Mathematical Circus Clown Tricks

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Abstract

The Mathematical Circus project was created in 2011 by the Ludus Association with the main goal of promoting the interest and motivation for learning mathematics. The Mathematical Circus team performs mathematical magic shows where complementary skills are merged to produce a high intervention capacity within a wide geographical range in Portugal. Some tricks are performed by a mathematical clown, a unique character in the circus world, who brings together the usual foolish characteristics of a clown and the rigorous mathematical knowledge. In this paper we will describe and explain some of the tricks of the Mathematical Circus repertoire involving the clown.

Keywords: mathematical magic, recreational mathematics, mathematical circus.

1 Introduction

The issue of academic failure in mathematics at schools is not new and has been debated for decades. Knowing its causes and finding ways to combat it is a priority at all levels of education. One of the problems identified in the failure of mathematics teaching is the lack of motivation and a negative attitude that students feel towards the subject (Hall and Pais, 2018). It is essential to reverse these feelings and motivate the students. Only then can we hope to achieve success in school mathematics.

One way of doing this is through recreational mathematics, which Singmaster (1992) defines as the mathematics that is fun, popular and with pedagogical value. In particular, the exploration of the relationship between magic and mathematics has enormous potential in the development of activities in schools or elsewhere, to effectively promote the interest for mathematics. In the classroom, the exploration and study of a magical effect, can become a mathematical problem and provide a source of reasoning and research. Outside the classroom, mathematical magic shows can promote a positive relationship with mathematics and stimulate the curiosity for understanding the tricks' explanation, and in turn, for learning mathematics. Mathematical magic is being used in Portugal as in other countries to engage students and promote enthusiasm for learning mathematics: see for instance Bastos (2015), Diaconis and Graham (2011), Gardner (1956), Rodrigues (2015), and Silva et al. (2017).

In March 2011, the Ludus Association created the Mathematical Circus project (https://circomatematico.wordpress.com/), coordinated by Jorge Nuno Silva, professor at the University of Lisbon. It is a project where complementary skills are merged to produce a high intervention capacity within a wide geographical range in the promotion of mathematics, an urgent task in Portugal.

Magic shows are well known throughout the world. Using mathematics in magic tricks is not a novelty but making an entire show out of mathematical magic is rare. Fernando Blasco in Spain is one the few people to do so (Blasco, 2011). The Mathematical Circus goes one step further, by adding to this setup a circus environment.

The performing arts landscape is evolving and diversifying in this ongoing 21^{st} century and the circus is one of the areas where this evolution is more noticeable, with contemporary circus getting the most out of creativity and joining together different art forms, high skilled acrobatics and technology. Interlacing mathematics with the circus environment is not common. In this context, the Mathematical Circus project is an innovative project which uses some of the circus elements to create an original combination: magic, beauty, surprise, laughter and wonder, all associated with the circus, are the key ingredients used by the Mathematical Circus. All the activities in the repertoire of the Circus have a mathematical content and although mathematical magic is the most relevant component, jokes and fooleries aren't less important and all based on mathematics.

The Mathematical Circus performs shows mostly in schools, but also in science centers and universities (for visiting schools), typically for students up to eighteen years old. Some shows are also performed in international conferences, such as the Recreational Mathematics Colloquium.

The Mathematical Circus has two teams, one in the north (hosted by the University of Aveiro) and one in the south (Lisbon). All authors of this paper are members of the north team. Figure 1 illustrates the Mathematical Circus shows in schools, by the north team. Typically, the shows consist of several magic tricks alternated with clown interventions, looking, as much as possible, like a live magic show with a circus atmosphere. The clown fools around in most of the tricks and has his own tricks and diversions.

In this paper we shall describe and explain some of the tricks of the Mathematical Circus repertoire involving the clown. Further information concerning the Mathematical Circus project can be found in Marques (2016).

Figure 1: Photos of the Mathematical Circus shows

2 Clown tricks

In this section we describe the performance of three tricks performed by the Mathematical Circus where the clown has a major role. The mathematical explanation of the tricks is given in sections 3.1, 3.2 and 3.3.

2.1 Guessing a card

In a magic show, very often, there are several tricks where the magician guesses a hidden card. In all cases the audience is unable to understand how the magician makes his guess and sometimes all that is needed is a piece of mathematics, unfamiliar to the public. The trick here addressed is performed as following.

The show host holds a set of cards which can be drawn from a usual pack of cards. For now we consider that the host holds a complete set of cards from one particular suit, 13 cards altogether. He asks for a volunteer and announces that the magician will guess one of the cards, chosen at random by the volunteer.

The clown appears and complains that he never gets the chance to guess. He begs for an opportunity to play as magician and the host proposes that both the clown and the magician will be given the opportunity to guess the card.

The host proceeds with the trick by asking the volunteer to: shuffle the cards, take one out, place it facing down on the table, and give six cards to the magician and the remaining six cards to the clown. No one knows which card is hiding on the table. The clown and the magician take a look at their cards and then the clown hives his hint followed by the magicians hint.

It is very likely that the clown will fail his guess, but that doesn't spoil the show since a clown is not expected to have magic powers as a true magician. However the magician's hint will be right for sure! In fact the clown's guess is merely a code that allows the magician to make the right guess. Only by coincidence will the clown's guess match the real answer.

2.2 Who wants to be a millionaire?

The clown asks the show host if he can present a contest, in the spirit of the well know "Who wants to be a millionaire?". He shows a box of sweets for the contest prize and the host agrees to the proposal.

The clown then displays a rope with 4 hanging objects, identified with letters A, B, C and D. He asks for a volunteer and tells him that he will make a question and only one of the four answers is correct. The question is very simple: "Which of the four hanging objects is the odd one out?"

The choice of the objects hanging out is unlimited but must follow a simple rule: with an appropriate argument, any of the four objects may be ruled out. For instance, one may choose to hang a set of four pairs of underpants such that:

- Three are children size and only one pair is for adults.
- Three pairs are briefs and only one pair are boxers.
- Three have the same colour (say white) and the other has a different color (and eventually with some pattern).
- Three are hanged out with clothes pegs of a certain colour and the other has pegs of a different colour.

Proceeding with the trick, the volunteer gives his answer and the clown firmly replies that the answer is wrong (whatever the answer may be). He then asks for a new volunteer who must provide a new answer. Again the answer is declared to be wrong by the clown, and a new volunteer is called on stage. Once more the volunteer's answer is declared wrong and the clown victoriously claims the prize to be his since no one gave the correct answer.

Next, the show host intervenes and asks each volunteer to provide an argument for his choice. Most likely all arguments are plausible and the host declares that the prize must be divided by all the winners: volunteers and clown.

2.3 The unfair division of the prize

Following the contest and faced with the need to share the prize, the clown tells how many sweets are contained in the prize box, say 24, and states that each of the winners is entitled to 15 sweets. He immediately starts to count his share of sweets out of the box. The host makes him stop and asks him to show how he did his calculations. Misusing arithmetics, he "proves" indeed that 24 divided by four is 15.

He starts by misusing the division algorithm (Figure 2). He says "four doesn't go into two, but four goes into four, once" (Figures 2a and 2b). He proceeds: "One times four is four (Figure 2c), subtracted from four is zero and we pull down the two (Figure 2d). Now, four doesn't go into two neither does it go into zero. But four goes into 20, five times (Figure 2e). Finally, four times five is 20 and subtracting 20 from 20 leaves us with a zero remainder!" (Figure 2f).

$$\begin{array}{r|l} 24 & \underline{\;4\;} \end{array} \qquad \begin{array}{r|l} 24 & \underline{\;4\;} \\ & 1 \end{array} \qquad \begin{array}{r|l} 24 & \underline{\;4\;} \\ 4 & 1 \end{array} \qquad \begin{array}{r|l} 24 & \underline{\;4\;} \\ \underline{-\;\;4} & 1 \\ 20 & \end{array} \qquad \begin{array}{r|l} 24 & \underline{\;4\;} \\ \underline{-\;\;4} & 15 \\ 20 & \end{array} \qquad \begin{array}{r|l} 24 & \underline{\;4\;} \\ \underline{-\;\;4} & 15 \\ 20 & \\ \underline{-\;20} & \\ 0 & \end{array}$$

(a) (b) (c) (d) (e) (f)

Figure 2: Misuse of the division algorithm (Portuguese notation)

The show host says this calculation doesn't look well. So, he asks the clown to prove the result by multiplication: if 24 divided by four were to be 15, then four times 15 ought to be 24. The clown says that indeed it is 24 and he performs the multiplication algorithm as in Figure 3 steps (a)–(d). On the way he says: "four times five is 20" and writes down this number; "four times one is four" and writes down four under the 20; "and 20 plus four is 24!"

$$\begin{array}{r} 15 \\ \underline{\times \quad 4} \end{array} \qquad \begin{array}{r} 15 \\ \underline{\times \quad 4} \\ 20 \end{array} \qquad \begin{array}{r} 15 \\ \underline{\times \quad 4} \\ 20 \\ \underline{+ \quad 4} \end{array} \qquad \begin{array}{r} 15 \\ \underline{\times \quad 4} \\ 20 \\ \underline{+ \quad 4} \\ 24 \end{array}$$

(a) (b) (c) (d)

Figure 3: Misuse of the multiplication algorithm (Portuguese notation)

As a last attempt to make things right, the host says to the clown that the calculation is not correct. He argues that if four times 15 were to be 24 then adding up 15 + 15 + 15 + 15 would be 24, which it isn't. He then says he will do the addition and sets up the algorithm as in Figure 4a.

The host starts doing the addition of the units digits and says "five plus five is ten, plus five is fifteen, plus five is twenty". The clown suddenly interrupts and proceeds by adding each "1" in the tens place to this sum, saying "... twenty-one, twenty-two, twenty-three, twenty-four!". So, the clown concludes the operation as in Figure 4b.

$$\begin{array}{r} 15 \\ 15 \\ 15 \\ +\ 15 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 15 \\ 15 \\ 15 \\ +\ 15 \\ \hline 24 \end{array}$$

(a) (b)

Figure 4: Misuse of the addition algorithm (Portuguese notation)

This diversion is based on a scene by Abbot and Costello (1941), where one of the actors misuses the division, multiplication and addition algorithms to conclude that seven goes 13 times into 28.

3 The math behind it

3.1 The failed magician, probably

This trick could be executed with many variations: the number of cards and suits may vary (being possible to choose a set of random cards too, but a previous offstage computation is required), as well as the number of *improvised* magicians who try to guess a card that was picked (and will probably fail). However, the clown is the perfect choice for this role: just lucky if he accidentally gets the right answer, or simply inept in any other case.

In the following, we are going to analyse a situation that mostly resembles the performance that took place during the conference presentation, as described in Section 2.1:

- out of a deck D with n distinct cards of a given suit (as usual, $n \leq 13$),
- a card $p \in \mathsf{D}$ is picked,
- a subset M of the cards left in the deck is handed to the magician,
- and the set C of all the remaining cards is given to the clown.

Clearly,

$$\mathsf{D} = \{p\} \cup \mathsf{M} \cup \mathsf{C},$$

where the unions are disjoint. Therefore, if we introduce the operator

$$\sigma_{\mathsf{A}} = \sum_{i \in \mathsf{A}} i, \qquad \mathsf{A} \subset \mathbb{N} \text{ (with finite size)},$$

that computes the sum of the cards in A (considering the natural association 'Ace'$\leftrightarrow$ 1, 'Jack'$\leftrightarrow$ 11, 'Queen'$\leftrightarrow$ 12, and 'King'$\leftrightarrow$ 13), then

$$\sigma_{\mathsf{D}} = p + \sigma_{\mathsf{M}} + \sigma_{\mathsf{C}} \qquad \Longleftrightarrow \qquad p = \sigma_{\mathsf{D}} - \sigma_{\mathsf{M}} - \sigma_{\mathsf{C}}. \tag{1}$$

Since σ_{D} is known in advance and the magician calculates σ_{M}, he only needs to know σ_{C} in order to determine the value of p. As we shall see, that information will be revealed by the clown's guess, but not directly. Indeed, σ_{C} is quite likely to be greater than n, while the clown may only say a card, i.e., a number between 1 and n. Nevertheless, this turns out to be enough, since the solution given by (1) still holds if we by employ arithmetics modulo n.

Actually, denote by $\overline{a}$ the equivalence class of integer numbers modulo n, such that $\overline{a} = \overline{b}$ if and only if n divides $a - b$. It is well known that the set $\overline{\mathsf{D}} = \mathbb{Z}/n\mathbb{Z} = \{\overline{1}, \overline{2}, \ldots, \overline{n} = \overline{0}\}$ is (at least) an additive group, whose operation (here simply denoted by '+') satisfies $\overline{a} + \overline{b} = \overline{a+b}$.

So, by identifying a card $a \in \mathsf{D}$ with the equivalence class $\overline{a} \in \overline{\mathsf{D}}$ and defining $\overline{\sigma_{\mathsf{A}}}$ to be the equivalence class associated with σ_{A}, we can rewrite (1) as

$$\overline{\sigma_{\mathsf{D}}} = \overline{p} + \overline{\sigma_{\mathsf{M}}} + \overline{\sigma_{\mathsf{C}}} \qquad \Longleftrightarrow \qquad \overline{p} = \overline{\sigma_{\mathsf{D}}} - \overline{\sigma_{\mathsf{M}}} - \overline{\sigma_{\mathsf{C}}}.$$

Thus, if the clown tells the value of $\overline{\sigma_{\mathsf{C}}}$, *disguised as a guess*, the magician will be able to calculate the picked card p.

Before giving a concrete example, we present some final observations. First, note that the value of $\overline{\sigma_{\mathsf{D}}}$ need not be explicitly computed, since its value is given by a simple rule:

$$\sigma_{\mathsf{D}} = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \begin{cases} \frac{n}{2}\,(n+1) & \text{if } n \text{ is even} \\ n\,\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases} \Rightarrow \ \overline{\sigma_{\mathsf{D}}} = \begin{cases} \overline{\frac{n}{2}}\,\overline{(n+1)} = \overline{\frac{n}{2}} & \text{if } n \text{ is even,} \\ \overline{n}\,\overline{\frac{n+1}{2}} = \overline{0} & \text{if } n \text{ is odd.} \end{cases}$$

Then, it may happen that the clown's hint for the magician corresponds to one of his own cards: for this reason, he should never show the cards he received.

The picked card too should not be revealed until the end of the trick: in this way, since the clown could unintentionally guess the right card, the magician would still be able to show his skills, confirming the clown's answer – that was right just for luck!

A real example. During the live performance, the following occurred:

- one card p was picked out of a deck with the $n = 13$ cards of a suit (hence $\overline{\sigma_{\mathsf{D}}} = \overline{0}$) and put face down on a table;
- the clown received the set of cards $\mathsf{C} = \{\mathrm{A}, 7, 9, 10, \mathrm{J}, \mathrm{K}\}$, thus *guessing* that the hidden card was
$$\overline{\sigma_{\mathsf{C}}} = \overline{1 + 7 + 9 + 10 + 11 + 13} = \overline{51} = \overline{12},$$
i.e., a Queen;
- the magician received the set of cards $\mathsf{M} = \{2, 3, 4, 6, 8, Q\}$ and computed
$$\overline{\sigma_{\mathsf{M}}} = \overline{2 + 3 + 4 + 6 + 8 + 12} = \overline{35} = \overline{9};$$
once he heard the clown's *guess*, he could determine that the hidden card was 5:
$$\overline{p} = \overline{\sigma_{\mathsf{D}}} - \overline{\sigma_{\mathsf{M}}} - \overline{\sigma_{\mathsf{C}}} = \overline{0} - \overline{9} - \overline{12} = -(\overline{9 + 12}) = -\overline{21} = -\overline{8} = \overline{5},$$
(since $\overline{8} + \overline{5} = \overline{13} = \overline{0}$). The reader can easily verify the result by looking at sets C and M.

"No, my dear clown, there's no chance you'll ever be a true magician: the card that was picked is not a Queen, but a 5!" said the magician, slowly lifting the card and showing it to the amazed public (applause).

3.2 Who wants to be a millionaire?

The mathematical circus's trick "Who wants to be a millionaire" starts with the problem of finding "which object is the odd one out" and is connected to mathematical logic. This kind of problems is one of the important topics in competitive exams or employment tests in which mathematics logical questions are being asked and the solution is expected to be unique. Nevertheless, while the criteria one uses to get the correct solution is the most obvious one, many times we could use different criteria and obtain another answer that is still based on logical thought.

For example, if one asks

Which one of the numbers $28, 49, 63, 23$ and 56 is the odd one out

The most obvious answer would be 23 because all the other numbers are divisible by 7, but any of the following answers would also be logically correct.

Only 63 has less units than tens.
Only 23 is prime
Only 49 has a O in its spelling
Only 28 has an H
Only 56 has two Fs (or two Is) in its spelling
Only 49 has two Ns in its spelling
Only 49 is a square
Only 23 has no digit with a closed shape $(8, 6, 9, 0, 4)$ in it
Only 49 has 2 digits with closed shapes
Only 49 has no digit in common with any of the other numbers.

In this trick, each volunteer is convinced that his is the correct answer, but after each of them explains his reasoning they all (and all the members of the audience) become aware of other possible logical explanations.

3.3 Clown's Divisions

The second part of the trick (clown's elementary arithmetic operations) serves to explain why writing an algorithm in the correct form is so important. This is one of our most successful tricks and everyone at the end smiles, even the students of ages between 7 and 10 who start by shouting "that is wrong!", "you don't know how to do it!", "No! The correct answer is ...".

When the clown "proves" that 24 divided by four is 15, he uses the following reasoning.

> "Four doesn't go into two, but four goes into four, once. One times four is four, subtracted from four is zero and we pull down the two. Now, four doesn't go into two neither does it go into zero. But four goes into 20, five times. Finally, four times five is 20 and subtracting 20 from 20 leaves us with a zero remainder."

So, one question that comes to mind is: For which numbers can we apply a similar reasoning and obtain a zero remainder? Any such example we will call a *clown division*.

Suppose we want to divide a two-digits number n, i.e. $n = n_1 \times 10 + n_0$, with $n_1 > 0$ by a one-digit number $d < 10$. If we follow the clown reasoning, we obtain the following conditions:

- $d > n_1$;
- $d \leq n_0$ and $n_0 = dq_1 + r$, where q_1 is a positive integer and $0 \leq r < d$;

- after pulling down n_1 one realizes that d can't go into n_1 nor into r, but it can go into $n_1 \times 10 + r$, q_0 times, that is $n_1 \times 10 + r = dq_0$.

Therefore, in order to perform the clown's "division algorithm", we must have

$$\begin{aligned} n_1 \times 10 + n_0 &= n_1 \times 10 + d \times q_1 + r \\ &= d \times q_0 + d \times q_1 \\ &= d \times (q_1 + q_0) \end{aligned}$$

with the conditions

$$10 > n_0 \geq d > n_1 > 0 \quad \text{and} \quad n_0 = d \times q_1 + r \quad \text{where} \quad q_1 > 0 \quad \text{and} \quad 0 \leq r < d.$$

The only numbers that satisfy all these conditions, i.e., the ones for which we can perform the clown's division, are given in Table 1.

$12 \div 2 = 15$	$15 \div 3 = 14$	$16 \div 4 = 13$
$14 \div 2 = 25$	$18 \div 3 = 24$	$24 \div 4 = 15$
$16 \div 2 = 35$	$24 \div 3 = 17$	$28 \div 4 = 25$
$18 \div 2 = 45$	$27 \div 3 = 27$	$36 \div 4 = 18$
$15 \div 5 = 12$	$18 \div 6 = 12$	$28 \div 7 = 13$
$25 \div 5 = 14$	$36 \div 6 = 15$	$49 \div 7 = 16$
$35 \div 5 = 16$	$48 \div 6 = 17$	$48 \div 8 = 15$
$45 \div 5 = 18$		

Table 1: All the possible clown's divisions. Some *exceedingly unreasonable* cases are highlighted: in red when the quotient is bigger than the dividend and in blue when they are equal.

Acknowledgements

This work was supported by the Portuguese Foundation for Science and Technology FCT (Fundação para a Ciência e a Tecnologia), through CIDMA (Center for Research and Development in Mathematics and Applications), within project UID/MAT/04106/2019 (CIDMA).

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