Universidade de Aveiro Departamento de Matemática
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PROBLEMAS DE RESISTÊNCIA AERODINÂMICA MÍNIMA

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tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica do Doutor Alexander Plakhov, Professor Auxiliar com Agregação do Departamento de Matemática da Universidade de Aveiro

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Doutor Alexander Plakhov
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palavras-chave optimização de forma, problemas de resistência mínima, teoria de dispersão clássica, aerodinâmica Newtoniana, corpos invisíveis em uma direção.

resumo

Nesta tese estamos preocupados com o problema da resistência mínima primeiro dirigida por I. Newton em seu *Principia* (1687): encontrar o corpo de resistência mínima que se desloca através de um médio. As partículas do médio não interagem entre si, bem como a interação das partículas com o corpo é perfeitamente elástica. Diferentes abordagens desse modelo foram feitas por vários matemáticos nos últimos 20 anos. Aqui damos uma visão geral sobre estes resultados que representa interesse independente, uma vez que os autores diferentes usam notações diferentes. Apresentamos uma solução do problema de minimização na classe de corpos de revolução geralmente não convexos e simplesmente conexos. Acontece que nessa classe existem corpos com resistência menor do que o mínimo da resistência na classe de corpos convexos de revolução. Encontramos o infimum da resistência nesta classe e construimos uma sequência regular de corpos que aproxima este infimum. Também apresentamos um corpo de resistência nula. Até agora ninguém sabia se tais corpos existem ou não, evidentemente o nosso corpo não pertence a nenhuma classe anteriormente analisado. Este corpo é não convexo e não simplesmente conexo; a forma topológica dele é um toro, parece um UFO extraterrestre. Apresentamos aqui várias famílias de tais corpos e estudamos as suas propriedades. Também apresentamos um corpo que é natural de chamar um corpo "invisíveis em uma direção", uma vez que a trajectória de cada partícula com a certa direcção coincide com a linha recta fora do invólucro convexo do corpo.

keywords

Shape optimization, problems of minimal resistance, classical scattering, Newtonian aerodynamics, bodies invisible in one direction.

abstract

In this thesis we are concerned with the problem of minimal resistance first addressed by I. Newton in his *Principia* (1687): find the body of minimal resistance moving through a medium. The medium particles do not mutually interact, and the interaction of particles with the body is perfectly elastic. Different approaches to that model have been tried by several mathematicians during the last 20 years. Here we give an overview of these results that represents interest in itself since all authors use different notations. We present a solution of the minimization problem in the class of generally non convex, simply connected bodies of revolution. It happens that in this class there are bodies with smaller resistance than the minimum in the class of convex bodies of revolution. We find the infimum of the resistance in this class, and construct a sequence of bodies which approximates this infimum. Also we present a body of zero resistance. Since earlier it was unknown if such bodies exists or not, evidently our body does not belong to any class previously examined. The zero resistance body found by us is non-convex and non-simply connected; topologically it is a torus, and it looks like an extraterrestrial UFO. We present here several families of such bodies and study their properties. We also present a body which is natural to call a body "invisible in one direction", since the trajectory of each particle with the given direction, outside the convex hull of the body, coincides with a straight line.

Contents

Chapter 1

Notation and introductory results

1.1 Historical overview

In 1687, I. Newton in his *Principia* [1] considered a problem of minimal resistance for a body moving in a homogeneous rarefied medium. In slightly modified terms, the problem can be expressed as follows.

A convex body is placed in a parallel flow of point particles. The density of the flow is constant, and velocities of all particles are identical. Each particle incident on the body makes an elastic reflection from its boundary and then moves freely again. The flow is very rare, so that the particles do not interact with each other. Each incident particle transmits some momentum to the body; thus, there is created a force of pressure on the body; it is called *aerodynamic resistance force*, or just *resistance*.

In this chapter we give a review on results concerning this topic. During the last 20 years there have been a quite high scientific activity on this problem [1]-[27] which involved several research groups who used different notations; this chapter plays the role of introduction to recent results in this theory. In the chapter 2 we present a solution of the resistance minimization problem in the class of generally non-convex simply-connected bodies of revolution. Using some tricks in the spirit of [19] the problem is reduced to a variational 1D problem in the class of convex curves which is studied using Pontryagin

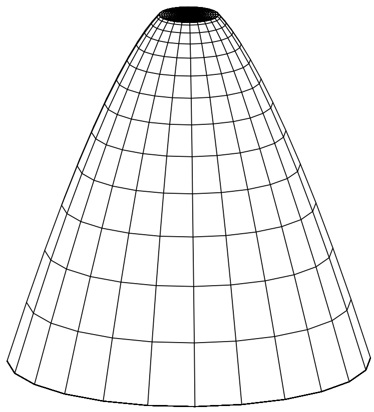


Figure 1.1: The Newton solution for $h = 2$.

variational principe. Chapter 3 is devoted to our discovery of a zero resistance body. We construct a one-dimensional family of such bodies and present the proof of the zero resistance property. We also present a modification of the construction which leads to a class of bodies which we call bodies *invisible in one direction.* They have the following property: the trajectory of each particle of the flow outside a prescribed bounded set coincides with a straight line. Indeed, such a body with mirror surface becomes invisible to an observer staying in the certain direction far enough from the body. The new results presented in chapters 2 and 3 are mainly based on the author's published papers [25],[26],[27].

Newton described (without proof) the body of minimal resistance in the class of convex and axially symmetric bodies of fixed length and maximal width, where the symmetry axis is parallel to the flow velocity. That is, any body from the class is inscribed in a right circular cylinder with fixed height and radius. A rigorous proof of the fact that the body described by Newton is indeed the minimizer was given two centuries later [2]. From now on, we suppose that the radius of the cylinder equals 1 and the height equals h, with h being a fixed positive number. The cylinder axis is vertical, and the flow falls vertically downwards. The body of least resistance for $h = 2$ is shown on fig. 1.1.

Time to time that problem attracted attention of scientists. Only in the 20th century engineers understood that this problem is adequate for aircrafts in the rarefied gas [22], for example, in the stratosphere or in space on low Earth orbits (from 100km to 1000km height). Since the early 1990s, there have been obtained new interesting results related to the problem of minimal resistance in various classes of admissible bodies [5]-[19]. In this chapter we will describe of all old and recent results.

1.1.1 Rigorous description of the model

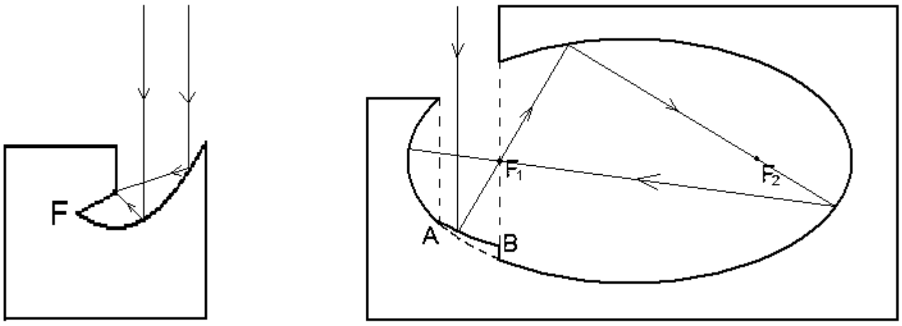
Consider a compact connected set $B \subset \mathbb{R}^3$ and choose an orthogonal reference system $Oxyz$ in such a way that the axis Oz is parallel to the flow direction; that is, the particles move vertically downwards with the velocity $(0, 0, -1)$. Suppose that a flow particle (or, equivalently, a billiard particle in $\mathbb{R}^3 \setminus B$) which initially moves according to $x(t) = x$, $y(t) = y$, $z(t) = -t$, falls on the body, makes a finite number of reflections at regular points of the boundary ∂B and moves freely afterwards. Denote by $v_B(x, y)$ the final velocity. If there are no reflections, put $v_B(x, y) = (0, 0, -1)$.

Thus, one gets the function $v_B = (v_B^x, v_B^y, v_B^z)$ taking values in S^2 and defined on a subset of $\mathbb{R}^2$. We impose the following condition.

Regularity condition. *v_B is defined on a full measure subset of $\mathbb{R}^2$.*

All convex sets B satisfy this condition; examples of non-convex sets violating it are given on figure 1.2. Both sets are of the form $B = G \times [0,\, 1] \subset \mathbb{R}^2_{x,z} \times R^1_y$, with G being shown on the figure. On fig. 1.2a, a part of the boundary is an arc of parabola with the focus F and with the vertical axis. Incident particles, after making a reflection from the arc, get into the singular point F of the boundary. On fig. 1.2b, one part of the boundary belongs to an ellipse with foci F_1 and F_2, and another part, AB, belongs to a parabola with the focus F_1 and with the vertical axis. After reflecting from AB, particles of the flow get trapped in the ellipse, making infinite number of reflections and approaching the line F_1F_2 as time goes to $+\infty$. In both cases, v_B is not defined on the corresponding positive-measure subsets of $\mathbb{R}^2$.

DEFINITION 1.1.1. *We denote by $\mathcal{B}$ the class of compact connected sets with piecewise smooth boundary satisfying the regularity condition.*



(a) (b)

Figure 1.2: (a) After reflecting from the arc of parabola, the particles get into the singular point F. (b) After reflecting from the arc of parabola AB, the particles get trapped in the ellipse.

Each particle interacting with the body B transmits to it the momentum equal to the particle mass times $((0,0,-1) - v_B(x,y))$. Summing up over all momenta transmitted per unit time, one obtains that the resistance of B equals $-\rho\,\mathrm{R}(B)$, where

$$\mathrm{R}(B) = \iint_{\mathbb{R}^2} (v_B^x,\, v_B^y,\, 1 + v_B^z)\, dx\, dy,$$

and ρ is the flow density. One is usually interested in minimizing the third component of $\mathrm{R}(B)$,[1]

$$\mathrm{R}_z(B) = \iint_{\mathbb{R}^2} (1 + v_B^z(x,y))\, dx\, dy. \tag{1.1.1}$$

If B is convex then the upper part of the boundary ∂B is the graph of a concave function $w(x,y)$. Besides, there is at most one reflection from the boundary, and the velocity of the reflected particle equals $v_B(x,y) = (1+|\nabla w|^2)^{-1}(-2w_x,\, -2w_y,\, 1-|\nabla w|^2)$. Therefore, the formula (1.1.1) takes the form

$$\mathrm{R}_z(w) = \iint_{\Omega} \frac{2}{1+|\nabla w(x,y)|^2}\, dx\, dy, \tag{1.1.2}$$

where Ω is the domain of w. Note that the same construction holds for the 2dimensional problem, therefore we keep the same notation $\mathcal{B}, R(B), R_z(B)$ for this case.

[1]Note that in the axisymmetric cases (i), (iv), and (v), the first and second components of $\mathrm{R}(B)$ are zeros, due to radial symmetry of the functions v_B^x and v_B^y : $\mathrm{R}_x(B) = 0 = \mathrm{R}_y(B)$.

Further, if B is a convex axially symmetric body then (in a suitable reference system) the function w is radial: $w(x, y) = f(\sqrt{x^2 + y^2})$, therefore one has

$$\mathrm{R}_z(f) = 2\pi \int \frac{2r}{1 + f'^2(r)}\, dr, \tag{1.1.3}$$

the integral being taken over the domain of f.

1.2 A new sight

1.2.1 Existence and uniqueness problems

So, in today's language Newton considered the problem of minimization of the functional (1.1.2), where $w : \Omega \to \mathbb{R}_0^+$ is a radial concave function and Ω is a plane disc. It is supposed that the function w describes the upper boundary of a 3D rotational body which has Ω as a base. We have to note that Newton in [1] did not state explicitly the convexity condition for w.

Legendre was the first after Newton who gave a look on the functional (1.1.3) and noted that it does not make physical sense for some obstacles. For example, if f is a zig-zag function with large values of derivative then (1.1.3) tends to zero, and of course this fact has nothing in common with resistance minimization. So he discussed other classes of admissible obstacles, in particular he proposed and described the problem for obstacles of revolution with prescribed length of the profile [3]. Recently M.Belloni and B.Kawohl [4] noted that actually Legendre solved another problem, namely the problem in the class of bodies with prescribed arclength

$$L = \int_0^1 \sqrt{1 + f_r^2(r)}\chi_{\{f_r \neq 0\}} dr$$

In both cases Buttazzo and Kawohl showed that the solution exists, and described trapezoidal form of the solution.

In 1993 G.Buttazzo and B.Kawohl [5, 1993] examined a nonsymmetric case with the rotational symmetry condition removed. They stated the following theorem.

Theorem 1.2.1. G.Buttazzo and B.Kawohl. *Let Ω be a bounded open convex subset of R^2 and let height $h > 0$ be given. Then the minimization problem of $R_z(\cdot)$ for w in the class*

$$C_h = \{w : \Omega \to [0, h],\ w \text{ is concave}\}$$

admits at least one solution, and every solution w has the property that $|\nabla w| \notin (0, 1)$.

The full proof was published in [6, 1995]. In [5, 1993] Buttazzo and Kawohl stated the following open problems:

BK.1 Is it true that the solution is zero on $\partial\Omega$?

BK.2 If the solution unique?

BK.3 Is there a flat region, that is, an open set $\Omega_0 \subset \Omega$ such that $u = h$ on Ω_0?

BK.4 Is the solution Lipschitz continuous up to the boundary $\partial\Omega$?

BK.5 Suppose that Ω is a symmetric set. Should the solution also be symmetric?

In [6, section 5, 1995] authors also considered the so-called "physical case". They introduced the class of bodies S_h:

$$S_h = \{0 \le w \le h, \quad \text{almost every particle hits the body at most once}\}. \tag{1.2.1}$$

They characterized this set by the condition

$$\forall x \in dom(\nabla w), \forall \tau > 0, \text{ such that } x - \tau \nabla w(x) \in \overline{\Omega},$$
$$\frac{w(x - \tau \nabla w(x)) - w(x)}{\tau} \le \frac{1}{2}\left(1 - |\nabla w(x)|^2\right).$$

They showed that it imply

$$|\nabla w(x)| \le \frac{h + \sqrt{h^2 + dist^2(x, \partial\Omega)}}{dist(x, \partial\Omega)}, \quad \forall x \in \Omega, \tag{1.2.2}$$

for all x where w is differentiable. Therefore they proved

Lemma 1.2.2. *For any $h \ge 0$ the set S_h is bounded in $W^{1,\infty}_{loc}(\Omega)$*

More open problems by G. Buttazzo, V. Ferone, B. Kawohl [6]

BFK.1 They noted that it is out of their possibilities to prove the existence of the minimal resistance problem in the class S_h.

BFK.2 Uniqueness of minimizers in the class of general convex domains.

BFK.3 They also noted the all standard methods to prove that "symmetry properties of Ω imply symmetry properties of minimizers" fail.

In [7, 1996] the authors answer questions BK.2 and BK.4; Namely, they considered the class of functions

$$CW_h = \{w \in W^{1,\infty}_{loc} | 0 \leq w(x) \leq h \text{ in } \Omega, w \text{ concave } \}$$

and proved

Theorem 1.2.3. *Let u be Newton's radial solution. Then u does not minimize (1.1.2) in CW_h*

So we see that the functions that minimize (1.1.2) in the class CW_h are not radial and therefore are not unique. (Any function obtained from a solution by rotating Ω around the center is also a solution.) "The proof consisted of remarking that the second derivative of the Newton functional calculated at Newton's function has a negative direction. The result naturally opened the hunt on the true form of the minimizer."[2].

The fact that optimal profiles with circular cross section do not need to be radially symmetric can be also proved by exhibiting nonsymmetric profiles which are more performant than the optimal radial one. This was first discovered by Guasoni in [8], who considered a body of the form obtained as the convex envelope of the set $(\Omega \times \{0\}) \cup (S \times \{h\})$ where S is a segment (see fig. 1.3).

In [9] there was considered the class of obstacles where the maximal cross section Ω is not fixed. Namely, three classes of obstacles $C_{K,Q}$, $C_{V,Q}$ and $S_{A,K,Q}$ were considered,

$$C_{K,Q} = \{E \text{ convex subset of } \mathbb{R}^3 \quad : \quad K \subset E \subset Q\}$$

$$C_{V,Q} = \{E \text{ convex subset of } \mathbb{R}^3 \quad : E \subset Q, |E| \geq V\}$$

$$S_{A,K,Q} = \{E \text{ convex close subset of } \mathbb{R}^3 \quad : K \subset E \subset Q, \sigma(E) \geq A\},$$

[2]Cited by [10]

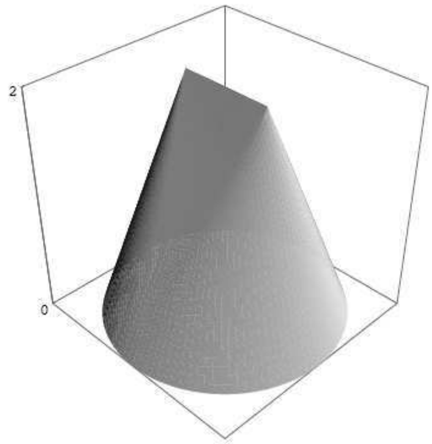


Figure 1.3: A nonradial profile better than the optimal radial one.

where K and Q are two compact subsets of $\mathbb{R}^3$ and V is a positive number, $\sigma(E)$ is a cross section, i.e. area of the projection of E onto a given plane. The authors proved [9, Th. 1] that the resistance minimization problem in classes $C_{K,Q}$ or $C_{V,Q}$ admits at least one solution. Moreover, they noted that the same result is true for classes $\widetilde{C}_{V,Q}$ and $\widetilde{S}_{A,K,Q}$ which differ from the original classes by substituting equalities $|E| \geq V, \sigma(E) \geq A$ for inequalities.

1.2.2 Properties of the solution

H.Berestycki [7],[10, 2001] noted that if u is in the class C^2 on some open subset $\Omega' \subset \Omega$ and satisfies $d^2u < 0$ on Ω', then u is not a minimizer of (1.1.2). Also he stated a conjecture that minimizers could be "affine by parts". In [10, 2001] the authors proved that whenever the Gaussian curvature on the surface of a minimizer u is finite, it is zero.

In [11, 2001] the same authors strengthened this result. Recall that "function f is strictly convex in U" means

$$\forall x, y \in U, \quad \forall t \in (0,1), \quad x \neq y \Rightarrow f(tx + (1-t)y) < tf(x) + (1-t)f(y).$$

Theorem 1.2.4. *[11, 2001] Let w be a minimizer of the problem (1.1.2) and let Ω_1 be an open subset of Ω. Then w is not strictly convex on Ω_1.*

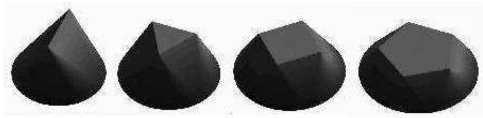


Figure 1.4: Several optimal shapes in $\mathcal{C}_d(h)$ for different values of h.

In [10, 2001] the problem (1.1.2) was considered in a class smaller than C_h. This reduction was motivated by the above mentioned result stating that the minimizer can not be strictly convex. Slightly changing the notation used in n [10, 2001] the definition of the class is as follows.

Let Ω be a disc. Consider the class $\mathcal{C}_d = \mathcal{C}_d(h)$ which contains all functions $w \in C_h$ such that their graph is the convex envelope in $\mathbb{R}^3$ of the set $\Omega \times \{0\}$ and the set $N_0 \times \{h\}$, where

$$N_0 = \{x \in \Omega : w(x) = h\}.$$

Theorem 1.2.5. *[10, 2001] Let $h > 0$ be given. If w is the minimizer in the class $\mathcal{C}_d$ then the set N_0 is a regular polygon centered in the center of Ω. Moreover, the number of sides of the polygon is a non-increasing function of the height h.*

1.3 Case of single impact assumption

1.3.1 Generally non symmetrical bodies

The authors of [13] examined the case of single impact assumption (1.2.2). Using the result of [6] that every function satisfying (1.2.2) must be locally Lipschitz, they considered the topology of $W^{1,\infty}_{loc}(\Omega) \cap C^0(\overline{\Omega})$, where Ω is a strictly convex domain. As already explained in [12], the physical meaning of the problem requires more regularity for the gradient than usually found for functions of $W^{1,\infty}_{loc}(\Omega)$, so they had to define a smaller vector space to work with. They considered

$$P(\Omega) = \{w \in C^0(\overline{\Omega}) : w \text{ polyhedral}\}$$

where "polyhedral" means that w is a result of a finite sequence of minimum or maximum operations on affine functions. The vector space of interest is

$$W(\Omega) := \overline{P(\Omega)},$$

where the closure operation is with respect to the topology of $W^{1,\infty}_{loc}(\Omega) \cap C^0(\overline{\Omega})$. It is easy to check that

$$C^1(\Omega) \cap C^0(\overline{\Omega}) \subsetneq W(\Omega) \subsetneq W^{1,\infty}_{loc}(\Omega) \cap C^0(\overline{\Omega})$$

Discussing radial case below we will show in example below that $W(\Omega) \neq W^{1,\infty}_{loc}(\Omega) \cap C^0(\overline{\Omega})$.

The set of admissible functions is

$$\mathcal{S}_h = \{w \in W(\Omega) : \quad 0 \leq w \leq h, \quad w \text{ satisfies } (1.2.1)\}$$

In fact, if $x \in \overline{\Omega}$ is such that w is not differentiable at x but its hypograph has a vertical tangent plane at $(x, w(x))$, we shall say that x is a *regular* point of w. We extend the notation of $dom(\nabla w)$ for points of the boundary as follows:

$$\mathrm{dom}(\nabla w) = \{x \in \overline{\Omega} \,:\, w \text{ is differentiable at } x\} \cup \{x \in \partial\Omega \,:\, x \text{ is a regular point for } w\}$$

Definition. A function $w \in \mathcal{S}_h$ is said to be "regular on the boundary" if

$$\exists\, \Gamma_w \subset \partial\Omega, \text{ nonempty and open in } \partial\Omega, \quad \Gamma_w \subset dom(\nabla w).$$

The authors proved the following theorem:

Theorem 1.3.1. *[13] 1. Let $w \in \mathcal{S}_h$ be regular on the boundary (in the sense of definition above). Then w is not a minimizer of (1.1.2) in the class $\mathcal{S}_h$.*

1.3.2 Analogue of Newton's problem for bodies with radial symmetry

Introduce the notations:

$$W_r = \{u \in W^{1,\infty}_{loc}(0,1) \cap C^0([0,1]) \text{ such that } w = u\left(\sqrt{x_1^2 + x_2^2}\right) \in W(\Omega)\}$$

and Ω is the unit disc of $\mathbb{R}^2$.

It is not obvious that W_r is actually smaller than $W^{1,\infty}_{loc}(0,1) \cap C^0([0,1])$. In order to show this, let us give an example of a function which does not belong to W_r.

Lemma 1.3.2. *[13, Lemma 3.1] the set $\mathcal{W}_r$ is equal to the set*

$$\{u \in W^{1,\infty}_{loc}(0,1) \cap C^0([0,1]) \text{ : } u' \textit{ has a right-limit and left-limit everywhere in } (0,1)\}$$

Consider a value $t_0 \in (0,1)$, and $\phi(t) = \phi_0(t - t_0)$ where $\phi_0(x) := x^2 \sin(1/x)$. Clearly $\phi \in W^{1,\infty}(0,1)$, since ϕ_0' is bounded near 0. On the other hand, ϕ_0' does not have a right or left limit at 0; hence $\phi \notin W_r$. Note that ϕ' cannot be approximated in $W^{1,\infty}$-topology by step functions (through it is obviously possible in $W^{1,p}$-topology, $p < \infty$);

Theorem 1.3.3. *[12],[13] there exists a minimum $w = u\left(\sqrt{x_1^2 + x_2^2}\right)$ of the minimization problem of (1.1.3) in class of*

$$\{u \in W_r(\Omega) : \quad 0 \le u \le h, \quad w = u\left(\sqrt{x_1^2 + x_2^2}\right) \textit{ satisfies (1.2.1)}\}$$

Moreover, there exists a critical value h^* such that for $h \ge h^*$ the minimizer is unique. And for $h < h^*$ the set of minimizers is not compact in $W^{1,p}(0,1) \cap W_r, \forall p > 0$.

As we can see, the minimum is attained in the radial class, but is not attained in the general class of non-radial functions. "This comes from the presence of a boundary, which induces special effects, and, in particular, allows oscillations near it"[3]...This leads us to the idea that without a boundary, the problem would be more "stable", and a global minimizer could exist with no radial symmetry assumption.

1.3.3 Bodies containing a half-space

Let us describe the result of [14]. Let Ω be a domain tiling the plane (meaning that there exists a finitely generated subgroup $\mathcal{G}_\Omega \subset \mathcal{O}_2$ such that $\cup_{g \in \mathcal{G}_\Omega} g(\overline{\Omega}) = \mathbb{R}^2$, and $g_1(\Omega) \cap g_2(\Omega) = \emptyset$, if $g_1 \neq g_2$); let $\overline{w} : \mathbb{R}^2 \to \mathbb{R}$ be a function with the same periodicity

[3]Cited by [14].

($\overline{w} \circ g = \overline{w}$ for all $g \in \mathcal{G}_\Omega$); and let w be the restriction of $\overline{w}$ to Ω. The authors looked for the body $\{(x,z) \in \mathbb{R}^3; z \leq \overline{w}(x)\}$ minimizing the mean value of (1.1.2), that is

$$F(w;\Omega) = \frac{1}{|\Omega|} \int_\Omega \frac{dx}{1 + |\nabla w(x)|^2} \tag{1.3.1}$$

with respect to all domains Ω tiling the plane and to all functions $w \in W(\Omega) : \Omega \to [0,h]$ having periodicity Ω.

Note that Ω is not well defined in general if only $\overline{w}$ is given. In order to fix the notations, we choose Ω such that $w(x) = 0$ for all $x \in \partial\Omega$ and $w > 0$ in Ω.

Theorem 1.3.4. *[14, Th. 1] Among all regular functions and regular domains tiling the plane, the minimum of F is attained in only two cases, up to a similitude (with the same minimal value):*

1. Ω is a square, $\Omega = (-a,a)^2$ with $a \leq 4h/3$, and w is the function

$$w(x_1,x_2) = max[\phi_a(|x_1|), \phi_a(|x_2|)], \text{ where } \phi_a(x) = \frac{(x+a)^2}{4a} - a.$$

2. Ω is a regular convex hexagon with diameter $4a/\sqrt{3}$, with center $O = (0,0)$ and two vertices $A = (a, a/\sqrt{3}), B = (a, -a/\sqrt{3})$; then w is the function invariant by rotation of $\pi/3$ whose restriction to the triangle OAB is $\phi_a(|x_1|)$.

In both cases, the optimal value for F is given by

$$F_{opt} = \pi + 12\ln 2 - 4\ln 5 - 4\arctan 2 \simeq 0.59330123. \tag{1.3.2}$$

The authors noted that resistance of the infinite tiling is less that 60 percents of the resistance of the plane (which has maximal values).

1.4 Arbitrary number of collisions

1.4.1 Generally non convex bodies and generally non symmetric bodies

By removing both assumptions of symmetry and convexity, one gets the (even wider) class of bodies inscribed in a given cylinder. A.Plakhov was the first [18] who considered the problem of minimization of (1.1.1) without imposing the single impact assumption. He proved the following theorem.

Denote by $\mathcal{O}_\varepsilon(\widehat{B})$ the open ε-neighborhood of $\widehat{B}$.

Theorem 1.4.1. *[18, A.Plakhov,] For any connected bounded set $\widehat{B} \subset \mathbb{R}^3$ and any $\varepsilon > 0$ there exists a connected set B such that the following is true.*
1. $\widehat{B} \subset B \subset \mathcal{O}_\varepsilon(\widehat{B})$.
2. $v_B(x, y)$ is defined and measurable on $\mathbb{R}^2$ and $|R(B)| < \varepsilon$.
3. If, additionally, $\widehat{B}$ is open and $\partial\widehat{B}$ is a two-dimensional C^2 manifold, then B can be chosen to be homeomorphic to $\widehat{B}$, and the corresponding homeomorphism shifts the points a distance smaller than ε.

Then Plakhov studied an analogue of the classical problem, considering generally nonconvex bodies inscribed in the cylinder and with a prescribed circular cross section: Denote $D_a = \{x \in \mathbb{R}^{d-1} : |x| \leq a\}, d = 2, 3$, then

$$\mathcal{G}_h^a = \{B \in \mathcal{B} : D_a \times \{0\} \subset B \subset D_a \times [0, h]\}, \quad a, h > 0.$$

In particular, he considered the following problem: "Determine the smallest m such that a set or a sequence of sets minimizing resistance could be chosen in such a way that the number of collisions with them would not exceed m". Note that case of single impact assumption (part 1.3) is about the case $m = 1$. Let

$$\rho(d) = \inf_{B \in \mathcal{G}_h^a} |R(B)|,$$

then the following theorem holds

Theorem 1.4.2. *[19, A.Plakhov,]*

$$\rho(3) = 0, \quad \rho(2) = 2a\left(1 - \frac{h}{\sqrt{a^2 + h^2}}\right);$$

besides for $d = 2$ and for $d = 3$, $h \geq a/2$, the greatest number of collisions for the corresponding family of sets equals 2, and this value cannot be improved, i.e., one cannot present a minimizing set or a family of sets with the number of collisions equal or less to 1. As $d = 3$ and $0 < h < a/2$, one has $m = 3$.

The author constructed corresponding sequences minimizing $|R(B)|$.

1.4.2 Bodies containing a half-space

Following the basic idea that the minimal resistance can be diminished by allowing multiple reflections, $m > 1$, A.Plakhov considered the problem of minimal specific resistance for the case of unbounded obstacles (see part 1.3.3). Let us describe Plakhov's result [20]: Let $n = (n_1, \dots, n_d) \in S^{d-1}$ such that $n_d > 0$ (d means dimension). Consider sets $B \subset \mathbb{R}^d$ such that

$$\{x \in \mathbb{R}^d : (x, n) < 0\} \subset B. \tag{1.4.1}$$

Denote $\mathbb{R}^d_- = \{x \in \mathbb{R}^d : x_d < 0\}$ and $\Pi = \partial\mathbb{R}^d_- = \{x : x_d = 0\}$. Denote by $\mathcal{B}_n$ the set of bodies $B \subset \mathbb{R}^d$ such that

1. B satisfies (1.4.1);

2. the vector function $v_B(x)$ is defined and Borel measurable on Π minus possibly a set of zero $(d-1)$-dimensional measure.

The total resistance corresponding to a Borel set $A \subset \Pi$ can be defined by

$$R(B, A) = \int_A \left(v_B^1,\, v_B^2, \dots, v_B^{d-1},\, 1 + v_B^d\right) dx, \quad B \in \mathcal{B}_n.$$

Note that scattering direction $v(x)$ should always be out of Π, that is why we have

$$n_d|A| \le |R(B, A)| \le 2|A|, \quad \forall \text{ Borel } A \subset \Pi, \quad B \in \mathcal{B}_n.$$

Evidently the upper bound is exact, to see it it suffices to consider the surface which is almost everywhere perpendicular to n. The following theorem shows that the lower bound is also exact.

Theorem 1.4.3. *[20] There exists a sequence of sets $B_k \in \mathcal{B}_n$ such that*

$$\lim_{k\to\infty} |R(B, A)| = n_d|A|,$$

for any Borel set $A \subset \Pi$ with $|A| < \infty$.

Let us compare this result with the result of (1.3.3). If $n = (0, 0, 1)$ then for

$$\frac{\inf_{B\in\mathcal{B}_n} |R(B, A)|}{|R(\mathbb{R}^d_-, A)|} = 0.5,$$

and in the case of single impact assumption the corresponding coefficient equals 0.593 (see (1.3.2)).

1.4.3 Case of symmetric but generally non convex bodies

By 2007 there have been studied the following cases:

(i) convex & axisymmetric (the classical Newton problem);

(ii) convex but generally non-symmetric (studied in [5]-[9]);

(iii) generally nonconvex and non-symmetric (studied in [18],[19]).

It was natural to consider the 4^{th} logically possible case:

(iv) axisymmetric but generally nonconvex bodies.

This problem was investigated in [25],[26], and the results are presented in the chapter 2. We should note that the authors impose another condition which was not mentioned in the title, since it did not seem to be significant. Actually, authors of [25],[26] considered simply connected surfaces only.

We should also notice that in the paper [12] there was considered the intermediate class of

(v) axially symmetric nonconvex bodies, under the additional so-called "single impact assumption".

On the contrary, multiple reflections are allowed in our setting; we only assume that the body's boundary is piecewise smooth and satisfies the regularity condition stated above in 1.1.1.

Let $\mathcal{G}_h$ be the class of compact connected sets $G \subset \mathbb{R}^2$ with piecewise smooth boundary that are inscribed in the rectangle $-1 \leq x \leq 1,\ 0 \leq z \leq h$. That is, belong to the rectangle and have nonempty intersection with each of its sides. Moreover it is symmetric with respect to the axis Oz, and satisfy the regularity condition. Denote by $\mathcal{G}_h^{conv}$ the class of convex sets from $\mathcal{G}_h$. One can easily see that if $G \in \mathcal{G}_h$ then $\operatorname{conv} G \in \mathcal{G}_h^{conv}$. For $G \subset \mathcal{G}_h^{conv}$ define the *modified law* of reflection as follows. A particle initially moves vertically downwards according to $x(t) = x,\ z(t) = -t$ and then reflects at a regular point of the boundary ∂G; at this point the velocity instantaneously changes to $\hat{v}_G(x) = (\hat{v}_G^x(x), \hat{v}_G^z(x))$, where $\hat{v}_G(x)$ is the unit vector tangent to ∂G such that $\hat{v}_G^z(x) \leq 0$ and $x \cdot \hat{v}_G^x(x) \geq 0$ (see fig. 1.5).

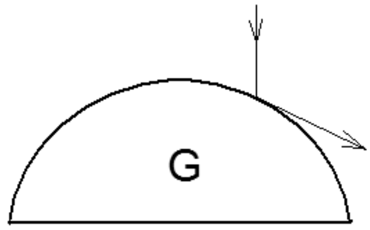


Figure 1.5: Modified reflection law.

The set $G \in \mathcal{G}_h^{conv}$ is bounded above by the graph of a concave even function $z = f_G(x)$. For $x > 0$, one has

$$\hat{v}_G(x) = \frac{(1, f'_G(x))}{\sqrt{1 + f'^2_G(x)}}\,. \tag{1.4.2}$$

The resistance of G under the modified reflection law equals $(0, -\widehat{R}(G))$, where

$$\widehat{R}(G) = \int_0^1 (1 + \hat{v}^z_G(x))\, x\, dx\,. \tag{1.4.3}$$

Taking into account (2.2.1), one gets

$$\widehat{R}(G) = \int_0^1 \left(1 + \frac{f'_G(x)}{\sqrt{1 + f'^2_G(x)}}\right) x\, dx; \tag{1.4.4}$$

the function f_G is concave, nonnegative, and monotone non-increasing, with $f_G(0) = h$.

The following theorem allows to restrict the problem of minimization of resistance R in the class of bodies of revolution to the problem of minimization of $\widehat{R}$ in the class of *convex* bodies of revolution.

Theorem 1. $\inf_{G\in\mathcal{G}_h} R(G) = \inf_{G\in\mathcal{G}_h^{conv}} \widehat{R}(G)$.

This theorem immediately follows from the next two lemmas, proved in the chapter 2.

Lemma 1.4.4. *Let $G \in \mathcal{G}_h$. Note that convex hull* $\operatorname{conv} G$ *belongs to G_h^{conv}. It holds* $\widehat{R}(\operatorname{conv} G) \le R(G)$.

Lemma 1.4.5. *Let $G \in \mathcal{G}_h^{conv}$. There exists a sequence of nonconvex sets $G_n \in \mathcal{G}_h$ such that* $\lim_{n\to\infty} R(G_n) = \widehat{R}(G)$.

The search for the minimum of $\widehat{R}$ is restricted now to minimization of $\int_0^1 (1 - \frac{f'(x)}{\sqrt{1+f'^2(x)}})\, x\, dx$ in the class of convex and nondecreasing functions f defined on $[0,\, 1]$, satisfying relations $f(0) = 0,\ f(1) \le h$. This minimization is done using Pontryagin's minimum principle; as a result we obtain the following theorem.

Theorem 2.

$$\inf_{G \in \mathcal{G}_h^{conv}} \widehat{R}(G) = \frac{1}{2} - \frac{1}{16}\left(8 - 2Z^{2/3} - 3Z^{4/3}\right)\sqrt{1 - Z^{2/3}} + \frac{3Z^2}{16} \ln\left(\frac{1 + \sqrt{1 - Z^{2/3}}}{Z^{1/3}}\right), \tag{1.4.5}$$

where $Z = Z(h)$ is a unique solution of the equation:

$$h = \int_z^1 \sqrt{\left(\frac{x}{z}\right)^{2/3} - 1}\, dx = -\frac{3}{8}\left((z^{1/3} - 2z^{-1/3})\sqrt{1 - z^{2/3}} - \ln\left(\frac{1 + \sqrt{1 - z^{2/3}}}{z^{1/3}}\right)\right). \tag{1.4.6}$$

Set $\widehat{G}_h \in \mathcal{G}_h^{conv}$, and is the minimizer of the functional $\widehat{R}$, is given by

$$\begin{cases} 0 \le z \le h, & \text{if } |x| \le Z \\ 0 \le z \le h - \int_Z^x \sqrt{\left(\frac{t}{Z}\right)^{2/3} - 1}\, dt, & \text{if } Z < |x| \le 1. \end{cases} \tag{1.4.7}$$

Denote by $\mathcal{R}(h) := \inf_{G \in \mathcal{G}_h} R(G)$ and $\mathcal{R}_N(h) := \inf_{G \in \mathcal{G}_h^{conv}} R(G)$ the minimal values for our problem and for Newton's one, respectively. The following expressions: $\mathcal{R}(0^+) = \frac{1}{2}\mathcal{R}_N(0^+) = 1/2$; $\mathcal{R}(h) = \frac{1}{4}\mathcal{R}_N(h)(1 + o(1)) = \frac{27}{128}\frac{1+o(1)}{h^2}$ as $h \to +\infty$.

1.4.4 Bodies of zero resistance and bodies invisible in one direction

The class $\mathcal{G}_h$ defined in the previous section 1.4.3 consists of compact connected sets $G \subset \mathbb{R}^2$ with piecewise smooth boundary that are inscribed in the rectangle $-1 \le x \le 1$, $0 \le z \le h$. It was additionally supposed that each body $G \in \mathcal{G}_h$ is symmetric with respect to the axis Oz and satisfies the regularity condition.

Let us now remove the condition that the set G should be connected. It was very surprising for us to learn that in the resulting wider class there exist bodies having exactly *zero resistance*! The 3D bodies obtained by rotating such sets about Oz, in turn, have zero resistance in $\mathbb{R}^3$. These bodies are topologically equivalent to a torus, and consequently, are connected, but not simply connected.

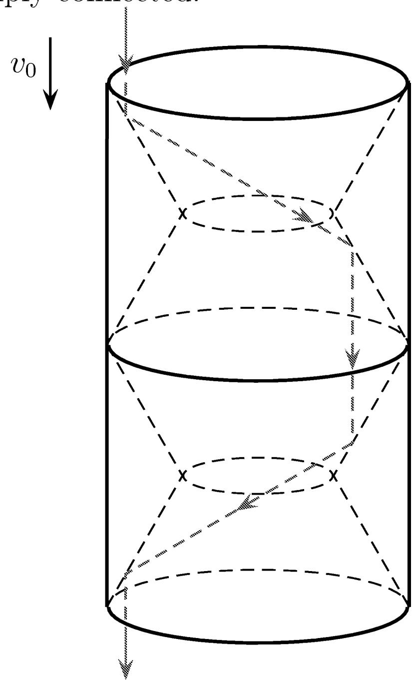


Figure 1.6: A body invisible in the direction v_0. It is obtained by taking 4 truncated cones out of the cylinder.

We have also constructed bodies that are invisible in one direction, say, $v_0 \in S^2$. This means that the trajectory of each particle with initial asymptotic velocity v_0 coincides with a straight line outside a prescribed bounded set. Such a body having mirror surface will really become invisible to an observer staying far enough from the body in this direction.

There have been presented several families of bodies of zero resistance, and their general properties have been studied. In particular, we proved that the maximal number of reflections of any individual particle from the body of zero resistance is more or equal to 2, and maximal number of reflections of any individual particle from the body invisible in one direction is more or equal to 4.

Recall for comparison that in 1.4.1 we considered a class of bodies for which the infimum of resistance is zero but cannot be attained. Actually, the properties of the bodies of small resistance constructed there are highly sensitive to variation of the incidence angle; namely, if the direction of the incident flow slightly changes, their resistance drastically increases. This is not the case for the bodies of zero resistance constructed by us in chapter 3: slight changing of the direction of incidence would imply only slight change of the resistance. More precisely, the following theorem holds.

Theorem 1.4.6. *Let B be a body having zero resistance in the direction $v_0 \in S^2$ presented in section 3.4. Then there exist positive constants C_1, C_2 and $\delta > 0$ such that*

$$C_1|v - v_0| \le R_v(B) \le C_2|v - v_0|, \quad \textit{if} \quad |v - v_0| \le \delta.$$

In the end of that chapter we discuss possible applications of bodies of zero resistance.

1.5 Averaged resistance over all directions

In subsection 1.1.1 we described resistance $R(B)$ supposing that the incident flow has velocity $(0, 0, -1) \in S^2$. Evidently one can define it for any arbitrary velocity vector $v \in S^2$. Corresponding function we will call $R(B, v), \quad v \in S^2$.

A.Plakhov introduced the averaged resistance

$$\widetilde{R}(B) = \int_{S^{d-1}} R(B, v) d\mu(v), \tag{1.5.1}$$

where $\mu(v)$ is the Lebesgue measure on the unit sphere S^{d-1}. He considered the problem of minimization $\widetilde{R}$ in two class of bodies:

$$P_g = \{B \ : \ vol(B) = 1\}$$

$$P_c = \{B \text{ convex} : \ vol(B) = 1\}$$

The following theorems were proved:

1.5.1 The convex problem

Theorem 1.5.1. *The least value of $\widetilde{R}$ in the class of bodies $B \in P_c$ is attained at the unit ball and is equal to*

$$\inf_{B \in P_c} \widetilde{R}(B) = \frac{c_d |S^{d-1}|}{B^{\frac{d-1}{d}}}, \quad c_d = \int_{S^{d-1}} |v_1|^3 d\mu(v),$$

where v_1 is the first coordinate of the vector v.

Remark 1 In particular, as d = 2, one has $c_2 = 8/3$, $|B^2| = \pi$, $|S^1| = 2\pi$, and therefore

$$\inf_{B\in P_c} \widetilde{R}(B) = \frac{16}{3}\sqrt{\pi}.$$

1.5.2 The non-convex problem for $d = 2$

Let $B \subset \mathbb{R}^2$ be a bounded set with piecewise smooth boundary of class C^2. A billiard motion $(x(t), v(t))$ in $\mathbb{R}^2\backslash B$ is called regular and asymptotically free (r.a.f.), if it is defined for all $t \in \mathbb{R}$ and has a finite number (maybe none) of reflections at regular points of ∂B. Obviously, any pair $(x, v) \in \mathbb{R}^2 \times S^1$ uniquely determines a billiard motion $(x(t), v(t))$ by the condition $x(t) = x + vt$ for t sufficiently small. Denote $C = \{(x, v) \in \mathbb{R}^2 \times S^1$: the corresponding billiard motion is not r.a.f.$\}$.

Statement[21, Part 3.]. C is a set of measure zero.

Let $(x(t), v(t))$ be a r.a.f. billiard motion, with $\lim_{t\to-\infty} v(t) = v, \lim_{t\to-\infty}(x(t) - vt) = x$; denote $\lim_{t\to\infty} v(t) =: v^+(x, v)$. This relation defines a function v^+ on the set $(\mathbb{R}^2\times S^1)\backslash C$. It is easy to see that $(\mathbb{R}^2\times S^1)\backslash C$ is open, and the function v^+ is continuous, bounded, and coincides with v for $|x|^2 - (x, v)^2$ sufficiently large. Hence for almost all values of v there exists the integral

$$R(B, v) = \int_{0<(x,v)<1} (v - v^+(x, v))dx,$$

which is a measurable function of v, therefore there also exists the integral $\widetilde{R}(B)$ (see (1.5.1)).

The author provided that if the direction of the flow is unknown, then the average resistance of minimizer in the class P_g is close to the resistance of the circle.

Theorem 1.5.2. *[21, Th. 2]*

$$0.9878 \le \frac{\inf_{B\in P_g} \widetilde{R}(B)}{\inf_{B\in P_c} \widetilde{R}(B)} \le 1$$

Here we stop our review and go on to the detailed exposition of our own results. Notice that several articles [24],[16] concerning extensions of the Newton's functional remain out of scope of our review.

Chapter 2

Generally non-convex bodies of revolution of minimal resistance

2.1 Description of the class of bodies

Let B be a compact connected set inscribed in the cylinder $x^2 + y^2 \le 1,\ 0 \le z \le h$ and possessing rotational symmetry with respect to the axis Oz. This set is uniquely defined by its vertical central cross section $G = \{(x, z) : (x, 0, z) \in B\}$. It is convenient to reformulate the problem in terms of the set G.

Consider the billiard in $\mathbb{R}^2 \setminus G$ and suppose that a billiard particle initially moves according to $x(t) = x,\ z(t) = -t$, then makes a finite number of reflections (maybe none) at regular points of ∂G, and finally moves freely with the velocity $v_G(x) = (v_G^x(x), v_G^z(x))$. The regularity condition now means that the so determined function v_G is defined for almost every x. One can see that $\nu_B^x(x, y) = (x/\sqrt{x^2 + y^2})v_G^x(\sqrt{x^2 + y^2}),\ \nu_B^y(x, y) = (y/\sqrt{x^2 + y^2})v_G^x(\sqrt{x^2 + y^2})$, and $\nu_B^z(x, y) = v_G^z(\sqrt{x^2 + y^2})$. It follows that $\mathrm{R}_x(B) = 0 = \mathrm{R}_y(B)$ and $\mathrm{R}_z(B) = 2\pi \int_0^1 (1 + v_G^z(x))\, x\, dx$. Thus, our minimization problem takes the form

$$\inf_{G \in \mathcal{G}_h} R(G), \quad \text{where } R(G) = \int_0^1 (1 + v_G^z(x))\, x\, dx \tag{2.1.1}$$

and $\mathcal{G}_h$ is the class of compact connected sets $G \subset \mathbb{R}^2$ with piecewise smooth boundary

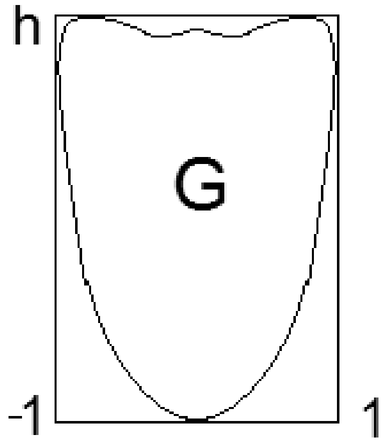


Figure 2.1: A set $G \in \mathcal{G}_h$.

that are inscribed in the rectangle $-1 \le x \le 1,\ 0 \le z \le h$,[1] are symmetric with respect to the axis Oz, and satisfy the regularity condition (see fig. 2.1).

The main results are stated in section 2: the minimization problem is solved and the solution is compared with the Newton solution (case (i)) and the single-impact solution (case (v)). Details of all proofs are put in section 3.

2.2 Statement of the results

Denote by $\mathcal{G}_h^{conv}$ the class of convex sets from $\mathcal{G}_h$. One can easily see that if $G \in \mathcal{G}_h$ then $\operatorname{conv} G \in \mathcal{G}_h^{conv}$. For $G \subset \mathcal{G}_h^{conv}$ define the *modified law* of reflection as follows. A particle initially moves vertically downwards according to $x(t) = x,\ z(t) = -t$ and reflects at a regular point of the boundary ∂G; at this point the velocity instantaneously changes to $\hat{v}_G(x) = (\hat{v}_G^x(x), \hat{v}_G^z(x))$, where $\hat{v}_G(x)$ is the unit vector tangent to ∂G such that $\hat{v}_G^z(x) \le 0$ and $x \cdot \hat{v}_G^x(x) \ge 0$ (see fig. 2.2).

The set $G \in \mathcal{G}_h^{conv}$ is bounded above by the graph of a concave even function $z = f_G(x)$. For $x > 0$, one has

$$\hat{v}_G(x) = \frac{(1, f_G'(x))}{\sqrt{1 + f_G'^2(x)}}\,. \tag{2.2.1}$$

[1]That is, belong to the rectangle and have nonempty intersection with each of its sides.

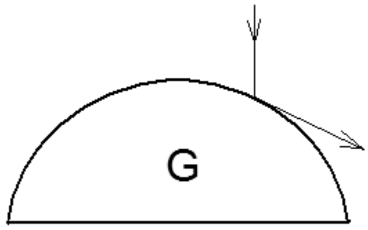


Figure 2.2: Modified reflection law.

The resistance of G under the modified reflection law equals $(0, -\widehat{R}(G))$, where

$$\widehat{R}(G) = \int_0^1 (1 + \hat{v}_G^z(x))\, x\, dx\,. \tag{2.2.2}$$

Taking into account (2.2.1), one gets

$$\widehat{R}(G) = \int_0^1 \left(1 + \frac{f'_G(x)}{\sqrt{1 + f'^2_G(x)}} \right) x\, dx; \tag{2.2.3}$$

the function f_G is concave, nonnegative, and monotone non-increasing, with $f(0) = h$.

Theorem 2.2.1.

$$\inf_{G \in \mathcal{G}_h} R(G) = \inf_{G \in \mathcal{G}_h^{conv}} \widehat{R}(G). \tag{2.2.4}$$

This theorem follows from the following lemmas 2.2.2 and 2.2.3 which will be proved in the next section.

Lemma 2.2.2. *For any $G \in \mathcal{G}_h$ one has*

$$R(G) \geq \widehat{R}(conv G).$$

Lemma 2.2.3. *Let $G \in \mathcal{G}_h^{conv}$. Then there exists a sequence of sets $G_n \in \mathcal{G}_h$ such that*

$$\lim_{n \to \infty} R(G_n) = \widehat{R}(G).$$

Indeed, lemma 2.2.2 implies that $\inf_{G \in \mathcal{G}_h} R(G) \geq \inf_{G \in \mathcal{G}_h^{conv}} \widehat{R}(G)$, and lemma 2.2.3 implies that $\inf_{G \in \mathcal{G}_h} R(G) \leq \inf_{G \in \mathcal{G}_h^{conv}} \widehat{R}(G)$.

Theorem 2.2.1 allows one to state the minimization problem (2.1.1) in an explicit form. Namely, taking into account (2.2.3) and putting $f = h - f_G$, one rewrites the right hand side of (2.2.4) as

$$\inf_{f\in\mathcal{F}_h} \int_0^1 \left(1 - \frac{f'(x)}{\sqrt{1+f'^2(x)}}\right) x\,dx, \tag{2.2.5}$$

where $\mathcal{F}_h$ is the set of convex monotone non-decreasing functions $f : [0,\, 1] \to [0,\, h]$ such that $f(0) = 0$. The solution of (2.2.5) is provided by the following general theorem.

Consider a positive piecewise continuous function $p = p(u)$ defined on $\mathbb{R}_+ := [0, +\infty)$ and converging to zero as $u \to +\infty$, and consider the problem

$$\inf_{f\in\mathcal{F}_h} \mathcal{R}[f], \quad \text{where} \quad \mathcal{R}[f] = \int_0^1 p(f'(x))\, x\, dx. \tag{2.2.6}$$

Denote by $\bar{p}(u)$, $u \in \mathbb{R}_+$ the greatest convex function that does not exceed $p(u)$. Put $\xi_0 = -1/\bar{p}'(0)$ and $u_0 = \inf\{u > 0 : \bar{p}(u) = p(u)\}$. One always has $\xi_0 \ge 0$; if $u_0 = 0$ and there exists $p'(0)$ then $\xi_0 = -1/p'(0)$, and if $u_0 > 0$ then $\xi_0 = u_0/(p(0) - p(u_0))$. Denote by $u = \upsilon(z)$, $z \ge \xi_0$ the generalized inverse of the function $z = -1/\bar{p}'(u)$, that is, $\upsilon(z) = \inf\{u \in \mathbb{R}_+ : -1/\bar{p}'(u) \ge z\}$. By Υ, denote the primitive of υ: $\Upsilon(z) = \int_{\xi_0}^z \upsilon(\xi) d\xi$, $z \ge \xi_0$. Finally, put $\mathcal{R}(h) := \inf_{f\in\mathcal{F}_h} \mathcal{R}[f]$.

Theorem 2.2.4. *For any $h > 0$ the solution f_h of the problem (2.2.6) exists and is uniquely determined by*

$$f_h(x) = \begin{cases} 0 & \text{if } 0 \le x \le x_0 \\ \frac{1}{Z}\,\Upsilon(Zx) & \text{if } x_0 < x \le 1\,, \end{cases} \tag{2.2.7}$$

where $Z = Z(h)$ is a unique solution of the equation

$$\Upsilon(z) = zh \tag{2.2.8}$$

and $x_0 = x_0(h) = \xi_0/Z(h)$. Further, one has $f_h'(x_0 + 0) = u_0$. The function $x_0(h)$ is continuous and $x_0(0) = 1$. The minimal resistance equals

$$\mathcal{R}(h) \;=\; \frac{1}{2}\left(\bar{p}(\upsilon(Z)) \;+\; \frac{\upsilon(Z) - h}{Z}\right); \tag{2.2.9}$$

in particular, $\mathcal{R}(0) = p(0)/2$.

If, additionally, the function p satisfies the asymptotic relation $p(u) = cu^{-\alpha}(1 + o(1))$ as $u \to +\infty$, $c > 0$, $\alpha > 0$ then

$$x_0(h) = c\alpha \left(\frac{\alpha+1}{\alpha+2}\right)^{\alpha+1} \xi_0 h^{-\alpha-1}(1 + o(1)), \quad h \to +\infty, \tag{2.2.10}$$

and

$$\mathcal{R}(h) = \frac{c}{2}\left(\frac{\alpha+1}{\alpha+2}\right)^{\alpha+1} h^{-\alpha}(1 + o(1)), \quad h \to +\infty. \tag{2.2.11}$$

Let us apply the theorem to the three cases under consideration.

1. First consider the *non-convex case.* The problem (2.2.5) we are interested in is a particular case of (2.2.6) with $p(u) = p_{\rm nc}(u) := 1 - u/\sqrt{1+u^2}$ (the subscript "nc" stands for "non-convex"). The function $p_{\rm nc}$ itself, however, is convex, hence $u_0 = 0$ and $\bar{p}_{\rm nc} \equiv p_{\rm nc}$. Further, one has $-1/\bar{p}'_{\rm nc}(u) = (1+u^2)^{3/2}$, therefore $\upsilon_{\rm nc}(z) = \sqrt{z^{2/3} - 1}$, $\xi_0^{\rm nc} = 1$, and

$$\Upsilon_{\rm nc}(z) = \frac{3}{8}(2z^{2/3} - 1)z^{1/3}\sqrt{z^{2/3}-1} - \frac{3}{8}\ln(z^{1/3} + \sqrt{z^{2/3}-1}). \tag{2.2.12}$$

The formulas (2.2.12), (2.2.8), and (2.2.7) with $x_0 = 1/Z$, determine the solution of (2.2.5). Notice that, as opposed to the Newton case, the solution is given by the *explicit* formulas. However, they contain the parameter Z to be defined implicitly from (2.2.8).

Further, according to theorem 2, $f'_h(x_0 + 0) = 0 = f'_h(x_0 - 0)$, $x_0 = x_0^{\rm nc}$, hence the solution f_h is differentiable everywhere in (0, 1). Besides, one has

$$x_0^{\rm nc}(h) = \frac{27}{64}\, h^{-3}(1 + o(1)) \quad \text{as} \;\; h \to +\infty. \tag{2.2.13}$$

The minimal resistance is calculated according to (2.2.9); after some algebra one gets

$$\mathcal{R}_{\rm nc}(h) \;=\; \frac{1}{2} + \frac{3 + 2Z^{2/3} - 8Z^{4/3}}{16Z^{5/3}}\sqrt{Z^{2/3}-1} + \frac{3}{16Z^2}\ln(Z^{1/3} + \sqrt{Z^{2/3}-1}).$$

One also gets from theorem 2.2.4 that $\mathcal{R}_{\rm nc}(0) = 0.5$ and

$$\mathcal{R}_{\rm nc}(h) = \frac{27}{128}h^{-2}(1 + o(1)) \quad \text{as} \;\; h \to +\infty. \tag{2.2.14}$$

2. The original *Newton problem* (case (i) in our classification) is also a particular case of (2.2.6), with $p(u) = p_N(u) := 2/(1+u^2)$. One has $u_0 = 1$ and $\bar{p}_N(u) =$

$\begin{cases} 2-u & \text{if } 0 \le u \le 1 \\ 2/(1+u^2) & \text{if } \quad u \ge 1 \end{cases}$, and after some calculation one gets that $\xi_0^N = 1$ and the function $\Upsilon_N(z)$, $z \ge 1$, in a parametric representation, is $\Upsilon_N = \frac{1}{4}\,(3u^4/4 + u^2 - \ln u - 7/4)$, $z = (1+u^2)^2/(4u)$, $u \ge 1$. From here one obtains the well-known Newton solution: if $0 \le x \le x_0$ then $f_h(x) = 0$, and if $x_0 < x \le 1$ then f_h is defined parametrically: $f_h = \frac{x_0}{4}\,(3u^4/4 + u^2 - \ln u - 7/4)$, $x = \frac{x_0}{4}\,\frac{(1+u^2)^2}{u}$, where $x_0 = 4u_*/(1+u_*^2)^2$ and u_* is determined from the equation $(3u_*^4/4 + u_*^2 - \ln u_* - 7/4)\,u_*/(1+u_*^2)^2 = h$. The function f_h is not differentiable at x_0: one has $f_h'(x_0+0) = 1$ and $f_h'(x_0-0) = 0$.

One also has $\mathcal{R}_N(0) = 1$,

$$\mathcal{R}_N(h) = \frac{27}{32}h^{-2}(1+o(1)) \quad \text{as } h \to +\infty. \tag{2.2.15}$$

and

$$x_0^N(h) = \frac{27}{16}\,h^{-3}(1+o(1)) \quad \text{as } h \to +\infty. \tag{2.2.16}$$

3. The minimal problem in the *single impact case* with $h > M^* \approx 0.54$ can also be reduced to (2.2.6), with $p(u) = p_{si}(u) := \begin{cases} p^* & \text{if } u = 0 \\ 2/(1+u^2) & \text{if } \quad u > 0 \end{cases}$, where $p^* = 8(\ln(8/5) + \arctan(1/2) - \pi/4) \approx 1.186$. This fact can be easily deduced from [12]; for the reader's convenience we put the details of derivation in the next section.[2] From the above formula one can calculate that $u_0 \approx 1.808$ and $\xi_0^{si} \approx 2.52$.

The asymptotic formulas here take the form

$$x_0^{si}(h) = \xi_0^{si} \cdot x_0^N(h)(1+o(1)) \quad \text{as } h \to +\infty \tag{2.2.17}$$

and

$$\mathcal{R}_{si}(h) = \frac{27}{32}h^{-2}(1+o(1)) \quad \text{as } h \to +\infty. \tag{2.2.18}$$

Finally, using the results of [12], one can show that $\mathcal{R}_{si}(0) = \pi/2 - 2\arctan(1/2) \approx 0.6435$. This will also be made in the next section.

[2] We would like to stress that the results presented here about the single impact case can be found in [12] or can be easily deduced from the main results of [12].

Now we are in a position to compare the solutions in the three cases. One obviously has $\mathcal{R}_{\rm nc}(h) \le \mathcal{R}_{\rm si}(h) \le \mathcal{R}_N(h)$. From the above formulas one sees that $\mathcal{R}_{\rm nc}(0) = 0.5$, $\mathcal{R}_N(0) = 1$, and $\mathcal{R}_{si}(0) \approx 0.6435$. Besides, one has $\lim_{h\to+\infty}(\mathcal{R}_{\rm nc}(h)/\mathcal{R}_N(h)) = 1/4$ and $\lim_{h\to+\infty}(\mathcal{R}_{\rm si}(h)/\mathcal{R}_N(h)) = 1$. Thus, for "short" bodies, the minimal resistance in the nonconvex case is two times smaller than in the Newton case, and 22% smaller, as compared to the single impact case. For "tall" bodies, the minimal resistance in the nonconvex case is four times smaller as compared th the Newton case, while the minimal resistance in the Newton case and in the single impact case are (asymptotically) the same.

In the three cases of interest, the convex hull of the three-dimensional optimal body of revolution has a flat disk of radius $x_0(h)$ at the front part of its boundary. One always has $x_0(0) = 1$. For "tall" bodies, one has $\lim_{h\to+\infty}(x_0^{\rm nc}(h)/x_0^N(h)) = 1/4$ and $\lim_{h\to+\infty}(x_0^{\rm si}(h)/x_0^N(h)) = \xi_0^{\rm si} \approx 2.52$; that is, the disk radius in the non-convex case and in the single impact case is, respectively, 4 times smaller and 2.52 times larger, as compared to the Newton case.

Besides, in the nonconvex case, the front part of the surface of the body's convex hull is smooth. On the contrary, in the Newton case, the front part of the body's surface has singularity at the boundary of the front disk.

2.3 Proofs of the results

Proof of lemma 2.2.2

It suffices to show that

$$v_G^z(x) \ge \hat{v}_{\operatorname{conv} G}^z(x) \ \text{ for any } \ x \in [0,\, 1]. \tag{2.3.1}$$

Consider two scenarios of motion for a particle that initially moves vertically downwards, $x(t) = x$ and $z(t) = -t$. First, the particle hits $\operatorname{conv} G$ at a point $r_0 \in \partial(\operatorname{conv} G)$ according to the modified reflection law and then moves with the velocity $\hat{v}_{\operatorname{conv} G}(x)$. Second, it hits G (possibly several times) according to the law of elastic reflection, and then

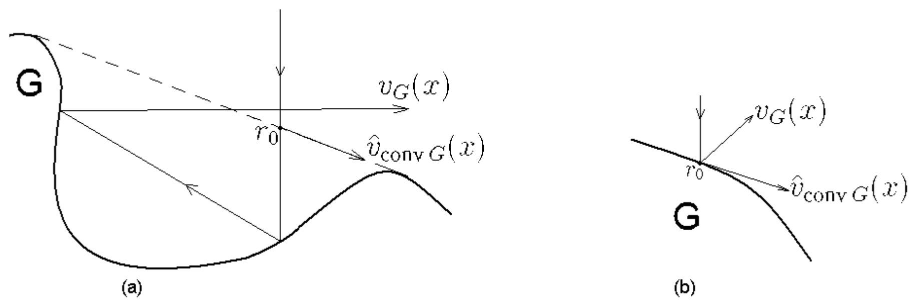


Figure 2.3: Two scenarios of reflection.

moves with the velocity $v_G(x)$. Denote by n the outer unit normal to $\partial(\text{conv}\, G)$ at r_0; on fig. 2.3 there are shown two possible cases: $r_0 \in \partial G$ and $r_0 \notin \partial G$.

It is easy to see that

$$\langle v_G(x), n \rangle \geq 0, \tag{2.3.2}$$

where $\langle \cdot\,, \cdot \rangle$ means the scalar product. Indeed, denote by $r(t) = (x(t), z(t))$ the particle position at time t. At some instant t_1 the particle intersects $\partial(\text{conv}\, G)$ and then moves outside $\text{conv}\, G$. The function $\langle r(t), n \rangle$ is linear and satisfies $\langle r(t), n \rangle \geq \langle r(t_1), n \rangle$ for $t \geq t_1$, therefore its derivative $\langle v_G(x), n \rangle$ is positive.

Let $\varphi_0 = \arcsin(\widehat{v}^z_{\text{conv}\, G}(x)) \in [-\pi/2, 0]$ be the angle between $\widehat{v}_{\text{conv}\, G}(x)$ and axe OX. Let $\widehat{\varphi} = \arcsin(v^z_G(x))$ be the angle between $v_G(x)$ and axe OX. From (2.3.2) we have $\widehat{\varphi} \in [\varphi_0, \varphi_0 + \pi]$. We get now (2.3.1) from the evident statement

$$\sin \varphi_0 = \min_{\varphi \in [\varphi_0, \varphi_0 + \pi]} \sin \varphi, \quad \varphi_0 \in [-\pi/2, 0].$$

Proof of lemma 2.2.3

Take a family of piecewise affine even functions $f_\varepsilon : [-1, 1] \to [0, h]$ such that f'_ε uniformly converges to f'_G as $\varepsilon \to 0^+$. Require also that the functions f_ε are concave and monotone decreasing as $x > 0$, and $f_\varepsilon(0) = h$, $f_\varepsilon(1) = f_G(1)$. Consider the family of convex sets $G_\varepsilon \in \mathcal{G}_h^{conv}$ bounded from above by the graph of f_ε and from below, by the segment $-1 \leq x \leq 1$, $z = 0$. Taking into account (2.2.3), one gets $\lim_{\varepsilon \to 0^+} \widehat{R}(G_\varepsilon) = \widehat{R}(G)$.

Below we shall determine a family of sets $G_{\varepsilon,\delta} \in \mathcal{G}_h$ such that $\lim_{\delta \to 0^+} R(G_{\varepsilon,\delta}) = \widehat{R}(G_\varepsilon)$ and next, using the diagonal method, select a sequence $\varepsilon_n \to 0$, $\delta_n \to 0$ such that

$\lim_{n\to\infty} R(G_{\varepsilon_n,\delta_n}) = \lim_{n\to\infty} \widehat{R}(G_{\varepsilon_n}) = \widehat{R}(G)$. This will finish the proof.

Fix $\varepsilon > 0$ and denote by $-1 = x_{-m} < x_{-m+1} < \ldots < x_0 = 0 < \ldots < x_m = 1$ the jump values of the piecewise constant function f'_ε. (One obviously has $x_{-i} = -x_i$.) For each $i = 1, \ldots, m$ we shall define a non self-intersecting curve $l^{i,\varepsilon,\delta}$ that connects the points $(x_{i-1}, f_\varepsilon(x_{i-1}))$ and $(x_i, f_\varepsilon(x_i))$ and is contained in the quadrangle $x_{i-1} \le x \le x_i$, $f_\varepsilon(x_i) \le z \le f_\varepsilon(x_{i-1}) + (f'_\varepsilon(x_{i-1}+0)+\delta)\cdot(x - x_{i-1})$. The curve $l^{-i,\varepsilon,\delta}$ is by definition symmetric to $l^{i,\varepsilon,\delta}$ with respect to the axis Oz. Let now $l^{\varepsilon,\delta} := \cup_{-m\le i\le m} l^{i,\varepsilon,\delta}$ and let $G_{\varepsilon,\delta}$ be the set bounded by the curve $l^{\varepsilon,\delta}$, by the two vertical segments $0 \le z \le f_\varepsilon(1),\ x = \pm 1$, and by the horizontal segment $-1 \le x \le 1,\ z = 0$.

For an interval $I \subset [0,\, 1]$, define

$$\widehat{R}_I(G_\varepsilon) := \int_I (1 + \hat{v}^z_{G_\varepsilon}(x))\, x\, dx \tag{2.3.3}$$

and

$$R_I(G_{\varepsilon,\delta}) := \int_I (1 + v^z_{G_{\varepsilon,\delta}}(x))\, x\, dx. \tag{2.3.4}$$

Denote $I_i = [x_{i-1},\, x_i]$; one obviously has $\widehat{R}(G_\varepsilon) = \sum_{i=1}^m \widehat{R}_{I_i}(G_\varepsilon)$ and $R(G_{\varepsilon,\delta}) = \sum_{i=1}^m R_{I_i}(G_{\varepsilon,\delta})$. Thus, it remains to determine the curve $l^{i,\varepsilon,\delta}$ and prove that

$$\lim_{\delta\to 0^+} R_{I_i}(G_{\varepsilon,\delta}) = \widehat{R}_{I_i}(G_\varepsilon). \tag{2.3.5}$$

This will complete the proof of the lemma.

Note that for $x \in I_i,\ i = 1, \ldots, m$ holds

$$\hat{v}^z_{G_\varepsilon} = \frac{f'_\varepsilon(x_{i-1}+0)}{\sqrt{1 + (f'_\varepsilon(x_{i-1}+0))^2}}. \tag{2.3.6}$$

Fix ε and i and mark the points $P = (x_{i-1}, f_\varepsilon(x_{i-1}))$, $P' = (x_i, f_\varepsilon(x_i))$, $Q = (x_{i-1}, f_\varepsilon(x_i))$, and $S = (x_i,\ f_\varepsilon(x_{i-1}) + (f'_\varepsilon(x_{i-1}+0)+\delta)\cdot(x_i - x_{i-1}))$; see fig. 2.4. Mark also the point $Q_\delta = (x_{i-1}+\delta, f_\varepsilon(x_i))$, which is located on the segment QP' at the distance δ from Q, and the points $P_\delta = (x_{i-1}+\delta, f_\varepsilon(x_{i-1}+\delta))$ and $S_\delta = (x_{i-1}+\delta, f_\varepsilon(x_{i-1}) + (f'_\varepsilon(x_{i-1}+0)+\delta)\cdot\delta)$, which have the same abscissa as Q_δ and belong to the segments PP' and PS, respectively. Denote by l the line that contains P_δ and is parallel to PS. Denote by Π_δ the arc of the parabola with vertex Q_δ and focus at P_δ (therefore its axis

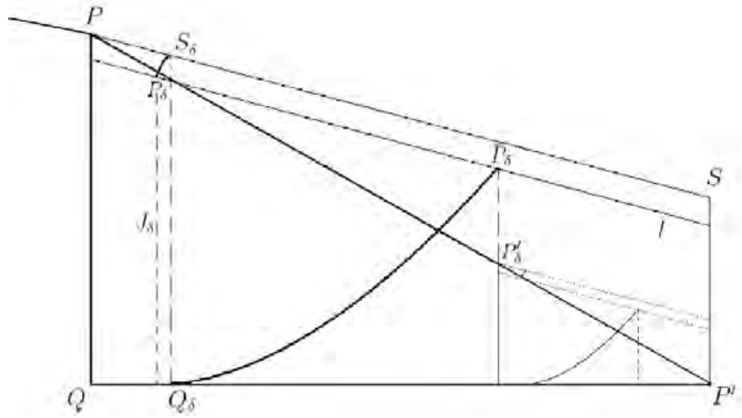


Figure 2.4: Constructing the curve $l^{i,\varepsilon,\delta}$: a detailed view.

is the vertical line $Q_\delta P_\delta$). This arc is bounded by the point Q_δ from the left, and by the point $\bar{P}_\delta$ of intersection of the parabola with l, from the right. Denote by x_i^δ the abscissa of $\bar{P}_\delta$ and denote by P'_δ the point that lies in the line PP' and has the same abscissa x_i^δ. Denote by π_δ the arc of the parabola with the same focus P_δ, the axis l, and the vertex situated on l to the left from P_δ. The arc π_δ is bounded by the vertex from the left, and by the point S'_δ of intersection of the parabola with the line $Q_\delta P_\delta$, from the right. There is an arbitrariness in the choice of the parabola; let us choose it in such a way that the arc π_δ is situated below the line PS. Finally, denote by J_δ the perpendicular dropped from the left endpoint of π_δ to QP', and denote by Q'_δ the base of this perpendicular.

If $x_i^\delta \geq x_i$, the curve $l^{i,\varepsilon,\delta}$ is the union (listed in the consecutive order) of the segments PS_δ and $S_\delta S'_\delta$, the arc π_δ, the segments J_δ and $Q'_\delta Q_\delta$, and the part of Π_δ located to the left of the line $P'S$.

If $x_i^\delta < x_i$, the definition of $l^{i,\varepsilon,\delta}$ is more complicated. Define the homothety with the center at P' that sends P to P'_δ, and define the curve $\tilde{l}^{i,\varepsilon,\delta}$ by the following conditions: (i) the intersection of $\tilde{l}^{i,\varepsilon,\delta}$ with the strip region $x_{i-1} \leq x \leq x_i^\delta$ is the union of PS_δ, $S_\delta S'_\delta$, π_δ, J_δ, $Q'_\delta Q_\delta$, Π_δ, and the interval $\bar{P}_\delta P'_\delta$; (ii) under the homothety, the curve $\tilde{l}^{i,\varepsilon,\delta}$ moves into itself. The curve $\tilde{l}^{i,\varepsilon,\delta}$ is uniquely defined by these conditions; it does not have self-intersections and connects the points P and P'. However, it is not piecewise smooth, since

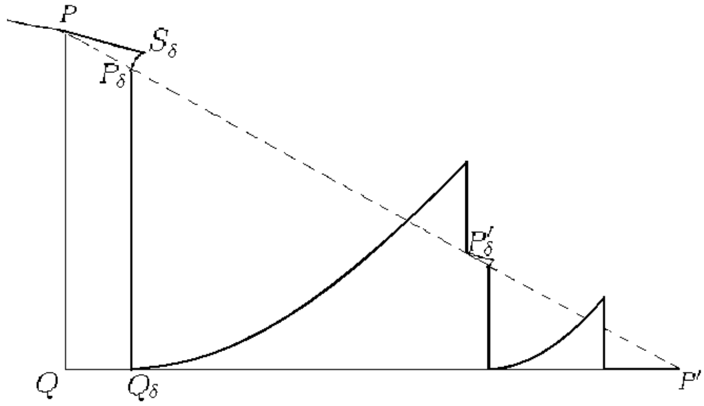


Figure 2.5: The curve $l^{i,\varepsilon,\delta}$, again.

it has infinitely many singular points near P'. In order to improve the situation, define the piecewise smooth curve $l^{i,\varepsilon,\delta}$ in the following way: in the strip $x_{i-1} \le x < x_i - \delta$, it coincides with $\tilde{l}^{i,\varepsilon,\delta}$, the intersection of $l^{i,\varepsilon,\delta}$ with the strip $x_i - \delta < x \le x_i$ is the horizontal interval $x_i - \delta < x \le x_i$, $z = f_\varepsilon(x_i)$, and the intersection of $l^{i,\varepsilon,\delta}$ with the vertical line $x = x_i - \delta$ is a point or a segment (or maybe the union of a point and a segment) chosen in such a way that the resulting curve $l^{i,\varepsilon,\delta}$ is continuous.

The particles of the flow falling on the arc Π_δ make a reflection from it, pass through the focus P_δ, then make another reflection from the arc π_δ, and finally move freely, the velocity being parallel to l. Choose $\delta < |f'_\varepsilon(0^+)|$ and $\delta < \min_{1\le i\le m-1}(f'_\varepsilon(x_{i-1}+0) - f'_\varepsilon(x_i + 0))$, then the particles after the second reflection will never intersect the other curves $l^{j,\varepsilon,\delta}$, $j \ne i$. Thus, for the corresponding values of x, the vertical component of the velocity of the reflected particle is

$$v^z_{G_{\varepsilon,\delta}}(x) = \frac{f'_\varepsilon(x_{i-1}+0)+\delta}{\sqrt{1+(f'_\varepsilon(x_{i-1}+0)+\delta)^2}} = \hat{v}^z_{G_\varepsilon}(x) + O(\delta), \quad \delta \to 0^+. \tag{2.3.7}$$

If $x_i^\delta \ge x_i$, the formula (2.3.7) is valid for $x \in [x_{i-1}+\delta,\, x_i]$. If $x_i^\delta < x_i$, it is valid for the values $x \in [x_{i-1}+\delta,\, x_i^\delta]$. Note, however, that (2.3.7) is also valid for values of x that belong to the iterated images of $x \in [x_{i-1}+\delta,\, x_i^\delta]$ under the homothety, but do not belong to $[x_i - \delta,\, x_i]$. Summarizing, (2.3.7) is true for $x \in [x_{i-1},\, x_i]$, except for a set of values of measure $O(\delta)$. Thus, taking into account (2.3.3), (2.3.4), (2.3.6), and (2.3.7), the convergence (2.3.5) is proved. Q.E.D.

Proof of theorem 2.2.4

Let us first state the following lemma.

Lemma 2.3.1. *Let $\lambda > 0$ and let the function $f_h \in \mathcal{F}_h$ satisfy the condition*

I$_\lambda$. *$f_h(1) = h$, and for almost all $x \in [0, 1]$ the value $u = f_h'(x)$ is a solution of the problem*

$$xp(u) + \lambda u \to \min, \quad u \in \mathbb{R}_+. \tag{2.3.8}$$

Then the function f_h is a solution of the problem (2.2.6) and any other solution satisfies the condition I_λ *with the same value of λ.*

Proof. This simple lemma is a direct consequence of the Pontryagin maximum principle. The proof we give here, however, is quite elementary and does not appeal to the maximum principle (cf. [22]).

For any $f \in \mathcal{F}_h$ one has

$$x\, p(f'(x)) + \lambda\, f'(x) \geq x\, p(f_h'(x)) + \lambda\, f_h'(x) \tag{2.3.9}$$

at almost every x. Integrating both sides of (2.3.9) over $x \in [0, 1]$, one gets

$$\frac{1}{2} \int_0^1 p(f'(x))\, dx^2 + \lambda\, (f(1) - f(0)) \geq$$

$$\geq \frac{1}{2} \int_0^1 p(f_h'(x))\, dx^2 + \lambda\, (f_h(1) - f_h(0)), \tag{2.3.10}$$

and using that $f(1) \leq h = f_h(1)$ and $f(0) = f_h(0) = 0$, one obtains that $\mathcal{R}(f) \geq \mathcal{R}(f_h)$.

Next, suppose that $f \in \mathcal{F}_h$ and $\mathcal{R}(f) = \mathcal{R}(f_h)$, then, using the relation (2.3.10) and the equality $f(0) = f_h(0)$, one gets that $f(1) \geq f_h(1) = h$, hence $f(1) = h$. Therefore the inequality in (2.3.10) becomes equality, which, in view of (2.3.9), implies that

$$x\, p(f'(x)) + \lambda\, f'(x) = x\, p(f_h'(x)) + \lambda\, f_h'(x)$$

for almost every x, hence $u = f'(x)$ is also a solution of (2.3.8), on a set of full measure. Thus, f satisfies the condition ($\mathbf{I}_\lambda$). □

Now we shall find the function f_h satisfying the condition I_λ for some positive λ. Let $x \in [0, 1]$ be the value for which I_λ is fulfilled. Then the value $u = f_h'(x)$ is also a minimizer for the function $x\bar{p}(u) + \lambda u$, and $p(u) = \bar{p}(u)$. This implies that (if the function f_h really exists then)

$$\mathcal{R}[f_h] = \int_0^1 \bar{p}(f_h'(x))\, x\, dx. \tag{2.3.11}$$

Besides, if $u > 0$ and $\bar{p}$ is differentiable at u then one has $\frac{d}{du}(x\bar{p}(u) + \lambda u) = 0$, hence

$$\frac{x}{\lambda} \;=\; -\frac{1}{\bar{p}'(u)}\,. \tag{2.3.12}$$

If $u > 0$ and $\bar{p}$ is not differentiable at u, then it has left and right derivatives at this point and

$$-\frac{1}{\bar{p}'(u-0)} \;\le\; \frac{x}{\lambda} \;\le\; -\frac{1}{\bar{p}'(u+0)}\,. \tag{2.3.13}$$

If, finally, $u = 0$ then one has

$$\frac{x}{\lambda} \;\le\; -\frac{1}{\bar{p}'(0)} \;=\; \xi_0. \tag{2.3.14}$$

Put $z = 1/\lambda$ and $x_0 = \xi_0/z$ and rewrite (2.3.12) and (2.3.13) in terms of the generalized inverse function: $v(zx - 0) \le u \le v(zx)$; thus the equality

$$u \;=\; v(zx), \tag{2.3.15}$$

is valid for almost all values $x \ge x_0$. Taking into account (2.3.14), substituting $u = f_h'(x)$, and integrating both parts of (2.3.15) with respect to x, one comes to (2.2.7). In particular, $f_h'(x_0 + 0) = v(\xi_0 + 0) = u_0$. Using that $f_h(1) = h$, one gets (2.2.8).

The function $\Upsilon(z)/z$ is continuous and monotone increasing; it is defined on $[\xi_0, +\infty)$ and takes the values from 0 to $+\infty$. Therefore the equation (2.2.8) uniquely defines Z as a continuous monotone increasing function of h; in particular, $z(0) = \xi_0$ and $x_0(0) = \xi_0/z(0) = 1$. The relations (2.2.7) and (2.2.8) define the function f_h solving the minimization problem (2.2.6). From the construction one can see that this function is uniquely defined.

Recall that $\mathcal{R}(h) = \mathcal{R}[f_h]$. Integrating by parts the right hand side of (2.3.11), one gets

$$\mathcal{R}(h) \;=\; \frac{\bar{p}(f_h'(1))}{2} \;-\; \int_0^1 \frac{x^2}{2}\, \bar{p}'(f_h'(x))\, df_h'(x).$$

Taking into account that $f_h'(1) = \upsilon(Z)$ and $x\bar{p}'(f_h'(x)) = -1/Z$, one obtains

$$\mathcal{R}(h) \;=\; \frac{\bar{p}(\upsilon(Z))}{2} + \frac{1}{2Z}\int_0^1 x\,df_h'(x),$$

and integrating by parts once again, one gets (2.2.9). Substituting in (2.2.9) $h = 0$ and using that $z(0) = \xi_0,\ \upsilon(\xi_0) = u_0,\ \Upsilon(\xi_0) = 0$, one obtains $\mathcal{R}(0) = (\bar{p}(u_0) + u_0/\xi_0)/2$, and using that $p(0) - \xi_0^{-1}u_0 = \bar{p}(u_0)$, one obtains $\mathcal{R}(0) = p(0)/2$.

Taking into account the asymptotic of $\bar{p}$ (which is the same as the asymptotics of p: $\bar{p}(u) = c\,u^{-\alpha}(1 + o(1)),\ u \to +\infty$), and the asymptotic of $\bar{p}'$: $\bar{p}'(u) = -c\alpha\,u^{-\alpha-1}(1 + o(1)),\ u \to +\infty,$ one comes to the formulas

$$\upsilon(\xi) \;=\; (C\alpha)^{\frac{1}{\alpha+1}}\,\xi^{\frac{1}{\alpha+1}}\,(1 + o(1)), \quad \xi \to +\infty,$$

$$\Upsilon(z) \;=\; \left(\frac{\alpha+1}{\alpha+2}\right)(c\alpha)^{\frac{1}{\alpha+1}}\,z^{\frac{\alpha+2}{\alpha+1}}\,(1 + o(1)), \quad z \to +\infty,$$

and

$$z \;=\; \frac{1}{c\alpha}\left(\frac{\alpha+2}{\alpha+1}\right)^{\alpha+1} h^{\alpha+1}\,(1 + o(1)), \quad h \to +\infty.$$

Substituting them into (2.2.9) and using the relation $x_0 = \xi_0/Z$, after a simple algebra one obtains (2.2.11) and (2.2.10). The theorem is proved.

Summarizing, the three-dimensional bodies of revolution minimizing the resistance are constructed as follows. First, we find the function f_h^{nc} minimizing the functional (2.2.5) and define the convex set $-1 \le x \le 1,\ 0 \le z \le h - f_h^{\mathrm{nc}}(|x|)$. Next, the upper part of its boundary (which is the graph of the function $z = h - f_h^{\mathrm{nc}}(|x|)$) is approximated by a broken line and then substituted with a curve with rather complicated behavior, according to lemma 2.2.3. The set bounded from above by this curve is "almost convex": it can be obtained from a convex set by making small hollows on its boundary. By rotating it around the axis Oz, one obtains the body of revolution B having nearly minimal resistance $\mathrm{R}_z(B)$.

The vertical central cross sections of optimal bodies in the Newton, single impact, and nonconvex cases, for $h = 0.8$, are presented on figure 9.

Derivation of the asymptotic relations in the single impact case

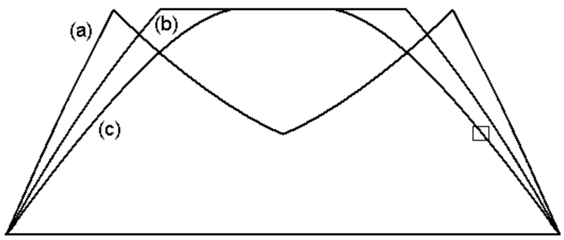


Figure 2.6: Profiles of optimal solutions in the single impact (a), Newton (b), and nonconvex (c) cases, for $h = 0.8$. In the nonconvex case, the profile is actually a zigzag curve with very small zigzags, as shown on the next figure.

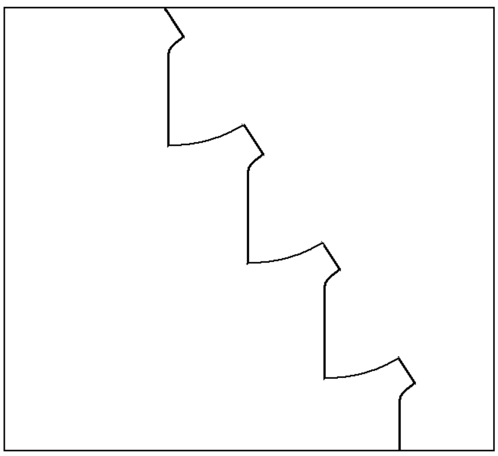


Figure 2.7: Detailed view of the zigzag curve.

For h small (namely, $h < M^* \approx 0.54$), a solution in the single impact case can be described as follows. There are marked several values $-1 < x_{-2n+1} < x_{-2n+2} < \ldots < x_{2n-2} < x_{2n-1} < 1$, $n \geq 2$ related to the singular points of the solution. As $h \to 0^+$, $n = n(h)$ goes to infinity. One has $x_{-k} = -x_k$ and $x_{2i} = (x_{2i-1} + x_{2i+1})/2$; thus $x_0 = 0$. Besides, one has $\max_k(x_k - x_{k-1}) = x_1 = 4h/3$. The vertical central cross section of the solution $G = G_h^{\rm si} \subset \mathbb{R}^2_{x,z}$ is bounded from above by the graph of a continuous non-negative piecewise smooth even function $f = f_h^{\rm si}$, and from below, by the segment $-1 \leq x \leq 1$, $z = 0$. This function has singularities at the points x_k, and the values of the function at the points x_{2i-1} coincide: $f(x_{2i-1}) = h$. On each interval $[x_{2i-1},\, x_{2i}]$, the graph of f is the arc of parabola with vertical axis and with the focus at (x_{2i+1}, h). Similarly, on $[x_{2i},\, x_{2i+1}]$ the graph of f is the arc of parabola with vertical axis and with the focus at (x_{2i-1}, h). The first parabola contains the focus of the second one, and vice versa. From this description one can see that on $[x_{2i-1},\, x_{2i}]$, the function equals $f(x) = \frac{(x-x_{2i+1})^2}{2(x_{2i+1}-x_{2i-1})} + y_i$, and on $[x_{2i},\, x_{2i+1}]$, $f(x) = \frac{(x-x_{2i-1})^2}{2(x_{2i+1}-x_{2i-1})} + y_i$, where $y_i = h - (x_{2i+1} - x_{2i-1})/2$. On the intervals $[-1,\, x_{-2n+1}]$ and $[x_{2n-1},\, 1]$ the graph of the function represents the so-called "Euler part" of the solution (see [12]).

Note that the solution is not unique. The values x_1 and x_{2n-1} are uniquely determined, but there is arbitrariness in choice of the intermediate values $x_3, \ldots, x_{2n-3}$ and also in the number n of the independent parameters.

After some calculation, one obtains the value of $1 + v_G^z$ for the figure G:

$$\text{if } x \in [x_{2i-1},\, x_{2i}], \qquad 1 + v_G^z(x) = \frac{2}{1 + \left(\frac{x_{2i+1}-x}{x_{2i+1}-x_{2i-1}}\right)^2};$$

$$\text{if } x \in [x_{2i},\, x_{2i+1}], \qquad 1 + v_G^z(x) = \frac{2}{1 + \left(\frac{x-x_{2i-1}}{x_{2i+1}-x_{2i-1}}\right)^2}.$$

Let us now calculate the integral $\int_{x_{2i-1}}^{x_{2i+1}} (1 + v_G^z(x))\, x\, dx$, $1 \leq i \leq n-1$. Since the function $1 + v_G^z(x)$, $x \in [x_{2i-1},\, x_{2i+1}]$ is symmetric with respect to $x = x_{2i}$, the integral equals $2x_{2i} \int_{x_{2i}}^{x_{2i+1}} (1 + v_G^z(x))\, dx$. Changing the variable $t = (x - x_{2i-1})/(x_{2i+1} - x_{2i-1})$ and taking into account that $2x_{2i} = x_{2i-1} + x_{2i+1}$, one comes to the integral $2x_{2i}(x_{2i+1} -$

$x_{2i-1}) \int_{1/2}^{1} 2/(1+t^2)\, dt = (x_{2i+1}^2 - x_{2i-1}^2)(\pi/2 - 2\arctan(1/2))$. Therefore

$$\int_{x_1}^{x_{2n-1}} (1 + v_G^z(x))\, x\, dx \;=\; (x_{2n-1}^2 - x_1^2)(\pi/2 - 2\arctan(1/2)).$$

Taking into account that $x_1 = 4h/3 \to 0$ and $x_{2n(h)-1} \to 1$ as $h \to 0^+$, one finally gets

$$R(G_h^{\mathrm{s}i}) \;=\; \int_0^1 (1 + v_{G_h^{\mathrm{s}i}}^z(x))\, x\, dx \;=\; \pi/2 - 2\arctan(1/2) + o(1), \qquad h \to 0^+,$$

that is, $\mathcal{R}_{\mathrm{s}i}(0) = \pi/2 - 2\arctan(1/2) \approx 0.6435$.

If $h > M^*$, the function $f = f_h^{\mathrm{s}i}$ has three singular points: $x_1 = x_1(h)$, 0, and $-x_1$. On the interval $[-x_1,\, x_1]$, the graph of f is the union of two parabolic arcs, as described above with $i = 0$. On the intervals $[-1,\, -x_1]$ and $[x_1,\, 1]$, the graph is the "Euler part" of the solution; on both intervals, f is a concave monotone function, with $f(\pm 1) = 0$ and $f(\pm x_1) = h$. The part of resistance of $G = G_h^{\mathrm{s}i}$ related to $[0,\, x_1]$ can be calculated:

$$\int_0^{x_1} (1 + v_G^z(x))\, x\, dx \;=\; x_1 p^*,$$

where $p^* = 8(\ln(8/5) + \arctan(1/2) - \pi/4) \approx 1.186$. That is, the convex hull of G represents the solution of the problem (2.2.6) with $p(u) = p_{\mathrm{s}i}(u) = \begin{cases} p^* & \text{if } u = 0 \\ 2/(1+u^2) & \text{if } u > 0 \end{cases}$.

Chapter 3

Bodies of zero resistance and bodies invisible in one direction

Earlier in the chapter 1, (in 1.1.1) the velocity of incidence onto the body B was set to be $(0, 0, -1)$. Evidently the same construction holds for arbitrary initial velocity $v_0 \in S^2$. So, to avoid misunderstanding, let us shortly repeat the description of our model for arbitrary v_0. Suppose that there is a parallel flow of non-interacting particles falling on B. Initially, the velocity of a particle equals $-v_0$; then it makes several reflections from B, and finally moves freely with the velocity $v_B^+(x, v_0)$, where x indicates the initial position of the particle. One can imagine that the flow is highly rarefied or consists of rays of light. (Equivalently, one can assume that the body translates at the velocity $-v_0$ through a highly rarefied medium of particles at rest.) The force of pressure of the flow on the body (or the force of resistance of the medium to the body's motion) is proportional to (compare with (1.1.1))

$$R_{v_0}(B) := \int_{\{v_0\}^\perp} (v_0 - v_B^+(x, v_0))\, dx, \tag{3.0.1}$$

where the ratio equals the density of the flow/medium and dx means the Lebesgue measure in $\{v_0\}^\perp$. Here we denoted by $\{v\}^\perp$ the orthogonal complement to the one-dimensional subspace $\{v\}$, that is, the plane that contains the origin and is orthogonal to v.

In this chapter we show that there exist bodies whose resistance is exactly zero.

More precisely, denoting by $v_0 \in S^2$ the initial flow velocity, we say that the body *has zero resistance in the direction* v_0, if the final velocity of almost every particle is also equal to v_0. We say that the body *leaves no trace* (or is *trackless*) *in the direction* v_0 if it has zero resistance in this direction and, additionally, the flow density behind the body is constant and coincides with the initial one. Further, we say that the body is *invisible in the direction* v_0 if the trajectory of each particle outside a prescribed bounded set coincides with a straight line. Indeed, such a body with mirror surface becomes invisible to an observer staying in the this direction far enough from the body. We prove that *there exist bodies of zero resistance, bodies leaving no trace, and bodies invisible in one direction.*

From the viewpoint of classical scattering by obstacle, we construct a body with zero total cross section. (The total cross section measures the density of scattered rays; see introduction to [28] for the definition.) Thus in this case, from the scattering data for a fixed angle of incidence, it is not possible even to say if an obstacle exists or not. A question arises, if a similar effect can take place in the wave scattering or in the non-relativistic quantum mechanics. We note in this regard that the wave and classical scattering at small wave length are closely connected (see, e.g., [29]).

This chapter is organized as follows. In section 3.1, we introduce the mathematical notation and give rigorous definitions for bodies of zero resistance, bodies that leave no trace, and bodies that are invisible in one direction. In section 3.2, we give an overview of the minimal resistance problem and put our result in this context. In section 3.3, we introduce families of zero resistance bodies, trackless bodies, and invisible bodies, discuss their properties, and state some open problems. Finally, in Section 3.7 possible applications of our models are discussed.

3.1 Notation and definitions

Here we introduce more notations to those were presented in the first chapter.

Let $B \subset \mathbb{R}^3$ be a bounded connected set with piecewise smooth boundary, and let

$v_0 \in S^2$. (B and v_0 represent the body and the flow direction, respectively.) Consider the billiard in $\mathbb{R}^3 \setminus B$. The scattering mapping $(x, v) \mapsto (x_B^+(x, v), v_B^+(x, v))$ from a full measure subset of $\mathbb{R}^3 \times S^2$ into $\mathbb{R}^3 \times S^2$ is defined as follows. Let the motion of a billiard particle $x(t),\ v(t)$ satisfy the relations $x(t) = \begin{cases} x + vt, & \text{if } t < t_1 \\ x^+ + v^+ t, & \text{if } t > t_2 \end{cases}$ and $v(t) = \begin{cases} v, & \text{if } t < t_1 \\ v^+, & \text{if } t > t_2 \end{cases}$ (here $t_1,\ t_2$ are a pair of real numbers depending on the particular motion); then $x^+ =: x_B^+(x, v),\ v^+ =: v_B^+(x, v)$.

Denote $\tilde{x}_B^+(x, v) = x^+ - \langle x^+, v^+ \rangle v^+$ and $t^* = t_B^*(x, v) = -\langle x^+, v^+ \rangle$, where $\langle \cdot\, , \cdot \rangle$ is the scalar product; then one has $x^+ + v^+ t = \tilde{x}^+ + v^+(t - t^*)$, and $\tilde{x}^+$ is orthogonal to v^+.

Definition.

D$_1$. *We say that B has* zero resistance in the direction v_0 *if $v_B^+(x, v_0) = v_0$ for almost every x.*

D$_2$. *We say that the body B* leaves no trace in the direction v_0 *if, additionally to* D$_1$, *the mapping $x \mapsto \tilde{x}_B^+(x, v_0)$ from a subset of $\{v_0\}^\perp$ into $\{v_0\}^\perp$ is defined almost everywhere in $\{v_0\}^\perp$ and preserves the two-dimensional Lebesgue measure.*

D$_3$. *We say that B is* invisible in the direction v_0 *if, additionally to* D$_2$, *one has $\tilde{x}_B^+(x, v_0) = x$.*

The condition D$_3$ is stronger than D$_2$, and D$_2$ is stronger than D$_1$. One easily sees that if B is invisible/leaves no trace in the direction v_0 then the same is true in the opposite direction $-v_0$.

In the case D$_1$ one has $R_{v_0}(B) = 0$. If the body has mirror surface then in the case D$_3$ it is invisible in the direction v_0. In the case D$_2$, if the body moves through a rarefied medium, the medium seems to be unchanged after the body has passed: the particles behind the body (actually, in the complement of the body's convex hull) are at rest and are distributed with the same density.

In section 3.7 we give examples of a body satisfying the condition D$_1$, but not satisfying D$_2$; a body satisfying D$_2$ but not D$_3$; and a body satisfying D$_3$. That is, there exists a body of zero resistance that leaves a trace (shown on Fig. 3.2a); a body leaving no trace but not invisible (Fig. 3.2b and 3.2c); and an invisible body (Fig. 3.3).

3.2 Problems of the body of minimal resistance

2. Consider the class of bodies B that are contained in the cylinder $\Omega \times [0,\, h]$ and contain a cross section $\Omega \times \{c\}$ with $c \in [0,\, h]$, $\ \Omega \times \{c\} \subset B \subset \Omega \times [0,\, h]$, and such that the integral $R_{v_0}(B)$ exists. For the sake of brevity, we shall call them *bodies inscribed in the cylinder.* Multiple reflections are allowed. If Ω is the unit circle then the infimum of resistance equals zero, $\inf_B |R_{v_0}(B)| = 0$ (see [19]). This result generalizes to the case of arbitrary Ω, so that the following statement holds true.

Conjecture. *For any Ω and h,* $\inf\{|R_{v_0}(B)| \,:\, B$ is inscribed in the cylinder $\Omega \times [0,\, h]\} = 0$. *If Ω is convex then the infimum is not attained. On the other hand, for some nonconvex Ω and some h, the infimum is attained; that is, there exist bodies of zero resistance.*

We call it conjecture, since the proof of the first assertion has never been published. The second assertion in the conjecture is reformulated and proved as Proposition 3.2. Several examples of zero resistance bodies, where Ω is a ring or a special kind of polygon with mutually orthogonal sides, are provided below in the text. This proves the third assertion.

Let Ω be a convex set with nonempty interior and let B be a body inscribed in the cylinder $\Omega \times [0,\, h]$ and such that the integral $R_{v_0}(B)$ exists. Then $R_{v_0}(B) \neq 0$.

Proof. The integral $R_{v_0}(B)$ exists, that is, the function $v_B^+(x, v_0)$ is defined for almost all $x \in \Omega$ and is measurable. Using that the particle trajectory does not intersect the section $\Omega \times \{c\}$ and Ω is convex, one concludes that the particle initially moves in the cylinder above this section, then intersects the lateral surface of the cylinder and moves freely afterwards. This implies that $v_B^+(x, v_0) \neq v_0$, hence $R_{v_0}(B) \neq 0$. □

3.3 Zero resistance bodies and invisible bodies

The main result of this paper is the following theorem. Fix $v_0 \in S^2$.

There exist (a) a body that has zero resistance in the direction v_0 but leaves a trace; (b) a body that leaves no trace in the direction v_0 but is not invisible; (c) a body invisible in the direction v_0.

Proof. (a) Consider two identical coplanar equilateral triangles ABC and A′B′C′, with C being the midpoint of the segment A′B′, and C′, the midpoint of AB. The vertical line CC′ is parallel to v_0. Let A″ (B″) be the point of intersection of segments AC and A′C′ (BC and B′C′, respectively); see Fig. 3.1. The body B generated by rotation of the triangle

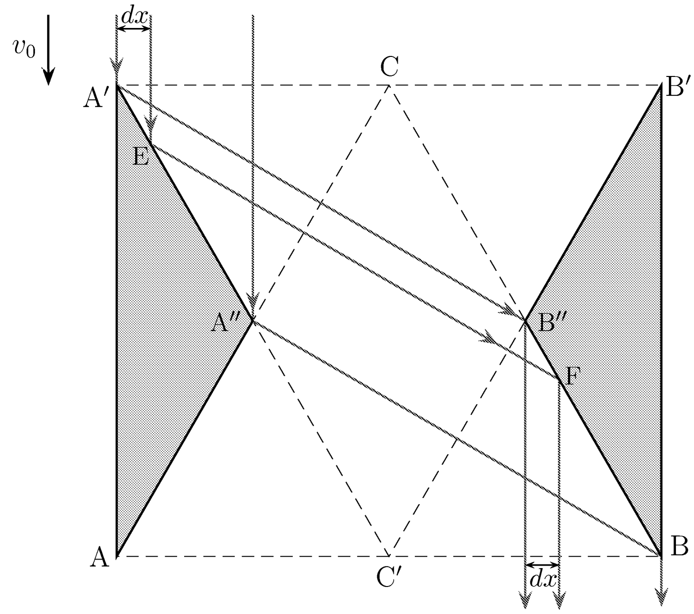


Figure 3.1: The basic construction.

AA′A″ (or BB′B″) around the axis CC′ is shown on Fig. 3.2a. It has zero resistance in the direction v_0. This can be better seen from Figure 3.1 representing a vertical central cross section of B.

If a particle initially belongs to this cross section, it will never leave it. Let the particle first hit the segment A′A″ at a point E. (If the particle first hits B′B″, the argument is the same.) After the reflection, the direction of motion forms the angle $\pi/3$ with the vertical. Next, the particle hits the segment B″B at the point F such that $|\text{A}'\text{E}| = |\text{B}''\text{F}|$, and after the second reflection moves vertically downward. That is, the final velocity equals v_0.

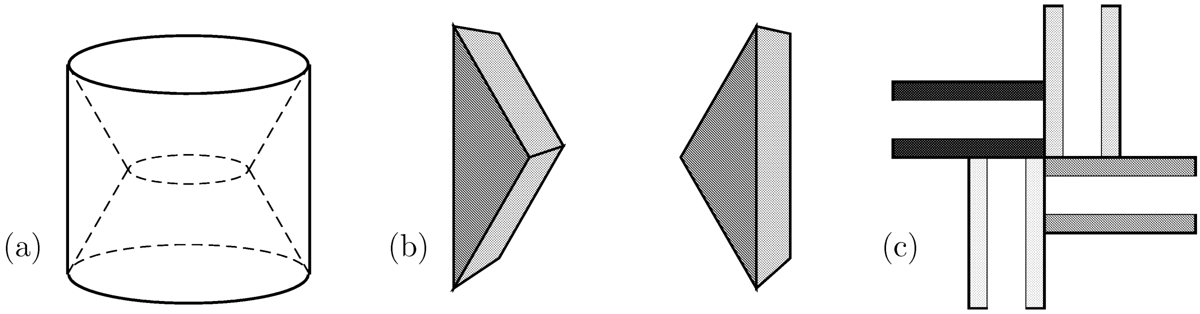


Figure 3.2: (a) A rotationally symmetric body of zero resistance. (b) A disconnected set leaving no trace. (c) The union of 4 sets identical to the one shown on fig. (b), the above view. It is simply connected and leaves no trace.

However, this body *does* leave a trace (and therefore is *not* invisible). Indeed, the particles that initially belong to a larger cylindrical layer of width dx (on Fig. 3.1 above), after two reflections get into a smaller layer of the same width dx (Fig. 3.1 below), and vice versa. Therefore, the density of the smaller layer gets larger below the body, and the density of the larger layer gets smaller. If dx is small then increase and decrease of the density is twofold.

(b) A set generated by translating the pair of triangles AA′A″ and BB′B″ along a segment orthogonal to their plane leaves no trace in the vertical direction v_0, but is not invisible. It is disconnected; however, by "gluing together" 4 copies of this set along the vertical faces, one can get a *connected* set (that is, a true body) leaving no trace. Figure 2c provides the above view of the resulting body.

(c) A body invisible in the direction v_0 can be obtained by doubling a zero resistance body; see Fig. 3.3.

Note that interior of this body is a disjoint union of two domains; this property can be undesirable. However, the construction can be improved as follows.

Consider a coordinate system $Ox_1x_2x_3$ such that the x_3-axis coincides with the symmetry axis of the body B shown on Fig. 3.2a, the upper half-space contains the body, and $v_0 = (0, 0, -1)$. Consider the body B' symmetric to B with respect to the horizontal plane $x_3 = 0$ and suppose that the distance $\mathrm{dist}(B, B') =: \varepsilon$ is small. Next, take the intersection of $B \cup B'$ with the set $x_1 x_2 \geq 0$ (this intersection is the disjoint union of 4 connected sets) and shift it vertically up or down on 2ε. The union of the shifted set with the remaining set $(B \cup B') \cap \{x_1 x_2 \leq 0\}$ is connected, that is, it is a true body invisible in the direction v_0.

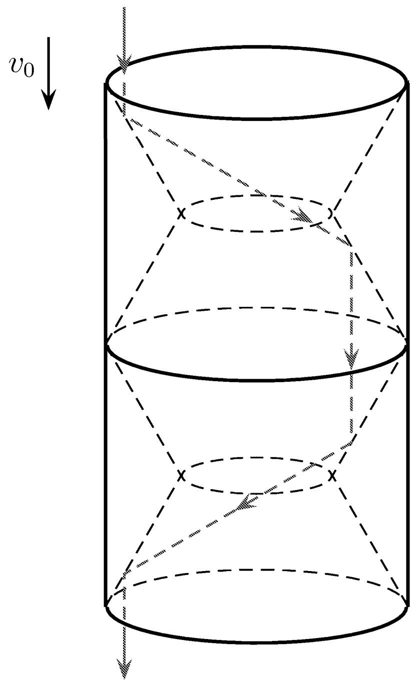


Figure 3.3: A body invisible in the direction v_0. It is obtained by taking 4 truncated cones out of the cylinder.

□

3.4 Several families of bodies invisible in one direction

Here we present our results on description of possible bodies of zero resistance or bodies invisible in one direction. Evidently, for two bodies Ω_1 and Ω_2 whose convex hulls do not overlap, holds the following: if both of them have zero resistance in the direction v_0, then their union $\Omega_1 \cup \Omega_2$ also has zero resistance in this direction, and if Ω_1 and Ω_2 are invisible in the direction v_0, then the union $\Omega_1 \cup \Omega_2$ is also invisible in the same direction.

Let us introduce some families of bodies having the desired properties.

3.4.1 Bodies based on isosceles triangles

First, consider a pair of isosceles triangles with the angles α, α, and $\pi - 2\alpha$, where $0 < \alpha < \pi/4$. The triangles are symmetric to each other with respect to a certain point. This point lies on the symmetry axis of each triangle, at the distance $(\tan 2\alpha - \tan\alpha)/2$ from its obtuse angle and at the distance $(\tan 2\alpha + \tan\alpha)/2$ from its base. The length of the base of each triangle equals 2. On Fig. 3.4 there are depicted two pairs of triangles,

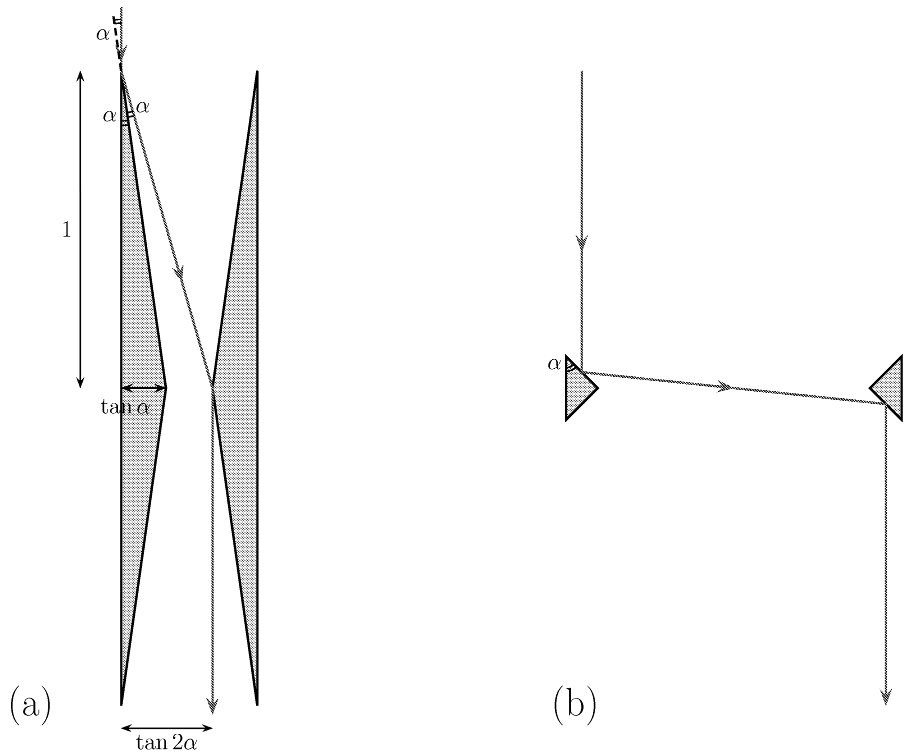


Figure 3.4: The central vertical cross section of the body B_α (a) with small α; (b) with α close to $\pi/4$.

with α small and α close to $\pi/4$.

As seen from the picture, this definition guarantees zero resistance in the direction v_0 parallel to the bases of the triangles. The zero resistance body, trackless body, and invisible body are created, respectively, by the procedures of rotation, translation with gluing, and doubling, applied to the pair of triangles.

Consider the one parameter family of zero resistance bodies B_α obtained by rotation of the pair of triangles. It contains the body $B = B_{\pi/6}$ constructed above. Before studying the properties of this family, introduce the following definition.

For a body $\mathcal{D}$, let $\kappa(\mathcal{D})$ be the relative volume of $\mathcal{D}$ in its convex hull, that is, $\kappa(\mathcal{D}) := \mathrm{Vol}(\mathcal{D})/\mathrm{Vol}(\mathrm{Conv}\mathcal{D})$. One obviously has $0 < \kappa(\mathcal{D}) \le 1$, and $\kappa(\mathcal{D}) = 1$ *iff* $\mathcal{D}$ is convex.

The convex hull of B_α is a cylinder of radius $L_\alpha = (\tan 2\alpha + \tan\alpha)/2$ and height $H = 2$; denote by h_α its relative height, $h_\alpha = H/L_\alpha$. One has $\mathrm{Vol}(B_\alpha) = \pi\tan\alpha(\tan 2\alpha + \tan\alpha/3)$. Now one easily derives the asymptotic relations for h_α and $\kappa_\alpha = \kappa(B_\alpha)$: as $\alpha \to 0$, one has $h_\alpha = \frac{4}{3\alpha}(1 + o(1)) \to \infty$ and $\kappa_\alpha \to 14/27 \approx 0.52$. For $\alpha = \pi/6$, one has $h_{\pi/6} = \sqrt{3}$ and $\kappa_{\pi/6} = 5/12 \approx 0.42$. Taking $\alpha = (\pi - \varepsilon)/4,\ \varepsilon \to 0^+$, one gets $h_{(\pi-\varepsilon)/4} = 2\varepsilon(1 + o(1))$ and $\kappa_\alpha = \varepsilon(1 + o(1))$.

3.4.2 Bodies obtained by intersecting a zero resistance body with several pairs of vertical stripes

Let Ω be a body of zero resistance composed of two isosceles triangles as described above in section 3.4.1. Notice that these triangles are mutually symmetric with respect to a straight line l_0 parallel to v_0.

Recall that the function $v_\Omega^+(x,v)$ and $x_\Omega^+(x,v)$ are defined by the following conditions: if a billiard trajectory equals $x(t) = x + vt$ for t sufficiently small, then $x(t) = x_\Omega^+(x,v) + v_\Omega^+(x,v)\,t$ for t sufficiently large. Notice that $v_\Omega^+(x,v_0) = v_0$ and denote the mapping ϱ in the space of vertical straight lines that takes the line $x + \mathbb{R}v_0$ to the line $x_\Omega^+(x,v_0) + \mathbb{R}v_0$. That is, if a particle initially moves along a vertical line l, then after reflections from Ω it will move along the vertical line $\varrho(l)$.

Now consider a set Ω_2 which is a union of a set of vertical lines intersecting Ω invariant with respect to ϱ and such that each line from the set intersects Ω. Actually, Ω_2 is just a union of an even number of vertical stripes: some of the to the left of l_0, and some to the right of l_0. The right stripes are obtained from the left ones by a horizontal shift by a fixed distance.

Lemma 3.4.1. *Let* Ω *be a body of zero resistance composed of two isosceles triangles and* Ω_2 *be a set described above. Then* $\Omega \cap \Omega_2$ *is a body of zero resistance.*

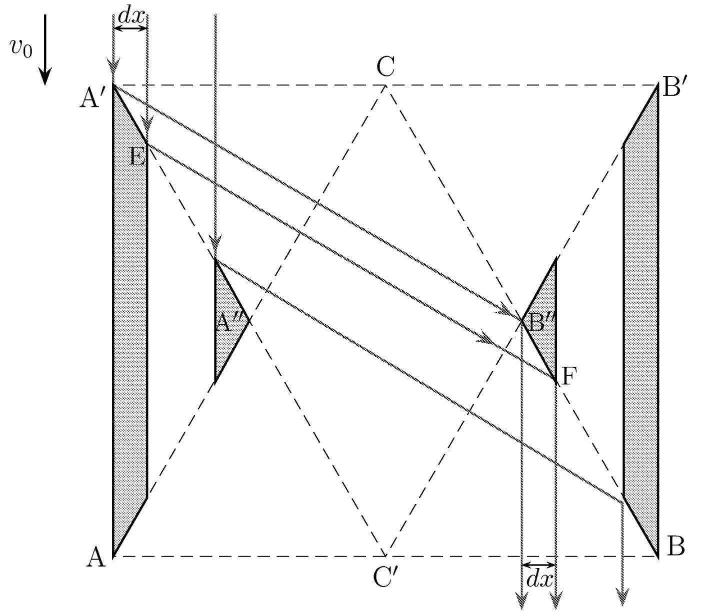


Figure 3.5: Body obtained by intersecting a zero resistance body with two pairs of vertical stripes.

Proof. If a particle initially moves along a line contained in Ω_2, then its interaction with $\Omega \cap \Omega_2$ is precisely the same as with Ω, and therefore, the final velocity equal v_0. Otherwise, the particle does not interact with $\Omega \cap \Omega_2$ and again, the vinal velocity equal v_0. See fig.3.4.2. □

3.4.3 Bodies of zero resistance based on isosceles trapezia

Now consider a more general construction based on the union of two isosceles trapezia ABCD and A′B′C′D′ (see Fig. 3.5). Take a billiard particle in $\mathbb{R}^2 \setminus (\mathrm{ABCD} \cup \mathrm{A'B'C'D'})$ with the initial velocity directed vertically downward and having at least one reflection from one of the trapezia. In this and the two next paragraphs we consider the part of the

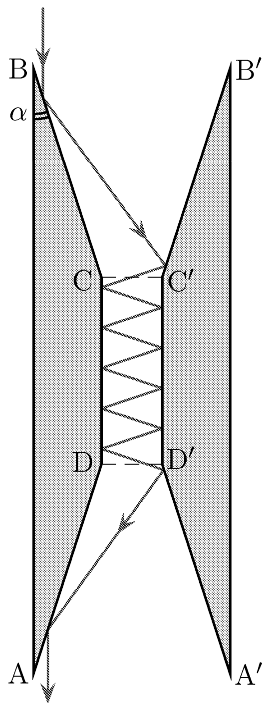


Figure 3.6: The vertical cross section of a zero resistance body of revolution.

trajectory contained in the trapezia BB′C′C (see Fig. 3.6). The particle gets in through the segment BB′, and after a finite number of alternating reflections from the sides BC and B′C′, it escapes through CC′ or BB′. Suppose without loss of generality that the first reflection takes place from BC, and apply the procedure of unfolding to the billiard trajectory. First, to both the trapezium BB′C′C and the part of the trajectory after the first reflection, apply the reflection from the line BC. As a result we obtain the trapezium B_1BCC_1 and a billiard trajectory in it, besides the first segment of the trajectory (between BC and the next point of reflection) belongs to the same vertical line as the initial part of the particle trajectory.

If the next reflection takes place from the side B_1C_1 (as on Fig. 3.7a) then to both the trapezium B_1BCC_1 and the rest of the trajectory (after this reflection and before escaping the trapezium) apply the reflection with respect to the line B_1C_1. As a result we obtain the trapezium $B_2B_1C_1C_2$ and a billiard trajectory in it, and again, the initial segment of this trajectory belongs to the same vertical line as above.

This procedure finishes in a finite number of steps — as a result we obtain a sequence of trapezia B_1BCC_1, $B_2B_1C_1C_2$,..., $B_kB_{k-1}C_{k-1}C_k$, $k \le \lfloor \pi/(2\alpha) + 1/2 \rfloor$ and the "unfolded" part of trajectory. This unfolded trajectory is a vertical segment whose initial endpoint belongs to BB', and the final endpoint, either (i) to the broken line $CC_1C_2 \ldots C_k$, or (ii) to the broken line $BB_1B_2 \ldots B_k$. The case (i) means that the original billiard trajectory intersects CC' and enters the rectangle $CDD'C'$. The case (ii) means that it eventually escapes $BB'C'C$ through the segment BB'.

Denote $r := |CC'|/|BB'|$ and $\alpha := \measuredangle ABC$ (and therefore $\alpha = \measuredangle BAD = \measuredangle A'B'C' = \measuredangle B'A'D'$); we assume that $\alpha < \pi/4$. Choose the parameters r and α in such a way that the broken line $CC_1C_2 \ldots C_{\lfloor \pi/(2\alpha)+1/2 \rfloor}$ touches the straight line AB, that is, intersects this line and is located to the right of it. It suffices to put $r = r(\alpha) = \sin\alpha / \sin(2\lfloor \pi/(4\alpha) \rfloor \alpha + \alpha)$. The function $r(\alpha)$ is continuous and monotonically increases from $r(0) = 0$ to $r(\pi/4) = 1$. With this choice, the case (ii) is excluded, that is, the particle always enters the rectangle $CDD'C'$.

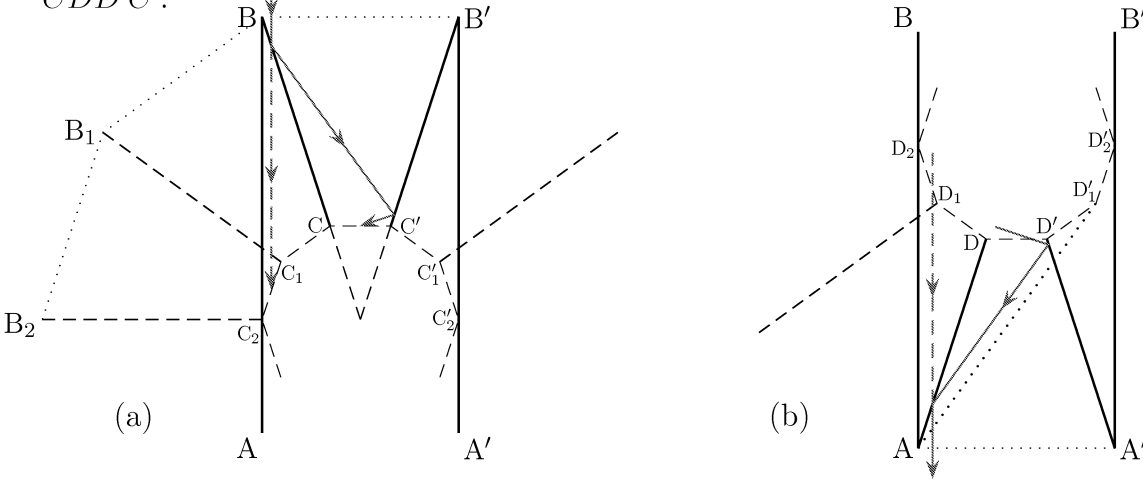


Figure 3.7: Unfolding of a billiard trajectory.

After the first reflection the particle velocity forms the angle 2α with the vertical direction $(0, -1)$ (we measure angles counterclockwise from the vertical); after the second reflection the angle becomes -4α, and so on. At the point of intersection with CC' the angle becomes $(-1)^{k-1} 2k\alpha$, where k is the number of reflections from BC and $B'C'$.

While the particle belongs to the rectangle $CC'D'D$, the modulus of the angle remains equal to $2k\alpha$, and when the particle makes reflections from the sides AD and $A'D'$, it decreases, taking successively the values $2(k-1)\alpha,\ 2(k-2)\alpha, \ldots$, and finally, after the last reflection, the angle becomes $2k'\alpha$, where k' is an integer, $|k'| \le k$.

Let us show that $k' = 0$ and therefore, the final velocity is vertical. To that end, let us apply the unfolding procedure again, this time to the part of the trajectory contained in the trapezium ADD′A′ (see Fig. 3.6b). Suppose without loss of generality that the point of last reflection belongs to AD. To both the part of the trajectory before that point and the trapezium, apply reflection with respect to the line AD. Repeating this procedure as described above, one obtains the "rectified" trajectory — an interval with the endpoints on the segment AA′ and on the broken line $\ldots \mathrm{D_2 D_1 D D' D'_1 D'_2} \ldots$ generated by the consecutive reflections of the unfolding procedure. This broken line touches the lines AB and A′B′.

We see that the tangents drawn from A to the broken line (the lines $\mathrm{AD_2}$ and $\mathrm{AD'_1}$ on Fig. 3.6b) form the angles 0 and -2α with the vertical. Analogously, the angles of the tangents drawn from A′ to that line are 2α and 0. This implies that both the tangents drawn from any point of the segment AA′ to that line have the angles greater than -2α and less than 2α. The same is true for the angle of inclination of the unfolded trajectory, that is, $|2k'\alpha| < 2\alpha$, and therefore, $k' = 0$.

The body of zero resistance is formed by rotation of the trapezia around the vertical symmetry axis. Its shape is determined by the two parameters α and $\gamma = |\mathrm{CD}|/|\mathrm{BC}|$. As $\alpha \to 0$ and $\gamma \to \infty$, the maximal number of reflections goes to infinity, the relative volume of the body in the cylinder ABB′A′ goes to 1, and the relative height of the cylinder goes to infinity.

By doubling this body, one obtains the body invisible in the direction v_0.

This result can be summarized as follows.

Let Ω be a ring $r^2 \le x_1^2 + x_2^2 \le 1$. For h sufficiently large, there exists a body inscribed in $\Omega \times [0,\, h]$ and invisible in the direction $v_0 = (0, 0, -1)$.

Remark. *This theorem is also true for the case where Ω is a special kind of polygon with mutually orthogonal sides; see, e.g., Fig. 3.2c.*

3.4.4 Four-parameter family of invisible bodies

Basing on the main idea of construction, one can provide various examples of multi-parameter families of zero resistance bodies. Here we will give an example of 4-parameter family.

Consider an arbitrary point A and an angle $\alpha \in (0, \pi/4)$, and take a point B such that vector AB forms the angle α with the direction of incidence v_0. (In fig. 3.3.5.1 the vector v_0 is identified with the vertical vector $(0, -1)$.) Put a point C below the point B, so that the line BC is vertical. Later we will derive an upper bound on the length $|BC|$. Choose an angle β such that

$$\alpha < \beta < \pi/4. \tag{3.4.1}$$

Draw two straight lines: one of them contains the point B and forms the angle 2α with the direction of incidence v_0. The other line contains the point C and forms the angle 2β with v_0. Let D be the point of intersection of these lines. Next we find the point E such that the vector ED is equal to the vector AB.

The condition we need is that the point E is located below the point C. If this condition is fulfilled, take a point F which is the intersection of two lines: one of them contains E and is perpendicular to the direction of incidence (that is, is horizontal), and the other line contains C and forms the angle β with the direction of incidence. The point G is calculated in such a way that the vector CF is equal to the vector DG.

The other points A', B', C', D', G' in figure 3.5.1 are obtained from the points A, B, C, D, G by vertical symmetry. More precisely, we obviously have that the points E and F are symmetric with respect to the vertical line l through the midpoint of the segment EF. Then by definition A', B', C', D', G' are, respectively, the points symmetric to A, B, C, D, G with respect to l.

Let us now derive the above mentioned upper bound for $|BC|$. First, using that $\measuredangle CBD = 2\alpha$ and $\measuredangle BDC = 2(\beta - \alpha)$, we have

$$\frac{|CD|}{\sin(2\alpha)} = \frac{|BC|}{\sin(2(\beta - \alpha))}. \tag{3.4.2}$$

The condition that E lies below C means that the projection of CD on the direction of incidence is larger than the projection of ED on the same direction, that is,

$$|CD|\cos(2\beta) > |AB|\cos\alpha.$$

Using (3.4.2), we get

$$|BC|\,\frac{\sin(2\alpha)}{\sin(2(\beta-\alpha))}\,\cos(2\beta) > |AB|\cos\alpha,$$

and finally,

$$|BC| > |AB|\,\frac{\cos\alpha}{\cos(2\beta)}\,\frac{\sin(2(\beta-\alpha))}{\sin(2\alpha)}. \tag{3.4.3}$$

Evidently, after fixing the points A, B, C one can always choose a $\beta > \alpha$ such that $\beta - \alpha$ is small enough to satisfy this condition.

From figure 3.5.1 one easily sees that the obtained body is invisible. Indeed, if an incident particle hits the segment AB at some point, then after reflection it goes inclined by the angle 2α, then it hits the segment ED and finally goes vertically down. Besides, the distance between A and the point of the first reflection is equal to the distance between E and the point of the second reflection. If an incident particle first hits the segment CF, then after reflection it goes inclined by the angle 2β, hits the segment DG and finally goes vertically down. The distance between C and the point of the first reflection is equal to the distance between D and the point of the second reflection. If a particle hits the segment $A'B'$ or the segment $C'E$, the consideration is completely similar.

Summarizing, we have constructed a 4-parameter family of invisible bodies, with the positive parameters α, β, $|AB|$, $|BC|$, satisfying the inequalities (3.4.1) and (3.4.3).

3.4.5 Body of zero resistance bounded by arcs of parabolas

Consider two parabolas P and P' with the same focus F and with common axis parallel to v_0. Designate by C and C' the vertices of the parabolas. The distances $|CF|$ and $|C'F|$ are arbitrary but different, besides $\overrightarrow{CF} \cdot \overrightarrow{C'F} > 0$. To be definite, suppose that $|C'F| > |CF|$.

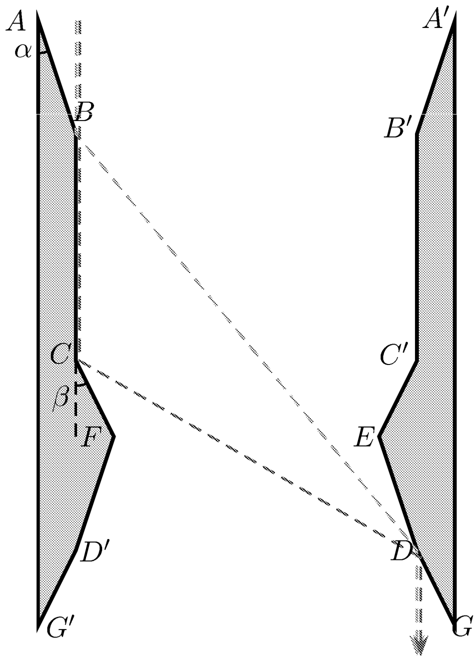


Figure 3.8: A 4-parameter family of zero resistance bodies.

In this section the vector v_0 is chosen to be vertical, $v_0 = (0, 1)$ (see the figure 3.4.5); thus the line CC' is also vertical.

From now on in this section we will consider only the points that lie to the left of the axis CC'. Consider an arbitrary point $A' \in P'$ such that $\overrightarrow{A'F} \cdot v_0 > 0$. Denote by A the point of intersection of the straight line $A'F$ with the parabola P. Next, we find the point $B \in P$ such that $\overrightarrow{A'B} \cdot v_0 = 0$. Denote by B' the point of intersection of P' with the straight line FB.

Denote by l the straight line parallel to v_0 through the midpoint of the segment $A'B$. Obviously, A' is symmetric to B with respect to l. Denote by $A''A'$ and BB'', respectively, the images of the arcs of parabolas AB and $A'B'$ under the symmetry with respect to l.

The body under consideration is the union of two sets. The first set is bounded by the arcs $A'A''$ and $A'B'$, by the vertical line through B', and by the horizontal line through A''. The second set is symmetric to the first one with respect to l.

Let us show that the obtained body has zero resistance. A particle with initial velocity v_0, incident on the arc $A'B'$, according to a focal property of parabola, after the

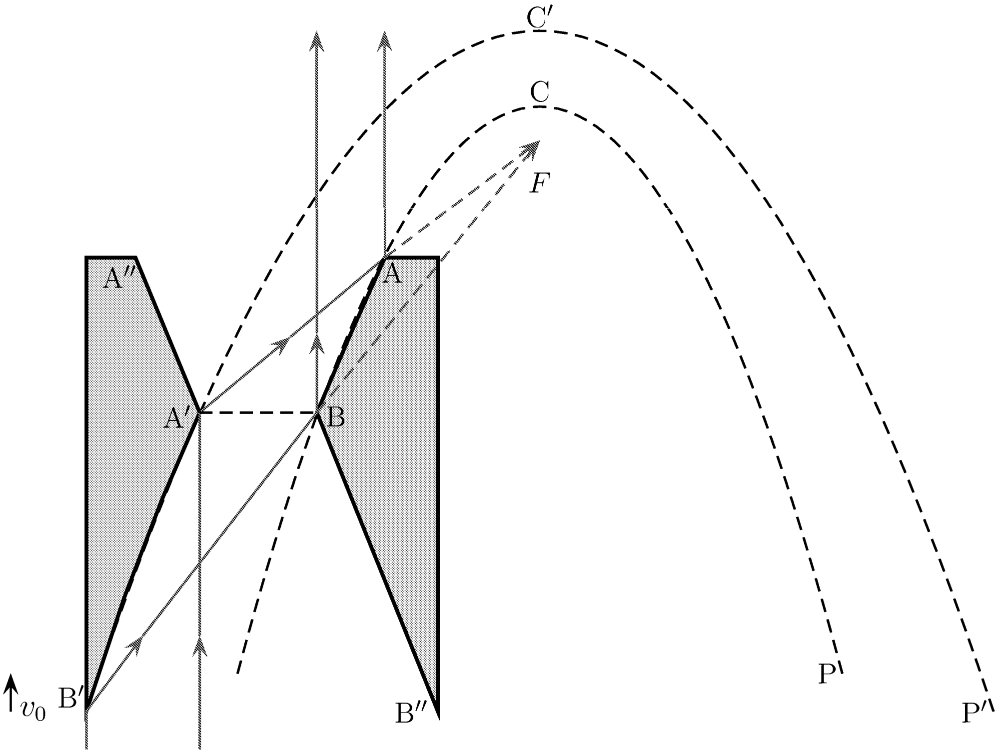


Figure 3.9: A body bounded by arcs of parabolas.

reflection after the reflection moves along a straight line passing through the focus F. Then it is reflected from the arc AB, and after the reflection, according to the same focal property, moves with the velocity v_0. If the particle is incident on BB'', the consideration is completely similar.

Note in addition that this body has zero resistance in the direction v_0, but does not in the direction $-v_0$.

3.5 General properties of invisible and zero resistance bodies

3.5.1 Minimal number of reflections

Denote by $m = m(B, v_0)$ the maximal number of reflections of an individual particle from the body.

Theorem 3.5.1. *(a) If the body B has zero resistance or leaves no trace in the direction v_0 then $m(B, v_0) \geq 2$.*

(b) If B is invisible in the direction v_0 then $m(B, v_0) \geq 4$. These inequalities are sharp: there exist zero resistance bodies and trackless bodies with exactly 2 reflections, and there exist invisible bodies with exactly 4 reflections.

Proof. (a) If $m = 1$ (that is, under the single impact assumption) then the final velocity of each particle does not coincide with the initial one, $v_B^+(x, v_0) \neq v_0$, therefore $R_{v_0}(B) \neq 0$. That is, a zero resistance body requires at least two reflections.

(b) Note that a thin parallel beam of particles changes the orientation under each reflection. To be more precise, let $x(t) = x + v_0 t,\ v(t) = v_0$ be the initial motion of a particle, and let $x(t) = x^{(i)}(x) + v^{(i)}(x)t,\ v(t) = v^{(i)}(x)$ be its motion between the ith and $(i+1)$th reflections, $i = 0, 1, \ldots, m$. Let the body be invisible in the direction v_0; then one has $v^{(0)} = v^{(m)} = v_0,\ x^{(0)} = x$, and $x^{(m)} - x \| v_0$. At each reflection and for any fixed x, the orientation of the triple $\big(\frac{\partial x^{(i)}}{\partial x_1}, \frac{\partial x^{(i)}}{\partial x_2}, v^{(i)}\big)$ changes. The initial and final orientations, $\big(\frac{\partial x^{(0)}}{\partial x_1}, \frac{\partial x^{(0)}}{\partial x_2}, v^{(0)}\big)$ and $\big(\frac{\partial x^{(m)}}{\partial x_1}, \frac{\partial x^{(m)}}{\partial x_2}, v^{(m)}\big)$, coincide, therefore m is even.

On the other hand, m cannot be equal to 2, as seen from Fig.3.10. Therefore, $m \geq 4$.

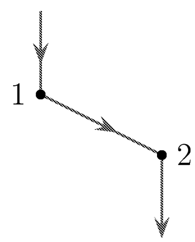


Figure 3.10: Two reflections are not enough for an invisible body.

From the examples of bodies discussed above one concludes that the inequalities in (a) and (b) are sharp. □

3.5.2 Doubling an arbitrary body of minimal resistance

Let $B \subset \mathbb{R}^3$, $v_0 \in S^2$, and let Π be a plane orthogonal to v_0, that is, $\Pi = \{x \in \mathbb{R}^3 : x \cdot v_0 = a\}$. Suppose also that B is contained in the closed half-space $\mathbb{R}^3_- = \{x \in \mathbb{R}^3 : x \cdot v_0 \le a\}$ bounded by Π. Denote by $\widehat{B}$ the body symmetric to B with respect to Π and let $\mathbf{B} = B \cup \widehat{B}$.

Theorem 3.5.2. *If B has zero resistance in the direction $v_0 \in S^2$, then $\mathbf{B}$ is invisible in this direction.*

To prove this theorem, consider first the interaction of billiard particles with B. Take an arbitrary billiard particle incident on B with the initial velocity v_0, and consider its trajectory $x(t)$, $t \le t_0$ until its intersection with the plane Π. In other words, for $t = t_0$ we have

$$x(t_0) \cdot v_0 = a \quad \text{and} \quad \dot{x}(t_0 - 0) \cdot v_0 > 0. \tag{3.5.1}$$

Since the particle moves freely for $t \ge t_0$, we have $\dot{x}(t_0 + 0) = v_0$.

Let us show that the point $x(t_0)$ does not belong to ∂B. Indeed, assuming that $x(t_0) \in \partial B$, we have to conclude that it is a regular point of ∂B. Since B lies on one side of Π and $x(t_0) \in \Pi$, we conclude that the outward normal to ∂B at the point $x(t_0)$ is equal to v_0. Since the particle makes a reflection at this point, we have

$$\dot{x}(t_0 - 0) \cdot v_0 < 0 \quad \text{and} \quad \dot{x}(t_0 + 0) \cdot v_0 > 0.$$

The first of these inequalities contradicts the second inequality in (3.5.1).

Let us now consider the broken line obtained by joining the line $x(t)$, $t \le t_0$ and its image under the symmetry about Π. This broken line can be parameterized as follows:

$$\hat{x}(t) = \begin{cases} x(t), & \text{if } \ t \le t_0 \\ x(2t_0 - t) + 2(a - x(2t_0 - t) \cdot v_0)v_0, & \text{if } \ t \ge t_0. \end{cases}$$

Consider the interaction of particles with the body $\mathbf{B}$. One easily sees that $\hat{x}(t)$, $t \in \mathbb{R}$ is the trajectory of a billiard particle interacting with $\mathbf{B}$. The moments and points of reflection of this trajectory from $\hat{B}$ are symmetric to the moments and points of reflection from B with respect to the point t_0 and to the plane Π, respectively. Besides, there is no reflection at the moment t_0. The final velocity of the particle, v_0, coincides with its initial velocity; moreover, the trajectory outside the convex hull of $\mathbf{B}$ belongs to a straight line. Since this arguments holds for any particle with initial velocity v_0, we conclude that the body $\mathbf{B}$ is invisible in this direction. The theorem is proved.

Recall that any body invisible in the direction v_0, is also invisible in the direction $-v_0$. This is not true for zero resistance bodies; see the example and figure from the section 3.4.4.

3.5.3 Resistance for small deviations of the flow direction

Let B be a body of zero resistance in a direction v_0. It is natural to study the behavior of the resistance for angles close to v_0. We will study this question in the particular case where B is one of the 2D bodies described in section 3.4. The following theorem holds true.

Theorem 3.5.3. *Let B be one of the bodies of zero resistance in the direction v_0 presented in section 3.4. Then there exists a nonzero vector r_0 such that*

$$R_v(B) = |v - v_0| r_0 + o(|v - v_0|^2) \quad \textit{as} \quad |v - v_0| \to 0.$$

Proof. For the sake of transparency we will prove this theorem supposing that B is the union of isosceles triangles with $\pi/6 < \alpha < \pi/4$ (see part 3.4.1).

Denote by $\varepsilon = \varepsilon(v)$ the angle between v and v_0. By $P_i = P_i(v), 1 \le i \le 7$ we denote the initial coordinates x such that the corresponding trajectory hits one of the vertices. One evidently has (see fig. 3.5.3, 3.5.3)

$$v_B^+(x, v) = v, \quad \text{if} \;\; x \notin ([P_1, P_2] \cup [P_3, P_4] \cup \{P_5\} \cup [P_6, P_7]),$$

and $v_B^+(x, v)$ is constant, if $x \in [P_1, P_2]$, or $x \in [P_3, P_4]$, or $x \in [P_6, P_7]$. Denote $v_B^+(x, v) := v_{(1,2)}$, if $x \in [P_1, P_2]$, $v_B^+(x, v) := v_{(3,4)}$, if $x \in [P_3, P_4]$, $v_B^+(x, v) := v_{(6,7)}$, if $x \in [P_6, P_7]$. Then

$$R_v(B) = (P_2(v) - P_1(v))(v - v_{(1,2)}) + (P_4(v) - P_3(v))(v - v_{(3,4)}) + (P_7(v) - P_6(v))(v - v_{(6,7)}).$$

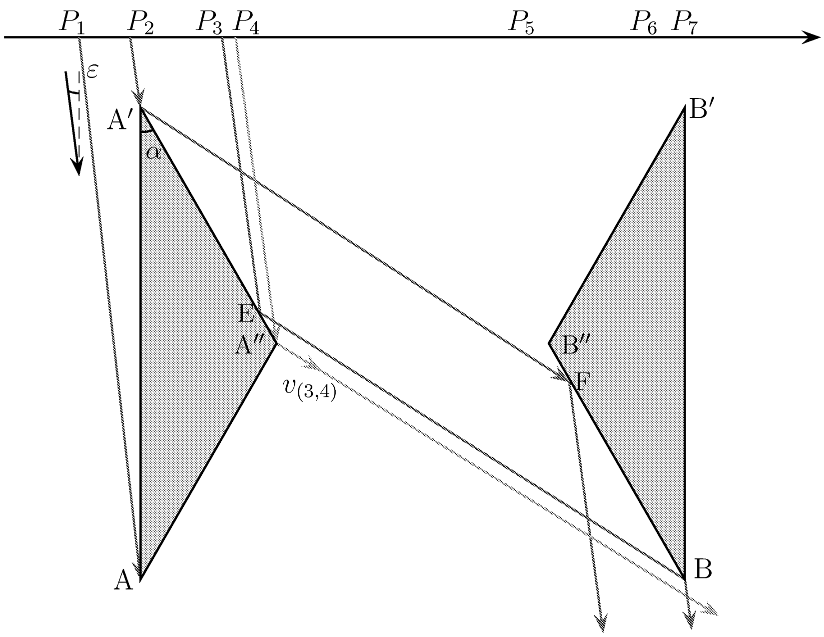


Figure 3.11: Incident angle has defect ε with v_0. Part 1.

First of all, note that $P_2(v) - P_1(v) = |AA'| \tan \varepsilon$ and $v_{(1,2)} = v + O(\varepsilon)$, therefore $(P_2(v) - P_1(v))(v - v_{(1,2)}) = O(\varepsilon^2)$.

Next, note that $\measuredangle A''EB = \alpha - \varepsilon$. Using law of sines we have

$$\frac{|EA''|}{\sin \varepsilon} = \frac{|A''B|}{\sin(\measuredangle A''EB)},$$

and therefore,

$$P_4(v) - P_3(v) = |EA''| \sin(\pi/2 - \alpha + \varepsilon) = \sin \varepsilon \frac{|A''B|}{\sin(\measuredangle A''EB)} \cos(\alpha - \varepsilon) =$$

$$\varepsilon \frac{|A''B|}{\sin \alpha} \cos \alpha + O(\varepsilon^2), \quad \text{as} \quad \varepsilon \to 0.$$

We then estimate $P_7(v) - P_6(v)$ in the same way,

$$\frac{|B'F'|}{\sin \varepsilon} = \frac{|A''B'|}{\sin(\measuredangle A''F'B')}.$$

Note that $\measuredangle A''F'B' = \pi - \measuredangle A''F'B'' = \pi - (\alpha + \varepsilon)$, therefore

$$P_7(v) - P_6(v) = |B'F'| \sin(\pi/2 - \alpha - \varepsilon) = \sin \varepsilon \frac{|A''B'|}{\sin(\measuredangle A''F'B')} \cos(\alpha + \varepsilon) =$$

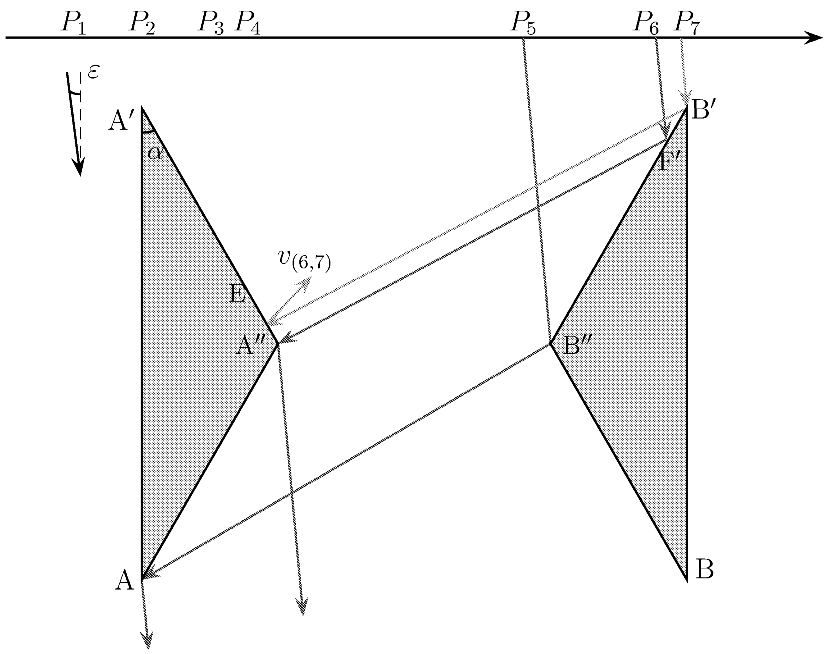


Figure 3.12: Incident angle has defect ε with v_0. Part 2.

$$= \varepsilon \frac{|A''B'|}{\sin\alpha} \cos\alpha + O(\varepsilon^2), \quad \text{as} \quad \varepsilon \to 0.$$

Analyzing the motion of the particle with $x \in [P_3, P_4]$, we see that the particle makes only one reflection and the final velocity $v_{(3,4)}$ makes the angle $2\alpha + O(\varepsilon)$ with v. Similarly, analyzing the motion with $x \in [P_6, P_7]$ and using that $\alpha > \pi/6$, we conclude that the particle makes two reflections and the final velocity $v_{(6,7)}$ makes the angle $4\alpha + O(\varepsilon)$ with v. Taking into account that $|A''B| = |A''B'|$, we get

$$R_v(B) == \varepsilon \frac{|A''B|}{\tan\alpha} (2v - (v_{(3,4)} + v_{(6,7)})) + O(\varepsilon^2).$$

Theorem 3.5.3 is proven.

3.6 Non-uniform motion of the zero-resistance body

We have shown that if the zero-resistance body (a union of two prisms; see the basic construction) moves uniformly in a homogeneous medium, no resistance force appears. However, in non-uniform conditions (e.g., non-uniform motion or non-homogeneous

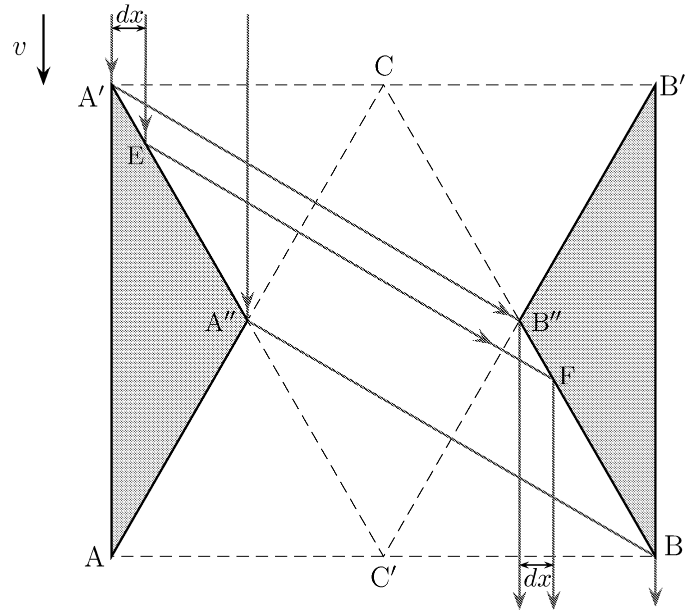


Figure 3.13: The basic construction.

medium), a force can appear. Below we consider two cases corresponding to (i) non-homogeneous medium and (ii) non-uniform motion, and calculate the resulting force.

3.6.1 Getting into a cloud and getting out of it

Consider a zero-resistance spacecraft composed of two prisms (see the basic construction, fig. 3.3.13). Denote by a, h and d the width, length and thickness of the spacecraft, respectively; that is, $a = |AB|$, $h = |AA'|$, and d is the height of each prism. Therefore, the convex hull of the body (spacecraft) is a parallelepiped with sizes a, h and d. One obviously has $h = \frac{\sqrt{3}}{2}\,a$.

Suppose that there is a flow falling on a face of the body, and the symmetric flow falling on the other face. The incidence angle (that is, the angle between the incident particles and the normal to the face) equals α, the flow velocity and density are v and ρ, respectively, and the area of the face exposed to the flow equals $S = \frac{a}{2} \times d$. Then the force of pressure on the face can be calculated as follows:

$$\text{force } = 2\rho v^2 S \sin^2 \alpha. \tag{3.6.1}$$

In order to calculate the resulting force acting on the body one needs to sum up the two forces, that is, multiply by 2, and take the projection on the vertical axis, that is, multiply by 1/2. Therefore, the resulting force will be given by the same formula (3.6.1).

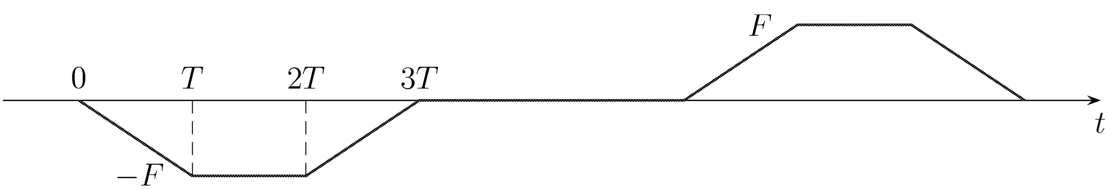
Now consider the spacecraft getting into an interstellar cloud and then getting out of it. The boundary of the cloud at the two points of intersection is perpendicular to the direction of motion. The spacecraft velocity equals v, and the density of the cloud equals ρ. An observer on board will see a parallel flow that starts falling on the upper face of the spacecraft. Then a force acting downwards on the body will be created. The force will increase linearly during the time

$$T = \frac{h}{2v}, \tag{3.6.2}$$

until it stabilizes at the maximal value F. This maximal force is calculated by substituting $S = \frac{a}{2}\, d$ and $\alpha = \pi/3$ into (3.6.1); that is,

$$F = 2\rho v^2\, \frac{a}{2}\, d\, \frac{1}{4}. \tag{3.6.3}$$

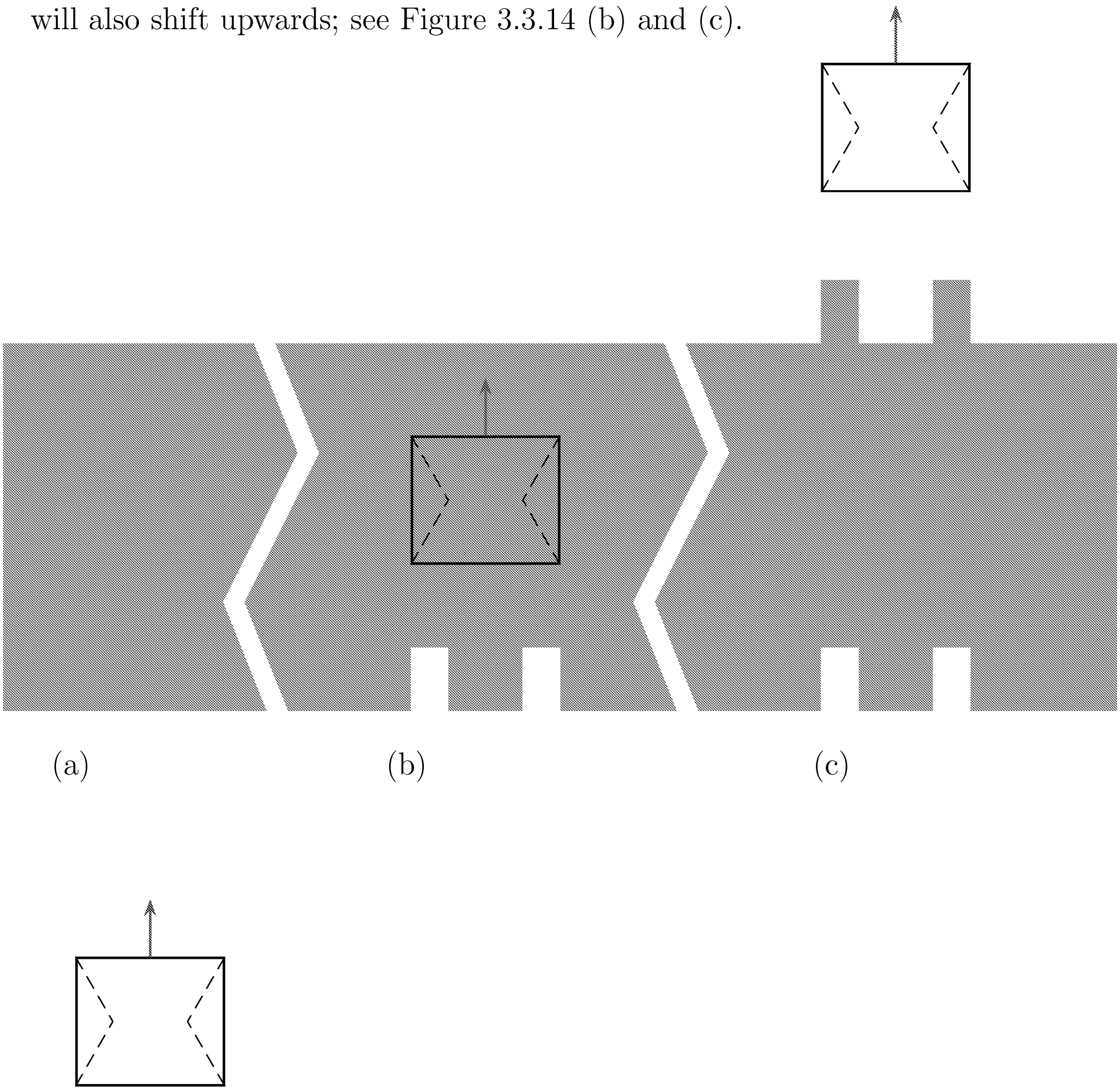
It takes the time $h/v = 2T$ for the first particles of the flow to get to the lower face of the prism. Then a compensating force acting upwards is created. It will linearly increase during the time T until it reaches the maximal value F. Since that moment the resulting force will be equal to zero; see the figure below.



That is, both times, when the spacecraft gets into the cloud and when it gets out, a force tending to expulse it out of the cloud is created.

It is convenient to choose the values F and T as the reference force and reference time, respectively.

Note that the incident particles, after two reflections from the body, shift upwards by the distance $h/2$. That is, after passing the body through a cloud, a part of the cloud will also shift upwards; see Figure 3.3.14 (b) and (c).



(a) (b) (c)

Figure 3.14: (a) Before entering the cloud. (b) The body moves through the cloud. (c) After passing the cloud.

3.6.2 The spacecraft stops and starts the motion

Consider now the case where a body (spacecraft) moves uniformly in a medium, and suddenly stops the motion. We are going to calculate the force acting on the body immediately after the moment of stopping.

The observer on the body will see that all the particles has suddenly stopped the motion, except for the set of particles that are traveling between the first and the second reflection. This set is the union of two subsets corresponding to particles traveling (i) from the left face to the right face, and (ii) from the right face to the left face. These subsets are mutually symmetric. On the figure below, the first subset is $A'B''BA''$.

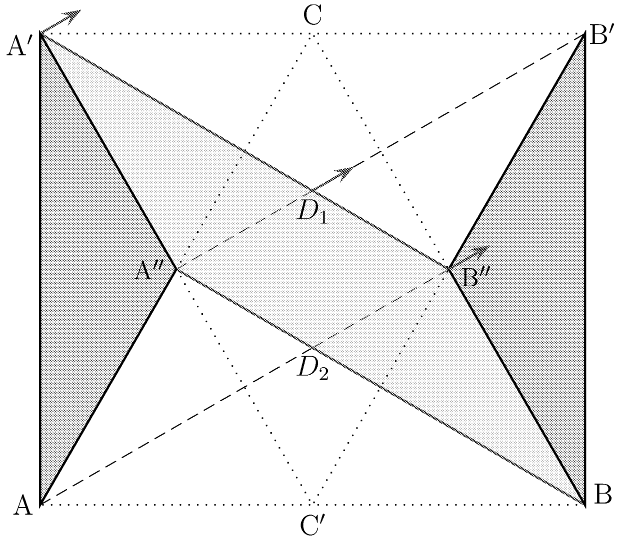
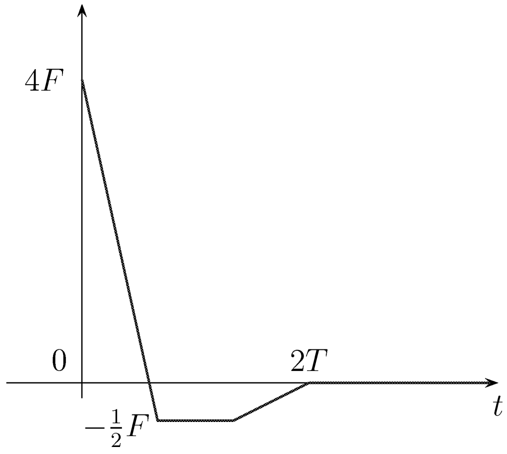


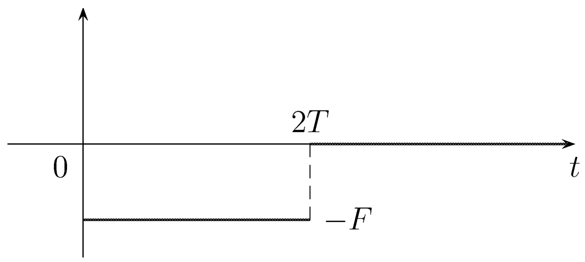
Figure 3.15: Stopping the motion.

The observer will see that the velocity of these particles has changed: initially it was parallel to the line $A'B''$, and now it is parallel to the line $A''B'$, and the modulus of the velocity remains equal to v. Notice that $|D_2B''| = h/3$ and D_2B'' is perpendicular to $B''B$. The width of the parallelogram $A'B''BA''$ equals $a/4$.

Initially, the pressure force on the lower face equals $2\rho v^2 \frac{a}{2} d = 4F$. During the time $h/(3v) = \frac{2}{3}T$ (the time needed for all the particles inside the triangle $D_2B''B$ to hit the lower face) this force linearly decreases to zero. On the contrary, the force on the upper face initially equals zero. During the same time $\frac{2}{3}T$ it linearly increases to $2\rho v^2 \frac{a}{4} d \frac{1}{4} = F/2$, then during the time $\frac{2}{3}T$ remains the same value, and finally, during the time $\frac{2}{3}T$ linearly decreases to 0. The resulting force on the body linearly increases from $4F$ to $-F/2$ during the time during the time $\frac{2}{3}T$, then remains equal to $-F/2$ during the same time, and finally, linearly decreases to 0 during $\frac{2}{3}T$; see the figure below.



Suppose now that the body initially stays at rest in the medium, and at a certain moment suddenly starts moving at the velocity v. The analysis of this case is easier than of the previous one. During the time $h/v = 2T$, the force F will be acting on the body, and then the particles will reach the lower face and hit it, thus creating the compensating force; therefore, the force acting on the body will sharply disappear; see the figure below.



3.7 Possible applications

We believe that the models proposed in this paper can find applications in optics and in aerodynamics of space flights.

A body of zero resistance with specular surface can be used, for example, as a constituent element of a structure (curtain) that lets light through only in one direction. By slightly modifying the construction, a surface can be designed that, like a lens, focuses sunlight onto one point. Bodies with mirror surface invisible in one direction may also be of interest.

Above 150 km, the atmosphere is so rarefied that the effect of intermolecular collisions is negligible [30]. As regards the body (union of two prisms) on Fig. 3.2a, the flow density in some zones between the prisms duplicates and triplicates as compared with the density outside the body, that is, remains sufficiently small. The bodies depicted on Figures 3.2a and 3.3 create infinite density along their symmetry axis; however this effect may be of little importance for practice, because of thermal motion of flow particles and not completely specular reflection from the body surface.

Our model is robust with respect to small changes of physical parameters. This means that in the case of slight thermal motion of gas molecules and nearly specular gas-surface interaction, the resistance is still small. The velocity of artificial satellites on low Earth orbits is much greates than the mean thermal motion of the atmospheric particles [31]. The gas-surface interaction is being intensively studied nowadays. It is very sensitive to many factors, including the spacecraft material, atmosphere composition (which in turn depends on the height), angle of incidence, velocity of the satellite, etc. It is commonly accepted now that the interaction of the atmospheric particles with the surface of existing spacecraft at heights between 150 and 300 km is mostly diffuse [30],[32]; however it is argued [31] that carefully manufactured clean smooth metallic surfaces would favour specular reflections.

Therefore we believe that spacecraft of the shapes indicated on Fig. 3.2a and 3.2b with suitably manufactured surface may experience reduced air resistance and, consequently, have increased lifetime and decreased deflection from the predicted trajectory.

Conclusion

The study of Newton's minimal resistance problem is attractive not only because of its classical flavor, but also because the model and the question are natural and simple and are closely connected with real world problems. It often happens that simple models need very complicated analysis and require different approaches. In the first chapter we give an overview of results and approaches, both very old and recent ones. This overview shows that a great variety of methods were used to attack this problem.

In chapter 2 we examine the class of simply connected, generally non-convex bodies of revolution and find the infimum of resistance in this class. We present a sequence of bodies which approximate the unattainable value of the resistance.

In chapter 3 we present a body of zero resistance. The construction is not a result of complicated mathematical calculations, but rather appears as a result of a deep analysis of unfoldings of billiard trajectories in funnels. We present reasons why the simplicity of the constructed body gives hope for future applications in aerodynamics, optics, and scattering theory. Even now this example represents an important object for study in mathematical disciplines like geometric optics, classical scattering theory, wave scattering theory, and inverse problems in general.

We should also note that the existence of a body of zero total cross section and invisible in one direction states many open questions:

1. Do there exist bodies invisible from a certain point?

2. Do there exist bodies invisible in more than one direction? The same question concerns bodies of zero resistance/leaving no trace.

Note that bodies having zero resistance in a set of directions of positive Lebesgue measure do not exist. The proof will be published elsewhere.

3. For which domains Ω (others than a ring) is Theorem 3.4.3 true?

4. The resistance of any convex body is nonzero. However, by taking a small portion of volume out of a convex body, one can get a body of zero resistance. Namely, there exists a sequence of zero resistance bodies B_n such that their relative volumes $\kappa(B_n)$ go to 1, $\lim_{n\to\infty} \kappa(B_n) = 1$. The maximal number of reflections for these bodies goes to infinity, $\lim_{n\to\infty} m(B_n, v_0) = \infty$. The question is: estimate the maximal relative volume of a zero resistance body B, given that the maximal number of reflections does not exceed a fixed value $m \geq 2$. In other words, estimate $\kappa_m := \sup\{\kappa(B) : R_{v_0}(B) = 0,\ m(B, v_0) \leq m\}$. It is already known that $\kappa_m \geq 14/27$ and $\lim_{m\to\infty} \kappa_m = 1$.

References

[1] I. Newton, *Philosophiae naturalis principia mathematica*, London: Streater, (1687) {1,5}

[2] A.Kneser, Ein Beitrag zur Frage nach der zweckmassigsten Gerstakt der Gescholll-spitzern. Arch. Math. Phys. 2, 267-278, (1902) {2}

[3] A.M.Legendre, Memoires de L'Academie royale de Sciences annee 1786, Paris: Chez Bachelier, Libraire, pp. 7-37, (1788). {5}

[4] M.Belloni, B.Kawohl, *A paper of Legendre revisited*, Forum Math. **9**, 655-667, (1997). {5}

[5] G. Buttazzo, B. Kawohl, *On Newton's problem of minimal resistance.* Math. Intell. **15**, No.4, 7-12 (1993). {3,5,6,15}

[6] G. Buttazzo, V. Ferone, B. Kawohl, *Minimum problems over sets of concave functions and related questions.* Math. Nachr. **173**, 71-89 (1995). {6,9}

[7] F. Brock, V. Ferone, B. Kawohl, *A symmetry problem in the calculus of variations.* Calc. Var. **4**, 593-599 (1996). {7,8}

[8] P. Guasoni, *Problemi di ottimizzazione di forma su classi di insiemi convessi,* Tesi di Laurea, Universit'a di Pisa, 1995-1996. {7}

[9] G. Buttazzo, P. Guasoni, *Shape optimization problems over classes of convex domains.* J. Convex Anal. **4**, No.2, 343-351 (1997). {7,8,15}

[10] Lachand-Robert, T.; Peletier, M. A.*An example of non-convex minimization and an application to Newton's problem of the body of least resistance.* Annales de l'institut Henri Poincare' (C) Analyse non line'aire, **18** no. 2 (2001), p. 179-198 {7,8,9}

[11] T. Lachand-Robert, M.A. Peletier, *Newton's problem of the body of minimal resistance in the class of convex developable functions.* Math. Nachr. **226**, 153-176 (2001). {8}

[12] M. Comte, T. Lachand-Robert, *Newton's problem of the body of minimal resistance under a single-impact assumption.* Calc. Var. **12**, 173-211 (2001). {9,11,15,26,36}

[13] M. Comte, T. Lachand-Robert. *Existence of minimizers for Newton's problem of the body of minimal resistance under a single-impact assumption.* J. Anal. Math. **83**, 313–335 (2001). {9,10,11}

[14] M. Comte, T. Lachand-Robert. *Functions and domains having minimal resistance under a single-impact assumption*, SIAM J. Math. Anal **34**, 101-120 (2002). {11,12}

[15] G. Carlier, T. Lachand-Robert. *Convex bodies of optimal shape*, Journal of Convex Analysis 10 (2003), No. 1, 265–273 {–}

[16] D. Horstmann, B. Kawohl, P. Villaggio, *Newton's aerodynamic problem in the presence of friction*, Nonlinear Differential Equations and applications, **9** 295-307, 2002. {20}

[17] T. Lachand-Robert and E. Oudet. *Minimizing within convex bodies using a convex hull method.* SIAM J. Optim. **16**, 368–379 (2006). {–}

[18] A. Yu. Plakhov. *Newton's problem of a body of minimal aerodynamic resistance.* Dokl. Akad. Nauk **390**, N°3, 314–317 (2003). {12,13,15}

[19] A. Yu. Plakhov. *Newton's problem of the body of minimal resistance with a bounded number of collisions.* Russ. Math. Surv. **58** N°1, 191-192 (2003). {1,3,13,15,42}

[20] *Newton's problem of minimal resistance for bodies containing a half-space* J. Dynam. Control Syst. **10**, 247-251 (2004). {14}

[21] *Newton's problem of the body of minimal averaged resistance* Sbornik: Mathematics 195, No7-8, 1017-1037 (2004).

[22] V. M. Tikhomirov. *Newton's aerodynamical problem.* Kvant, no. 5, 11–18 (1982). {20} {2,32}

[23] Alexander Plakhov and Delfim Torres. *Newton's aerodynamic problem in media of chaotically moving particles.* Sbornik: Math. **196**, 885-933 (2005). {−}

[24] A. Wagner, *A remark on Newton;s Resistance formula*, Z. Angew. Math. Mech. **79** 6, 423-427 (1999). {20}

[25] A. Aleksenko, A.Plakhov, *On the Newton aerodynamic problem for non-convex bodies.* Russ. Math. Surv. **63**, No. 5, 959-961 (2008). {2,15}

[26] A. Plakhov, A.Aleksenko, *The problem of the body of revolution of minimal resistance,* ESAIM: Control, Optimisation and Calculus of Variations, **16**, 206-220 (2010). {2,15}

[27] A.Aleksenko, A. Plakhov, *Bodies of zero resistance and bodies invisible in one direction*, Nonlinearity **22**, 1247-1258, (2009). {1,2}

[28] M. Reed and B. Simon. *Metods of Modern Mathematical Physics, Vol. 3, Scattering Theory.* Academic Press, New York (1979). {40}

[29] V. Petkov. *High frequency asymptotics of the scattering amplitude for non-convex bodies*, Comm. Partial Diff. Eq. **5**, 293-329 (1980). {40}

[30] I K. Harrison and G. G. Swinerd. *A free molecule aerodynamic investigation using multiple satellite analysis.* Planet. Space Sci. **44**, 171-180 (1996). {66}

[31] P. D. Fieseler. *A method for solar sailing in a low Earth orbit.* Acta Astronautica **43**, 531-541 (1998). {66}

[32] K. Moe and M. M. Moe. *Gas-surface interactions and satellite drag coefficients.* Planet. Space Sci. **53**, 793-801 (2005). {66}