42]:

$$\langle \tau_{ij} \rangle = \frac{3L^2}{16\pi G} g_{(3)ij} = \langle \tau_{ij}^{(0)} \rangle + \langle \tau_{ij}^{(s)} \rangle, \tag{56}$$

being expressed as the sum of a background part plus a Maxwell contribution (which possesses a nontrivial (θ, φ) -dependence),

$$\langle \tau_{ij}^{(0)} \rangle dx^i dx^j = \frac{1}{16\pi G} \left(\frac{r_H^3}{L^2} + \frac{q^2 + r_H^2}{r_H} \right) \left(d\theta^2 + \sin^2 \theta d\varphi^2 + \frac{2}{L^2} dt^2 \right), \tag{57}$$

$$\langle \tau_{ij}^{(s)} \rangle dx^i dx^j = \frac{3}{16\pi G} \frac{1}{L^2} \bigg(f_{33}(\theta,\varphi) d\theta^2 + 2s_{33}(\theta,\varphi) \sin\theta d\theta d\varphi + f_{33}(\theta,\varphi) \sin^2\theta d\varphi^2 - \frac{1}{L^2} f_{03}(\theta,\varphi) dt^2 \bigg).$$
(58)

As expected, this stress-tensor is finite and traceless. Also, it satisfies the conservation law in the presence of a background electric field

$$\langle \tau^{ab} \rangle_{;a} + j_a \mathcal{F}^{ab} = 0, \tag{59}$$

a relation which holds via (50). Here $j = v_1 dt$ is the current density on the boundary and $\mathcal{F} = d\mathcal{A}$ (with $\mathcal{A} = v_0 dt$) is the boundary electromagnetic field.

References

- [1] N. S. Manton and P. Sutcliffe, 'Topological solitons', Cambridge University Press, 2004;
 - L. H. Kauffman, Knots and physics, River Ridge, New York, 2000;
 - E. Radu and M. S. Volkov, Phys. Rept. 468 (2008) 101 [arXiv:0804.1357 [hep-th]];
 - Y. M. Shnir, 'Topological and Non-Topological Solitons in Scalar Field Theories', Cambridge University Press, 2018.

- [2] C. J. Houghton and P. M. Sutcliffe, Commun. Math. Phys. 180 (1996) 343 [hep-th/9601146].
- [3] L. D. Faddeev and A. J. Niemi, Nature **387** (1997) 58 [hep-th/9610193].
- [4] R. A. Battye and P. Sutcliffe, Proc. Roy. Soc. Lond. A 455 (1999) 4305 [hep-th/9811077].
- [5] R. A. Battye and P. M. Sutcliffe, Phys. Rev. Lett. 79 (1997) 363 [hep-th/9702089].
- [6] P. T. Chrusciel, J. Lopes Costa and M. Heusler, Living Rev. Rel. 15 (2012) 7 doi:10.12942/lrr-2012-7 [arXiv:1205.6112 [gr-qc]].
- [7] A. Lichnerowicz, Theories Relativistes de la Gravitation et de l'Electromagnetisme, Masson, Paris, 1955.
- [8] C. A. Herdeiro and J. M. Oliveira, Class. Quant. Grav. 36 (2019) no.10, 105015 [arXiv:1902.07721 [gr-qc]].
- [9] W. Israel, Commun. Math. Phys. 8 (1968) 245.
- [10] W. Israel, Phys. Rev. 164 (1967) 1776.
- [11] I. Pena and D. Sudarsky, Class. Quant. Grav. 14 (1997) 3131.
- [12] D. Kastor and J. H. Traschen, Phys. Rev. D 46 (1992) 5399 [hep-th/9207070].
- [13] M. S. Volkov, Phys. Lett. B 524 (2002) 369 [hep-th/0103038].
- [14] S. A. Ridgway and E. J. Weinberg, Phys. Rev. D 52 (1995) 3440 [gr-qc/9503035].
- [15] B. Kleihaus and J. Kunz, Phys. Rev. Lett. 79 (1997) 1595 [gr-qc/9704060].
- [16] B. Hartmann, B. Kleihaus and J. Kunz, Phys. Rev. D 65 (2002) 024027 [hep-th/0108129].
- [17] B. Kleihaus, J. Kunz, E. Radu and B. Subagyo, Phys. Lett. B 725 (2013) 489 [arXiv:1306.4616 [gr-qc]].
- [18] T. Ioannidou, B. Kleihaus and J. Kunz, Phys. Lett. B **635** (2006) 161 [gr-qc/0601103].
- [19] C. A. R. Herdeiro and E. Radu, Phys. Rev. Lett. 117 (2016) no.22, 221102 [arXiv:1606.02302 [gr-qc]].
- [20] C. A. R. Herdeiro, E. Radu, N. Sanchis-Gual and J. A. Font, Phys. Rev. Lett. 121 (2018) no.10, 101102 [arXiv:1806.05190 [gr-qc]].
- [21] M. Headrick, S. Kitchen and T. Wiseman, Class. Quant. Grav. 27 (2010) 035002 [arXiv:0905.1822 [gr-qc]].
- [22] A. Adam, S. Kitchen and T. Wiseman, Class. Quant. Grav. 29 (2012) 165002 [arXiv:1105.6347 [gr-qc]].
- [23] T. Wiseman, arXiv:1107.5513 [gr-qc].
- [24] O. J. C. Dias, J. E. Santos and B. Way, arXiv:1510.02804 [hep-th].
- [25] L. Smarr, Phys. Rev. D 7 (1973) 289.
- [26] G. W. Gibbons, C. A. R. Herdeiro and C. Rebelo, Phys. Rev. D 80 (2009) 044014 [arXiv:0906.2768 [gr-qc]].
- [27] W. Schönauer and R. Weiß, J. Comput. Appl. Math. 27, 279 (1989) 279;
- M. Schauder, R. Weiß and W. Schönauer, *The CADSOL Program Package*, Universität Karlsruhe, Interner Bericht Nr. 46/92 (1992).
- [28] C. Herdeiro and E. Radu, Phys. Lett. B 749 (2015) 393 [arXiv:1507.04370 [gr-qc]].
- [29] C. Herdeiro and E. Radu, Phys. Lett. B **757** (2016) 268 [arXiv:1602.06990 [gr-qc]].
- [30] M. S. Costa, L. Greenspan, M. Oliveira, J. Penedones and J. E. Santos, arXiv:1511.08505 [hep-th].
- [31] V. Balasubramanian and P. Kraus, Commun. Math. Phys. 208 (1999) 413 [hep-th/9902121].
- [32] C. A. Herdeiro, J. M. Oliveira and E. Radu, Eur. Phys. J. C 80 (2020) no.1, 23 [arXiv:1910.11021 [gr-qc]].
- [33] H. O. Silva, J. Sakstein, L. Gualtieri, T. P. Sotiriou and E. Berti, Phys. Rev. Lett. 120 (2018) no.13, 131104 [arXiv:1711.02080 [gr-qc]].
- [34] D. D. Doneva and S. S. Yazadjiev, Phys. Rev. Lett. 120 (2018) no.13, 131103 [arXiv:1711.01187 [gr-qc]].
- [35] G. Antoniou, A. Bakopoulos and P. Kanti, Phys. Rev. Lett. **120** (2018) no.13, 131102 [arXiv:1711.03390 [hep-th]].
- [36] S. Hod, Phys. Lett. B 798 (2019), 135025 [arXiv:2002.01948 [gr-qc]].
- [37] P. G. S. Fernandes, C. A. R. Herdeiro, A. M. Pombo, E. Radu and N. Sanchis-Gual, Class. Quant. Grav. 36 (2019) no.13, 134002 Erratum: [Class. Quant. Grav. 37 (2020) no.4, 049501] [arXiv:1902.05079 [gr-qc]].
- [38] L. G. Collodel, B. Kleihaus, J. Kunz and E. Berti, Class. Quant. Grav. 37 (2020) no.7, 075018 [arXiv:1912.05382 [gr-qc]].
- [39] Y. Brihaye, B. Kleihaus and J. Kunz, Phys. Rev. D 47 (1993) 1664.
- [40] H. Lu, A. Perkins, C. N. Pope and K. S. Stelle, Phys. Rev. Lett. 114 (2015) no.17, 171601 [arXiv:1502.01028 [hep-th]].
- [41] A. Ashtekar and S. Das, Class. Quant. Grav. 17 (2000) L17 [hep-th/9911230].
- [42] S. de Haro, S. N. Solodukhin and K. Skenderis, Commun. Math. Phys. 217 (2001) 595 [hep-th/0002230].