# Global solution of the initial value problem for the focusing Davey-Stewartson II system 

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#### Abstract

We consider the two dimensional focusing Davey-Stewartson II system and construct the global solution of the Cauchy problem for a dense in $L^{2}(\mathbb{C})$ set of initial data. We do not assume that the initial data is small. So, the solutions may have singularities. We show that the blow-up may occur only on a real analytic variety and the variety is bounded in each strip $t \leq T$.


Key words: Davey-Stewartson, $\bar{\partial}$-method, Dirac equation, exceptional point, inverse problem.

## 1 Introduction

Let $q_{0}(z), z=x+i y,(x, y) \in \mathbb{R}^{2}$, be a compactly supported (or fast decaying) sufficiently smooth function. Consider the two dimensional focusing Davey-Stewartson II (DSII) system of equations for unknown functions $q=q(z, t), \phi=\phi(z, t),(x, y) \in \mathbb{R}^{2}, t \geq 0$ :

$$
\begin{array}{r}
q_{t}=2 i q_{x y}-4 q(\bar{\varphi}-\varphi), \\
\partial \varphi=\bar{\partial}|q|^{2}, \\
q(z, 0)=q_{0}(z) . \tag{1}
\end{array}
$$

A smooth solution of (11) exists for all $t>0$ if $q_{0}$ is small enough, [1, 19, 20, 21, 3]. We will call this solution classical. If $q_{0}$ is not small, the solution was constructed locally in our previous work [14] via the IST (inverse scattering transform) using the $\bar{\partial}$-method that has been generalized in [12], [13] to the case when exceptional points may be present.

[^0]The solution was obtained in a neighbourhood of any point $\left(z_{0}, t_{0}\right)$ for generic initial data $q_{0}$ that depend on the point. The main objectives of this article are to obtain the solution for an arbitrary initial data from a specific set and to get the global solution defined in the whole space, including a description of the set where the solution blows up. It will be shown that the latter set is a real analytic variety that is bounded in every strip $0 \leq t \leq T$.

Let us recall that the focusing DSII equation may have a finite time blow-up (e.g., [16]). While the uniqueness is known for smooth (in some sense) solutions, see [8], [9], one has to be careful with the definition of the solution that has singularities. We understand these solutions in the following sense. Let us multiply the initial data $q_{0}$ by a positive parameter $a \in(0,1]$. We will show that the classical solution that exists when $a \ll 1$ allows an analytic continuation into a complex neighborhood of $(0,1]$, and this analytic continuation will be used to single out the solution with singularities when $a=1$.

Let us mention some recent articles on DSII: [11, [17], [18].

## 2 The inverse nonlinear Fourier transform

### 2.1 The solution of the Cauchy problem, main results

Let $q_{0}(z) \in L^{2}(\mathbb{C})$. Denote

$$
Q_{0}(z)=\left(\begin{array}{cc}
0 & q_{0}(z)  \tag{2}\\
-q_{0}(z) & 0
\end{array}\right), \quad z \in \mathbb{C} .
$$

Let $\bar{\partial}=\frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i \frac{\partial}{\partial y}\right)$, and let the $2 \times 2$ matrix $\psi(\cdot, k), k \in \mathbb{C}$, be a solution of the following problem for the Dirac equation in $\mathbb{C}$ :

$$
\begin{equation*}
\frac{\partial \psi}{\partial \bar{z}}=Q_{0} \bar{\psi}, \quad \psi(z, k) e^{-i \bar{k} z / 2} \rightarrow I, \quad z \rightarrow \infty \tag{3}
\end{equation*}
$$

The corresponding Lippmann-Schwinger equation has the following form:

$$
\begin{equation*}
\psi(z, k)=e^{i \bar{k} z / 2} I+\int_{\mathbb{C}} G\left(z-z^{\prime}, k\right) Q_{0}\left(z^{\prime}\right) \bar{\psi}\left(z^{\prime}, k\right) d \sigma_{z^{\prime}}, \quad G(z, k)=\frac{1}{\pi} \frac{e^{i \bar{k} z / 2}}{z} \tag{4}
\end{equation*}
$$

where $d \sigma_{z^{\prime}}=d x^{\prime} d y^{\prime}$. Here and below we use the same notation for functional spaces, irrespectively of whether those are the spaces of matrix-valued or scalar-valued functions. After the substitution,

$$
\begin{equation*}
\mu(z, k)=\psi(z, k) e^{-i \bar{k} z / 2}, \quad \mu(z, k)-I \rightarrow 0, \quad z \rightarrow \infty \tag{5}
\end{equation*}
$$

equation (4) takes the form

$$
\begin{equation*}
\mu(z, k)=I+\frac{1}{\pi} \int_{\mathbb{C}} \frac{e^{i \Re(\bar{k} z)}}{z-z^{\prime}} Q_{0}\left(z^{\prime}\right) \bar{\mu}\left(z^{\prime}, k\right) d \sigma_{z^{\prime}}, \tag{6}
\end{equation*}
$$

and becomes Fredholm in $L^{q}(\mathbb{C}), q>2$, after the additional substitution $\nu=\mu-I$ (see, e.g., [15, lemma 5.3]).

Solutions $\psi$ of (4) are called the scattering solutions, and the values of $k$ such that the homogeneous equation (6) has a non-trivial solution are called exceptional points. The set of exceptional points will be denoted by $\mathcal{E}$. Thus the scattering solution may not exist if $k \in \mathcal{E}$. Note that the operator in equation (6) is not analytic in $k$, and $\mathcal{E} \subset \mathbb{C}$ may contain one-dimensional components. There are no exceptional points in a neighborhood of infinity (e.g., [19, lemma 2.8], [2, lemma C]). Let us choose $A \gg 1$ and $k_{0} \in \mathbb{C}$ such that all the exceptional points are contained in the disk

$$
\begin{equation*}
D=\{k \in \mathbb{C}: 0 \leq|k|<A\} \tag{7}
\end{equation*}
$$

and $k_{0}$ belongs to the same disc $\bar{D}$ and is not exceptional.
The generalized scattering data (an analogue of the scattering amplitude in the standard scattering problem) are defined by the following integral (when the integral converges)

$$
\begin{equation*}
h_{0}(\varsigma, k)=\frac{1}{(2 \pi)^{2}} \int_{\mathbb{C}} e^{-i \bar{\tau} / 2} Q_{0}(z) \bar{\psi}(z, k) d \sigma_{z}, \varsigma \in \mathbb{C}, k \in \mathbb{C} \backslash \mathcal{E} \tag{8}
\end{equation*}
$$

In fact, from the Green formula, it follows that $h_{0}$ can be determined without using the potential $Q_{0}$ or the solution $\psi$ of the Dirac equation (3) if the Dirichlet data at $\partial \Omega$ are known for the solution of (3) in a bounded region $\Omega$ containing the support of $Q_{0}$ :

$$
\begin{equation*}
h_{0}(\varsigma, k)=\frac{-i}{8 \pi^{2}} \int_{\partial \mathcal{O}} e^{-i \bar{\zeta} z / 2} \bar{\psi}(z, k) d z, \varsigma \in \mathbb{C}, k \notin \mathcal{E} \tag{9}
\end{equation*}
$$

Note that $h_{0}$ is continuous when $k \notin D$ under minimal assumptions on $Q$ [19], [20], and moreover,

$$
\begin{equation*}
h_{0}=h_{0}(\varsigma, k) \in C^{\infty} \quad \text { when } \quad|k| \geq A \tag{10}
\end{equation*}
$$

if $Q$ is bounded and decays faster then any power at infinity. This follows from the fact that (6) admits differentiation in $z$ nd $k$ when $k \notin D$.

The inverse problem (recovery of $Q$ when $h_{0}$ is given) was solved using $\bar{\partial}$-method in [1], [19, 20, 21] when the potential is small enough to guarantee the absence of exceptional points. When $\mathcal{E} \neq \varnothing$, the inverse problem was solved in a generic sense in [13]. The latter results were applied in [14] to construct solutions of the focusing DSII system. Let us recall some results obtained in [14.

Consider the space

$$
\begin{equation*}
\mathcal{B}^{s}=\left\{u \in L^{s}(\mathbb{C} \backslash D) \bigcap C(D)\right\}, \quad s>2 \tag{11}
\end{equation*}
$$

where $C(D)$ is the space of analytic functions in $D$ with the norm $\|u\|=\sup _{D}|u|$. Here and below, we use the same space notation for matrices as for their entries.

Let operator $T_{z}: \mathcal{B}^{s} \rightarrow \mathcal{B}^{s}, s>2$, be defined as follows:

$$
\begin{array}{r}
T_{z} \phi(k)=\frac{1}{\pi} \int_{\mathbb{C} \backslash D} e^{i(\bar{\zeta} z+\bar{z} \varsigma) / 2} \bar{\phi}(\varsigma) \Pi^{o} h(\varsigma, \varsigma) \frac{d \sigma_{\varsigma}}{\varsigma-k} \\
+\frac{1}{2 \pi i} \int_{\partial D} \frac{d \varsigma}{\varsigma-k} \int_{\partial D}\left[e^{i / 2\left(\varsigma \bar{z}+\overline{\varsigma^{\prime}} z\right)} \overline{\phi^{-}\left(\varsigma^{\prime}\right)} \Pi^{o}+e^{i / 2\left(\varsigma-\varsigma^{\prime}\right) \bar{z}} \phi^{-}\left(\varsigma^{\prime}\right) \Pi^{d} \mathbf{C}\right]\left[\operatorname{Ln} \frac{\overline{\varsigma^{\prime}}-\bar{\varsigma}}{\overline{\varsigma^{\prime}}-\overline{k_{0}}} h\left(\varsigma^{\prime}, \varsigma\right) \overline{d \varsigma^{\prime}}\right],( \tag{12}
\end{array}
$$

where $d \sigma_{\varsigma}=d \varsigma_{R} d \varsigma_{I}, z \in \mathbb{C}, \phi \in \mathcal{B}^{s}, \phi^{-}$is the boundary trace of $\phi$ from the interior of $D$, C is the operator of complex conjugation, $\Pi^{o} M$ is the off-diagonal part of a matrix $M$, $\Pi^{d} M$ is the diagonal part. Let us specify the logarithmic function in (12). Let us shift the coordinates in $\mathbb{C}$ and move the origin to the point $\overline{\varsigma^{\prime}} \in \partial D$. Then we rotate the plane in such a way that the direction of the $x$-axis is defined by the vector from $\overline{\varsigma^{\prime}}$ to $-\overline{\varsigma^{\prime}}$. Then $\left|\arg \left(\overline{\varsigma^{\prime}}-\bar{\zeta}\right)\right|<\pi / 2, \varsigma^{\prime} \neq \varsigma$, and $\left|\arg \left(\overline{\varsigma^{\prime}}-\overline{k_{0}}\right)\right| \leq \pi / 2$, i.e.,

$$
\left|\arg \frac{\overline{\varsigma^{\prime}}-\bar{\varsigma}}{\overline{\varsigma^{\prime}}-\overline{k_{0}}}\right|<\pi, \quad \varsigma^{\prime}, \varsigma \in \partial D, \quad \varsigma^{\prime} \neq \varsigma .
$$

This defines the values of the logarithmic function uniquely.
It turns out that, after the substitution $w=v-I \in \mathcal{B}^{s}, s>2$, the equation

$$
\left(I+T_{z}\right) v=I
$$

becomes Fredholm in $\mathcal{B}^{s}$, and the potential $q_{0}$ can be expressed explicitly in terms of $v$ (see [12, 13]).

In order to solve the DSII problem (11), we apply this reconstruction procedure to a specially chosen scattering data. We start with the scattering data $h_{0}$ defined by $q_{0}$ and extend it in time as follows:

$$
\begin{equation*}
h(\varsigma, k, t):=e^{-t\left(k^{2}-\bar{\varsigma}^{2}\right) / 2} \Pi^{o} h_{0}(\varsigma, k)+e^{-t\left(\bar{k}^{2}-\bar{\varsigma}^{2}\right) / 2} \Pi^{d} h_{0}(\varsigma, k), \varsigma \in \mathbb{C}, k \in \mathbb{C} \backslash \mathcal{E}, t \geq 0 \tag{13}
\end{equation*}
$$

For $t \geq 0$, we define the operator

$$
\begin{array}{r}
T_{z, t} \phi(k)=\frac{1}{\pi} \int_{\mathbb{C} \backslash D} e^{i(\bar{\varsigma} z+\bar{z} \varsigma) / 2} \bar{\phi}(\varsigma) \Pi^{o} h(\varsigma, \varsigma, t) \frac{d \sigma_{\varsigma}}{\varsigma-k} \\
+\frac{1}{2 \pi i} \int_{\partial D} \frac{d \varsigma}{\varsigma-k} \int_{\partial D}\left[e^{\left.i / 2\left(\varsigma \bar{z}+\overline{\left.\varsigma^{\prime} z\right)}\right) \overline{\phi^{-}\left(\varsigma^{\prime}\right)} \Pi^{o}+e^{i / 2\left(\varsigma-\varsigma^{\prime}\right) \bar{z}} \phi^{-}\left(\varsigma^{\prime}\right) \Pi^{d} \mathbf{C}\right]\left[\operatorname{Ln} \frac{\overline{\varsigma^{\prime}}-\bar{\varsigma}}{\overline{\varsigma^{\prime}}-\overline{k_{0}}} h\left(\varsigma^{\prime}, \varsigma, t\right) \overline{d \varsigma^{\prime}}\right] .( } .\right. \tag{14}
\end{array}
$$

Theorem 2.1. ([14]) Let $q_{0}(\cdot)$ a function with compact support. Assume that it is 6 times differentiable in $x$ and $y$. Alternatively, this condition can be replaced by the superexponential decay of $q_{0}$ :

$$
\begin{equation*}
\lim _{z \rightarrow \infty} e^{\widetilde{A}|z|} \partial_{x}^{i} \partial_{y}^{j} q_{0}(z)=0 \text { for each } \widetilde{A}>0, i+j \leq 6 \tag{15}
\end{equation*}
$$

Then, for each $s>2$, the following statements are valid.

[^1]- The operator $T_{z, t}$ is compact in $\mathcal{B}^{s}$ for all $z \in \mathbb{C}, t \geq 0$, and depends continuously on $z$ and $t \geq 0$. The same property holds for its first derivative in time and all the derivatives in $x, y$ up to the third order, where the derivatives are defined in the norm convergence. The function $T_{z, t} I$ belongs to $\mathcal{B}^{s}$ for all $t \geq 0$.
- Let the kernel of $I+T_{z, t}$ in the space $\mathcal{B}^{s}$ be trivial for $(z, t)$ in an open or half open ${ }^{2}$ set $\omega \subset \mathbb{R}^{3}$. Let $v_{z, t}=w_{z, t}+I$, where $w_{z, t} \in \mathcal{B}^{s}$ is the solution of the equation

$$
\begin{equation*}
\left(I+T_{z, t}\right) w_{z, t}=-T_{z, t} I \tag{16}
\end{equation*}
$$

Then functions $q, \varphi$ defined by

$$
\begin{array}{r}
\left(\begin{array}{cc}
\varphi(z, t) & q(z, t) \\
-q(z, t) & \varphi(z, t)
\end{array}\right):=\frac{-i}{2}\left(\Pi^{o}+\bar{\partial} \Pi^{d}\right)\left(\frac{1}{\pi} \int_{\mathbb{C} \backslash D} e^{i(\varsigma z+\overline{z s}) / 2} \overline{v_{z, t}}(\varsigma) \Pi^{o} h(\varsigma, \varsigma, t) d \sigma_{\varsigma}\right. \\
\left.-\frac{1}{2 \pi i} \int_{\partial D} d \varsigma \int_{\partial D}\left[e^{i / 2\left(\varsigma \bar{z}+\varsigma^{\prime} z\right)} \overline{v_{z, t}^{-}\left(\varsigma^{\prime}\right)} \Pi^{o}-e^{i / 2\left(\varsigma-\varsigma^{\prime}\right) \bar{z}} v_{z, t}^{-}\left(\varsigma^{\prime}\right) \Pi^{d} \mathbf{C}\right]\left[\operatorname{Ln} \frac{\overline{\varsigma^{\prime}}-\bar{\varsigma}}{\overline{\varsigma^{\prime}}-\overline{k_{0}}} h\left(\varsigma^{\prime}, \varsigma, t\right) \overline{d \varsigma^{\prime}}\right]\right), \tag{17}
\end{array}
$$

satisfy all the relations (1) in the classical sense when $(z, t) \in \omega$,. In particular, $q(z, 0)=q_{0}(z)$.

- Consider a set of initial data $a q_{0}(z)$ that depend on $a \in(0,1]$. Then equation (16) with $Q^{0}$ replaced by a $Q^{0}$ ( $Q^{0}$ is fixed) is uniquely solvable for almost every $(z, t, a) \in$ $\mathbb{R}^{2} \times \mathbb{R}^{+} \times(0,1]$. Moreover, 3 for each $(z, t)$, the solution of (16) is meromorphic in $a \in[0,1]$ and has at most a finite set of poles $a=a_{j}(z, t)$.

Remark. All the exceptional points are located in a disk whose radius depends only on the norm of $a q_{0}$. Hence $D$ and $k_{0}$ can be chosen independently of $a \in[0,1]$ (see [13, Lemma 5.1]). From now on, we assume that the disk $D$ is fixed and contains the exceptional points for all the potentials $a q_{0}, a \in[0,1]$.

In order to state the main results of the present paper, we need to recall the construction (e.g., [19]) of the global solution of (11) when $q_{0}$ is small. The latter expression (" $q_{0}$ is small") will be used below only for problem (1) with initial data $a q_{0}$ where $q_{0}$ is infinitely smooth and satisfies (15), and $0<a \ll 1$. Let us recall that the scattering problem (3) and the Lippmann-Schwinger equation (4) are uniquely solvable for all $k$ when $q_{0}$ is small, i.e., there are no exceptional points in this case and $h_{0}(k, k)$ is defined for all the values of $k$. Operator $T_{z, t}$ is needed only with $D=\emptyset$ if $q_{0}$ is small. Hence, only the first term is present in the right-hand side of (14). Moreover, $\left\|T_{z, t}\right\|<1$ when $q_{0}$ is small, and therefore equation (16) is uniquely solvable for all $z \in \mathbb{C}, t \geq 0$. Then $(q, \phi)$ given by (17) with $D=\emptyset$ is a smooth global solution of problem (1) with the small initial data. We will call this solution classical. It exists under a weaker assumption on the decay of $q_{0}$ than in Theorem 2.1.

[^2]We will consider analytic continuations of functions $h_{0}, q_{0}$, and we need some notation. Let $\gamma=\left(\gamma_{1}, \gamma_{2}\right) \in \mathbb{C}^{2}$ and let $A_{\gamma}: \mathbb{C} \rightarrow \mathbb{C}^{2}$ be the map defined by $A_{\gamma} z=A_{\gamma}(x+i y)=$ $\left(x+\gamma_{1}, y+\gamma_{2}\right) \in \mathbb{C}^{2}$, i.e., the map $A_{\gamma}$ shifts real points $x, y$ into complex planes. If a function $f=f(z)$ is analytic in $(x, y)$, then $B_{\gamma} f(z):=f\left(A_{\gamma} z\right)$ is the value of the analytic continuation of $f$ at point $A_{\gamma} z$. We will use notation $A_{\sigma}^{\prime}, B_{\sigma}^{\prime}, \sigma \in \mathbb{C}^{2}$, for the same operations applied to a function of $k \in \mathbb{C}$, and $A_{\eta}^{\prime \prime}, B_{\eta}^{\prime \prime}, \eta \in \mathbb{C}^{2}$, if they are applied to a function of $\varsigma \in \mathbb{C}$.

The main result of the paper is obtained under the following condition on the initial data that must hold for large enough $R$ :

Condition $Q(R)$. The initial data $q_{0}$ admits analytic continuation in $(x, y)$ and, for a given $R>0$, there exist a $C=C(R)$ such that

$$
\left|B_{\gamma} q_{0}(z)\right| \leq C e^{-R|z|}, \quad z \in \mathbb{C}, \quad \text { when } \quad|\gamma| \leq R
$$

Remark 2.2. Clearly, linear combinations of Gaussian functions satisfy Condition $Q(R)$ for all $R>0$, and it was shown in [4, 5] that these combinations form a dense set in $L^{2}(\mathbb{C})$.

We will show that Condition $Q(R)$ implies a similar behavior of the scattering data, i.e., the validity of the following assumption.

Condition $H(R)$. For a given $R>0$, the estimate

$$
\left|h_{0}(\varsigma, \varsigma)\right| \leq e^{-R|\varsigma|} \quad \text { as } \quad|\varsigma| \rightarrow \infty
$$

holds, and there exists $C=C(R)$ and $a_{0}(R)$ such that the scattering data $h_{0}(\varsigma, \varsigma)$ for the potential $a q_{0}$ with $a \in\left(0, a_{0}\right)$ admits analytic continuation $B_{\eta}^{\prime \prime} h_{0}(\varsigma, \varsigma),|\eta| \leq R$, with respect to variables $\varsigma_{i}$, and

$$
\left|B_{\eta}^{\prime \prime} h_{0}(\varsigma, \varsigma)\right| \leq C(R) e^{-R|\varsigma|}, \quad \varsigma \in \mathbb{C}, \quad \text { when } \quad|\eta| \leq R .
$$

We will show that there is a duality between these two conditions. To be more exact, the validity of $Q(R)$ implies the validity of $H(R-\varepsilon)$. Conversely, if the initial data is small, Condition $H(R)$ holds, and $h_{0}$ is extended in time according to (13), then the potential $q(z, t)$ that corresponds to the extended data $h(\varsigma, \varsigma, t)$ satisfies Condition $Q(R)$ with a smaller $R$ that depends on $t$. These results will be obtained in the next section. Note that they are an analogue of similar results of L. Sung ([20, Cor. 4.16]) who established a duality of the non-linear Fourier transform in the Schwartz class. We need a refined result to study the more complicated form (14) of operator $T_{z, t}$ that appears in the presence of exceptional points.

Below is the main statement of the present paper.
Theorem 2.3. Let us fix an arbitrary disk $D$ containing all the exceptional points for the potentials aq$q_{0}(z), a \in[0,1]$. Let Condition $Q(R)$ hold with $R>(1+2 T) A$, where $A$ is the radius of the disk $D$. Then

1) for each point $(z, t), 0 \leq t \leq T$, the classical solution $(q, \phi)$ of problem (1) with the initial data aq and small enough $a>0$ admits a meromorphic continuation with respect to $a$ in a neighborhood of the segment $[0,1]$. This meromorphic continuation is given by (17) with an arbitrary choice of the disk $D$ and an arbitrary choice of point $k_{0} \in \partial D^{4}$.
2) when $a=1$, the analytic continuation of $(q, \phi)$ is infinitely smooth and satisfies (11) everywhere, except possibly a set $S$ that is bounded in the strip $0 \leq t \leq T,(x, y) \in R^{2}$, and is such that $S_{t}=S \bigcap\{t=$ const $\}$ is a bounded $1 D$ real analytic variety.

Remark. The theorem implies that the local solutions found in Theorem 2.1 are analytic continuations in $a$ of the global classical solutions (under the assumption that condition $Q(R)$ holds). At the same time, the theorem does not prohibit the solution from blowing up at an arbitrarily small time $t>0$ (see the recent paper [10] and citations there on instantaneous blow-ups). We can't say anything about relation between our global solution and local solutions found in [8].

Two important technical improvements of the previous results will be used in the proof of Theorem 2.3, First, we will show that the Hilbert space $\mathcal{B}^{2}$ can be used in Theorem 2.1 instead of the Banach space $\mathcal{B}^{s}, s>2$. The space $\mathcal{B}^{2}$ is defined as follows:

$$
\begin{equation*}
\mathcal{B}^{2}=\left\{u \in\left(L^{2}(\mathbb{C} \backslash D) \oplus \mathbb{C}^{1}\right) \bigcap L_{+}^{2}(\partial D)\right\} . \tag{18}
\end{equation*}
$$

Here $\mathbb{C}^{1}$ is the one-dimensional space of functions of the form $\frac{c \beta(k)}{k}$, where $c$ is a complex constant, $\beta \in C^{\infty}$ is a fixed function that vanishes in a neighbourhood of the disk $D$ and equals one in a neighbourhood of infinity. By $L_{+}^{2}(\partial D)$ we denote the space of analytic functions $u=\sum_{n \geq 0} c_{n} z^{n}$ in $D$ with the boundary values in $L^{2}(\partial D)$ and the norm

$$
\|u\|_{L_{+}^{2}(\partial D)}=\left(\sum_{n \geq 0} A^{2 n}\left|c_{n}\right|^{2}\right)^{1 / 2}
$$

where $A$ is the radius of the disk $D$.
Secondly, we will simplify the form of the operator $T_{z, t}$ by writing the second term in (12) and (14) without the logarithmic factor. We also will allow $k_{0}$ to be on $\partial D$, and not necessarily in $D$, and show that formula (14) in the latter case can be written as

$$
\begin{array}{r}
T_{z, t} \phi(k)=\frac{1}{\pi} \int_{\mathbb{C} \backslash D} e^{i(\varsigma z+\bar{z} \varsigma) / 2} \bar{\phi}(\varsigma) \Pi^{o} h(\varsigma, \varsigma, t) \frac{d \sigma_{\varsigma}}{\varsigma-k}- \\
i \int_{\partial D} \frac{d \varsigma}{\varsigma-k} \int_{\widehat{k_{0}, \varsigma}}\left[e^{i\left(\varsigma \bar{z}+\bar{\varsigma}^{\top} z\right) / 2} \overline{\phi^{-}\left(\varsigma^{\prime}\right)} \Pi^{o}+e^{i\left(\varsigma-\varsigma^{\prime}\right) \bar{z} / 2} \phi^{-}\left(\varsigma^{\prime}\right) \Pi^{d} \mathbf{C}\right]\left[h\left(\varsigma^{\prime}, \varsigma, t\right) \overline{d \varsigma^{\prime}}\right] \tag{19}
\end{array}
$$

where $\widehat{k_{0}, \varsigma}$ is the arc on $\partial D$ between points $k_{0}$ and $\varsigma$ with the counter clock-wise direction on it.

[^3]The following two difficulties were resolved in the paper. We show that if one starts with a small potential $q_{0}$ and its scattering data $h_{0}(\varsigma, \varsigma)$, and extends $h_{0}(\varsigma, \varsigma)$ in time according to (13), then the solution $q(z, t)$ of the inverse scattering problem with the scattering data (13) decays exponentially at infinity, and the scattering data (8) for this potential $q(z, t)$ coincides with the scattering data $h(\varsigma, \varsigma, t)$ from which the potential was obtained (this will be done in the next section). Another difficulty concerns the proof of the invertibility of operator $I+T_{z, t}$ for large $|z|$ in spite of the exponential growth of the integrands in the second terms of (14) and (19) as $|z| \rightarrow \infty$ (see section 5).

## 3 Exponential decay of the scattering data and of $q(z, t)$

Lemma 3.1. Let

$$
I(z)=\int_{\mathbb{C}} \frac{f\left(z_{1}\right)}{z-z_{1}} d \sigma_{z_{1}}, \quad J(z)=\int_{\mathbb{C}} \frac{f\left(z_{1}\right)}{\bar{z}-\bar{z}_{1}} d \sigma_{z_{1}}, \quad z \in \mathbb{C}
$$

where $f(z)$ is analytic in $(x, y)$, and

$$
\left|f\left(A_{\gamma} z\right)\right|,\left|\nabla_{\gamma} f\left(A_{\gamma} z\right)\right| \leq \frac{C(\gamma)}{1+x^{2}+y^{2}}
$$

Then $I(z), J(z)$ admit analytic continuation in $(x, y)$, and

$$
B_{\gamma} I(z)=\int_{\mathbb{C}} \frac{f\left(A_{\gamma} z_{1}\right)}{z-z_{1}} d \sigma_{z_{1}}, \quad B_{\gamma} J(z)=\int_{\mathbb{C}} \frac{f\left(A_{\gamma} z_{1}\right)}{\bar{z}-\bar{z}_{1}} d \sigma_{z_{1}} .
$$

Proof. Let us rewrite $I(z)$ in the form

$$
I(z)=-\int_{\mathbb{C}} \frac{f\left(z+z_{1}\right)}{z_{1}} d \sigma_{z_{1}} .
$$

This immediately implies that $I(z)$ is analytic in $(x, y)$, and

$$
B_{\gamma} I(z)=\int_{\mathbb{C}} \frac{B_{\gamma} f\left(z+z_{1}\right)}{-z_{1}} d \sigma_{z_{1}}=\int_{\mathbb{C}} \frac{f\left(x+\gamma_{1}+x_{1}, y+\gamma_{2}+y_{2}\right)}{-z_{1}} d \sigma_{z_{1}}=\int_{\mathbb{C}} \frac{f\left(A_{\gamma} z_{1}\right)}{z-z_{1}} d \sigma_{z_{1}} .
$$

The statement for $J$ can be proved absolutely similarly.
Let us provide some examples of analytic continuations of functions from $\mathbb{C}$ into $\mathbb{C}^{2}$ : (1) If $f(z)=z=x+i y$, then $B_{\gamma} f(z)=x+\gamma_{1}+i\left(y+\gamma_{2}\right)=z+\gamma^{\prime}, \gamma^{\prime}=\gamma_{1}+i \gamma_{2} \in \mathbb{C}$. (2) If $f(z)=\bar{z}=x-i y$, then $B_{\gamma} f(z)=x+\gamma_{1}-i\left(y+\gamma_{2}\right)=\bar{z}+\gamma^{\prime \prime}$, $\gamma^{\prime \prime}=\gamma_{1}-i \gamma_{2} \in \mathbb{C}$ (note that $\gamma^{\prime} \neq \overline{\gamma^{\prime \prime}}$ since $\gamma_{i}$ are complex.) (3) if $f(z)=\Re(k \bar{z})=k_{1} x+k_{2} y$, then $B_{\gamma} f(z)=$ $\Re(k \bar{z})+k_{1} \gamma_{1}+k_{2} \gamma_{2}$ and $B_{\sigma} \Re(k \bar{z})=\Re(k \bar{z})+\sigma_{1} x+\sigma_{2} y$.

Lemma 3.2. Let the potential be aq$q_{0}(z)$ where $q_{0}$ satisfies Condition $Q(R)$ for some $R>0$. Then there exists $a_{0}=a_{0}(R)$ such that function $\bar{\mu}=\bar{\mu}(z, k)$ defined by (4) via the solution of the Lippmann-Schwinger equation with the potential aq$q_{0}, a \in\left(0, a_{0}\right)$, admits analytic continuation to $\mathbb{C}^{4}$ with respect to variables $x, y, k_{1}, k_{2}$, and

$$
\begin{equation*}
\left|B_{\sigma}^{\prime} B_{\gamma} \bar{\mu}(z, k)\right|<C(R, \varepsilon) \quad \text { when } \quad z, k \in \mathbb{C}, \quad|\sigma|,|\gamma| \leq R-\varepsilon, \quad a<a_{0} \tag{20}
\end{equation*}
$$

The statement remains valid if $a=1$, but $|k| \geq \rho(R)$ with large enough $\rho$.
Proof. We will prove the statement of the lemma for the component $\mu_{11}$ of the matrix $\mu$. Other components can be treated similarly. Let us iterate equation (6). The following equation is valid for the first component:

$$
\begin{equation*}
\overline{\mu_{11}}=1+\frac{1}{\pi^{2}} \int_{\mathbb{C}} d \sigma_{z_{1}} \int_{\mathbb{C}} d \sigma_{z_{2}} \frac{e^{i \Re\left(k \bar{z}_{1}\right)}}{\bar{z}-\bar{z}_{1}} \bar{Q}_{12}\left(z_{1}\right) \frac{e^{-i \Re\left(k \bar{z}_{2}\right)}}{z_{1}-z_{2}} Q_{21}\left(z_{2}\right) \overline{\mu_{11}}\left(z_{2}, k\right), \tag{21}
\end{equation*}
$$

where $Q_{21}$ and $Q_{12}$ are entries of the matrix $Q_{0}$. Denote $Q=Q_{12}=-Q_{21}$. Assume that the analytic continuation $B_{\gamma} \bar{\mu}_{11}$ exists. Then from Lemma 3.1 and (21) it follows that

$$
\begin{gathered}
B_{\gamma} \bar{\mu}_{11}=1-\frac{1}{\pi^{2}} \int_{\mathbb{C}} d \sigma_{z_{1}} \frac{B_{\gamma} e^{i \Re\left(k \bar{z}_{1}\right)}}{\bar{z}-\overline{z_{1}}} B_{\gamma} \bar{Q}\left(z_{1}\right) B_{\gamma} \int_{\mathbb{C}} d \sigma_{z_{2}} \frac{e^{-i \Re\left(k \bar{z}_{2}\right)}}{z_{1}-z_{2}} Q\left(z_{2}\right) \overline{\mu_{11}}\left(z_{2}, k\right) \\
=1-\frac{1}{\pi^{2}} \int_{\mathbb{C}} d \sigma_{z_{1}} \frac{e^{i \Re\left(k \bar{z}_{1}\right)+i<k, \gamma>}}{\bar{z}-\overline{z_{1}}} B_{\gamma} \bar{Q}\left(z_{1}\right) \int_{\mathbb{C}} \frac{e^{-i \Re\left(k \bar{z}_{2}\right)-i<k, \gamma>}}{z_{1}-z_{2}} B_{\gamma} Q\left(z_{2}\right) B_{\gamma} \overline{\mu_{11}}\left(z_{2}, k\right) d \sigma_{z_{2}} \\
=1-\frac{1}{\pi^{2}} \int_{\mathbb{C}} d \sigma_{z_{1}} \frac{e^{i \Re\left(k \bar{z}_{1}\right)}}{\overline{z-\overline{z_{1}}}} B_{\gamma} \bar{Q}\left(z_{1}\right) \int_{\mathbb{C}} \frac{e^{-i \Re\left(k \bar{z}_{2}\right)}}{z_{1}-z_{2}} B_{\gamma} Q\left(z_{2}\right) B_{\gamma} \overline{\mu_{11}}\left(z_{2}, k\right) d \sigma_{z_{2}} .
\end{gathered}
$$

Hence, if the analytic continuation $\Psi:=B_{\sigma}^{\prime} B_{\gamma} \bar{\mu}_{11}$ exists, then it satisfies the equation

$$
\Psi(z, k)=1-\frac{1}{\pi^{2}} \int_{\mathbb{C}} d \sigma_{z_{1}} \frac{e^{i \Re\left(k \bar{z}_{1}\right)+i<\sigma, z_{1}>}}{\bar{z}-\overline{z_{1}}} B_{\gamma} \bar{Q}\left(z_{1}\right) \int_{\mathbb{C}} \frac{e^{-i \Re\left(k \bar{z}_{2}\right)-i<\sigma, z_{2}>}}{z_{1}-z_{2}} B_{\gamma} Q\left(z_{2}\right) \Psi\left(z_{2}, k\right) d \sigma_{z_{2}} .
$$

Denote by $K^{ \pm}=K_{k, \sigma, \gamma}^{ \pm}$the integral operators given by the exterior and interior integrals above, respectively. Their norms in the space $L^{\infty}(\mathbb{C})$ can be estimated from above by the norms of the potential (see [19]):

$$
\left\|K^{-}\right\|<C\left(\left\|e^{-i<\sigma, z_{2}>} B_{\gamma} Q\left(z_{2}\right)\right\|_{L^{p}(\mathbb{C})}+\left\|e^{-i<\sigma, z_{2}>} B_{\gamma} Q\left(z_{2}\right)\right\|_{L^{q}(\mathbb{C})}\right), \quad 1<p<2<q<\infty
$$

and a similar estimate is valid for $K^{+}$. Thus the assumption $a_{0} \ll 1$ and Condition Q imply that $\left\|K^{ \pm}\right\|<1, \Psi$ exists, and

$$
|\Psi|<C(R) \quad \text { when } \quad|\Im \sigma|,|\gamma| \leq R, a<a_{0} .
$$

Moreover, the derivatives of $K^{ \pm}$with respect to complex variables $\sigma_{i}, \gamma_{j}$ also have small norms, i.e., $\Psi=\Psi(z, k, \sigma, \gamma)$ is analytic in $\left(\sigma_{1}, \sigma_{2}, \gamma_{1}, \gamma_{2}\right)$. One can easily see that $\Psi=$ $\Psi(z+\gamma, k+\sigma)$. Hence $\Psi$ is the analytic continuation of $\bar{\mu}$. The proof of (20) is complete.

In order to prove the statement of Lemma 3.2 concerning $a=1$, one needs only to show that operator $K:=K^{+} K^{-}$and its derivatives in $\sigma_{i}, \gamma_{j}$ are small (less than one) as $|k| \rightarrow \infty$. This can be done by a standard procedure: one splits $K$ into two terms $K=K_{1}+K_{2}$, where $K_{1}$ is obtained by adding the factor $\alpha\left(\frac{z_{1}-z}{\varepsilon}\right) \alpha\left(\frac{z_{1}-z_{2}}{\varepsilon}\right)$ in the integral kernel of $K$. Here $\alpha=\alpha(z)$ is a cut-off function that is equal to one when $|z|<1$ and vanishes when $|z|>2$. Then $\left\|K_{1}\right\| \rightarrow 0$ as $\varepsilon \rightarrow 0$, and $\left\|K_{2}\right\|=O\left(|k|^{-1}\right)$ as $|k| \rightarrow \infty$. The latter can be shown by appropriate integration by parts in $x_{1}, y_{1}$.

Theorem 3.3. If Condition $Q(R)$ holds for some $R>0$, then Condition $H(R-\varepsilon)$ holds for each $\varepsilon>0$.

Proof. Recall that

$$
h_{0}(\varsigma, \varsigma)=\frac{1}{(2 \pi)^{2}} \int_{\mathbb{C}} e^{-i \Re(\varsigma z)} Q_{0}(z) \bar{\mu}(z, \varsigma) d \sigma_{z}, k, \varsigma \in \mathbb{C} .
$$

We shift the complex plane $\mathbb{C}$ in the integral above by vector $\gamma=-i \frac{\left(\varsigma_{1}, \varsigma_{2}\right)}{|\varsigma|}(R-\varepsilon)$, and then apply operator $B_{\eta}$. This leads to

$$
\left|B_{\eta}^{\prime \prime} h_{0}\right|<\frac{1}{(2 \pi)^{2}} \int_{\mathbb{C}}\left|e^{-i(\langle\eta, z>+<\varsigma, \gamma>+<\eta, \gamma>)} Q_{0}\left(A_{\gamma} z\right) B_{\gamma} B_{\eta}^{\prime \prime} \bar{\mu}(z, \varsigma)\right| d \sigma_{z} .
$$

It remains to use Lemma 3.2 and Condition $Q(R)$.

Theorem 3.4. Let Condition $Q(R)$ hold for some $R>0$ and let the scattering data $h_{0}$ be defined by the potential aq $q_{0}, 0<a<a_{0}(R)$, where $a_{0}(R)$ is defined in Lemma 3.2. Then the time dependent scattering data $h(\varsigma, \varsigma, t), 0 \leq t \leq T$, given by (13), admits an analytic continuation in $\left(\varsigma_{1}, \varsigma_{2}\right)$, and

$$
\left|B_{\eta}^{\prime \prime} h(\varsigma, \varsigma, t)\right| \leq C(R, \varepsilon) e^{\left(-\frac{R}{1+2 T}+\varepsilon\right)|\varsigma|}, \quad|\eta| \leq \frac{R}{1+2 T}
$$

The statement remains valid if $a=1$, but $|\varsigma|>\rho$, where $\rho=\rho(R)$ is large enough.
Proof. The statement follows immediately from Theorem 3.3 and formula (13). One needs only to combine the upper bound $C e^{(-R+\varepsilon)|\varsigma|}$ for the analytic continuation of $h_{0}$ obtained in Theorem 3.3 with the upper bound $C e^{\frac{2 T R}{1+2 T}|\varsigma|}$ for the time-dependent factor in (13).

Let us recall again the procedure to obtain the classical solution of the focusing DSII equation with initial data $a q_{0}$ and a very small $a$ such that there are no exceptional points. As the first step, one needs to solve the equation $\left(I+T_{z, t}\right) v=I$, where $T_{z, t}$ is given by (14) with $D=\emptyset$, i.e., the equation for $v=v_{z, t}$ has the form

$$
\begin{equation*}
v_{z, t}(k)+\frac{1}{\pi} \int_{\mathbb{C}} e^{i(\bar{\varsigma}+\overline{\bar{\varsigma}} \varsigma) / 2} \overline{v_{z, t}}(\varsigma) \Pi^{o} h(\varsigma, \varsigma, t) \frac{d \sigma_{\varsigma}}{\varsigma-k}=I, \quad w_{z, t}(\cdot)=v_{z, t}(\cdot)-I \in \mathcal{B}^{s} \tag{22}
\end{equation*}
$$

Then the solution of the focusing DSII equation with initial data $a q_{0}$ is given by (17). In particular,

$$
\begin{equation*}
q(z, t)=\frac{1}{2 \pi i} \int_{\mathbb{C}} e^{i(\varsigma z+\bar{z} \varsigma) / 2}\left(\overline{v_{z, t}}\right)_{11}(\varsigma) h_{12}(\varsigma, \varsigma, t) d \sigma_{\varsigma} \tag{23}
\end{equation*}
$$

Theorem 3.5. Let Condition $Q(R)$ hold for $q_{0}$, and let the potential $q(z, t), 0 \leq t \leq T$, in (23) be constructed from the initial data $a q_{0}(z)$ with $0<a<a_{1} \ll 1$. Then there exists $a_{1}=a_{1}(R, T)$ such that Condition $Q\left(\frac{R}{1+2 T}-\varepsilon\right)$ holds for the potential (23) for all $t \in[0, T]$.

Proof. There is a complete duality (e.g. [20, Th. 4.15]) between the nonlinear Fourier transform given by (6), (8) and the inverse transform (22), (23). Function $h$ in (22) plays the role of the potential $Q_{0}$ in (6). Theorem 3.4 implies that the Condition $Q\left(R^{\prime}\right)$ holds for $h$ with $R^{\prime}=\frac{R}{1+2 T}-\frac{\varepsilon}{2}$. From Lemma 3.2 applied to (22) instead of (6), it follows that $v$ has the same properties as the properties of $\mu$ established in Lemma 3.2. One needs only to take $a$ small enough to guarantee that the analogues of operators $K^{ \pm}$have norms that do not exceed one. Then

$$
\left|B_{\sigma}^{\prime} B_{\gamma} \bar{v}(z, k)\right|<C\left(R^{\prime}, \varepsilon\right) \quad \text { when } \quad z, k \in \mathbb{C}, \quad|\sigma|,|\gamma| \leq R^{\prime}-\frac{\varepsilon}{2}, \quad a \ll 1
$$

Then the statement of the theorem can be obtained similarly to the proof of Theorem 3.3, i.e., by using the shift of the complex plane $\mathcal{C}$ in (23) by the vector $\eta=i \frac{(x, y)}{|z|}\left(R^{\prime}-\frac{\varepsilon}{2}\right)$.

## 4 Proof of the first statement of Theorem 2.3

Consider problem (11) with $q_{0}$ replaced by $a q_{0}, a \in(0,1]$. Let $D$ be a disk containing all the exceptional points for problems (3), (4) for all $a \in(0,1]$. Let $k_{0} \in \partial D$ be a nonexceptional point for all $a \in(0,1]$. We will use notation $v^{1}$ for the solution of (16) and $\left(q^{1}, \varphi^{1}\right)$ for the pair defined by (17) when the operator $T_{z, t}$ is defined using the disk $D$. We preserve the notations $v,(q, \varphi)$ for the same objects when there are no exceptional points and $D=\emptyset$. Since $q^{1}, \varphi^{1}$ are meromorphic in $a$ in a neighbourhood of $(0,1]$ (see Theorem 2.1), the first statement of Theorem 2.3 will be proved if we show that $\left(q^{1}, \varphi^{1}\right)=(q, \varphi)$ when $a>0$ is small and $t>0$.

From (51), (8) and Condition $Q(R)$ with $R>(1+2 T) A>A$, it follows that the scattering data $h_{0}=h_{0}(\varsigma, k)$ is defined for all the potentials $a q_{0}$ when $|\varsigma|,|k| \leq A$ (i.e., $\varsigma, k \in \bar{D}$ ) and also for all $\varsigma=k$. We define $h(\varsigma, k, t)$ (extension of $h_{0}$ in $t$ ) according to (13). Let $v=v_{z, t}=w_{z, t}+I$, where $w_{z, t} \in \mathcal{B}^{s}, s>2$, is the solution of (16) with $T_{z, t}$ given by (14) with $D=\emptyset$ (i.e., the right-hand side in (14) contains only the first term, see equation (22)). Then ( $q, \phi$ ) given by (17) with $D=\emptyset$ solves the DSII equation (1) (see [7]), and

$$
\begin{equation*}
\psi=\psi(z, k, t):=\Pi^{d} \bar{v} e^{i \bar{k} z / 2}+e^{-i \bar{z} k / 2} \Pi^{o} v, \varsigma, k \in \mathbb{C}, t \geq 0 \tag{24}
\end{equation*}
$$

is the solution of the scattering problem (3) (and the Lippmann-Schwinger equation (4)) with the potential $Q_{t}(z)=\left(\begin{array}{cc}0 & q(z, t) \\ -q(z, t) & 0\end{array}\right)$ instead of $Q_{0}$.

Consider now the scattering data

$$
\begin{equation*}
\widehat{h}(\varsigma, k, t):=\frac{1}{(2 \pi)^{2}} \int_{\mathbb{C}} e^{-i \bar{\varsigma} z / 2} Q_{t}(z) \bar{\psi}(z, k, t) d \sigma_{z} \tag{25}
\end{equation*}
$$

defined by the solution $\psi$ of the Lippmann-Schwinger equation (4) with the potential $Q_{t}(z)$. If $0 \leq t \leq T$, then from Theorem 3.5 (it is assumed there that $R>(1+2 T) A$ ) it follows that integral (25) converges when $|\varsigma|,|k| \leq A$ (i.e., $\varsigma, k \in \bar{D})$ and when $\varsigma=k$. Moreover, $\widehat{h}(\varsigma, k, t)=\widehat{h}(k+\alpha, k, t)$ is an anti-analytic continuation of $\widehat{h}(k, k, t)$ in $\alpha$. We will prove that $\widehat{h}$ coincides with the scattering data $h(\varsigma, k, t)$ defined in (13). We also will prove that there exists an analytic in $k$ function $\widehat{v}_{1}^{+}=\widehat{v}_{1}^{+}(k, t), k \in D$, such that

$$
\begin{equation*}
\left.\left(v-\widehat{v}_{1}^{+}\right)\right|_{\varsigma \in \partial D}=\int_{\partial D}\left[e^{i / 2\left(\varsigma \bar{z}+\overline{\varsigma^{\prime}} z\right)} \overline{\widehat{v}_{1}^{+}\left(\varsigma^{\prime}\right)} \Pi^{o}-e^{i / 2\left(\varsigma-\varsigma^{\prime}\right) \bar{z}} \widehat{v}_{1}^{+}\left(\varsigma^{\prime}\right) \Pi^{d} \mathbf{C}\right]\left[\operatorname{Ln} \frac{\overline{\varsigma^{\prime}}-\bar{\varsigma}}{\overline{\varsigma^{\prime}}-\overline{k_{0}^{1}}} \widehat{h}_{t}\left(\varsigma^{\prime}, \varsigma\right) d \varsigma^{\prime}\right] . \tag{26}
\end{equation*}
$$

From these two facts and the $\bar{\partial}$-equation (see [19])

$$
\begin{equation*}
\frac{\partial}{\partial \bar{k}} v(z, k, t)=e^{i(\bar{k} z+\bar{z} k) / 2} \bar{v}(z, k, t) \Pi^{o} h(k, k, t), \quad k \in \mathbb{C} \backslash D, \tag{27}
\end{equation*}
$$

it follows (see [13, Lemma 3.3]) that the function

$$
v^{\prime}(z, k):=\left\{\begin{array}{lc}
v(z, k), & k \in \mathbb{C} \backslash D  \tag{28}\\
\widehat{v}_{1}^{+}(z, k), & k \in D
\end{array}\right.
$$

satisfies the integral equation (16), where operator $T_{z, t}$ is constructed using the scattering data $\widehat{h}$. Equation (16) has a unique solution when $a$ is small enough. Under the assumption that $\widehat{h}=h$, we have $v^{1} \equiv v^{\prime}$. Therefore $v^{1}(z, k)=v(z, k)$ when $k \in \mathbb{C} \backslash D$. Solution $(q, \phi)$ of the DSII equation can be determined via the asymptotics of $v$ at large values of $k$ (e.g., [19, (1.17)], [14, Lemma 3.3]). Hence $\left(q^{1}, \phi^{1}\right)=(q, \phi)$ for small $a$. Thus the first statement of the theorem will be proved as soon as we show that $\widehat{h}=h, t>0$, and that $\widehat{v}^{+}$exists.

Justification of the equality $\widehat{h}=h, t>0$. Everywhere below, till the end of the section, we omit mentioning the parameter $a$ and assume that the initial data $q_{0}$ is small. Let us recall (see [14, Lemmas 4.1, 4.2]) that the symmetry of the matrix $Q_{0}$ (see (24)) implies that $h_{11}=h_{22}, h_{12}=-h_{21}$, and the same relations hold for matrix $v$ determined from the integral equation (16) and related to $\psi$ by (24).

Let us introduce functions

$$
\left(\begin{array}{cc}
a & b \\
-b & a
\end{array}\right)=h_{0}(k+\alpha, k),
$$

and note that

$$
\begin{aligned}
& b(\alpha, k)= \frac{1}{(2 \pi)^{2}} \int_{\mathbb{C}} e^{-\bar{\alpha} z / 2} e^{-i(k \bar{z}+\bar{k} z) / 2} q_{0}(z) v_{11}(z, k) d \sigma_{z} \\
& a(\alpha, k)=\frac{1}{(2 \pi)^{2}} \int_{\mathbb{C}} e^{-\bar{\alpha} z / 2} q_{0}(z) \bar{v}_{12}(z, k) d \sigma_{z}
\end{aligned}
$$

Now define

$$
\left(\begin{array}{cc}
a(\alpha, k, t) & b(\alpha, k, t) \\
-b(\alpha, k, t) & a(\alpha, k, t)
\end{array}\right)=h(k+\alpha, k, t)
$$

where $h$ is given by (13). Similar quantities $\widehat{a}, \widehat{b}$ are defined via the solutions $v(\cdot, k, t)$ :

$$
\begin{aligned}
\widehat{b}(\alpha, k, t):= & \frac{1}{(2 \pi)^{2}} \int_{\mathbb{C}} e^{-\bar{\alpha} z / 2} e^{-i(k \bar{z}+\bar{k} z) / 2} q(z, t) v_{11}(z, k, t) d \sigma_{z}, \\
& \widehat{a}(\alpha, k, t):=\frac{1}{(2 \pi)^{2}} \int_{\mathbb{C}} e^{-\bar{\alpha} z / 2} q(z, t) \bar{v}_{12}(z, k, t) d \sigma_{z} .
\end{aligned}
$$

These quantities are well defined due to Theorem 3.5, Let

$$
\widehat{h}=\left(\begin{array}{cc}
\widehat{a}(\alpha, k, t) & \widehat{b}(\alpha, k, t) \\
-\widehat{b}(\alpha, k, t) & \widehat{a}(\alpha, k, t)
\end{array}\right) .
$$

Consider solution $\psi(z, k, t)$ of (3) with potential $Q_{0}$ replaced by $Q_{t}$, and let $v$ be defined by (24). From (24) it follows that $v \rightarrow I$ uniformly on each compact with respect to the variable $z$ when $k \rightarrow \infty$. Therefore, from Theorem 3.5 it follows that

$$
\begin{equation*}
\widehat{a}(\alpha, k, t) \rightarrow 0, k \rightarrow \infty . \tag{29}
\end{equation*}
$$

Obviously (see (13)), the same relation holds for $a(\alpha, k, t)$.
The $\bar{\partial}$-equation (27) implies that the following rules are valid when $t=0$ :

$$
\begin{equation*}
\frac{\partial b}{\partial \bar{k}}=\frac{\partial b}{\partial \bar{\alpha}}+a b_{0}, \quad \frac{\partial a}{\partial k}=b \overline{b_{0}}, \quad \text { where } \quad b_{0}=b(0, k), \quad|\alpha| \leq A, \quad k \in \mathbb{C} . \tag{30}
\end{equation*}
$$

Due to Theorem 3.5, the same relations are valid for $\widehat{a}(\alpha, k, t), \widehat{b}(\alpha, k, t)$ :

$$
\begin{equation*}
\frac{\partial \widehat{b}}{\partial \bar{k}}=\frac{\partial \widehat{b}}{\partial \bar{\alpha}}+\widehat{a} \widehat{b}_{0}, \quad \frac{\partial \widehat{a}}{\partial k}=\widehat{\widehat{b}_{0}}, \quad \widehat{b}_{0}=\widehat{b}(0, k, t), \quad|\alpha| \leq A, \quad k \in \mathbb{C}, \quad 0 \leq t \leq T \tag{31}
\end{equation*}
$$

From (13), (30), and the obvious relations

$$
e^{-t\left(\bar{k}^{2}-(\overline{k+\alpha})^{2}\right) / 2}=e^{-t\left(k^{2}-(\overline{k+\alpha})^{2}\right) / 2} \overline{e^{-i \Im k^{2}}}, \quad \frac{\partial}{\partial \bar{\alpha}} e^{-t\left(k^{2}-(\overline{k+\alpha})^{2}\right) / 2}=\frac{\partial}{\partial \bar{k}} e^{-t\left(k^{2}-(\overline{k+\alpha})^{2}\right) / 2}
$$

it follows that (31) holds for $a(\alpha, k, t), b(\alpha, k, t)$ :

$$
\begin{equation*}
\frac{\partial b}{\partial \bar{k}}=\frac{\partial b}{\partial \bar{\alpha}}+a b_{0}, \quad \frac{\partial a}{\partial k}=b \overline{b_{0}}, \quad b_{0}=b(0, k, t), \quad|\alpha| \leq A, \quad k \in \mathbb{C}, \quad 0 \leq t \leq T \tag{32}
\end{equation*}
$$

Now we note that $\widehat{b}(0, k, t)=b(0, k, t)$ (see [21, Theorem 5.3]). The second relations in (31), (32) with $\alpha=0$ imply that $\left.(\widehat{a}-a)\right|_{\alpha=0}$ is anti-analytic in $k$. Then the maximum principle, together with (29) for both $\widehat{a}$ and $a$, imply that $\left.\widehat{a}\right|_{\alpha=0}=\left.a\right|_{\alpha=0}$. Now from the first relations in (31), (32), with $\alpha=0$, it follows that $\left.\frac{\partial b}{\partial \bar{\alpha}}\right|_{\alpha=0}=\left.\frac{\partial b}{\partial \bar{\alpha}}\right|_{\alpha=0}$. Then we differentiate the second relations in (31), (32) in $\bar{\alpha}$ and put $\alpha=0$ there. This leads to the anti-analyticity in $k$ of $\left.\frac{\partial \widehat{a}}{\partial \bar{\alpha}}\right|_{\alpha=0}-\left.\frac{\partial a}{\partial \bar{\alpha}}\right|_{\alpha=0}$. The maximum principle with (29) imply that $\left.\frac{\partial \widehat{a}}{\partial \bar{\alpha}}\right|_{\alpha=0}=\left.\frac{\partial a}{\partial \bar{\alpha}}\right|_{\alpha=0}$. After the differentiation in $\bar{\alpha}$ of the first relations in (31), (32), we obtain that $\left.\frac{\partial^{2} \widehat{b}}{\partial \bar{\alpha}^{2}}\right|_{\alpha=0}=\left.\frac{\partial b^{2}}{\partial \bar{\alpha}^{2}}\right|_{\alpha=0}$, and so on. Hence all the derivatives in $\bar{\alpha}$ of the vectors $(\widehat{a}, \widehat{b})$ and $(a, b)$ coincide when $\alpha=0$. Since both vectors are anti-analytic in $\alpha$, they are identical, i.e., $\widehat{h}=h, t>0$.

The existence of $\widehat{v}^{+}$can be shown similarly to the proof of same statement in [13], where the potential was assumed to be compactly supported. Namely, consider the following analogue of the Lippmann-Schwinger equation with different values $k_{0}, k \in \bar{D}$ of the spectral parameter in the operator and in the free term of the equation:

$$
\begin{equation*}
\psi^{+}(z, k)=e^{i \frac{\bar{k} z}{2}} I+\int_{z \in \mathbb{C}} G\left(z-z^{\prime}, k_{0}\right) Q_{t}\left(z^{\prime}\right) \overline{\psi^{+}}\left(z^{\prime}, k\right) d \sigma_{z^{\prime}}, \quad G(z, k)=\frac{1}{\pi} \frac{e^{i \bar{k} z / 2}}{z} \tag{33}
\end{equation*}
$$

We substitute here $\psi^{+}=\mu^{+} e^{i \overline{k_{0}} z / 2}$ and rewrite the equation in terms of

$$
\begin{equation*}
w^{+}=\mu^{+}(z, k)-e^{i\left(\overline{k-k_{0}}\right) z / 2} I \in L_{z}^{\infty}\left(L_{k}^{p}\right), p>1 \tag{34}
\end{equation*}
$$

The equation takes the form
$w^{+}(z, k)-\int_{z \in \mathbb{C}} \frac{e^{-i \Re\left(\overline{k_{0}} z^{\prime}\right)}}{z-z^{\prime}} Q_{t}\left(z^{\prime}\right) \overline{w^{+}}\left(z^{\prime}, k\right) d \sigma_{z^{\prime}}=\int_{z \in \mathbb{C}} \frac{e^{-i \Re\left(\overline{k_{0}} z^{\prime}\right)}}{z-z^{\prime}}\left[Q_{t}\left(z^{\prime}\right) e^{-i \overline{z^{\prime}}\left(k-k_{0}\right) / 2}\right] d \sigma_{z^{\prime}}$.
Theorem 3.5implies that function $\left[Q_{t}\left(z^{\prime}\right) e^{-i \overline{z^{\prime}}\left(k-k_{0}\right) / 2}\right]$ decays exponentially as $z \rightarrow \infty$, and $|k|,\left|k_{0}\right| \leq A$. The unique solvability of the problem (35) is obvious since the potential is small.

Function $\widehat{v}^{+}$is defined by $\psi^{+}$in the same way as $v$ is defined by $\psi$ in (24). The analyticity of $\hat{v}^{+}$and (26) are proved in Lemmas 3.1 and 3.5 of [13].

## 5 Proof of statement 2 of the Theorem 2.3.

Reduction to Theorem 5.2 and Lemma 5.3. Theorem 3.5 immediately implies that the operator $T_{z, t}: \mathcal{B}^{s} \rightarrow \mathcal{B}^{s}, s>2$, is analytic in $x$ and $y$ in a complex neighborhood of $\mathbb{R}^{2}$. In order to use the multidimensional analytic Fredholm theory ([23, Th. 4.11, 4.12] or [22]) and obtain a decay of operator norm $\left\|T_{z, t}^{2}\right\|$ as $|z| \rightarrow \infty$, we would like to consider this operator in the Hilbert space $\mathcal{B}^{2}$ instead of the Banach space $\mathcal{B}^{s}, s>2$. All
the previous and new results mentioned in this paper remain valid if $s>2$ is replaced by $s=2$ (with the appropriate definition of the space $\mathcal{B}^{2}$ given in (18)). In order to justify the latter statement, one needs to show that the properties of the operator $T_{z, t}$ are preserved when $s>2$ is replaced by $s=2$. This will be done in Theorem 5.2 below (we will not discuss the properties that obviously are $s$-independent), but we will show that operator $T_{z, t}: \mathcal{B}^{2} \rightarrow \mathcal{B}^{2}, 0 \leq t \leq T$, is compact, continuous in $(z, t)$, and analytic in $(x, y)$ in a complex neighborhood of $\mathbb{R}^{2}$. After that, we will show (Lemma (5.3) the invertibility of $I+T_{z, t}$ at large values of $|z|$. Then the second statement of the theorem will be a simple consequence of the first statement and the analytic Fredholm theory. Note that the invertibility of operator $I+T_{z, t}$ will be proved for $z$ on each ray $\arg z=\psi=$ const, $|z| \geq Z_{0}$, with $\psi$-independent $Z_{0}$ and with $T_{z, t}$ defined (see (14)) using a special value of $k_{0}=k_{0}(\psi)$. Since the solution $(q, \phi)$ of problem (1) does not depend on the choice of $k_{0}$ (see Theorem 2.3), it remains only to prove Theorem 5.2 and Lemma 5.3 ,

### 5.1 Compactness of operator $T$

We will need the following lemma.
Lemma 5.1. Let operator $M: \mathcal{B}^{2} \rightarrow \mathcal{B}^{2}$ have the form

$$
(M f)(k)=\int_{\mathbb{C} \backslash D} \frac{g(\varsigma)}{\varsigma-k} f(\varsigma) d \sigma_{\varsigma}, \quad k \in \mathbb{C}
$$

where function $g_{\delta}=g(\varsigma)(1+|\varsigma|)^{\delta}$ has the following properties

$$
\left|g_{\delta}\right|<a_{1}<\infty, g_{\delta} \rightarrow 0 \text { as } \varsigma \rightarrow \infty, \text { and }\left\|g_{\delta}\right\|_{L^{2}(\mathbb{C} \backslash D)}=a_{2}<\infty
$$

for some $\delta>0$. Then $M$ is compact and $\|M\| \leq C\left(a_{1}+a_{2}\right)$.
Proof. Let $P$ be the following operator in $\mathcal{B}^{2}$ of rank one:

$$
\begin{equation*}
P f=-\frac{\beta(k)}{k} \int_{\mathbb{C} \backslash D} g(\varsigma) f(\varsigma) d \sigma_{\varsigma}, \tag{36}
\end{equation*}
$$

where $\beta$ is the function introduced in the definition of the space $\mathcal{B}^{2}$. Since $\operatorname{Pf}=0$ in a neighborhood of $D$, and

$$
\int_{\mathbb{C} \backslash D} g(\varsigma) f(\varsigma) d \sigma_{\varsigma} \leq a_{2} \int_{\mathbb{C} \backslash D}\left|\frac{f(\varsigma)}{(1+|\varsigma|)^{\delta}}\right|^{2} d \sigma_{\varsigma} \leq C a_{2}\|f\|_{\mathcal{B}^{2}},
$$

it is enough to prove the statement of the lemma for operator $M-P=M_{1}+M_{2}$, where $M_{i} f=\int_{\mathbb{C} \backslash D} K_{i}(k, \varsigma) f(\varsigma) d \sigma_{\varsigma}, \quad K_{1}(k, \varsigma)=\frac{\alpha(\varsigma-k)}{\varsigma-k} g(\varsigma), \quad K_{2}(k, \varsigma)=\left[\frac{\beta(\varsigma-k)}{\varsigma-k}+\frac{\beta(k)}{k}\right] g(\varsigma)$, and $\alpha:=1-\beta$ is a cut-off function which is equal to one in a neighborhood of $D$.

Let $M_{i}^{\prime}$ be the operator defined by the same formulas as operators $M_{i}$, but considered as operators in $L^{2}(\mathbb{C})$. Let us show that operators $M_{i}^{\prime}$ are compact and their norms do not exceed $C\left(a_{1}+a_{2}\right)$.

Since $|g| \leq a_{1}$, we have

$$
\sup _{k \in \mathbb{C}} \int_{\mathbb{C}}\left|K_{1}(k, \varsigma)\right| d \sigma_{\varsigma}+\sup _{\varsigma \in \mathbb{C}} \int_{\mathbb{C}}\left|K_{1}(k, \varsigma)\right| d \sigma_{k} \leq C a_{1} .
$$

Hence, from the Young theorem, it follows that $\left\|M_{i}^{\prime}\right\| \leq C a_{1}$. Similarly, using the decay of $g_{1}$ at infinity, we obtain that $M_{1}^{\prime}=\lim _{R \rightarrow \infty} M_{1, R}^{\prime}$, where $M_{1, R}^{\prime}$ are operators in $L^{2}(\mathbb{C})$ with the integral kernels $K_{1}(k, \varsigma) \alpha(\varsigma / R)$. Operators $M_{1, R}$ are pseudo-differential operators of order -1 (they increase the smoothness of functions by one) defined in a bounded domain. Hence operators $M_{1, R}^{\prime}$ and their limit $M_{1}^{\prime}$ are compact operators in $L^{2}(\mathbb{C})$.

The boundedness (with the upper bound $C a_{2}$ ) and compactness of the operator $M_{2}$ will be proved if we show that

$$
\int_{\mathbb{C}} \int_{\mathbb{C}}\left|K_{2}(k, \varsigma)\right|^{2} d \sigma_{k} d \sigma_{\varsigma} \leq C a_{2}
$$

We split the interior integral in two parts: over region $|k|<2|\varsigma|$ and over region $|k|>2|\varsigma|$, and estimate each of them separately. We have

$$
\int_{|k|<2|\varsigma|}\left|K_{2}(k, \varsigma)\right|^{2} d \sigma_{k} \leq 2|g(\varsigma)|^{2} \int_{|k|<2|\varsigma|}\left[\frac{\beta^{2}(\varsigma-k)}{|\varsigma-k|^{2}}+\frac{\beta^{2}(k)}{|k|^{2}}\right] d \sigma_{k} \leq C|g(\varsigma)|^{2}(1+|\varsigma|)^{\delta} .
$$

A better estimate with a logarithmic factor is valid, but we do not need this accuracy. Next,

$$
\int_{|k|>2|\varsigma|}\left|K_{2}(k, \varsigma)\right|^{2} d \sigma_{k}=|g(\varsigma)|^{2} \int_{|k|>2|\varsigma|} \frac{|k \beta(\varsigma-k)+(\varsigma-k) \beta(k)|^{2}}{|(\varsigma-k) k|^{2}} d \sigma_{k}
$$

The denominator of the integrand can be estimated from below by $\frac{1}{4}|k|^{4}$. The numerator, denoted by $n$, has the following properties. If $|\varsigma|$ is large enough, than both beta functions in $n$ are equal to one, and $n=|\varsigma|^{2}$. The same is true if $|\varsigma|$ is bounded and $|k|$ is large. If both variables are bounded, than $|n|$ is bounded. Thus $|n|<(C+|\varsigma|)^{2}$, and the integrand above does not exceed $C \frac{1+|s|^{2}}{|k|^{4}}$. Obviously, the integrand vanishes when $|k|$ is small enough. Thus there is a constant $c>0$ such that

$$
\begin{aligned}
& \int_{|k|>2|\varsigma|}\left|K_{2}(k, \varsigma)\right|^{2} d \sigma_{k} \leq C|g(\varsigma)|^{2} \\
\leq & C|g(\varsigma)|^{2} \int_{|k|>\max (c, 2|\varsigma|)} \frac{1+|\varsigma|^{2}}{|k|^{4}} d \sigma_{k} \\
& \frac{1}{|k|^{4}} d \sigma_{k}+C|g(\varsigma)|^{2} \int_{|k|>2|\varsigma|} \frac{|\varsigma|^{2}}{|k|^{4}} d \sigma_{k}=C_{1}|g(\varsigma)|^{2} .
\end{aligned}
$$

Hence

$$
\int_{\mathbb{C}} \int_{\mathbb{C}}\left|K_{2}(k, \varsigma)\right|^{2} d \sigma_{k} d \sigma_{\varsigma} \leq C \int_{\mathbb{C}}|g(\varsigma)|^{2}(1+|\varsigma|)^{\delta} d \sigma_{\varsigma} \leq C^{\prime} a_{2}
$$

Thus, operators $M_{i}^{\prime}: L^{2}(\mathbb{C}) \rightarrow L^{2}(\mathbb{C})$ are compact and $\left\|M_{i}^{\prime}\right\| \leq C\left(a_{1}+a_{2}\right)$.
Denote by $M_{i}^{\prime \prime}: \mathcal{B}^{2} \rightarrow L^{2}(\mathbb{C})$ operators with the same integral kernels $K_{i}$ as for operators $M_{i}^{\prime}$, but with the domain $\mathcal{B}^{2}$ instead of $L^{2}(\mathbb{C})$. Compactness of these operators will be proved if we show the boundedness of $M_{i}^{\prime}$ on the one-dimensional space of functions of the form $f_{c}(\varsigma)=c \frac{\beta(\varsigma)}{\varsigma}, c=$ const. The upper estimate on $\left\|M_{i}^{\prime \prime} f_{c}\right\|$ can be obtained by repeating the arguments above used to estimate $\left\|M_{i}^{\prime}\right\|$. One needs only to replace $f_{c}$ by the function $f=f_{c} /|\varsigma|^{\delta / 2} \in L^{2}(\mathbb{C})$ and replace the kernel $K_{i}$ by $K_{i}|\varsigma|^{\delta / 2}$. Hence, operators $M_{i}^{\prime \prime}$ are compact and $\left\|M_{i}^{\prime}\right\| \leq C\left(a_{1}+a_{2}\right)$.

Obviously, for each $f \in \mathcal{B}^{2}$, the function $\left(M_{1}+M_{2}\right) f$ is analytic in $D$. Consider its trace on $\partial D$. Let $M_{D}: \mathcal{B}^{2} \rightarrow L^{2}(\partial D)$ be the operator that maps each $f \in \mathcal{B}^{2}$ into the trace of $\left(M_{1}+M_{2}\right) f$ on $\partial D$. In order to complete the proof of the lemma, it remains to show that operator $M_{D}$ is well defined, compact, and $\left\|M_{D}\right\| \leq C\left(a_{1}+a_{2}\right)$. To prove these properties of $M_{D}$, we split the operator into two terms $M_{D}=M_{D} \phi+M_{D}(1-\phi)$, where $\phi$ is the operator of multiplication by the indicator function of a disk $D_{1}$ of a larger radius than the radius of $D$. Then $M(1-\phi) f$ is analytic in $D_{1}$, and

$$
\|M(1-\phi) f\|_{L^{2}\left(D_{1}\right)} \leq\|M f\|_{L^{2}(\mathbb{C})} \leq C\left(a_{1}+a_{2}\right)\|f\|_{\mathcal{B}^{2}}
$$

From a priori estimates for elliptic operators, it follows that

$$
\|M(1-\phi) f\|_{H^{s}(D)} \leq C_{s}\|M(1-\phi) f\|_{L^{2}\left(D_{1}\right)} \leq C_{s}\left(a_{1}+a_{2}\right)\|f\|_{\mathcal{B}^{2}}
$$

where $H^{s}$ is the Sobolev space and $s$ is arbitrary. Hence $\|M(1-\phi) f\|_{H^{s-1 / 2}(\partial D)} \leq C\left(a_{1}+\right.$ $\left.a_{2}\right)\|f\|_{\mathcal{B}^{2}}$. This implies that operator $M_{D}(1-\phi)$ is compact and its norm does not exceed $C\left(a_{1}+a_{2}\right)$. We will take $D_{1}$ not very large, so that function $\beta$ vanishes on $D_{1}$. Then $M \phi f$ is the convolution of $1 / k$ and $\phi g f$, i.e., $M \phi f=\frac{1}{k} *(\phi g f)$. The latter expression is a pseudo differential operator of order -1 applied to the function $\phi g f$ with a compact support. Thus,

$$
\|M \phi f\|_{H^{1}(D)} \leq C\|\phi g f\|_{L^{2}\left(D_{1}\right)} \leq C a_{1}\|f\|_{\mathcal{B}^{2}}
$$

and therefore $\left\|M_{D} \phi f\right\|_{H^{1 / 2}(D)} \leq C a_{1}\|f\|_{\mathcal{B}^{2}}$. Hence, operator $M_{D} \phi$ is compact and its norm does not exceed $C a_{1}$.

Theorem 5.2. Let conditions of Theorem 2.3 hold. Then operator $T_{z, t}: \mathcal{B}^{2} \rightarrow \mathcal{B}^{2}, 0 \leq$ $t \leq T$, is compact, continuous in $(z, t)$, and analytic in $(x, y)$ in a complex neighborhood of $\mathbb{R}^{2}$. The same properties are valid for derivatives of $T_{z, t}$ of any order in $t, x, y$.

Remark. $T_{z, t}$ is analytic in $x, y$ in the region $|\Im x|^{2}+|\Im y|^{2} \leq R^{2}$.
Proof. The operator $T_{z, t}$ can be naturally split into two terms: $T_{z, t}=\mathcal{M}+\mathcal{D}$, where $\mathcal{M}$ involves integration over $\mathbb{C} \backslash D$ and $\mathcal{D}$ involves integration over $\partial D$. In particular,

$$
\mathcal{M} \phi=\frac{1}{\pi} \int_{\mathbb{C} \backslash D} \frac{e^{i \Re((\bar{z})} \bar{\phi}(\varsigma) \Pi^{o} h(\varsigma, \varsigma, t)}{\varsigma-k} d \sigma_{\varsigma}
$$

The statements of the theorem are valid for operator $\mathcal{M}$ due to (10), Lemma 5.1 and Theorem 3.4. Indeed, the compactness and continuity of $M$ in $(z, t)$ is proved in Lemma 5.1. The analyticity in $(x, y)$ follows from the fast decay of $h$ at infinity which is established in Theorem 3.4.

Let us show that the same properties are valid for $\mathcal{D}$. We write $\mathcal{D}$ in the form $D=I_{1} I_{2}$, where operator $I_{2}: L^{2}(\partial D) \rightarrow C^{\alpha}(\partial D)$ is defined by the interior integral in the expression for $\mathcal{D}$ in (14), and operator $I_{1}: C^{\alpha}(\partial D) \rightarrow \mathcal{H}^{s}$ is defined by the exterior integral in the same expression. Here $C^{\alpha}(\partial D)$ is the Holder space and $\alpha$ is an arbitrary number in $(0,1 / 2)$. The integral kernel of operator $I_{2}$ has a logarithmic singularity at $\varsigma=\varsigma^{\prime}$, i.e., $I_{2}$ is a pseudo differential operator of order -1 , and therefore $I_{2}$ is a bounded operator from $L^{2}(\partial D)$ into the Sobolev space $H^{1}(\partial D)$. Thus it is compact as operator from $C(\partial D)$ to $C^{\alpha}(\partial D), \alpha \in(0,1 / 2)$, due to the Sobolev embedding theorem. Thus the compactness of $\mathcal{D}$ will be proved as soon as we show that $I_{1}$ is bounded.

For each $\phi \in C^{\alpha}(\partial D)$, function $I_{1} \phi$ is analytic outside of $\partial D$ and vanishes at infinity. Due to the Sokhotski-Plemelj theorem, the limiting values $\left(I_{1} \phi\right)_{ \pm}$of $\left(I_{1} \phi\right)$ on $\partial D$ from inside and outside of $D$, respectively, are equal to $\frac{ \pm \phi}{2}+P . V \cdot \frac{1}{2 \pi i} \int_{\partial D} \frac{\phi(\varsigma) d \varsigma}{\varsigma-\lambda}$. Thus

$$
\max _{\partial D}\left|\left(I_{1} \phi\right)_{ \pm}\right| \leq C\|\phi\|_{C^{\alpha}(\partial D)} .
$$

From the maximum principle for analytic functions, it follows that the same estimate is valid for function $I_{1} \phi$ on the whole plane. Taking also into account that $I_{2} \phi$ has the following behavior at infinity $I_{2} \phi \sim c / k+O\left(|k|^{2}\right)$, we obtain that operator $I_{1}$ is bounded. Hence operator $\mathcal{D}$ is compact. Since $h$ decays superexponentially at infinity, the arguments above allow one to obtain not only the compactness of $\mathcal{D}$, but also its smoothness in $t, x, y$ and analyticity in $(x, y)$.

### 5.2 The invertibility of $I+T_{z, t}$ at large values of $z$

We will prove the following lemma.
Lemma 5.3. The following relation is valid for operator norm of $T_{z, t}^{2}$ in $\mathcal{B}^{2}$ :

$$
\max _{0 \leq t \leq T}\left\|T_{z, t}^{2}\right\| \rightarrow 0, \quad z \in \mathbb{C}, z \rightarrow \infty
$$

Hence the operator $I+T_{z, t}$ is invertible when $z \in \mathbb{C},|z| \gg 1$.
We split operator $T_{z, t}$ into two terms $T_{z, t}=\mathcal{M}+\mathcal{D}$ that correspond to the integration over $\mathcal{C} \backslash D$ and $D$, respectively, in (14). The entries $M^{i j}, D^{i j}, i, j=1,2$, of the matrix operators $\mathcal{M}$ and $\mathcal{D}$ are

$$
M^{11}=M^{22}=0, \quad M^{12} \phi=-M^{21} \phi=\frac{1}{\pi} \int_{\mathbb{C} \backslash D} \frac{e^{i \Re(\varsigma \bar{z})-t\left(\varsigma^{2}-\bar{\varsigma}^{2}\right) / 2} \bar{\phi}(\varsigma) h_{12}(\varsigma, \varsigma)}{\varsigma-k} d \sigma_{\varsigma},
$$

$$
\begin{aligned}
& D^{11} \phi=D^{22} \phi=\frac{1}{2 \pi i} \int_{\partial D} \frac{d \zeta}{\zeta-k} \int_{\partial D} \overline{\operatorname{Ln} \overline{\overline{\varsigma^{\prime}}-\bar{\varsigma}} \overline{\varsigma^{\prime}}-\overline{k_{0}}} h_{11}\left(\varsigma^{\prime}, \varsigma\right) e^{\frac{i}{2}\left(\varsigma-\varsigma^{\prime}\right) \bar{z}+\frac{t}{2}\left(\varsigma^{\prime 2}-\varsigma^{2}\right)} \phi\left(\varsigma^{\prime}\right) d \zeta^{\prime}, \\
& D^{12} \phi=-D^{21} \phi=\frac{1}{2 \pi i} \int_{\partial D} \frac{d \zeta}{\zeta-k} \int_{\partial D} \operatorname{Ln} \frac{\overline{\varsigma^{\prime}}-\bar{\varsigma}}{\overline{\varsigma^{\prime}}-\overline{k_{0}}} h_{12}\left(\varsigma^{\prime}, \varsigma\right) e^{\frac{i}{2}\left(\varsigma \bar{z}+\bar{\varsigma}^{\prime} z\right)+\frac{t}{2}\left(\bar{\varsigma}^{2}-\varsigma^{2}\right)} \bar{\phi}\left(\varsigma^{\prime}\right) \overline{d \varsigma^{\prime}} .
\end{aligned}
$$

We used here the relations $h_{12}=-h_{12}, h_{11}=h_{22}$ for the entries of $h_{0}$ that were established, for example, in [14, Lemma 4.1].

Lemma 5.1 implies the uniform boundedness of $M^{21}, M^{12}$ when $0 \leq t \leq T, z \in \mathbb{C}$. Thus Lemma 5.3 will be proved if we show that operator norms of $M^{21} \overline{M^{12}}$ and $D^{i j}, i, j=$ 1,2 , vanish as $z \rightarrow \infty$. Let us prove the statement about $D^{i j}$.

Lemma 5.4. For each $T>0$, there exists a constant $C_{T}$ such that

$$
\|D \varphi\|_{\mathcal{B}^{2}} \leq \frac{C_{\alpha, T}}{1+|z|^{1 / 4}}\|\varphi\|_{\mathcal{B}^{2}}, \quad z \in \mathbb{C}, \quad 0 \leq t \leq T
$$

if $k_{0}$ in the definition of operator $\mathcal{D}$ is chosen to belong to $\partial D$ and equal to $k_{0}=-i A e^{i \psi}$, where $\psi=\arg z$ and $A$ is the radius of the disk $D$.

Proof. We will prove the estimate for the component $D^{12}$ of the matrix $D$. Other components of $D$ can be estimated similarly. Consider the interior integral in $D^{12}$ :

$$
\begin{equation*}
R^{12} \phi=\int_{\partial D} \operatorname{Ln} \frac{\overline{\varsigma^{\prime}}-\bar{\varsigma}}{\overline{\varsigma^{\prime}}-\overline{k_{0}}} h_{12}\left(\varsigma^{\prime}, \varsigma\right) e^{\frac{i}{2}\left(\varsigma \bar{z}+\overline{\varsigma^{\prime}} z\right)+\frac{t}{2}\left(\bar{\varsigma}^{2}-\varsigma^{2}\right)} \bar{\phi}\left(\varsigma^{\prime}\right) d \overline{\varsigma^{\prime}}, \quad \varsigma \in \partial D, \quad \phi \in \mathcal{B}^{2} \tag{37}
\end{equation*}
$$

Our goal is to show that

$$
\begin{equation*}
\left\|R^{12} \phi\right\|_{L^{\infty}(\partial D)} \leq \frac{C_{T}}{1+|z|^{1 / 4}}\|\phi\|_{L^{2}(\partial D)}, \quad \phi \in \mathcal{B}^{2} \tag{38}
\end{equation*}
$$

The integrand in (37) is anti-holomorphic in $\varsigma^{\prime} \in D$ with logarithmic branching points at $k_{0}$ and $\varsigma$. If $k_{0}$ is strictly inside $D$, then the integration over $\partial D$ in (37) can be replaced by the integration over two sides of the segment $\left[k_{0}, \varsigma\right]$, which are passed in the counter clock-wise direction. The values of the logarithm on these sides differ by the constant $2 \pi$. This leads to an alternative form of the operator $\mathcal{D}$ :

If $k_{0} \in \partial D$, the contour of integration above can be replaced by $\operatorname{arc}\left[k_{0}, \varsigma\right]$. Thus

$$
R^{12} \phi=i \int_{\widehat{k_{0}, \varsigma}} h_{12}\left(\varsigma^{\prime}, \varsigma\right) e^{\frac{i}{2}\left(\varsigma \bar{z}+\overline{\varsigma^{\prime}} z\right)+\frac{t}{2}\left({\overline{\varsigma^{2}}}^{2}-\varsigma^{2}\right)} \phi\left(\varsigma^{\prime}\right) d \overline{\zeta^{\prime}}, \quad \varsigma \in \partial D, \quad \phi \in L^{2}(\partial D)
$$

Consider the following function (from the exponent in the integrand above): $\Phi=$ $\Re\left[\frac{i}{2} \varsigma \bar{z}\right]$. This function is linear in $\varsigma$, and for each fixed $z=|z| e^{i \psi}, \psi \in[0,2 \pi)$, it has
the unique global maximum on $D$. The maximum occurs on the boundary at the point $\varsigma_{0}=-i A e^{i \psi}$, which depends only on the argument of $z$. Due to Theorem [2.3, point $k_{0} \in \partial D$ can be chosen arbitrarily. We choose $k_{0}=\varsigma_{0} \in \partial D$, and we get that

$$
\left|R^{12} \phi\right| \leq C\left(\int_{\widehat{\varsigma_{0}, \varsigma}} \exp 2\left(\Phi(\varsigma)-\Phi\left(\varsigma^{\prime}\right)\right)\left|d \varsigma^{\prime}\right|\right)^{1 / 2}\|\phi\|_{L^{2}}
$$

Let us estimate the integral above. Let $\varsigma=-i A e^{i(\psi+\varphi)},|\varphi| \leq \pi$. For $\varsigma^{\prime} \in \widehat{\varsigma_{0}, \varsigma}$, we have

$$
\Phi\left(\varsigma^{\prime}\right)=A|z|\left(\cos \varphi^{\prime}\right) / 2, \quad \Phi(\varsigma)=A|z|(\cos \varphi) / 2
$$

and the integral is equal to

$$
\int_{0}^{\varphi} e^{A|z|\left(\cos \varphi-\cos \varphi^{\prime}\right) / 2} d \varphi^{\prime}=O\left(\frac{1}{\sqrt{|z|}}\right), z \rightarrow \infty
$$

This justifies (38).
Let us show now that the following statement holds.

## Lemma 5.5.

$$
\begin{equation*}
\max _{0 \leq t \leq T}\left\|M^{21} \overline{M^{12}}\right\|_{\mathcal{B}^{2}} \rightarrow 0, \quad z \in \mathbb{C}, z \rightarrow \infty \tag{39}
\end{equation*}
$$

Proof. Kernels of $M^{12}, M^{21}$ are smooth, see (10). From Theorem 3.4 it follows that the kernels and rapidly decaying functions in $\mathbb{C}$. Therefore, Lemma 5.1 implies that operators $M^{12}, M^{21}$ can be approximated in $\mathcal{B}^{2}$ by operators with function $h_{12}$ replaced by a compactly supported one. Therefore, without loss of the generality, we will assume below that the supports of $h_{12}, h_{21}$ belong to a bounded domain $\mathcal{O}$.

We will use the notation $P$ for the one-dimensional operator defined in (36) with the density $g=e^{i \Re(\varsigma \bar{z})-t\left(\varsigma-\bar{\varsigma}^{2}\right) / 2} h_{12}(\varsigma, \varsigma)$. Let $\widehat{M}:=\left(M^{12}-P\right) \overline{\left(M^{21}-P\right)}$. We will prove that

$$
\begin{equation*}
\max _{0 \leq t \leq T}\|\widehat{M}\|_{\mathcal{B}^{2}} \rightarrow 0, \quad z \in \mathbb{C}, z \rightarrow \infty \tag{40}
\end{equation*}
$$

The other three terms $M^{12} \overline{\left.M^{21}-P\right)},\left(M^{12}-P\right) \overline{M^{21}}$, and $P \bar{P}$ can be treated in the same way. We have

$$
\widehat{M} \varphi=\frac{1}{\pi^{2}} \int_{\mathcal{O} \backslash D} A\left(z, \varsigma, \varsigma_{2}\right) \overline{h_{21}}\left(\varsigma_{2}, \varsigma_{2}\right) e^{-i \Re\left(\varsigma_{2} \bar{z}\right)+t\left(\varsigma_{2}-\bar{\varsigma}^{2}\right) / 2} \varphi\left(\varsigma_{2}\right) d \sigma_{\varsigma_{2}},
$$

where

$$
\begin{equation*}
A\left(z, \varsigma, \varsigma_{2}\right):=\int_{\mathcal{O} \backslash D} e^{i \Re\left(\varsigma_{1} \bar{z}\right)-t\left(\varsigma_{1}-\bar{\varsigma}^{2}\right) / 2} h_{12}\left(\varsigma_{1}, \varsigma_{1}\right)\left(\frac{1}{\varsigma_{1}-\varsigma}+\frac{\beta(\varsigma)}{\varsigma}\right) \overline{\left(\frac{1}{\varsigma_{2}-\varsigma_{1}}+\frac{\beta\left(\varsigma_{1}\right)}{\varsigma_{1}}\right)} d \sigma_{\varsigma_{1}} . \tag{41}
\end{equation*}
$$

The Minkovsky inequality in the integral form implies the following two estimates.

$$
\begin{aligned}
\|\widehat{M} f\|_{L^{2}(\mathbb{C} \backslash D)} & \leq \int_{\mathcal{O} \backslash D}\left[\int_{\mathcal{O} \backslash D} \mid A\left(z, \varsigma_{,}\right) \varsigma_{2} d \sigma_{\varsigma}\right]^{1 / 2}\left|h_{21}\left(\varsigma_{2}, \varsigma_{2}\right) f\left(\varsigma_{2}\right)\right| d \sigma_{\varsigma_{2}}, \quad f \in \mathcal{B}^{2} . \\
\|\widehat{M} f\|_{L^{2}(\partial D)} & \leq \int_{\mathcal{O} \backslash D}\left[\int_{\partial D}\left|A\left(z, \varsigma, \varsigma_{2}\right)\right|^{2}|d \varsigma|\right]^{1 / 2}\left|h_{21}\left(\varsigma_{2}, \varsigma_{2}\right) f\left(\varsigma_{2}\right)\right| d \sigma_{\varsigma_{2}}, \quad f \in \mathcal{B}^{2} .
\end{aligned}
$$

Since the norm of the operator $L^{2}(\mathbb{C} \backslash D) \rightarrow L^{1}(\mathbb{C} \backslash D)$ of multiplication by $h_{21}$ can be estimated by a constant, the validity of (40) will follow from the estimates above if we show that

$$
\sup _{\varsigma_{2} \in \mathbb{C} \backslash D} \int_{\mathcal{O} \backslash D}\left|A\left(z, \varsigma, \varsigma_{2}\right)\right|^{2} d \sigma_{\varsigma} \rightarrow 0, \quad \sup _{\varsigma_{2} \in \mathbb{C} \backslash D} \int_{\partial D}\left|A\left(z, \varsigma, \varsigma_{2}\right)\right|^{2}|d \varsigma| \rightarrow 0, \text { as } z \rightarrow \infty .
$$

We will prove only the first inequality above, since the second one can be proved similarly. Note that, uniformly in $\varsigma_{2} \in \mathcal{O}$,

$$
\begin{gathered}
\int_{\mathcal{O} \backslash D}\left|A\left(z, \varsigma, \varsigma_{2}\right)\right|^{2} d \sigma_{\varsigma} \\
\leq \int_{\mathcal{O} \backslash D}\left|\int_{\mathcal{O} \backslash D} h_{12}\left(\varsigma_{1}, \varsigma_{1}\right)\left(\frac{1}{\varsigma_{1}-\varsigma}+\frac{\beta(\varsigma)}{\varsigma}\right) \overline{\left(\frac{1}{\varsigma_{2}-\varsigma_{1}}+\frac{\beta\left(\varsigma_{1}\right)}{\varsigma_{1}}\right)} d \sigma_{\varsigma_{1}}\right|^{2} d \sigma_{\varsigma}<C
\end{gathered}
$$

The boundedness follows from the fact that the internal integral is $O\left(\ln \left|\varsigma-\varsigma_{2}\right|\right), \varsigma-\varsigma_{2} \rightarrow 0$. Let $A^{s}$ be given by (41) with the extra factor $\left.\eta_{s}:=\eta\left(s\left|\varsigma-\varsigma_{1}\right|\right) \eta\left(s\left|\varsigma_{1}-\varsigma_{2}\right|\right)\right), s>0$, in the integrand, where $\eta \in C^{\infty}(\mathbb{R}), \eta=1$ outside of a neighborhood of the origin, and $\eta$ vanishes in a smaller neighborhood of the origin.

For each $\varepsilon$, there exists $s=s_{0}(\varepsilon)$ such that

$$
\int_{\mathcal{O} \backslash D}\left|A-A^{s_{0}}\right|^{2} d \sigma_{\varsigma}<\varepsilon
$$

for all the values of $\varsigma_{2} \in \mathcal{O}, z \in \mathbb{C}$. Denote by $R^{s_{0}}$ the function $A^{s_{0}}$ with the potential $h_{12}$ replaced by its $L_{1}$-approximation $\widetilde{h}_{12} \in C_{0}^{\infty}(\mathbb{C} \backslash D)$. We can choose this approximation in such a way that

$$
\int_{\mathcal{O} \backslash D}\left|A^{s_{0}}-R^{s_{0}}\right|^{2} d \sigma_{\varsigma}<\varepsilon
$$

for all the values of $\varsigma_{2}, z$. Now it is enough to show that

$$
\left|R^{s_{0}}\left(\varsigma, \varsigma_{2}, z\right)\right| \rightarrow 0 \quad \text { as } \quad z \rightarrow \infty
$$

uniformly in $\varsigma_{,} \varsigma_{2} \in \mathcal{O}$. The latter can be obtained by integration by parts in
$R_{z}^{s_{0}}\left(\varsigma, \varsigma_{2}\right):=\int_{\mathbb{C} \backslash D}\left(1-\eta_{s}\right) e^{i \Re\left(\varsigma_{1} \bar{z}\right)-t\left(\varsigma_{1}-\bar{\varsigma}_{1}^{2}\right) / 2} \widetilde{h_{12}}\left(\varsigma_{1}, \varsigma_{1}\right)\left(\frac{1}{\varsigma_{1}-\varsigma}+\frac{\beta(\varsigma)}{\varsigma}\right) \overline{\left(\frac{1}{\varsigma_{2}-\varsigma_{1}}+\frac{\beta\left(\varsigma_{1}\right)}{\varsigma_{1}}\right)} d \sigma_{\varsigma_{1}}$
(integrating $e^{i \Re\left(\varsigma_{1} \bar{z}\right)}$ and differentiating the complementary factor). This completes the proof of (39).

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[^1]:    ${ }^{1}$ This fact was not mentioned in the paper, but it can be easily checked

[^2]:    ${ }^{2}$ We will say that a set $\omega$ of points $(z, t)$ in $\mathbb{R}_{+}^{3}=\mathbb{R}^{3} \bigcap\{t \geq 0\}$ is half-open if $\omega$ contains points where $t=0$ and, for each point $\left(z_{0}, 0\right) \in \omega$, there is a ball $B_{0}$ centered at this point such that $B_{0} \bigcap\{t \geq 0\} \subset \omega$.
    ${ }^{3}$ that statement can be found in [13, lemma 5.1]

[^3]:    ${ }^{4}$ all the exceptional points are inside $D$, i.e., all the points on $\partial D$ are non-exceptional.

