Ana Pedro Lemos Paião

Controlo Ótimo e Modelos Matemáticos em Epidemiologia

Optimal Control and Mathematical Models in Epidemiology

Universidade de Aveiro Departamento de Matemática
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Tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática Aplicada, realizada sob a orientação científica do Doutor Delfim Fernando Marado Torres, Professor Catedrático do Departamento de Matemática da Universidade de Aveiro, e da Doutora Cristiana João Soares da Silva, Investigadora do Centro de Investigação e Desenvolvimento em Matemática e Aplicações (CIDMA) do Departamento de Matemática da Universidade de Aveiro.

Thesis submitted to the University of Aveiro in fulfilment of the requirements for the degree of Doctor in Applied Mathematics, under the supervision of Professor Delfim Fernando Marado Torres, Full Professor at the Department of Mathematics of University of Aveiro, and of Cristiana João Soares da Silva, CIDMA's Researcher at the Department of Mathematics of University of Aveiro.

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Dedico este trabalho à avó Paulinha, aos meus pais (Catarina e Pedro) e aos meus irmãos (João David e Luís Miguel).

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Palavras Chave

Tempos de atraso, controlo ótimo, condições suficientes de otimalidade, modelos matemáticos para a transmissão da cólera, estabilidade local assintótica.

Resumo

Nesta tese de doutoramento, provamos condições suficientes de otimalidade para problemas de controlo ótimo com tempos de atraso, transformando-os em problemas equivalentes sem tempos de atraso. Tal transformação é feita considerando uma técnica proposta por Guinn em 1976 e mais tarde promovida por Maurer e seus colaboradores. Deste modo, somos capazes de usar condições suficientes de otimalidade conhecidas para problemas de controlo ótimo sem tempos de atraso e voltar aos que consideram tempos de atraso. Propomos e estudamos vários modelos que traduzem a propagação da cólera e que consideram diferentes tipos de tratamento ou de medidas de prevenção. Problemas de controlo ótimo correspondentes são formulados e estudados. Tais análises teóricas são depois aplicadas a epidemias de cólera reais que ocorreram no Haiti e no Iémen.

Keywords Time delays, optimal control, sufficient optimality conditions, mathematical models for cholera transmission, local asymptotic stability.

Abstract In this PhD thesis, we prove sufficient optimality conditions for delayed optimal control problems, by transforming them into equivalent non-delayed problems. Such transformation is done by considering a technique proposed by Guinn in 1976 and later promoted by Maurer and his collaborators. In this way, we are able to use well-known sufficient optimality conditions for non-delayed optimal control problems and to return to the delayed ones. We propose and study several models that can translate the spread of cholera and that consider different types of treatment or prevention measures. Corresponding optimal control problems are formulated and studied. Such theoretical analysis are then applied to real cholera outbreaks that occured in Haiti and Yemen.

2010 Mathematics Subject Classification: 34C60; 49K15; 92D30.

“La matematica è l’alfabeto con cui Dio ha scritto l’universo.”
– Galileo Galilei

Contents

Introduction

This PhD thesis addresses two fields of Mathematics: *Mathematical Models in Epidemiology* and *Optimal Control Theory*. In the last mentioned, we usually consider a control system, where the dynamics are described by a mathematical model, whether it can be, for example, a system of ordinary differential equations, partial differential equations, or discrete difference equations (see e.g. [110]). Here, we only consider systems of ordinary differential equations with, or without, time delays. Such systems have several state variables and they can be controlled through control functions. The heart of the matter is how to control them in order to obtain the "best" outcome, as restricted by some specific goals. As Lenhart and Workman wrote in [110]:

> "The mathematical theory behind answering these questions, often called *Optimal Control Theory* or *Dynamic Optimization*, has found application in a myriad of fields, from the Biological Sciences, to Economics, to Business and Management, to Physics and Engineering."

In this work, such systems are studied with the purpose to analyse the transmission dynamics of infectious diseases. In such analysis, it is important to know what is an equilibrium point and the basic reproduction number, as well as how we can determine them. An equilibrium point of a differential system is a point for which the system is in equilibrium, that is, for which the state functions of the system do not vary. The basic reproduction number is the expected number of new infections due to a contact between only one typical infective individual and a completely susceptible population. In the context of Epidemiology, we usually compute the disease-free and the endemic equilibrium points. In general, we can denote them by equilibria. Moreover, we carry out stability studies with respect to equilibria. Then, we formulate and study non-delayed and delayed optimal control problems that consider such systems. In applications, we provide optimal, or extremal, control measures to curtail the spread of infectious diseases by considering real situations, where the endemic equilibrium is locally asymptotic stable.

The thesis is divided into two parts: *State of the Art* and *Original Results*. In the first one, we recall important stability concepts and theoretical

results associated with delayed, or non-delayed, *ordinary differential equations*; as well as the definition of *compartmental models* and other related definitions and theoretical results. We also give some well-known examples of compartmental models (see Chapter 1). Still in the first part, we define a non-delayed optimal control problem for which we recall well-known necessary and sufficient optimality conditions, following the approaches used in [56, 103, 140]. We also define an optimal control problem with constant time delays in state and control variables for which we recall necessary optimality conditions derived in [56] (see Chapter 2). In the second part of this thesis (*Original Results*), we firstly give answer to an open question by proving sufficient optimality conditions for optimal control problems with constant time delays in state and control variables (see Chapter 3). In the proof of our main results, we transform delayed optimal control problems to equivalent non-delayed problems, considering the technique proposed by Guinn in [59] and used by Göllmann et al. in [56, 57]. This allows us to use well-known theoretical results that ensure sufficient optimality conditions for non-delayed optimal control problems, recalled in Chapter 2. We remark that all the contents of Chapter 3 are already published in international journals (see [108, 109]). In Chapter 4, we explain an infectious disease that remains a global threat to public health and an indicator of inequity and lack of social development – cholera (see [106, 107, 192]). The number of cholera cases reported by World Health Organization (WHO) has continued to be high over the last few years. During 2017, 1227391 cases were notified from 34 countries, including 5654 deaths (see [192]). Moreover, we give a general idea about what has already been done to understand the dynamics of cholera through the study of mathematical models. We formulate and explain several models that can translate the spread of cholera, considering different types of treatment or prevention measures. Chapter 4 is crucial, since the branches of Mathematics approached in this thesis are applied to cholera, in Chapters 5 and 6, using the proposed cholera models of Chapter 4. In Chapters 5 and 6, we study the formulated cholera models by determining the equilibrium points and the basic reproduction numbers for each one. Stability analysis of equilibria is also carried out. Then, we formulate and study some non-delayed and delayed optimal control problems, using some of these models. While in Chapter 5 we consider applications related with the cholera outbreak that occurred in the Department of Artibonite – Haiti, in 2010; in Chapter 6, we apply theoretical studies to the biggest cholera outbreak of world's history that began on 27th April 2017, in Yemen. We end this thesis with some conclusions and open questions.

Part I

State of the Art

Chapter 1

Mathematical Models in Epidemiology

We begin this chapter by recalling some basic definitions and propositions of Linear Algebra and Mathematical Analysis, since they will be used throughout this thesis. Some academic and practical examples are given. In second place, we present stability definitions and theoretical results for systems of ordinary differential equations (ODE) and ordinary delayed differential equations (ODDE). Again, some illustrative examples are provided. Next, we recall the definition of compartmental models and some associated theoretical results. We finish this chapter by giving some examples of compartmental models.

1.1 Introduction

Frequently, a phenomenon is not analysed directly, but indirectly through a model of it. A model is a representation, often trough a mathematical view, of what is considered important and crucial to translate a phenomenon. Many researchers have manipulated such models with the purpose to obtain new knowledge about the modelled phenomena without the danger, cost, or inconvenience of manipulating the real phenomena itself. Most of the real life modelling requires knowledge in Mathematics. Important information of many physical phenomena can be described numerically. Moreover, the relations between different features of real life can be translated by equations or inequalities. In a particular way, quantities associated with, for example, Natural Sciences and Engineering can be explained mathematically. Examples of such quantities can be: mass, position, velocity, acceleration, force, number of a specific type of individuals and concentration of a bacterium. To provide a modelling approach successfully, it is indispensable to know the modelled phenomena and properties of such models. In Mathematical Systems Theory, the dynamic behaviour of these phenomena is very im-

portant. In other words, it is essential to know how characteristic features develop over time and what are the relationships, that are also described as functions of time. Mathematical Systems Theory incorporates the basic knowledge for technical areas such as Automatic Control and Networks. Furthermore, it is also the starting point for Optimal Control Theory from which some important theoretical results will be recalled in Chapter 2 (see [132]).

As Ma and Li write in [113], the spread of infectious diseases has always been a threat to public health. It has hampered the survival of human beings and other species, as well as it has created barriers to the economic and social development of the human society. Although relevant prevention and control measures have been developed to stop the spread of infectious diseases, some of such illnesses still continue killing many people around the world. To curtail more effectively the propagation of these type of diseases, firstly it is essential to fully understand the transmission dynamics of these illnesses. Epidemic dynamics study is a way to obtain this knowledge. It formulates mathematical models to translate the mechanisms of disease transmissions and dynamics of infectious agents. Such formulations require information associated with population dynamics, behaviour of disease transmissions, features of the infectious agents and the connections with other social and physiologic factors. The study of such models can incorporate, for example, quantitative and qualitative analysis, sensitivity analysis and numeric simulations. Then, we are able to understand better the spread of infectious diseases, to find principles that command the transmission dynamics and to determine the more sensitive parameters. Consequently, we can provide useful and reliable predictions, as well as guidance in order to establish better control strategies.

Although the mathematical research on infectious diseases (through deterministic models), as a discipline, actually began in the XX century, *Daniel Bernoulli* (1700–1782) already used mathematical models to analyse the spread of smallpox, in 1760 (see [12, 113]). Later, in 1906, a discrete-time model for the propagation of measles was proposed (see [61, 113]). In 1911, *Ronald Ross* (1857–1932) considered a differential equation model to describe the transmissions of malaria between humans and mosquitoes (see [113, 146]). He determined that there is a threshold of the size of mosquitoes below which the spread of malaria can be controlled. Due to his brilliant contributions in the research of the transmission dynamics of malaria, *Ross* was awarded his second Nobel Prize in Medicine. Later, in 1927, *Anderson Gray McKendrick* (1876–1943) and *William Ogilvy Kermack* (1898–1970) created a well-known and well-recognized SIR (Susceptible–Infectious–Recovered) compartmental model with the purpose to analyse the outbreak of Black Death in London (1665–1666) and the epidemic of plague in Mumbai (1906). For more details see [87, 113]. In 1932, they also formulated a SIS (Susceptible–Infectious–Susceptible) compartment model (see [88]). With

the study of such model, they formally introduced the concept of thresholds that allow us to know whether a disease spreads in a certain population. Most of the deterministic models incorporate ordinary differential equations. Nevertheless, first and second order partial differential equations and delayed differential equations have been also considered (see [113]). With the study of such deterministic mathematical models, we can conclude if their solutions make sense in a certain reality and, moreover, we can analyse the existence and stability of steady states, which characterize the spread of such diseases. As it is crucial to know some properties associated with the models in study, we recall here some important concepts with respect to models that are translated by a differential system. Some of such concepts are: equilibrium point, local stability and local asymptotic stability. Moreover, we are going to state and illustrate theoretical results that allow us to obtain stability conclusions. We remark that we are going to approach such concepts for ODE and ODDE. Although ODE have been an important tool for the study of population dynamics, more realistic models should consider some of the past information (see [97]). As Kuang wrote in [97, p. ix]:

> "ideally, a real system should be modeled by differential equations with time delays."

In this chapter, we give some preliminaries needed throughout this thesis. We begin with Section 1.2 by defining some important concepts of Linear Algebra that are essential in the study of mathematical models: *eigenvalue*, *eigenvector*, *eigenspace*, *characteristic polynomial*, *characteristic equation*, *algebraic multiplicity* and *geometric multiplicity*. An example is given with the purpose to illustrate all these concepts. At the end of Section 1.2, we also recall some well-known definitions and propositions of Mathematical Analysis associated with *continuity* and *differentiability* for vectorial functions of several variables. In Section 1.3, we define *equilibrium point* and its *stability* for non-delayed systems of ordinary differential equations (ODE). This section is divided into four sections. Sections 1.3.1 and 1.3.2 are, respectively, devoted to the stability study of linear and non-linear differential systems. Furthermore, several illustrative examples are solved in these two sections. In most cases, these stability studies depend on the roots of a polynomial. Nevertheless, the determination of these roots is not always easy. Thus, we sometimes can resort to the Routh–Hurwitz Criterion or to Descartes' Rule of Signs, presented in Sections 1.3.3 and 1.3.4, respectively. The Routh–Hurwitz Criterion gives us a necessary and sufficient condition for all roots of a given polynomial to have negative real part, only using the values of their coefficients. Descartes' Rule of Signs allow us to know the maximum number of positive real roots of a polynomial, only using the signs of their coefficients. In Section 1.4, we define *equilibrium point* and its *stability*, when we consider a system of ordinary delayed differential equations (ODDE). Sections 1.4.1 and 1.4.2 are, respectively, devoted to the stability

study of linear and non-linear differential systems. Finally, with the purpose to study mathematically the propagation of infectious diseases in a heterogeneous population, in Section 1.5 we recall the definition of compartmental models and some associated theoretical results, following the approach used in [176]. We finish the current chapter with Section 1.6, giving several examples of compartmental models, and Section 1.7 of conclusion.

1.2 Preliminaries

In this section we recall some basic definitions of Linear Algebra and Mathematical Analysis, following the approaches used in [46, 94, 132]. Below are the definitions of *eigenvalue*, *eigenvector*, *eigenspace*, *characteristic polynomial*, *characteristic equation*, *algebraic multiplicity* and *geometric multiplicity*.

Definition 1.1 (See p. 408 of [94]). *Let A be a $n \times n$ real matrix. The number $\lambda \in \mathbb{C}$ is an* eigenvalue *of A if there is a vector*

$$v = [v_1 \quad \cdots \quad v_n]^T \in \mathbb{C}^n \setminus \{0_{\mathbb{C}^n}\}$$

such that

$$Av = \lambda v.$$

If this vector v exists, then it is called by eigenvector *associated with the* eigenvalue *λ.*

Definition 1.2 (See p. 417 of [94]). *Let A be a $n \times n$ real matrix and $\lambda \in \mathbb{C}$ be an* eigenvalue *of A. The* eigenspace *$\mathscr{U}_\lambda$ associated with λ is defined by*

$$\mathscr{U}_\lambda = \left\{v \in \mathbb{C}^n : \ (A - \lambda I_n)v = 0_{\mathbb{C}^n}\right\}.$$

Definition 1.3 (See p. 412 of [94]). *Let A be a $n \times n$ real matrix. The* characteristic polynomial *of A, with degree equal to n, is defined by*

$$p_A(\lambda) = \det(A - \lambda I_n).$$

Moreover, the characteristic equation *is given by*

$$p_A(\lambda) = \det(A - \lambda I_n) = 0.$$

In practice, the eigenvalues of A are the roots of the characteristic polynomial p_A (see Theorem 8.2 of [94]). Assuming that A has k different eigenvalues $\lambda_1, \ldots, \lambda_k$, with $k \in \mathbb{N}$ and $k \leq n$, then we can write $p_A(\lambda)$ as follows

$$p_A(\lambda) = (\lambda_1 - \lambda)^{n_1} \times \cdots \times (\lambda_k - \lambda)^{n_k},$$

where $n_1 + \cdots + n_k = n$ and $n_i \in \mathbb{N}$ for all $i = 1, \ldots, k$.

Definition 1.4 (See p. 34 of [132]). *Let A be a $n \times n$ real matrix with the different eigenvalues $\lambda_1, \ldots, \lambda_k$ ($k \in \mathbb{N}$ and $k \leq n$) and* characteristic polynomial

$$p_A(\lambda) = (\lambda_1 - \lambda)^{n_1} \times \cdots \times (\lambda_k - \lambda)^{n_k}.$$

The algebraic multiplicity *and the* geometric multiplicity *of the* eigenvalue *λ_i is, respectively, equal to n_i and to the dimension of $\mathscr{U}_{\lambda_i}$ for $i = 1, \ldots, k$. Moreover, λ_i is called* simple eigenvalue *when $n_i = 1$ for $i = 1, \ldots, k$.*

Let us give an example which involves all the previous definitions.

Example 1.5. *Consider the following 3×3 real matrix given by*

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

The respective characteristic polynomial *is*

$$p_A(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 4-\lambda & 1 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (4-\lambda)^2(3-\lambda).$$

Thus, we can conclude that A has two different eigenvalues*: $\lambda_1 = 4$ and $\lambda_2 = 3$. Moreover, the* algebraic multiplicity *of λ_1 and λ_2 is, respectively, $n_1 = 2$ and $n_2 = 1$. Throughout this example we are considering that*

$$v = [v_1 \quad v_2 \quad v_3]^T \in \mathbb{C}^3.$$

Let us determine the eigenspace *$\mathscr{U}_4$:*

$$(A - 4I_3)v = 0_{\mathbb{C}^3} \Leftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v_1,$$

where $v_1 \in \mathbb{C}$. Then, we have that

$$\mathscr{U}_4 = \{v \in \mathbb{C}^3 : \ v_2 = v_3 = 0\}.$$

Consequently, the geometric multiplicity *of λ_1 is equal to $\dim(\mathscr{U}_4) = 1$. Now, let us find the* eigenspace *$\mathscr{U}_3$:*

$$(A - 3I_3)v = 0_{\mathbb{C}^3} \Leftrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow v = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} v_2,$$

where $v_2 \in \mathbb{C}$. Then, we have that

$$\mathscr{U}_3 = \{v \in \mathbb{C}^3 : \ v_1 = v_3 = -v_2\}.$$

Consequently, the geometric multiplicity *of λ_2 is equal to $\dim(\mathscr{U}_3) = 1$.*

We end this section by recalling some well-known concepts and results associated with *continuity* and *differentiability* for vectorial functions of several variables, since they will be used throughout this thesis.

Definition 1.6 (Continuity – see p. 34 of [46]). *Let $h : D_h \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function and $\check{x}$ be an interior point of D_h. We say that h is continuous at $x = \check{x}$ if $\lim_{x \to \check{x}} h(x) = h(\check{x})$.*

Proposition 1.7 (See Proposition 2.5 of [46]). *Being h and $\check{x}$ as in Definition 1.6, we have that h is continuous at $x = \check{x}$ if and only if each of its components h_i is continuous at $x = \check{x}$ for $i \in \{1, \ldots, m\}$.*

Definition 1.8 (Differentiability – see p. 128 of [46]). *Let $h : D_h \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function and $\check{x}$ be an interior point of D_h. We say that h is differentiable at $x = \check{x}$ if there is a linear transformation[1] L (depending on $\check{x}$) such that*

$$\lim_{k \to 0_{\mathbb{R}^n}} \left(\frac{h(\check{x} + k) - h(\check{x})}{|k|} - L(k) \right) = 0_{\mathbb{R}^n}.$$

Proposition 1.9 (See Proposition 4.3 of [46]). *Being h and $\check{x}$ as in Definition 1.8, we have that*

i) h is differentiable at $x = \check{x}$ if and only if each of its components h_i is differentiable at $x = \check{x}$ for $i \in \{1, \ldots, m\}$;

ii) if h is differentiable at $x = \check{x}$, then the matrix of the linear transformation L is the matrix of partial derivatives $\frac{\partial h_i}{\partial x_j}(\check{x})$ for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

Next, we define the functions of class $\mathscr{C}^q$, where $q \in \mathbb{N}_0$.

Definition 1.10 (See p. 91 and 131 of [46]). *Let $h : D_h \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function, where D_h is an open set. If all of the qth order partial derivatives ($q \in \mathbb{N}_0$) of each component h_i exist and are continuous at every $x \in D_h$ for $i \in \{1, \ldots, m\}$, then h is a function of class $\mathscr{C}^q$. Moreover, we then write that $h \in \mathscr{C}^q(D_h, \mathbb{R}^m)$.*

Note that a function f is of class $\mathscr{C}^q$ if and only if f is q times continuously differentiable.

1.3 Stability for non-delayed differential systems

Following the approach of [132], we begin this section by recalling some important definitions and stability results for non-delayed systems of ordinary differential equations (ODE) given by

$$\dot{x} = f(x) \tag{1.1}$$

[1] For more details about linear transformations, one can see [46, p. 119–124].

with the initial condition $x(a) = x_a$, where $a \in \mathbb{R}_0^+$ is the fixed initial time, $x(t) \in \mathbb{R}^n$ for all $t \geq a$ and f is continuously differentiable as many times as we need. Next, we define equilibrium point and its stability.

Definition 1.11 (See Definition 4.1 of [132]). *An* equilibrium point *of system* (1.1) *is a vector* $\bar{x} \in \mathbb{R}^n$ *that satisfies the equation* $f(\bar{x}) = 0$.

The designations *fixed point*, *stationary point*, *rest point*, *singularity*, *steady state* and *critical point* are synonymous of *equilibrium point*.

Definition 1.12 (See Definition 4.1 of [132]). *For each* $t \geq a$, *let* $\bar{x}$ *and* $x(t)$ *be an* equilibrium point *and the solution of system* (1.1) *with the initial condition* $x(a) = x_a$, *respectively. The* equilibrium point $\bar{x}$ *is called*

i) locally stable *if, for all* $t \geq a$,

$$\forall \varepsilon > 0,\ \exists \delta > 0\ :\ ||x_a - \bar{x}|| < \delta \implies ||x(t) - \bar{x}|| < \varepsilon;$$

ii) locally asymptotic stable *if it is* stable *and*

$$\exists \delta > 0\ :\ ||x_a - \bar{x}|| < \delta \implies \lim_{t \to +\infty} ||x(t) - \bar{x}|| = 0;$$

iii) unstable *if it is not* locally stable.

Note that $||\cdot||$ represents an arbitrary norm in the previous definition. The *Euclidean norm* is used frequently. Let x_a be an initial point in a neighbourhood of a certain equilibrium point $\bar{x}$. Intuitively, $\bar{x}$ is locally stable if and only if the solution of system (1.1) with the initial condition $x(a) = x_a$ remains in a neighbourhood of $\bar{x}$. On the other side, when $\bar{x}$ is locally asymptotic stable, we have that the solution remains in a neighbourhood of $\bar{x}$ and moreover it converges to $\bar{x}$, ensuring that the initial point x_a is sufficiently close to $\bar{x}$. If $\bar{x}$ is unstable, then even when the initial point x_a is arbitrarily close to $\bar{x}$, it always exists a solution $x_u(\cdot)$ of $\dot{x} = f(x)$ for which it is not possible to find any neighbourhood of $\bar{x}$ where $x_u(\cdot)$ could remain. So, $x_u(\cdot)$ "explodes" or "diverges away" from $\bar{x}$.

1.3.1 Linear differential systems

In this section, we consider that system (1.1) is linear, i.e.,

$$\dot{x} = f(x) = Ax, \tag{1.2}$$

where A is a $n \times n$ real matrix. Obviously, $\bar{x} = 0_{\mathbb{R}^n}$ is an equilibrium point of $\dot{x} = Ax$. Nevertheless, there will be others if $\det(A) = 0$. Following the approach of [132], we recall a stability result for the linear differential system (1.2) only with respect to the fixed point $\bar{x} = 0_{\mathbb{R}^n}$. So, the system $\dot{x} = Ax$ (or the matrix A) is stable, locally asymptotic stable or unstable if $\bar{x} = 0_{\mathbb{R}^n}$ is stable, locally asymptotic stable or unstable, respectively.

Theorem 1.13 (See Theorem 4.2 of [132]). *Let A be a $n \times n$ real matrix with k different eigenvalues $\lambda_1, \ldots, \lambda_k$, with $k \in \mathbb{N}$ and $k \leq n$. Consider the linear differential system $\dot{x} = Ax$ with an equilibrium point $\bar{x} = 0_{\mathbb{R}^n}$, where $x \in \mathbb{R}^n$. Then, $\bar{x} = 0_{\mathbb{R}^n}$ is*

i) locally stable *if and only if $\Re(\lambda_i) \leq 0$ for all $i = 1, \ldots, k$ and the algebraic and geometric multiplicities are equal for each eigenvalue λ_i with $\Re(\lambda_i) = 0$;*

ii) locally asymptotic stable *if and only if $\Re(\lambda_i) < 0$ for all $i = 1, \ldots, k$;*

iii) unstable *if and only if $\Re(\lambda_i) > 0$ for some $i = 1, \ldots, k$ or for some eigenvalue λ_i with $\Re(\lambda_i) = 0$ the algebraic multiplicity is larger than the geometric multiplicity.*

Remark 1.14. *If a matrix A is such that $\det(A) = 0$, then system (1.2) has, at least, two different equilibrium points. One of them is clearly $\bar{x}_1 = 0_{\mathbb{R}^n}$ and the other is $\bar{x}_2 \in \mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}$. When we apply the change of variable*

$$x = y - \bar{x}_2$$

in system (1.2), we obtain

$$\dot{x} = Ax \Leftrightarrow \dot{y} - \dot{\bar{x}}_2 = Ay - A\bar{x}_2 \Leftrightarrow \dot{y} = Ay.$$

Observe that $y = 0_{\mathbb{R}^n}$ is an equilibrium point of $\dot{y} = Ay$ and furthermore $y = 0_{\mathbb{R}^n} \Leftrightarrow x = \bar{x}_2$. So, the stability conclusions about $\bar{x}_1 = 0_{\mathbb{R}^n}$ and $\bar{x}_2 \in \mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}$ are the same.

Now, we define direct sum.

Definition 1.15 (See p. 34 of [132]). *A linear space $\mathscr{V}$ is the* direct sum *of two linear subspaces $\mathscr{V}_1$ and $\mathscr{V}_2$, if each $v \in \mathscr{V}$ can uniquely be decomposed as*

$$v = v_1 + v_2,$$

where $v_1 \in \mathscr{V}_1$ and $v_2 \in \mathscr{V}_2$. Moreover, $\mathscr{V}$ can be denoted by

$$\mathscr{V} = \mathscr{V}_1 \oplus \mathscr{V}_2.$$

Next we define stable, unstable and center subspaces associated with the linear differential system (1.2).

Definition 1.16 (See Definition 4.5 of [132] and p. 30 of [181]). *Let A be a $n \times n$ real matrix with k different eigenvalues $\lambda_1, \ldots, \lambda_k$, with $k \in \mathbb{N}$ and $k \leq n$. Assume that each eigenvalue λ_i has algebraic multiplicity n_i and*

$$\mathscr{N}_{\lambda_i} = \left\{ v \in \mathbb{C}^n : \ (A - \lambda_i I_n)^{n_i} v = 0_{\mathbb{C}^n} \right\}$$

for $i = 1, \ldots, k$. Moreover, consider the linear differential system $\dot{x} = Ax$ for $x \in \mathbb{R}^n$. The stable subspace *for this system is the real subspace of the direct sum of linear subspaces $\mathscr{N}_{\lambda_i}$ that correspond to eigenvalues of A with negative real part. The* unstable subspace *and the* center subspace *are defined similarly, then corresponding to eigenvalues of A with, respectively, positive and zero real parts.*

Remark 1.17 (See Theorem 3.9 of [132]). *Considering the notions of Definitions 1.15 and 1.16, one can prove that $\mathbb{C}^n = \mathscr{N}_{\lambda_1} \oplus \mathscr{N}_{\lambda_2} \oplus \ldots \oplus \mathscr{N}_{\lambda_k}$.*

Remark 1.18 (Stable, unstable and center manifold). *As in [181, p. 28–30], instead of defining deeply the concept of a manifold, we describe only the portion of the vast theory that we will need. So, as Wiggins writes in [181, p. 28–30], roughly speaking, a* manifold *is a set that, locally, has the structure of an Euclidean space. With these considerations, it is important to state that the stable, unstable and center subspaces rise the stable, unstable and center manifolds, respectively.*

Next, we present some illustrative examples of Theorem 1.13 and Definition 1.16.

Example 1.19. *Let us study the stability of the equilibrium point $\bar{x} = 0_{\mathbb{R}^3}$ of the linear system $\dot{x} = Ax$, where*

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 3 & -4 & 0 \\ 7 & 2 & -2 \end{bmatrix}.$$

The characteristic polynomial is given by

$$p_A(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} -1-\lambda & 0 & 0 \\ 3 & -4-\lambda & 0 \\ 7 & 2 & -2-\lambda \end{vmatrix} = -(\lambda+4)(\lambda+2)(\lambda+1).$$

As A has only eigenvalues with negative real part ($\lambda_1 = -4$, $\lambda_2 = -2$ and $\lambda_3 = -1$), then the equilibrium point $\bar{x} = 0_{\mathbb{R}^3}$ of $\dot{x} = Ax$ is locally asymptotic stable. We can also state that $\bar{x} = 0_{\mathbb{R}^3}$ is locally stable, by Definition 1.12. The stable subspace S is the real subspace of $\mathscr{N}_{-4} \oplus \mathscr{N}_{-2} \oplus \mathscr{N}_{-1} = \mathbb{C}^3$. It means that $S = \mathbb{R}^3$. Obviously, the center subspace C and unstable subspace U are such that $C = U = \emptyset$.

Example 1.20. *Let us study the stability of the equilibrium point $\bar{x} = 0_{\mathbb{R}^3}$ of the linear system $\dot{x} = Ax$, where*

$$A = \begin{bmatrix} 0 & 4 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

The characteristic polynomial is given by

$$p_A(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} -\lambda & 4 & 1 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{vmatrix} = -\lambda(\lambda+2)(\lambda+1).$$

The eigenvalues of A *are* $\lambda_1 = 0$, $\lambda_2 = -2$ *and* $\lambda_3 = -1$ *with algebraic multiplicities* $n_1 = n_2 = n_3 = 1$. *In order to take conclusions about the stability of* $\bar{x} = 0_{\mathbb{R}^3}$, *we need to find the geometric multiplicity of* $\lambda_1 = 0$. *Throughout this example, we are considering that* $v = [v_1 \quad v_2 \quad v_3]^T$. *Let us determine the eigenspace* $\mathscr{U}_0$:

$$Av = 0_{\mathbb{C}^3} \Leftrightarrow v = [1 \quad 0 \quad 0]^T v_1,$$

where $v_1 \in \mathbb{C}$. *So, we obtain that*

$$\mathscr{U}_0 = \{v \in \mathbb{C}^3 : \ v_2 = v_3 = 0\}.$$

Consequently, the geometric multiplicity of λ_1 *is equal to* $\dim(\mathscr{U}_0) = 1$. *As the matrix* A *has two eigenvalues with negative real part and the algebraic and geometric multiplicities of* $\lambda_1 = 0$ *are equal* ($n_1 = \dim(\mathscr{U}_0) = 1$), *then the equilibrium point* $\bar{x} = 0_{\mathbb{R}^3}$ *of* $\dot{x} = Ax$ *is locally stable, but it is not locally asymptotic stable.*

Now, let us determine the center and stable subspaces. In this example, we have that $\mathscr{N}_\lambda = \mathscr{U}_\lambda$ *for all* $\lambda \in \{-2, -1, 0\}$, *because all eigenvalues are simple. So, the center subspace is given by*

$$C = \{v \in \mathbb{R}^3 : \ v_2 = v_3 = 0\}.$$

Let us determine the subspace $\mathscr{N}_{-2} = \left\{v \in \mathbb{C}^3 : \ (A + 2I_3)v = 0_{\mathbb{C}^3}\right\}$:

$$(A + 2I_3)v = 0_{\mathbb{C}^3} \Leftrightarrow v = [-2 \quad 1 \quad 0]^T v_2,$$

with $v_2 \in \mathbb{C}$. *Then, we have that*

$$\mathscr{N}_{-2} = \left\{v \in \mathbb{C}^3 : v_1 + 2v_2 = 0 \wedge v_3 = 0\right\}.$$

Let us determine the subspace $\mathscr{N}_{-1} = \left\{v \in \mathbb{C}^3 : \ (A + I_3)v = 0_{\mathbb{C}^3}\right\}$:

$$(A + I_3)v = 0_{\mathbb{C}^3} \Leftrightarrow v = [-5 \quad 1 \quad 1]^T v_3,$$

with $v_3 \in \mathbb{C}$. *Then, we have that*

$$\mathscr{N}_{-1} = \left\{v \in \mathbb{C}^3 : v_1 + 5v_3 = 0 \wedge v_2 = v_3\right\}.$$

One can check that $\mathscr{N}_{-2} \cap \mathscr{N}_{-1} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$. *Thus, the stable subspace* S *is composed by all real linear combinations of* $[-2 \quad 1 \quad 0]^T$ *and* $[-5 \quad 1 \quad 1]^T$. *It*

means that a vector $s = [s_1 \quad s_2 \quad s_3]^T \in S$ *if and only if there are* $c_1, c_2 \in \mathbb{R}$ *such that*

$$[s_1 \quad s_2 \quad s_3]^T = c_1[-2 \quad 1 \quad 0]^T + c_2[-5 \quad 1 \quad 1]^T.$$

This is equivalent to say that the system

$$\begin{cases} -2c_1 - 5c_2 = s_1 \\ c_1 + c_2 = s_2 \\ c_2 = s_3 \end{cases} \tag{1.3}$$

is possible. We can write the previous system in the following way:

$$\begin{bmatrix} -2 & -5 & | & s_1 \\ 1 & 1 & | & s_2 \\ 0 & 1 & | & s_3 \end{bmatrix} \underset{L_2 \leftarrow L_2 - L_3}{\sim} \begin{bmatrix} -2 & -5 & | & s_1 \\ 1 & 0 & | & s_2 - s_3 \\ 0 & 1 & | & s_3 \end{bmatrix}$$

$$\underset{L_1 \leftarrow L_1 + 2L_2 + 5L_3}{\sim} \begin{bmatrix} 0 & 0 & | & s_1 + 2s_2 + 3s_3 \\ 1 & 0 & | & s_2 - s_3 \\ 0 & 1 & | & s_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & s_2 - s_3 \\ 0 & 1 & | & s_3 \\ 0 & 0 & | & s_1 + 2s_2 + 3s_3 \end{bmatrix}.$$

Thus, system (1.3) *is possible if and only if* $s_1 + 2s_2 + 3s_3 = 0$. *Concluding, the stable subspace is given by*

$$S = \{s \in \mathbb{R}^3 : s_1 + 2s_2 + 3s_3 = 0\}.$$

Obviously, the unstable subspace is $U = \emptyset$.

Example 1.21. *Let us study the stability of the equilibrium point* $\bar{x} = 0_{\mathbb{R}^4}$ *of the linear system* $\dot{x} = Ax$, *where*

$$A = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial is given by

$$\begin{aligned} p_A(\lambda) = \det(A - \lambda I_4) &= \begin{vmatrix} -\lambda & -1 & 1 & 0 \\ 1 & -\lambda & 0 & 1 \\ 0 & 0 & -\lambda & -1 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} \\ &= -\begin{vmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 0 & -1 \end{vmatrix} - \lambda \begin{vmatrix} -\lambda & -1 & 1 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = \lambda^2 + 1 + \lambda^2(\lambda^2 + 1) \\ &= (\lambda^2 + 1)^2 = (\lambda + i)^2(\lambda - i)^2. \end{aligned}$$

So, the eigenvalues of A *are* $\lambda_1 = -i$ *and* $\lambda_2 = i$. *Then,* $\bar{x} = 0_{\mathbb{R}^4}$ *is not locally asymptotic stable. Let us see if* $\bar{x} = 0_{\mathbb{R}^4}$ *is locally stable. Both eigenvalues*

have algebraic multiplicity equal to 2 ($n_1 = n_2 = 2$). As $\Re(\lambda_1) = \Re(\lambda_2) = 0$, then we have to determine the respective geometric multiplicities. Throughout this example we are considering that

$$v = [v_1 \quad v_2 \quad v_3 \quad v_4]^T \in \mathbb{C}^4.$$

Let us determine the eigenspace $\mathscr{U}_{-i}$:

$$(A + iI_4)v = 0_{\mathbb{C}^4} \Leftrightarrow \begin{bmatrix} i & -1 & 1 & 0 \\ 1 & i & 0 & 1 \\ 0 & 0 & i & -1 \\ 0 & 0 & 1 & i \end{bmatrix} v = 0_{\mathbb{C}^4} \Leftrightarrow \; v = [-i \quad 1 \quad 0 \quad 0]^T v_2,$$

where $v_2 \in \mathbb{C}$. Then, we have that

$$\mathscr{U}_{-i} = \{v \in \mathbb{C}^4 : \; v_1 = -iv_2 \text{ and } v_3 = v_4 = 0\}.$$

Consequently, the geometric multiplicity of λ_1 is equal to $\dim(\mathscr{U}_{-i}) = 1$. So, the algebraic and geometric multiplicities of λ_1 are not equal, because

$$n_1 = 2 \neq \dim(\mathscr{U}_{-i}) = 1.$$

So, we can conclude that the equilibrium point $\bar{x} = 0_{\mathbb{R}^4}$ of $\dot{x} = Ax$ is unstable and we do not need to determine $\dim(\mathscr{U}_i)$. The center subspace C is the real subspace of $\mathscr{N}_{-i} \oplus \mathscr{N}_i = \mathbb{C}^4$. It means that $C = \mathbb{R}^4$. Obviously, the stable subspace S and unstable subspace U are such that $S = U = \emptyset$.

Example 1.22. *Considering the matrix A of Example 1.5, the equilibrium point $\bar{x} = 0_{\mathbb{R}^3}$ of the linear system $\dot{x} = Ax$ is unstable, because A only has positive eigenvalues. The unstable subspace U is the real subspace of $\mathscr{N}_3 \oplus \mathscr{N}_4 = \mathbb{C}^3$. It means that $U = \mathbb{R}^3$. Obviously, the center subspace C and stable subspace S are such that $C = S = \emptyset$.*

1.3.2 Non-linear differential systems

In general, many phenomena of real life are translated by non-linear differential systems and we can not use Theorem 1.13. Alternatively, it is common to linearise these systems. For this it is important to know how the linearisation of system (1.1) can be done (see Chapter 3.1 of [132]). Let $\tilde{x}(\cdot)$ and $\tilde{x}(\cdot) + y(\cdot)$ be solutions of system (1.1) with the initial condition $x(a) = x_a$, where $y(\cdot)$ is small enough. It means that

$$\begin{cases} \dot{\tilde{x}} = f(\tilde{x}), \\ \tilde{x}(a) = x_a, \\ \dot{\tilde{x}} + \dot{y} = f(\tilde{x} + y), \\ \tilde{x}(a) + y(a) = x_a. \end{cases}$$

The last equation of the previous system is equivalent to

$$y(a) = x_a - \tilde{x}(a) = 0_{\mathbb{R}^n}.$$

We have already assumed that f is continuously differentiable as many times as we need. Consequently, by Taylor's Theorem, we can expand $f(\tilde{x}+y)$ as follows:

$$f(\tilde{x}+y) = f(\tilde{x}) + Df(\tilde{x})y + \mathscr{O}, \tag{1.4}$$

where $Df(\tilde{x}) = \frac{\partial f}{\partial x}(\tilde{x})$ is the *Jacobian matrix* of f applied in $\tilde{x}$ and $\mathscr{O}$ represents the higher order terms. As $f(\tilde{x}+y) = \dot{\tilde{x}} + \dot{y}$ and $f(\tilde{x}) = \dot{\tilde{x}}$, then equation (1.4) is equivalent to

$$\dot{\tilde{x}} + \dot{y} = \dot{\tilde{x}} + Df(\tilde{x})y + \mathscr{O} \Leftrightarrow \dot{y} = Df(\tilde{x})y + \mathscr{O}.$$

If the higher order terms are ignored, we can obtain the *linearised system* given by

$$\dot{y} = Df(\tilde{x})y$$

with initial condition $y(a) = 0_{\mathbb{R}^n}$. Note that the matrix $Df(\tilde{x})$ is also called the *linearisation matrix* of system (1.1) around the solution $\tilde{x}(\cdot)$.

The following result allow us to know if an equilibrium point $\bar{x}$ of system (1.1) is locally asymptotic stable or unstable.

Theorem 1.23 (See Theorem 1.6 of [156]). *Let us consider system* (1.1) *with equilibrium point* $\bar{x}$ *and the linearisation matrix* $Df(\bar{x})$ *with* k *different eigenvalues* $\lambda_1, \ldots, \lambda_k$, *with* $k \in \mathbb{N}$ *and* $k \leq n$. *Then,* $\bar{x}$ *is*

i) locally asymptotic stable *if* $\Re(\lambda_i) < 0$ *for all* $i = 1, \ldots, k$;

ii) unstable *if* $\Re(\lambda_i) > 0$ *for some* $i = 1, \ldots, k$.

An illustrative example of Theorem 1.23 will be given at the end of Section 1.3 (see Example 1.32).

Note that Theorem 1.23 can only allow us to take stability conclusions of an equilibrium point, when the linearisation matrix around it only has eigenvalues with non-zero real parts. Next, we give a result from which we can obtain a stability study of an equilibrium point, when the linearisation matrix has a simple eigenvalue with zero real part. First, it is important to define *hyperbolic equilibrium point*.

Definition 1.24 (See Definition 1.2.6 of [181]). *Consider system* (1.1) *with equilibrium point* $\bar{x}$. *Let* $Df(\bar{x})$ *be the linearisation matrix of system* (1.1) *around* $\bar{x}$. *The equilibrium point* $\bar{x}$ *is* hyperbolic *if all eigenvalues of* $Df(\bar{x})$ *have non-zero real parts.*

Next we present a result that gives some stability conclusions about a non-hyperbolic equilibrium point $\hat{x}$ of a non-linear differential system similar to (1.1). It gives rise to the center manifold of $\hat{x}$ that is composed by orbits whose behaviour around $\hat{x}$ is not controlled by either the attraction of the *stable manifold* or the repulsion of the *unstable manifold* associated with the linearisation matrix $Df(\hat{x})$. More details about *Center Manifold Theory* can be found in [24, 58, 181]. In [25], the authors describe a theory that not only can determine the local stability of a non-hyperbolic equilibrium, but also settles the question of existence of another equilibrium (bifurcated from the non-hyperbolic equilibrium). The results proved in [25] are also based on the general *Center Manifold Theory* and we are going to present them in the following theorem.

Theorem 1.25 (See Theorem 4.1 of [25]). *Consider a differential system with a parameter $\varrho \in \mathbb{R}$ given by*

$$\dot{x} = f(x, \varrho), \tag{1.5}$$

where $a \in \mathbb{R}_0^+$ is the fixed initial time, $x(t) \in \mathbb{R}^n$ for all $t \geq a$ and f is continuously differentiable as many times as we need. Let us assume that

i) $\hat{x}$ is an equilibrium point for system (1.5) for all values of the parameter ϱ: $f(\hat{x}, \varrho) \equiv 0$ for all $\varrho \in \mathbb{R}$;

ii) $\lambda = 0$ is a simple eigenvalue of A_l and all other eigenvalues of A_l have negative real parts, where $A_l = \frac{\partial f}{\partial x}(\hat{x}, 0)$ is the linearisation matrix of system (1.5) around $\hat{x}$ with $\varrho = 0$;

iii) matrix A_l has a non-negative right eigenvector w and a left eigenvector v corresponding to the eigenvalue $\lambda = 0$.

Let k be the k-th component of f and

$$a = \sum_{i,j,k=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i x_j}(\hat{x}, 0) \quad \text{and} \quad b = \sum_{i,k=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \varrho}(\hat{x}, 0).$$

The local dynamics of (1.5) around $\hat{x}$ are totally determined by a and b. If

i) $a > 0$ and $b > 0$, then

- *$\hat{x}$ is locally asymptotic stable and there exists a positive unstable equilibrium, when $\varrho < 0$ with $|\varrho| \ll 1$;*
- *$\hat{x}$ is unstable and there exists a negative and locally asymptotic stable equilibrium, when $0 < \varrho \ll 1$;*

ii) $a < 0$ and $b < 0$, then

- *$\hat{x}$ is unstable, when $\varrho < 0$ with $|\varrho| \ll 1$;*

- *$\hat{x}$ is locally asymptotic stable and there exists a positive unstable equilibrium, when $0 < \varrho \ll 1$;*

iii) *$a > 0$ and $b < 0$, then*

- *$\hat{x}$ is unstable and there exists a locally asymptotic stable negative equilibrium, when $\varrho < 0$ with $|\varrho| \ll 1$;*
- *$\hat{x}$ is stable and a positive unstable equilibrium appears, when $0 < \varrho \ll 1$;*

iv) *$a < 0$ and $b > 0$, then $\hat{x}$ changes its stability from stable to unstable, when ϱ changes from negative to positive. Correspondingly, a negative unstable equilibrium becomes positive and locally asymptotic stable.*

Remark 1.26 (See Remark 1 of [25]). *The requirement that w is non-negative in Theorem 1.25 is not necessary. We still can apply Theorem 1.25, even when w has some negative components. Considering that $\hat{x}$ is a non-negative equilibrium point of interest, if $\hat{x}_j > 0$, then w_j does not need to be positive.*

Theorem 1.25 is used in the proof of Theorems 5.6, 6.9 and 6.15 of Sections 5.2.3, 6.2.4 and 6.3.3, respectively.

1.3.3 Routh–Hurwitz Criterion

As we have already seen in Section 1.2, the eigenvalues of a $n \times n$ real matrix A are the roots of its characteristic polynomial given by

$$p_A(\lambda) = \det(A - \lambda I_n) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

with $a_n \neq 0$. Nevertheless, sometimes the determination of their roots is not easy. So, we can resort to *Routh–Hurwitz Criterion* that give us a necessary and sufficient condition for all roots of p_A to have negative real part, only using the coefficients a_i, with $i = 0, \ldots, n$. The Routh Test was proposed in 1876 by the English mathematician *Edward John Routh* (1831–1907). It allows to know if all the roots of the characteristic polynomial of a linear system have negative real part (see [147] for more details). Later (in 1895), the German mathematician *Adolf Hurwitz* (1859–1919), independently, proposed to arrange the coefficients of the polynomial into a matrix. Moreover, he showed that the polynomial is stable if and only if the sequence of determinants of its principal sub-matrices are all positive (see [74] and its translation [11, p. 70–82] for more details). Although the two techniques are equivalent, the Routh Test provide a more efficient way to compute the Hurwitz determinants than computing them directly. Those two procedures originated the *Routh–Hurwitz Criterion* that consists in arranging the coefficients a_i,

with $i = 0, \ldots, n$, in the so-called *Routh–Hurwitz matrix* (see [132, p. 55–57]) given by

$$\begin{bmatrix} a_n & a_{n-2} & a_{n-4} & \cdots \\ a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ b_{n-2} & b_{n-4} & b_{n-6} & \cdots \\ c_{n-3} & c_{n-5} & c_{n-7} & \cdots \\ d_{n-4} & d_{n-6} & d_{n-8} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1.6}$$

where the coefficients b_i, c_i, d_i, etc., are defined by

$$b_{n-2} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}, \quad b_{n-4} = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}, \quad \ldots,$$

$$c_{n-3} = \frac{b_{n-2}a_{n-3} - a_{n-1}b_{n-4}}{b_{n-2}}, \quad c_{n-5} = \frac{b_{n-2}a_{n-5} - a_{n-1}b_{n-6}}{b_{n-2}}, \quad \ldots,$$

$$d_{n-4} = \frac{c_{n-3}b_{n-4} - b_{n-2}c_{n-5}}{c_{n-3}}, \quad d_{n-6} = \frac{c_{n-3}b_{n-6} - b_{n-2}c_{n-7}}{c_{n-3}}, \quad \ldots$$

$$\vdots \qquad\qquad \vdots$$

In the construction of matrix (1.6), all coefficients with negative index are equal to zero. When we are computing the elements of a row of matrix (1.6), all of them are obtained through a quotient where the denominator is the first element of the previous row. So, the last row of matrix (1.6) is the first row that has zero in the first position. Summarizing, the construction of the Routh–Hurwitz matrix ends, when it is obtained the first zero in the first column. As we are considering that $a_n \neq 0$, then the minimum number of rows are two, when $a_{n-1} = 0$. The maximum number of rows is $n+1$, when the first $n+1$ elements of the first column are different to zero.

Theorem 1.27 (Routh–Hurwitz Criterion – see Theorem 1.2.9 of [181])**.** *All roots of the polynomial*

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

with $a_n \neq 0$ have negative real part if and only if all $n+1$ elements in the first column of the Routh–Hurwitz matrix (1.6) *are non-zero and have the same sign.*

Next we present an illustrative example of Routh–Hurwitz Criterion.

Example 1.28. *Consider the polynomial $p(\lambda) = \lambda^3 + 3\lambda^2 + 3\lambda + 9 = 0$ and let us write the Routh–Hurwitz matrix:*

$$\begin{bmatrix} a_3 & a_1 \\ a_2 & a_0 \\ b_1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \\ \dfrac{a_2a_1 - a_3a_0}{a_2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \\ 0 & 0 \end{bmatrix}.$$

Actually, the Routh–Hurwitz matrix is simply given by

$$\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}.$$

Since the first column of the previous matrix does not have $n+1 = 4$ non-null elements, then we can conclude that there is at least one root of p that does not have negative real part, by Routh–Hurwitz Criterion (Theorem 1.27). Now, let us to determine the roots of p with the purpose to confirm the previous conclusions. One root of p is $\lambda = -3$. Using the Ruffini Rule*, we obtain*

	1	3	3	9
-3		-3	0	-9
	1	0	3	0

and, consequently, we also have that

$$p(\lambda) = (\lambda + 3)(\lambda^2 + 3) = (\lambda + 3)(\lambda + \sqrt{3}i)(\lambda - \sqrt{3}i).$$

Thus, the roots of the polynomial p are $\lambda_1 = -3$, $\lambda_2 = -\sqrt{3}i$ and $\lambda_3 = \sqrt{3}i$. Actually, two roots of p (λ_2 and λ_3) do not have negative real part.

1.3.4 Descartes' Rule of Signs

There is other result that allow us to know the maximum number of positive real roots of a polynomial, only using the signs of their coefficients. Consequently, with this information we can take stability conclusions. This theoretical result, called by *Descartes' Rule of Signs*, was firstly described by the French mathematician *René Descartes* (1596–1650) in his work *La Geometrie*, published in 1637 (see [40]).

Theorem 1.29 (Descartes' Rule of Signs – see Theorem 1.2.8 of [181])**.** *Consider the polynomial*

$$p_A(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

with $a_n \neq 0$ and the sequence of their coefficients as follows:

$$a_n,\ a_{n-1},\ \ldots,\ a_1,\ a_0,$$

omitting the null coefficients. Let j be the total number of sign changes from one coefficient to the next in the previous sequence. Then, the number of positive real roots of the polynomial p_A is either equal to j, or j minus a positive even integer.

Remark 1.30. *With respect to Theorem 1.29, if*

i) $j = 0$, then the polynomial p_A does not have positive real roots;

ii) $j = 1$, *then the polynomial* p_A *has exactly one positive real root.*

Next, we give an illustrative example for Descartes' Rule of Signs.

Example 1.31. *By Descartes' Rule of Signs, we can conclude that the polynomial* p *given by*

$$p(\lambda) = \lambda^3 - 2\lambda^2 - \lambda + 2$$

has either two, or zero positive real roots, because there are two sign changes. The first change is from the coefficient $a_3 = 1$ *to* $a_2 = -2$ *and the second is from the coefficient* $a_1 = -1$ *to* $a_0 = 2$. *Let us compute the roots of* p. *Clearly, one can observe that* $p(1) = 0$. *So, by* Ruffini Rule, *we obtain*

$$\begin{array}{c|cccc} & 1 & -2 & -1 & 2 \\ 1 & & 1 & -1 & -2 \\ \hline & 1 & -1 & -2 & 0 \end{array}$$

and, consequently, we have that

$$p(\lambda) = (\lambda - 1)(\lambda^2 - \lambda - 2).$$

Doing simple calculations, we obtain that

$$p(\lambda) = (\lambda - 1)(\lambda + 1)(\lambda - 2).$$

Concluding, p *has two positive real roots (*1 *and* 2*).*

Now, we are going to give an example that illustrates the linearisation method, Theorem 1.23, Routh–Hurwitz Criterion and Descartes' Rule of Signs presented previously.

Example 1.32. *Let us consider the following non-linear differential system:*

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) = -2x_1 + x_2^2 + 2x_2 + 1, \\ \dot{x}_2 = f_2(x_1, x_2) = -x_2 - 1. \end{cases} \tag{1.7}$$

The unique equilibrium point of (1.7) *is* $(\bar{x}_1, \bar{x}_2) = (0, -1)$. *The Jacobian matrix associated with the function* $f = [f_1 \quad f_2]^T$ *is given by*

$$Df(x_1, x_2) = \begin{bmatrix} -2 & 2x_2 + 2 \\ 0 & -1 \end{bmatrix}.$$

Consequently, the Jacobian matrix applied in $(0, -1)$ *is given by*

$$Df(0, -1) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}.$$

Considering that $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$, *the linearised system associated to* (1.7) *is given by*

$$\dot{y} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} y.$$

As $Df(0,-1)$ is a diagonal matrix, it is easy to conclude that their eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -1$. So, by Theorem 1.23, we can assert that the equilibrium point $(0,-1)$ of (1.7) is locally asymptotic stable, since all the eigenvalues of $Df(0,-1)$ have negative real part. Let us take the same conclusions using Routh–Hurwitz Criterion and Descartes' Rule of Signs. For this, we have to compute the characteristic polynomial of the matrix $Df(0,-1)$. It is given by

$$p(\lambda) = \begin{vmatrix} -2-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = (\lambda+2)(\lambda+1) = \lambda^2 + 3\lambda + 2.$$

The Routh–Hurwitz matrix is given by

$$\begin{bmatrix} a_2 & a_0 \\ a_1 & 0 \\ b_0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ \dfrac{a_1 a_0 - a_2 \times 0}{a_1} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 2 & 0 \end{bmatrix}.$$

Since the first column of the previous matrix has $n+1 = 3$ positive elements, then Routh–Hurwitz Criterion allow us to confirm that all the roots of p have negative real part and, consequently, the equilibrium point $(0,-1)$ of (1.7) is locally asymptotic stable. By Descartes' Rule of Signs, we can also conclude that the number of roots of p with positive real part is equal to zero, because there is no sign change in the following sequence:

$$a_2,\ a_1,\ a_0 \quad \to \quad 1,\ 3,\ 2.$$

1.4 Stability for delayed differential systems

In this section, we recall some important definitions and theoretical results about the stability of a system of ordinary delayed differential equations (ODDE) given by

$$\dot{x} = f(x, x_r) \tag{1.8}$$

with initial condition

$$x(t) = \phi(t) \quad \text{for} \quad t \in [a-r, a], \tag{1.9}$$

where $a \in \mathbb{R}_0^+$ is a fixed time, $[a-r, a]$ is the initial time interval, $r \in \mathbb{R}^+$ is the fixed time delay, $x(t) \in \mathbb{R}^n$ for all $t \geq a-r$, $x_r(t) = x(t-r)$ for all $t \geq a$ and f is continuously differentiable as many times as we need. Next, we define equilibrium point and its stability.

Definition 1.33 (See p. 121 of [156]). *An* equilibrium point *of system* (1.8) *is a vector $\bar{x} \in \mathbb{R}^n$ that satisfies the equation $f(\bar{x}, \bar{x}) = 0$.*

Definition 1.34 (See Definitions 2.3 and 2.4 of [18]). *For each $t \geq a - r$, let $\bar{x}$ and $x(t)$ be an* equilibrium point *and the solution of system* (1.8) *with initial condition* (1.9)*, respectively. The point $\bar{x}$ is*

i) locally stable *if, for all $t' \in [a - r, a]$ and $t > a$,*

$$\forall \varepsilon > 0,\ \exists \delta > 0\ :\ ||\phi(t') - \bar{x}|| < \delta \implies ||x(t) - \bar{x}|| < \varepsilon;$$

ii) locally asymptotic stable *if it is* stable *and, for all $t' \in [a - r, a]$,*

$$\exists \delta > 0\ :\ ||\phi(t') - \bar{x}|| < \delta \implies \lim_{t \to +\infty} ||x(t) - \bar{x}|| = 0;$$

iii) unstable *if it is not* locally stable.

1.4.1 Delayed linear differential systems

In this section we follow the approach of [164]. Let us suppose that system (1.8) is linear, i.e.,

$$\dot{x} = f(x, x_r) = Ax + Bx_r, \tag{1.10}$$

where A and B are $n \times n$ real matrices. Clearly, $\bar{x} = 0$ is an equilibrium point of (1.10). The corresponding *characteristic equation* is

$$\det(\lambda I_n - A - e^{-\lambda r} B) = 0. \tag{1.11}$$

Next we state a stability result for the delayed linear system (1.10) only with respect to the fixed point $\bar{x} = 0$ (see a particular case of Theorem 4.3 of [164]).

Theorem 1.35 (See Theorem 4.3 of [164]). *Let us consider the delayed linear system* (1.10)*. For $k \in \mathbb{N}$ and $k \leq n$, suppose that the* characteristic equation (1.11) *has k different solutions: $\lambda_1, \ldots, \lambda_k$. The* equilibrium point $\bar{x} = 0_{\mathbb{R}^n}$ *is*

i) locally asymptotic stable *if $\Re(\lambda_i) < 0$ for all $i = 1, \ldots, k$;*

ii) unstable *if $\Re(\lambda_i) > 0$ for some $i = 1, \ldots, k$.*

Next, we present two illustrative examples of Theorem 1.35.

Example 1.36. *Let us consider the following delayed linear differential system:*

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + x_1(t-1) + x_2(t-1), \\ \dot{x}_2(t) = -2x_1(t) - 3x_2(t) - x_1(t-1) - x_2(t-1), \end{cases} \tag{1.12}$$

which is equivalent to the following matrix equation:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = A\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + B\begin{bmatrix}x_1(t-1)\\ x_2(t-1)\end{bmatrix},$$

where

$$A = \begin{bmatrix}-1 & 0\\ -2 & -3\end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}.$$

Clearly, $\bar{x} = [0 \quad 0]^T$ *is an equilibrium point of system* (1.12). *The respective* characteristic equation $\det(\lambda I_2 - A - e^{-\lambda}B) = 0$ *is equivalent to*

$$\begin{vmatrix}\lambda + 1 - e^{-\lambda} & -e^{-\lambda}\\ 2 + e^{-\lambda} & \lambda + 3 + e^{-\lambda}\end{vmatrix} = 0 \Leftrightarrow (\lambda+1)(\lambda+3) = 0 \Leftrightarrow \lambda = -1 \vee \lambda = -3.$$

As all the solutions of the characteristic equation *have negative real part* ($\lambda = -1 \vee \lambda = -3$), *then, by Theorem 1.35,* $\bar{x}$ *is locally asymptotic stable.*

Example 1.37. *Let us consider the following delayed linear differential system:*

$$\begin{cases}\dot{x}_1(t) = -x_1(t) - 2x_2(t) + x_1(t-3) + x_2(t-3),\\ \dot{x}_2(t) = x_2(t) - x_1(t-3) - x_2(t-3),\end{cases} \tag{1.13}$$

which is equivalent to the following matrix equation:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = A\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + B\begin{bmatrix}x_1(t-3)\\ x_2(t-3)\end{bmatrix},$$

where

$$A = \begin{bmatrix}-1 & -2\\ 0 & 1\end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}.$$

Clearly, $\bar{x} = [0 \quad 0]^T$ *is an equilibrium point of system* (1.13). *The respective* characteristic equation $\det(\lambda I_2 - A - e^{-3\lambda}B) = 0$ *is equivalent to*

$$\begin{vmatrix}\lambda + 1 - e^{-3\lambda} & 2 - e^{-3\lambda}\\ e^{-3\lambda} & \lambda - 1 + e^{-3\lambda}\end{vmatrix} = 0 \Leftrightarrow (\lambda+1)(\lambda-1) = 0 \Leftrightarrow \lambda = -1 \vee \lambda = 1.$$

As the characteristic equation *has, at least, a solution with positive real part* ($\lambda = 1$), *then, by Theorem 1.35,* $\bar{x}$ *is unstable.*

1.4.2 Delayed non-linear differential systems

As we have mentioned before, most of the phenomena of real life are translated by non-linear differential systems and in these cases we can not use Theorem 1.35. In order to be possible to take some stability conclusions for the delayed non-linear differential system (1.8), we are going to proceed with the linearisation of system (1.8) (see Chapter 3.1 of [132] and p. 121–122 of [156]).

Let $\tilde{x}(\cdot)$ and $\tilde{x}(\cdot)+y(\cdot)$ be solutions of system (1.8), where $y(\cdot)$ is small enough. It means that

$$\begin{cases} \dot{\tilde{x}} = f(\tilde{x}, \tilde{x}_r), \\ \tilde{x}(t) = \phi(t) \quad \text{for} \quad t \in [a-r, a], \\ \dot{\tilde{x}} + \dot{y} = f(\tilde{x}+y, \tilde{x}_r + y_r), \\ \tilde{x}(t) + y(t) = \phi(t) \quad \text{for} \quad t \in [a-r, a]. \end{cases}$$

Note that $\tilde{x}_r(t) + y_r(t) = \tilde{x}(t-r) + y(t-r)$ for all $t \geq a$. Moreover, for all $t \in [a-r, a]$ the last equation of the previous system is equivalent to

$$y(t) = \phi(t) - \tilde{x}(t) = 0_{\mathbb{R}^n}.$$

We have already assumed that f is continuously differentiable as many times as we need. Consequently, by Taylor's Theorem, we can expand $f(\tilde{x}+y, \tilde{x}_r + y_r)$ as follows:

$$f(\tilde{x}+y, \tilde{x}_r + y_r) = f(\tilde{x}, \tilde{x}_r) + D_x f(\tilde{x}, \tilde{x}_r)y + D_{x_r} f(\tilde{x}, \tilde{x}_r)y_r + \mathscr{O}, \quad (1.14)$$

where $D_x f(\tilde{x}, \tilde{x}_r) = \frac{\partial f}{\partial x}(\tilde{x}, \tilde{x}_r)$ and $D_{x_r} f(\tilde{x}, \tilde{x}_r) = \frac{\partial f}{\partial x_r}(\tilde{x}, \tilde{x}_r)$ are the *Jacobian matrices* of f with respect to variables x and x_r, respectively, and $\mathscr{O}$ represents the higher order terms. As $f(\tilde{x}+y, \tilde{x}_r + y_r) = \dot{\tilde{x}} + \dot{y}$ and $f(\tilde{x}, \tilde{x}_r) = \dot{\tilde{x}}$, then equation (1.14) is equivalent to

$$\begin{aligned} &\dot{\tilde{x}} + \dot{y} = \dot{\tilde{x}} + D_x f(\tilde{x}, \tilde{x}_r)y + D_{x_r}(\tilde{x}, \tilde{x}_r)y_r + \mathscr{O} \\ \Leftrightarrow\ &\dot{y} = D_x f(\tilde{x}, \tilde{x}_r)y + D_{x_r}(\tilde{x}, \tilde{x}_r)y_r + \mathscr{O}. \end{aligned}$$

If the higher terms are ignored, we can obtain the *linearised system* given by

$$\dot{y} = D_x f(\tilde{x}, \tilde{x}_r)y + D_{x_r}(\tilde{x}, \tilde{x}_r)y_r$$

with initial condition $y(t) = 0_{\mathbb{R}^n} \quad \text{for} \quad t \in [a-r, a]$.

The following theorem allow us to know if an equilibrium point $\bar{x}$ of system (1.8) is locally asymptotic stable or unstable.

Theorem 1.38 (See Theorem 4.8 of [164]). *Let us consider system* (1.8). *Assume that $\bar{x}$ is an equilibrium point of $\dot{x} = f(x, x_r)$, $A = D_x f(\bar{x}, \bar{x})$ and $B = D_{x_r} f(\bar{x}, \bar{x})$. For $k \in \mathbb{N}$ and $k \leq n$, suppose that the following* characteristic equation

$$\det(\lambda I_n - A - e^{-\lambda r}B) = 0 \quad (1.15)$$

has k different solutions: $\lambda_1, \ldots, \lambda_k$. The equilibrium point $\bar{x}$ *is*

i) locally asymptotic stable *if $\Re(\lambda_i) < 0$ for all $i = 1, \ldots, k$;*

ii) unstable *if $\Re(\lambda_i) > 0$ for some $i = 1, \ldots, k$.*

An illustrative example of Theorem 1.38 is given in [164, p. 55–56].

Usually, the left-side of equation (1.11) is not so simple as in Examples 1.36 and 1.37. Sometimes, it has the form

$$p_1(\lambda) + e^{-\lambda r} p_2(\lambda),$$

where p_1 and p_2 are polynomials in λ with real coefficients. We denote the degree of p_j by $m_j \in \mathbb{N}_0$ for $j = 1, 2$. In these situations it is not easy to determine the solutions of equation (1.11). Thus, we follow the method for dealing with equation

$$p_1(\lambda) + e^{-\lambda r} p_2(\lambda) = 0$$

for arbitrary m_1 and $m_2 < m_1$, established in [34] and corrected in [16]. Sometimes, the introduction of a time delay can destabilize the system, but that is not always true. If we vary the value of the positive time delay r, it may happen that the solutions of equation (1.11) cross the imaginary axis. Consequently, we can change from a stability situation to an instability one or vice versa. If it occurs, then we say that there has been a stability switch or reversal. Next, we present the main result of [34] (see Theorem 1) that was corrected in [16] (see also Theorem 4.1 of [97]). Such result allows to determine how the increasing delay affects stability.

Theorem 1.39 (See [16, 34] and Theorem 4.1 of [97] in p. 83)**.** *Consider the equation*

$$p_1(\lambda) + e^{-\lambda r} p_2(\lambda) = 0,$$

where p_1 and p_2 are analytic functions in $\Re(\lambda) > 0$ that satisfy the following conditions:

i) p_1 *and* p_2 *have no common imaginary root;*

ii) $\overline{p_1(-iy)} = p_1(iy)$ *and* $\overline{p_2(-iy)} = p_2(iy)$ *for real* y*;*

iii) $p_1(0) + p_2(0) \neq 0$*;*

iv) $\lim\limits_{|\lambda|\to\infty,\ \Re(\lambda)\geq 0} \sup\left\{\left|\frac{p_2(\lambda)}{p_1(\lambda)}\right|\right\} < 1$*;*

v) $F(y) \equiv |p_1(iy)|^2 - |p_2(iy)|^2$ *has at most a finite number of real zeros, for real* y*.*

Then, the following statements are true:

a) if $F(y) = 0$ *has no positive roots, then no stability switch may occur; if an equilibrium point is* locally asymptotic stable *for* $r = 0$*, then it remains* locally asymptotic stable *for all* $r > 0$*; whereas if it is* unstable *for* $r = 0$*, it remains* unstable *for all* $r > 0$*;*

b) if $F(y) = 0$ has, at least, one positive root and each of them is simple, then as r increases, stability switches *may occur: there exists a positive number r^* such that* instability *occurs for all $r > r^*$; as r varies from 0 to r^*, at most a finite number of* stability switches *may occur.*

Remark 1.40. *Note that*

$$\lim_{|\lambda|\to\infty,\ \Re(\lambda)\geq 0} \sup\left\{\left|\frac{p_2(\lambda)}{p_1(\lambda)}\right|\right\}$$

denotes the limit superior of $\left|\frac{p_2(\lambda)}{p_1(\lambda)}\right|$ as $|\lambda| \to \infty$ and $\Re(\lambda) \geq 0$.

We use Theorem 1.39 in Section 5.3.3.

Remark 1.41. *Note that while Theorems 1.35 and 1.38 allow us to take some conclusions about stability of an equilibrium point for a fixed and positive time delay r, the stability conclusions obtained by Theorem 1.39 are in function of $r \in \mathbb{R}^+$.*

1.4.3 Non-negativity of solutions

We are going to present a result that ensures the non-negativity of the solutions of a differential system if non-negative initial conditions are considered (see [199]).

Lemma 1.42 (See Lemma 2 of [199])**.** *Let us consider the differential equations*

$$\dot{x}_i(t) = f_i\big(t, x_1(t), \ldots, x_n(t)\big), \quad i = 1, \ldots, n, \tag{1.16}$$

supposing that $\Phi \subset \mathbb{R} \times \mathscr{C}\big([-r, 0], \mathbb{R}^n\big)$ and $f_i \in \mathscr{C}\big(\Phi, \mathbb{R}\big)$ with $i = 1, \ldots, n$. If

$$f_i\big|_{\xi(x_i)} \geq 0, \quad i = 1, \ldots, n,$$

then $\mathscr{C}\big([-r, 0], \big(\mathbb{R}_0^+\big)^n\big)$ is the invariant domain of equations (1.16) for $t \geq a$, where $r \in \mathbb{R}_0^+$ and

$$\xi(x_i) = \Big\{x_i(t) = 0 \text{ and } \big(x_1(\cdot), \ldots, x_n(\cdot)\big) \in \mathscr{C}\big([-r, 0], \big(\mathbb{R}_0^+\big)^n\big)\Big\}.$$

Note that Lemma 1.42 is used in Sections 5.2.1, 5.3.1, 6.2.1 and 6.3.1.

Remark 1.43. *If we are studying an autonomous differential system, then we trivially have, for all $t \geq a$, that*

$$f_i\big(t, x_1(t), \ldots, x_n(t)\big) = f_i\big(x_1(t), \ldots, x_n(t)\big), \quad i = 1, \ldots, n$$

and, consequently, we can also use Lemma 1.42.

1.5 Compartmental Models

In this section we recall the definition of compartmental models, without time delays, and some associated theoretical results, following the approach used in [176].

When we intend to study mathematically the propagation of an epidemic in a heterogeneous population, we divide it into n homogeneous compartments/classes. The individuals can be distinguished by age, behaviour, spatial position and/or stage of the disease (see [176]). Let $x_i \in \mathbb{R}_0^+$ be the number of individuals in the compartment/class i, with $i = 1, \ldots, n$. Then, we define $x = (x_1, \ldots, x_n)$. In this section, we assume that the first m compartments are related to infective individuals. We can only decide if a class corresponds to infective individuals after an epidemiological interpretation. Moreover, for some models there are more than one interpretation. Let us consider the following definitions in order to obtain a clear explanation of compartmental models.

Remark 1.44 (Infected/infectious/infective individual)**.** *In this work, an individual who is infected by a virus/bacteria and does not transmit the disease is called an* infected individual*. On the other hand, an individual who is infected by a virus/bacteria and is able to transmit the infection is called an* infectious individual*. In general, an* infective individual *can be* infected *and/or* infectious*.*

Definition 1.45 (See Section 2 of [176])**.** *The set of all* disease-free states *is defined by*

$$X_s = \left\{ x \in \left(\mathbb{R}_0^+\right)^n : \ x_i = 0, \ i = 1, \ldots, m \right\}.$$

Next we define compartmental model.

Definition 1.46 (See Section 2 of [176])**.** *A* compartmental model*, which translates the transmission of an infectious disease, is composed by non-negative initial conditions and by the differential system given by*

$$\dot{x} = f(x) = \mathscr{F}(x) - \mathscr{V}(x) = \begin{bmatrix} \mathscr{F}_1(x) - \mathscr{V}_1(x) \\ \vdots \\ \mathscr{F}_n(x) - \mathscr{V}_n(x) \end{bmatrix}, \tag{1.17}$$

where $\mathscr{F}_i(x)$ is the rate of appearance of new infections in compartment i, $\mathscr{V}_i^+(x)$ is the rate of transfer of individuals into compartment i by all other means, $\mathscr{V}_i^-(x)$ is the rate of transfer of individuals out of compartment i and $\mathscr{V}_i(x) = \mathscr{V}_i^-(x) - \mathscr{V}_i^+(x)$, for all $i = 1, \ldots, n$. Furthermore, it is assumed that each function mentioned before is continuously differentiable at least twice in each variable and the following assumptions are satisfied:

(A1) if $x \geq 0$, *then* $\mathscr{F}_i, \mathscr{V}_i^+, \mathscr{V}_i^- \geq 0$ *for all* $i = 1, \ldots, n$;

(A2) if $x_i = 0$, *then* $\mathscr{V}_i^- = 0$ *for all* $i = 1, \ldots, n$;

(A3) if $i > m$, *then* $\mathscr{F}_i = 0$;

(A4) if $x \in X_s$, *then* $\mathscr{F}_i(x) = \mathscr{V}_i^+(x) = 0$ *for all* $i = 1, \ldots, m$;

(A5) if $\mathscr{F}(x)$ *is set to zero, then all eigenvalues of the Jacobian matrix* $Df(x_0)$ *have negative real parts; considering that there is only one disease-free equilibrium (DFE)* x_0 *of* (1.17) *(equilibrium point of* (1.17) *restrict to* X_s*).*

All the considerations done in the Definition 1.46 imply the following result.

Lemma 1.47 (See Lemma 1 of [176])**.** *Let us consider the compartmental model of Definition 1.46. Then the Jacobian matrices* $D\mathscr{F}(x_0)$ *and* $D\mathscr{V}(x_0)$ *are partitioned as follows:*

$$D\mathscr{F}(x_0) = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad D\mathscr{V}(x_0) = \begin{bmatrix} V & 0 \\ J_3 & J_4 \end{bmatrix},$$

where F *and* V *are the* $m \times m$ *matrices defined by*

$$F = \left[\frac{\partial \mathscr{F}_i}{\partial x_j}(x_0)\right] \quad \textit{and} \quad V = \left[\frac{\partial \mathscr{V}_i}{\partial x_j}(x_0)\right]$$

with $1 \leq i, j \leq m$. *Moreover,* F *is non-negative,* V *is an invertible matrix and all eigenvalues of* J_4 *have positive real part.*

According to Theorem 1.23, the disease-free equilibrium x_0 of (1.17) is locally asymptotic stable if all eigenvalues of $Df(x_0)$ have negative real part and unstable if there is, at least, one eigenvalue of $Df(x_0)$ with positive real part. Due to Lemma 1.47, the eigenvalues of $Df(x_0)$ can be partitioned into two sets. One of them is composed by the eigenvalues of $F - V$ and the other by those of $-J_4$. The stability of x_0 is completely determined by the eigenvalues of $F - V$, since all eigenvalues of $-J_4$ have negative real part by Lemma 1.47. Moreover, we define basic reproduction number that is a threshold parameter for the stability of x_0, being in this way an important measure in the mathematical study of infectious disease's propagation (see [41, 176]). Its value allows us to know if an epidemic will spread, or no.

Definition 1.48 (See Section 3 of [176])**.** *The* basic reproduction number*, usually denoted by* R_0*, is the expected number of new infections due to a contact between only one typical infective individual and a completely susceptible population.*

Let us consider that there is an unique specific infective individual in a completely susceptible population. Over the course of its infectious period we can obtain, at least, two scenarios: $R_0 < 1$ or $R_0 > 1$. If $R_0 < 1$, then the unique infective individual causes, on average, less than one new infection. Consequently, in this case the disease will die out naturally. If $R_0 > 1$, then the unique infective individual causes, on average, more than one new infection and so the disease will spread and grow. Following [41, 176], FV^{-1} is the *next generation matrix* for the considered model and

$$R_0 = \rho\left(FV^{-1}\right), \tag{1.18}$$

where $\rho(A)$ denotes the spectral radius of a matrix A, that is, $\rho(A)$ is equal to the greatest eigenvalue of matrix A. The following result asserts that R_0 is a threshold parameter for the stability of the disease-free equilibrium x_0 of (1.17).

Theorem 1.49 (See Theorem 2 of [176]). *Let us consider the compartmental model of Definition 1.46. If x_0 is a disease-free equilibrium of this model, then x_0 is locally asymptotic stable if $R_0 < 1$ or unstable if $R_0 > 1$, where R_0 is defined by* (1.18).

1.6 Examples of Compartmental Models

In this section we provide some examples of compartmental models defined in Section 1.5. Most dynamical models for infectious diseases are based on compartmental structures firstly proposed by *Anderson Gray McKendrick* (1876–1943) and *William Ogilvy Kermack* (1898–1970) in 1927 and 1932 (see [113]). These type of models have been developed and improved by many other biomathematicians. McKendrick was a Scottish military physician and epidemiologist and Kermack was a Scottish biochemist (see [183, 185]). They developed the first mathematical studies of epidemics spread, establishing connections between environmental factors and some specific diseases (see [185]). *Joseph Oscar Irwin* (1898–1982), a British statistician, even commented on the quality of McKendrick's work as follows (see [183, 184]):

> "Although an amateur, he was a brilliant mathematician, with a far greater insight than many professionals."

Next, we present the two first models proposed by McKendrick and Kermack and others based on these ones, following the approach used in [113]. From now on, we are going to present examples of compartmental models that consider several compartments/classes and parameters. Their descriptions can be found in Tables 1.1 and 1.2.

1.6.1 Kermack–McKendrick SIR compartmental model

In 1927, the compartmental model proposed by Kermack and McKendrick divided the population into three compartments:

i) susceptible compartment, labelled by S, which is composed by all the susceptible individuals who become infective if they contact with an infectious disease;

ii) infective compartment, labelled by I, which is composed by all infective individuals;

iii) removed compartment, labelled by R, in which are all the removed or recovered individuals.

The number of individuals in compartments S, I and R at time $t \in \mathbb{R}_0^+$ are denoted by $S(t)$, $I(t)$ and $R(t)$, respectively. Moreover, they considered the following assumptions:

i) for all time $t \in \mathbb{R}_0^+$, the total population has a non-negative constant size: $C \equiv S(t) + I(t) + R(t)$, since it is assumed that the disease spreads in a closed environment, i.e., there is no inflow of individuals due to, for example, emigration, immigration, births and deaths;

ii) the number of susceptible individuals who are infected by an infectious individual, per unit of time, is proportional to $S(t)$ with the proportional transmission coefficient rate β and incidence rate $\beta I(t)$ at time $t \in \mathbb{R}_0^+$, being the number of new infections given by $\beta S(t)I(t)$;

iii) the number of removed/recovered individuals from the compartment I, per unit of time, is $\gamma I(t)$ at time $t \in \mathbb{R}_0^+$, where γ is the removal/recovery rate coefficient;

iv) the removed/recovered individuals gain permanent immunity.

Note that after the period of time $\frac{1}{\gamma}$, all the individuals of compartment I move to R. Concluding, $\frac{1}{\gamma}$ is actually the mean infection period without death. Therefore, the mathematical model is translated by the following non-linear differential system:

$$\begin{cases} \dot{S}(t) = -\beta S(t)I(t), \\ \dot{I}(t) = \beta S(t)I(t) - \gamma I(t), \\ \dot{R}(t) = \gamma I(t). \end{cases} \tag{1.19}$$

The SIR model (1.19) is appropriate to, for example, influenza, measles and chickenpox, because for these infectious diseases the recovered individuals, in general, gain immunity to the same virus.

1.6.2 Kermack–McKendrick SIS compartmental model

For bacterial diseases such as encephalitis and gonorrhoea, the recovered individuals gain no immunity and can be reinfected. Due to this, Kermack and Mckendrick suggested only two compartments for these type of diseases: S and I. Therefore, they proposed in 1932 the SIS mathematical model given by

$$\begin{cases} \dot{S}(t) = -\beta S(t)I(t) + \tilde{\gamma} I(t), \\ \dot{I}(t) = \beta S(t)I(t) - \tilde{\gamma} I(t), \end{cases} \tag{1.20}$$

where the population size is also constant.

Remark 1.50. *In the two previous Kermack-McKendrick models the incidence rate is a linearly increasing function of the number of infective individuals: $\beta I(t)$ for each $t \in \mathbb{R}_0^+$. As Capasso and Serio write in [21], this incidence rate can make sense when the number of infective individuals is small. Nevertheless, the number of contacts of a susceptible individual, per unit of time $t \in \mathbb{R}_0^+$, can not always increase linearly with respect to $I(t)$. The authors of [21] considered that it is much more realistic to use a general non-linear bounded function to represent the incidence rate. From now on, we are going to represent the incidence rate by a function $f_{ir}(x_1, \ldots, x_m)$, where $x_i \in \mathbb{R}_0^+$ denotes the number of individuals in the compartment/class i with $i = 1, \ldots, m$. It is important to recall that the first m compartments are related to infective individuals, as we have already considered in Section 1.5. Moreover, different types of general incidence rates can be found, for instance, in [21, 85, 111, 120, 148] and some references cited therein.*

1.6.3 Compartmental models without vital dynamics

When the infectious disease in study spreads quickly, sometimes the vital dynamics (birth, natural death and disease-induced death) can be omitted. Examples of this type of diseases are influenza, measles, rubella and chickenpox. With respect to models without vital dynamics, we can also consider or not latent periods. We do not consider them, when infected individuals become infectious immediately.

Without latent periods

Some examples of models without vital dynamics and without latent periods can be:

i) the SI model – the individuals of compartment I can not recover from infection. A differential system with these characteristics can be

$$\begin{cases} \dot{S}(t) = -f_{ir}\big(I(t)\big)S(t), \\ \dot{I}(t) = f_{ir}\big(I(t)\big)S(t). \end{cases}$$

ii) the SIS compartmental model – the individuals of compartment I can recover from infection, but they do not gain immunity after recovery. It means that after recovery the individuals become susceptible immediately. A differential system with these characteristics can be the system (1.20).

iii) the SIR compartmental model – the individuals of compartment I can recover from infection and gain permanent immunity after recovery. A differential system with these characteristics can be the system (1.19).

iv) the SIRS compartmental model – the individuals of compartment I can recover from infection and gain temporary immunity. It means that the recovered individuals will become susceptible again at rate coefficient ω_1. A differential system with these characteristics can be

$$\begin{cases} \dot{S}(t) = -f_{ir}\big(I(t)\big)S(t) + \omega_1 R(t), \\ \dot{I}(t) = f_{ir}\big(I(t)\big)S(t) - \gamma I(t), \\ \dot{R}(t) = \gamma I(t) - \omega_1 R(t). \end{cases}$$

Note that $\frac{1}{\omega_1}$ is the mean immunity period.

v) the SIRI compartmental model – the individuals of compartment I can recover temporarily, having the possibility to be reinfected under some conditions, at recurrence rate coefficient $\tilde{\omega}$. This type of models are suitable for tuberculosis, for example. A differential system with these characteristics can be

$$\begin{cases} \dot{S}(t) = -f_{ir}\big(I(t)\big)S(t), \\ \dot{I}(t) = f_{ir}\big(I(t)\big)S(t) - \gamma I(t) + \tilde{\omega} R(t), \\ \dot{R}(t) = \gamma I(t) - \tilde{\omega} R(t). \end{cases}$$

With latent periods

Sometimes it makes sense to assume that there is a latent period. Because of this, it can be considered an exposed compartment, labelled by E, which is composed by exposed individuals, i.e., by infected individuals who do not have symptoms and are not infectious yet. Note that $E(t)$ represents the number of individuals in compartment E for each time $t \in \mathbb{R}_0^+$. Consider that $\upsilon \in \mathbb{R}_0^+$ is the progression rate coefficient for individuals from compartments E to I. Thus, some examples of dynamical models with mean latent period $\frac{1}{\upsilon}$ can be:

i) the SEI compartmental model represented by

$$\begin{cases} \dot{S}(t) = -f_{ir}\big(I(t)\big)S(t), \\ \dot{E}(t) = f_{ir}\big(I(t)\big)S(t) - \upsilon E(t), \\ \dot{I}(t) = \upsilon E(t); \end{cases}$$

ii) the SEIS compartmental model represented by

$$\begin{cases} \dot{S}(t) = -f_{ir}\big(I(t)\big)S(t) + \tilde{\gamma} I(t), \\ \dot{E}(t) = f_{ir}\big(I(t)\big)S(t) - \upsilon E(t), \\ \dot{I}(t) = \upsilon E(t) - \tilde{\gamma} I(t); \end{cases}$$

iii) the SEIR compartmental model represented by

$$\begin{cases} \dot{S}(t) = -f_{ir}\big(I(t)\big)S(t), \\ \dot{E}(t) = f_{ir}\big(I(t)\big)S(t) - \upsilon E(t), \\ \dot{I}(t) = \upsilon E(t) - \gamma I(t), \\ \dot{R}(t) = \gamma I(t); \end{cases}$$

iv) the SEIRS compartmental model represented by

$$\begin{cases} \dot{S}(t) = -f_{ir}\big(I(t)\big)S(t) + \omega_1 R(t), \\ \dot{E}(t) = f_{ir}\big(I(t)\big)S(t) - \upsilon E(t), \\ \dot{I}(t) = \upsilon E(t) - \gamma I(t), \\ \dot{R}(t) = \gamma I(t) - \omega_1 R(t); \end{cases}$$

v) the SEIRE compartmental model represented by

$$\begin{cases} \dot{S}(t) = -f_{ir}\big(I(t)\big)S(t), \\ \dot{E}(t) = f_{ir}\big(I(t)\big)S(t) - \upsilon E(t) + \tilde{\omega} R(t), \\ \dot{I}(t) = \upsilon E(t) - \gamma I(t), \\ \dot{R}(t) = \gamma I(t) - \tilde{\omega} R(t). \end{cases}$$

1.6.4 Compartmental models with vital dynamics

In this section, we are going to present some mathematical models with vital dynamics. In first place, we are going to consider a constant population size and secondly a variable one. For both cases, we can assume that there

is, or not, vertical transmission. Note that to assume the existence of vertical transmission is equivalent to assume that the disease is inherited from parents to their new generations. For example, it makes sense to consider vertical transmission for AIDS, hepatitis B and hepatitis C. Here latent periods are not considered, but the corresponding dynamical models with them are obtained in a similar way to what is done in Section 1.6.3.

Constant population size

With the purpose to consider a constant population size $C \in \mathbb{R}_0^+$ in a closed environment, we suppose that

i) the birth (Λ_b) and death (μ) rates are equal during the epidemic period of the disease;

ii) disease-induced death, immigration and emigration do not exist.

Thus, if we consider a SIRS model, we have that

$$\dot{S}(t) + \dot{I}(t) + \dot{R}(t) = 0 \quad \Leftrightarrow \quad S(t) + I(t) + R(t) = C$$

for all $t \in \mathbb{R}_0^+$. Let $p \in [0,1]$ be the fraction of newborns who are infective and $\bar{p} = 1 - p$. The rest is only susceptible to the disease. A SIRS dynamical model with these characteristics is given by

$$\begin{cases} \dot{S}(t) = \mu\big(S(t) + \bar{p}I(t) + R(t)\big) - f_{ir}\big(I(t)\big)S(t) + \omega_1 R(t) - \mu S(t), \\ \dot{I}(t) = f_{ir}\big(I(t)\big)S(t) + \mu p I(t) - \gamma I(t) - \mu I(t), \\ \dot{R}(t) = \gamma I(t) - \omega_1 R(t) - \mu R(t). \end{cases}$$

Variable population size

On the other hand, let us suppose that

i) the birth and death rates are not equal;

ii) disease-induced death and immigration exist.

Then, the population size is not constant. Let $A \in \mathbb{R}_0^+$ be the input rate of the total population due to immigration, $\alpha_1 \in \mathbb{R}_0^+$ be the disease-induced death rate and the remaining parameters as previously. Thus, a SIRS dynamical model with these assumptions is given by

$$\begin{cases} \dot{S}(t) = A + \Lambda_b\big(S(t) + \bar{p}I(t) + R(t)\big) - f_{ir}\big(I(t)\big)S(t) + \omega_1 R(t) - \mu S(t), \\ \dot{I}(t) = f_{ir}\big(I(t)\big)S(t) + \Lambda_b p I(t) - \gamma I(t) - (\alpha_1 + \mu)I(t), \\ \dot{R}(t) = \gamma I(t) - \omega_1 R(t) - \mu R(t). \end{cases}$$

Remark 1.51. *Note that, either for constant or for variable population size, if it does not make sense to consider vertical transmission, we just have to assume that* $p = 0$.

1.6.5 Compartmental models with treatment

In order to curtail the spread of infectious diseases, we can treat some infective individuals, at treatment rate coefficient $\delta \in \mathbb{R}_0^+$, until they recover; while the rest of the infective do not get treatment and recover at rate coefficient γ, as it was considered before. To all the previous models we can introduce a compartment T which is composed by all the treated individuals. The number of individuals in compartment T at time $t \in \mathbb{R}_0^+$ is denoted by $T(t)$. Considering, for example, the previous SIRS model with variable population size, the corresponding SITRS model is given by

$$\begin{cases} \dot{S}(t) = A + \Lambda_b\big(S(t) + \bar{p}I(t) + R(t)\big) - f_{ir}\big(I(t)\big)S(t) + \omega_1 R(t) - \mu S(t), \\ \dot{I}(t) = f_{ir}\big(I(t)\big)S(t) + \Lambda_b pI(t) - \gamma I(t) - \delta I(t) - (\alpha_1 + \mu)I(t), \\ \dot{T}(t) = \delta I(t) - \varepsilon T(t) - (\mu + \alpha_2)T(t), \\ \dot{R}(t) = \gamma I(t) + \varepsilon T(t) - \omega_1 R(t) - \mu R(t), \end{cases}$$

where $\alpha_2 \in \mathbb{R}_0^+$ is the disease-induced death rate in treatment. Generally, we have that $\mu < \alpha_2 < \alpha_1$. Note that $\frac{1}{\varepsilon}$ is the mean treatment period without death.

Remark 1.52. *For some infectious diseases it is usual to consider a treatment through quarantine, i.e., some infective individuals are quarantined with an appropriate medication until complete recovery. In [45, 68, 127, 128, 131, 195], one can find examples of dynamical models with quarantine.*

1.6.6 Compartmental models with vaccination

With the purpose to prevent an infectious disease spread, we can propose vaccination for some susceptible individuals. As this measure avoids an increase of the number of new infections since the beginning of a possible epidemic spread, vaccination is considered to be one of the most effective and cost-effective method of preventing infectious diseases (see e.g. [113]).

We can use a SIRS model to describe the transmission dynamics and we assume to vaccine a fraction $q \in [0, 1]$ of susceptible individuals who become immune, permanently or not. For this, we introduce a fifth class V which is composed by all the vaccinated individuals. The number of individuals in compartment V at time $t \in \mathbb{R}_0^+$ is denoted by $V(t)$. When the immunity due to vaccination is temporary, vaccinated individuals can become susceptible again at rate coefficient $\omega_2 \in \mathbb{R}^+$. In this case, the mean immunity period for vaccinated individuals is $\frac{1}{\omega_2}$. On the other hand, if immunity due to vaccination is permanent, then we assume $\omega_2 = 0$ and vaccinated individuals will not become susceptible ever more. Considering a population with variable size, disease-induced death and vertical transmission, a SIRVS dynamical

model is given by

$$\begin{cases} \dot{S}(t) = A + \Lambda_b\big(S(t) + \bar{p}I(t) + R(t)\big) - f_{ir}\big(I(t)\big)S(t) - qS(t) \\ \qquad\quad + \omega_1 R(t) + \omega_2 V(t) - \mu S(t), \\ \dot{I}(t) = f_{ir}\big(I(t)\big)S(t) + \Lambda_b pI(t) - \gamma I(t) - (\alpha_1 + \mu)I(t), \\ \dot{R}(t) = \gamma I(t) - \omega_1 R(t) - \mu R(t), \\ \dot{V}(t) = qS(t) - \omega_2 V(t) - \mu V(t). \end{cases}$$

Remark 1.53. *To all the previous compartmental models one can introduce constant time delays. In Section 5.3, we study a compartmental model with a constant time delay.*

Compartment	Description
S	Compartment of susceptible individuals
E	Compartment of exposed/latent individuals
I	Compartment of infective individuals
T	Compartment of individuals in treatment
V	Compartment of vaccinated individuals
R	Compartment of removed/recovered individuals

Table 1.1: Description of the compartments/classes used in the examples of compartmental models of Section 1.6.

1.7 Conclusion

In this chapter we provided a stability study of mathematical models that can translate the spread of some infectious diseases. In the next chapter, we are going to define some optimal control problems whose goal is to minimize/maximize a cost functional subject to a differential system. For optimal control problems without time delays we will recall necessary and sufficient optimality conditions. Such problems involve dynamical systems similar to those of Sections 1.3, 1.5 and 1.6. On the other hand, for optimal control problems with time delays, we will only recall necessary optimality conditions. These problems consider differential systems similar to those studied in Section 1.4.

Parameter	Description
A	Input rate due to immigration
Λ_b	Birth rate coefficient
μ	Natural death rate coefficient
q	Fraction of susceptible individuals who are vaccinated
β	Transmission rate coefficient
p	Fraction of infective newborns
υ	Rate coefficient at which infected individuals become infectious
γ	Removal/recovery rate coefficient without treatment
$\tilde{\gamma}$	Removal rate coefficient in compartment I
δ	Treatment rate coefficient
ε	Recovery rate coefficient for treated individuals
$\tilde{\omega}$	Reinfection rate coefficient in compartment R
ω_1	Immunity waning rate coefficient in compartment R
ω_2	Immunity waning rate coefficient in compartment V
α_1	Disease-induced death rate coefficient in compartment I
α_2	Disease-induced death rate coefficient in compartment T

Table 1.2: Description of the parameters used in the examples of compartmental models of Section 1.6.

Chapter 2

Optimal Control Theory

Generally, an arbitrary optimal control problem consists to minimize or maximize a cost functional subject to a system and maybe to some initial or final conditions. We begin this chapter with a brief introduction about the Optimal Control Theory. Next, our attention is devoted to non-delayed optimal control problems. We define them and recall well-known necessary and sufficient optimality conditions, following the approaches used in [56, 103, 140]. We also define an optimal control problem with constant time delays in state and control variables for which we recall the necessary optimality conditions derived in [56]. Later, the necessary optimality conditions recalled here are going to be used in Sections 5.2.4, 5.3.4 and 6.3.4. On the other hand, the sufficient optimality conditions presented here are going to be generalized for delayed optimal control problems, in Chapter 3 of original results.

2.1 Introduction

The origin of the Calculus of Variations was in the XVII century due to the contributions of *Pierre de Fermat* (1607–1665), *Isaac Newton* (1643–1727), *Gottfried Wilhelm Leibniz* (1646–1716), *Jacob Bernoulli* (1655–1705) and *Johann Bernoulli* (1667–1748). Mathematicians as *Héctor J. Sussmann* (1946–) and *Jan Camiel Willems* (1939–2013) defend that the origin of Optimal Control coincides with the birth of the Calculus of Variations, in 1697 – year of the publication of the solution of the *brachistochrone problem* by *Johann Bernoulli* (see [158, 169]). The word *brachistochrone* comes from the Ancient Greek *brákhistos khrónos* and it means the "shortest time". The main goal of such problem is to find the curve between two points, on a vertical plane, that a sphere without friction covers in the shortest time. The sphere starts at the initial point A with zero velocity and is constrained to move along the curve until the final point B, under the action of gravity force and supposing that there is no friction (see Figure 2.1). The *brachis-*

tochrone problem was also studied by *Galileo Galilei* (1564–1642), in 1638. The shortest time path between these two points is not an arc of circle, as *Galileo* believed. Nevertheless, this mathematician had already remarked that the straight line is not the solution and he was correct. In 1696, *Johann Bernoulli* challenged the best mathematicians of his time to solve this problem. Consequently, *Johann Bernoulli* himself discovered the solution, as well as his brother *Jacob Bernoulli*, *Newton*, *Leibniz* and *Guillaume François Antoine* – Marquis de l'Hôpital (1661–1704). The solution is a cycloid arc starting with a vertical tangent (see Figure 2.1 and [112, 158, 169]). Note that *skateboarding ramps* and the fastest decreases of *aqua-parks* have the form of a cycloid. Some authors consider that *Newton's problem of aero-dynamical*

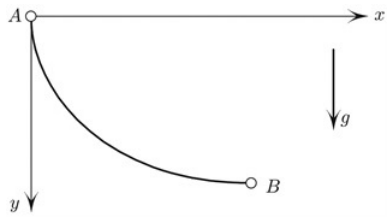


Figure 2.1: Solution to the *brachistochrone problem*.

resistance, formulated and solved by *Newton* in his *Principia Mathematica*, in 1686, is a typical optimal control problem (see [158, 160, 173]). Actually, Optimal Control Theory is a recent branch of Mathematics that can also be seen as an extension of the Calculus of Variations.

During Second World War (1939–1945), Optimal Control Theory has been recognized as advantageous and useful. Namely, in the fields of Engineering, Aeronautics and Flight Dynamics. Then, in the beginning of the Cold War (1945–1991) the USA and the USSR gave great importance to mathematicians and their theories in order to create defence techniques. Therefore, several mathematicians developed solution methods for problems which nowadays are known as optimal control problems. The minimum time interception problems for fighter aircraft are examples of this. So, the conventional wisdom asserts that Optimal Control Theory has emerged due to the formulation and proof of *Pontryagin's Maximum Principle* carried out by *Lev Semenovich Pontryagin* (1908–1988), a Russian mathematician, and his group of collaborators, in 1956 (see [140]). Different approaches of Optimal Control Theory can be found in [3, 5, 86, 103, 140, 163], for example.

Pontryagin and his colleagues generalized the theory of the Calculus of Variations to curves that take values on closed sets (with boundary). Furthermore, *Pontryagin's Maximum Principle* also generalizes the necessary conditions of *Euler–Lagrange* (1707–1783 and 1736–1813) and *Karl Weierstrass* (1815–1897). Important discoveries associated with this field of Mathematics are, for example (see [158, 174, 175]) the:

i) dynamic programming method;

ii) introduction of functional analysis to the theory of optimal systems;

iii) connections between the solutions of an optimal control problem and the results on stability of *Aleksandr Lyapunov* (1857–1918) theory.

Later appeared the foundations of Stochastic Control and Filtering in Dynamic Systems, Game Theory, Control of Partial Differential Equations and Hybrid Control Systems. They are some of the many branches of current research (see [2, 158, 166]).

In this chapter, we recall well-known optimality conditions for some non-autonomous optimal control problems. Such conditions are provided for non-delayed and delayed optimal control problems in Sections 2.2 and 2.3, respectively. With the purpose to recall the meaning of necessary and sufficient optimality conditions, we denote an admissible solution for an optimal control problem (OC) by (AS); necessary optimality condition by (NC) and sufficient optimality condition by (SC). So, a (NC) and a (SC) are statements of the following types, respectively:

i) if (AS) is an optimal solution of (OC), then (AS) must satisfy (NC);

ii) if (AS) satisfies (SC), then (AS) is an optimal solution of (OC).

This chapter is organised as follows. In Section 2.2.1, we begin with a general idea of optimal control applications. Then, we define, on a fixed finite time interval, a non-autonomous optimal control problem without time delays and with a fixed initial state denoted by (OCP). In Section 2.2.2, we recall well-known necessary optimality conditions for (OCP). Both Sections 2.2.1 and 2.2.2 are based on the approaches used in [56, 140]. We finish Section 2.2 recalling well-known sufficient optimality conditions for (OCP) and for a particular problem of it (see Section 2.2.3). This particular problem of (OCP), denoted by ($\mathrm{OCP_L}$), considers a differential system that is linear with respect to the state variable and a cost functional that can be written as a sum of two functions. The first only depends on time and state and the second only depends on time and control. We remark that Section 2.2.3 is based on [103]. In Section 2.3.1, we define, on a fixed finite time interval, a non-autonomous optimal control problem with constant time delays in state and control variables. Such problem, denoted by ($\mathrm{OCP_D}$), also considers a fixed initial state. We finish Section 2.3 recalling well-known necessary optimality conditions for ($\mathrm{OCP_D}$) (see Section 2.3.2). Note that Section 2.3 is entirely based on [56]. We finish the current chapter with Section 2.4 of conclusion.

2.2 Non-delayed optimal control problems

We begin this section by defining a non-autonomous optimal control problem without time delays and with a fixed initial state, on a fixed finite time duration, and some related concepts.

2.2.1 Statement of the optimal control problem

As one can read in [140, p. 9–11], in most applications of Optimal Control Theory we consider the behaviour of an object or individuals whose state at each instant of time $t \in [a,b]$ ($a \geq 0$) can be represented by real numbers $x_1(t), \ldots, x_n(t)$. For instance, these values may be coordinates or velocities. From a mathematical point of view, the behaviour of the object or individuals change with the time. The phase space of the object/individuals under consideration is $X \subseteq \mathbb{R}^n$, i.e., $x(t) = \begin{bmatrix} x_1(t) & \cdots & x_n(t) \end{bmatrix}^T \in X$ for $t \in [a,b]$. Moreover, it is assumed that the behaviour of an object or individuals depends on certain controls. The position of them is represented by a control $u(t) = \begin{bmatrix} u_1(t) & \cdots & u_m(t) \end{bmatrix}^T$ with range in a certain control region U for each instant of time $t \in [a,b]$ under consideration, i.e., $u(t) \in U$ for $t \in [a,b]$. Although we can simply consider that U is any set such that $U \subseteq \mathbb{R}^m$, in applications it makes sense to consider that $U \subseteq \mathbb{R}^m$ is a closed and bounded set. For instance, the quantity of fuel being supplied to a motor, temperature, current, voltage, fraction of sick people who get medication/treatment, etc., can not take arbitrarily large values.

Using the approach of [56, 140], we define below a non-autonomous optimal control problem without time delays and with a fixed initial state, on a fixed finite time interval.

Definition 2.1 (See Section 2 of [56] and p. 66–67 of [140]). *An optimal control problem without time delays and with a fixed initial state, on a fixed finite time interval $[a,b]$, is denoted by* (OCP) *and consists in*

$$\min \quad C\big(x(\cdot),u(\cdot)\big) = g^0\big(x(b)\big) + \int_a^b f^0\big(t,x(t),u(t)\big)dt$$

subject to the differential system

$$\dot{x}(t) = f\big(t,x(t),u(t)\big) \tag{2.1}$$

with boundary conditions

$$x(a) = x_a \quad \textit{and} \quad x(b) \in \Pi \subseteq \mathbb{R}^n, \tag{2.2}$$

where

i) the state trajectory is $x(t) \in \mathbb{R}^n$ for all $t \in [a,b]$;

ii) the control is $u(t) \in U \subseteq \mathbb{R}^m$ *for all* $t \in [a,b]$*;*

iii) $f = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix}^T$.

Obviously, we assume that Π *and* U *are non-empty.*

In what follows we define admissible pair for (OCP).

Definition 2.2 (See Section 2 of [56]). *We say that* $\big(x(\cdot), u(\cdot)\big)$ *is an* admissible pair *for* (OCP) *if it respects the following conditions:*

i) $\big(x(\cdot), u(\cdot)\big) \in W^{1,\infty}\big([a,b], \mathbb{R}^n\big) \times L^{\infty}\big([a,b], \mathbb{R}^m\big)$*;*

ii) $\big(x(\cdot), u(\cdot)\big)$ *satisfies the conditions* (2.1) *and* (2.2)*;*

iii) $\big(x(t), u(t)\big) \in \mathbb{R}^n \times U$ *for all* $t \in [a,b]$.

Next we define local and global minimizer for (OCP).

Definition 2.3 (Local and global minimizer – see Section 2 of [56]). *An admissible pair* $\big(\hat{x}(\cdot), \hat{u}(\cdot)\big)$ *is called a* local minimizer *of* (OCP) *(see Definition 2.1) if*

$$C\big(\hat{x}(\cdot), \hat{u}(\cdot)\big) \leq C\big(x(\cdot), u(\cdot)\big)$$

for all admissible pair $\big(x(\cdot), u(\cdot)\big)$ *in a neighbourhood of* $\big(\hat{x}(\cdot), \hat{u}(\cdot)\big)$ *with* $||x(t) - \hat{x}(t)||$, $||u(t) - \hat{u}(t)|| < \varepsilon$ *for all* $t \in [a,b]$ *and* $\varepsilon > 0$ *sufficiently small. In contrast, an admissible pair* $\big(\hat{x}(\cdot), \hat{u}(\cdot)\big)$ *is called a* global minimizer *of* (OCP) *if*

$$C\big(\hat{x}(\cdot), \hat{u}(\cdot)\big) \leq C\big(x(\cdot), u(\cdot)\big)$$

for all admissible pair $\big(x(\cdot), u(\cdot)\big)$.

With the purpose to avoid any misunderstanding, in the following remark we provide all types of minimality/maximality conditions that, from now on, can be found on the necessary and sufficient optimality conditions.

Remark 2.4. *Consider an optimal control problem denoted by* (P) *which consists in minimizing/maximizing the cost functional given by*

$$C\big(x(\cdot), u(\cdot)\big) = g^0\big(x(b)\big) + \int_a^b f^0\big(t, x(t), u(t)\big)dt$$

subject to

$$\begin{aligned} &\dot{x}(t) = f\big(t, x(t), u(t)\big) \\ &x(a) = x_a \quad \textit{and} \quad x(b) \in \Pi \subseteq \mathbb{R}^n, \end{aligned}$$

where $U \subseteq \mathbb{R}^m$ *and* $\big(x(t), u(t)\big) \in \mathbb{R}^n \times U$ *for all* $t \in [a,b]$. *The so-called* Hamiltonian *associated with* (P) *is defined by*

$$H(t, x, u, \eta_0, \eta) = \eta_0 f^0(t, x, u) + \eta f(t, x, u), \tag{2.3}$$

where $t, \eta_0 \in \mathbb{R}$; $x, \eta^T \in \mathbb{R}^n$ *and* $u \in \mathbb{R}^m$. *Note that the dimensions of* x, u *and* η *are* $n \times 1$, $m \times 1$ *and* $1 \times n$, *respectively. The classes of functions in which* g^0, f^0 *and* f *belong will be specify later in each theorem. Depending on the type of problem (minimization or maximization) and the value of* η_0, *we can obtain different types of optimality conditions as it is summarized in Table 2.1. Note that the* Minimality Condition *and the* Maximality Condition

The objective of (P)	η_0 in (2.3)	Type of optimality condition
min $C\big(x(\cdot), u(\cdot)\big)$	1	Minimality condition
min $C\big(x(\cdot), u(\cdot)\big)$	-1	Maximality condition
max $C\big(x(\cdot), u(\cdot)\big)$	1	Maximality condition
max $C\big(x(\cdot), u(\cdot)\big)$	-1	Minimality condition

Table 2.1: The type of the optimality condition depends on Hamiltonian's expression and the type of problem.

are, respectively, equivalent to

$$\min_{u \in U} H\big(t, x^*(t), u, \eta_0(t), \eta(t)\big) = H\big(t, x^*(t), u^*(t), \eta_0(t), \eta(t)\big)$$

and

$$\max_{u \in U} H\big(t, x^*(t), u, \eta_0(t), \eta(t)\big) = H\big(t, x^*(t), u^*(t), \eta_0(t), \eta(t)\big),$$

where $\big(x^*(\cdot), u^*(\cdot)\big)$ *is a minimizer/maximizer of* (P) *to which it corresponds the multipliers* $\big(\eta_0, \eta(\cdot)\big)$.

2.2.2 Necessary optimality condition

In the following theorem, we recall a well-known necessary optimality condition, also known as *Pontryagin's Maximum Principle* (PMP), associated with (OCP) (see Definition 2.1). Note that along this work we use the notation $\partial_i f$ to denote the partial derivative of a certain function f with respect to its ith argument. For example, $\partial_2 f(t, x, u) = \dfrac{\partial f}{\partial x}(t, x, u)$.

Theorem 2.5 (PMP – see Theorem 3.1 of [56] or Theorem 7 of [140])**.** *Consider* (OCP) *and assume that*

i) the functions f^0 *and* f *are of class* $\mathscr{C}^1$ *with respect to all their arguments;*

ii) $\Pi = \mathbb{R}^n$.

If $\big(x^*(\cdot), u^*(\cdot)\big)$ *is a minimizer of* (OCP), *then there is a non-zero function* $\eta(\cdot)^T \in W^{1,\infty}\big([a, b], \mathbb{R}^n\big)$ *that verifies the:*

i) transversality condition

$$\eta(b)^T = \frac{\partial g^0}{\partial x}\big(x^*(b)\big);$$

ii) adjoint system

$$\dot{\eta}(t) = -\partial H_2\big(t, x^*(t), u^*(t), \eta(t)\big), \quad \textit{a.e. } t \in [a,b];$$

iii) maximality condition

$$\max_{u \in U} H\big(t, x^*(t), u, \eta(t)\big) = H\big(t, x^*(t), u^*(t), \eta(t)\big), \quad \textit{a.e. } t \in [a,b];$$

where $H(t,x,u,\eta) = -f^0(t,x,u) + \eta f(t,x,u)$ *is the* Hamiltonian *associated with* (OCP).

Remark 2.6. *Note that when we are deriving the previous necessary optimality condition using a* minimality condition *instead of a* maximality condition *(see Remark 2.4 and Theorem 2.5), we are going to use the denomination* Pontryagin's Minimum Principle *instead of* Pontryagin's Maximum Principle.

Remark 2.7 (Adjoint function)**.** *The vector function* $\eta(\cdot)$ *of the previous Theorem 2.5 is known as the* adjoint function.

Note that Theorem 2.5 is used in the proofs of Theorems 5.8 and 6.16 of Sections 5.2.4 and 6.3.4, respectively.

Strict bang-bang property

Consider that $\big(x^*(\cdot), u^*(\cdot)\big)$ is a local minimizer of (OCP). Assuming that $k \in \mathbb{N}$ and $a = t_0 < t_1 < \cdots < t_k < b = t_{k+1}$, we denote the finite set of all possible discontinuity points of the control $u^*(\cdot)$ by

$$\Theta = \{t_1, \ldots, t_k\} \subset \Big[\tilde{a}, \tilde{b}\Big] \subset [a,b].$$

In the following discussion, let us suppose that $m = 1$, $U = [u_{\min}, u_{\max}] \subset \mathbb{R}$ and the Hamiltonian is linear with respect to the control. With the previous assumption, we have that

$$H(t,x,u,\eta) = \phi(t,x,\eta)u.$$

Function $\phi(t,x,\eta) = \partial_3 H(t,x,u,\eta)$ is called the *switching function*. In order to simplify the notation, from now on we simply denote $\phi\big(t, x(t), \eta(t)\big)$ by $\phi(t)$. Consequently, the maximality condition of Theorem 2.5 is equivalent to

$$\max_{u \in U} \phi(t)u = \phi(t)u^*(t) \tag{2.4}$$

for almost all $t \in [a,b]$ (see Section 4.1 of [118]). As $u_{\min} \leq u \leq u_{\max}$, then we obtain

$$\begin{cases} \phi(t)u_{\min} \leq \phi(t)u \leq \phi(t)u_{\max}, & \text{if} \quad \phi(t) > 0; \\ \phi(t)u_{\max} \leq \phi(t)u \leq \phi(t)u_{\min}, & \text{if} \quad \phi(t) < 0. \end{cases}$$

As we have a maximality condition, we obtain the control law

$$u^*(t) = \begin{cases} u_{\min}, & \text{if} \quad \phi(t) < 0; \\ u_{\max}, & \text{if} \quad \phi(t) > 0; \\ singular, & \text{if} \quad \phi(t) = 0, \quad \forall t \in \left[\tilde{a}, \tilde{b}\right] \subset [a,b]. \end{cases} \tag{2.5}$$

As it is written in Section 4.1 of [118], u is called *bang-bang control* on an interval $\left[\tilde{a}, \tilde{b}\right]$, if the switching function ϕ has only isolated zeros on $\left[\tilde{a}, \tilde{b}\right]$, that is,

$$\phi(t) = 0, \quad \forall t \in \Theta.$$

In this case, we have the so-called *strict bang-bang property* (see [117, 118, 133]):

$$\begin{aligned} &\left\{\phi(t) > 0 \ \text{ for } \ t \in]t_{i-1}, t_i[\ \wedge \ \dot{\phi}(t_i) < 0 \ \wedge \ \phi(t) < 0 \ \text{ for } \ t \in]t_i, t_{i+1}[\right\} \\ \vee &\left\{\phi(t) < 0 \ \text{ for } \ t \in]t_{i-1}, t_i[\ \wedge \ \dot{\phi}(t_i) > 0 \ \wedge \ \phi(t) > 0 \ \text{ for } \ t \in]t_i, t_{i+1}[\right\} \end{aligned}$$

for all $i \in \{1, \ldots, k\}$. It means that if the control has a switch from $u_{\max}$ to $u_{\min}$ at the switching time $t_i \in \Theta$, then we have that

$$\phi(t) > 0 \ \text{ for } \ t \in]t_{i-1}, t_i[\ \wedge \ \dot{\phi}(t_i) < 0 \ \wedge \ \phi(t) < 0 \ \text{ for } \ t \in]t_i, t_{i+1}[. \tag{2.6}$$

If the control has a switch from $u_{\min}$ to $u_{\max}$ at the switching time $t_i \in \Theta$, then we have that

$$\phi(t) < 0 \ \text{ for } \ t \in]t_{i-1}, t_i[\ \wedge \ \dot{\phi}(t_i) > 0 \ \wedge \ \phi(t) > 0 \ \text{ for } \ t \in]t_i, t_{i+1}[. \tag{2.7}$$

On the other hand, u is called a *singular control*, if the switching function ϕ vanishes identically on an interval $\left[\tilde{a}, \tilde{b}\right]$, that is,

$$\phi(t) = 0, \quad \forall t \in \left[\tilde{a}, \tilde{b}\right].$$

Note that if we define the Hamiltonian by

$$H(t, x, u, \eta) = f^0(t, x, u) + f(t, x, u),$$

then the maximality condition of Theorem 2.5 is replaced by the *minimality condition*

$$\min_{u \in U} H\big(t, x^*(t), u, \eta(t)\big) = H\big(t, x^*(t), u^*(t), \eta(t)\big), \quad \text{a.e. } t \in [a,b].$$

Consequently, the maximality condition (2.4) is also replaced by the *minimality condition*

$$\min_{u\in U}\phi(t)u = \phi(t)u^*(t).$$

Furthermore, the control law (2.5) is then replaced by

$$u^*(t) = \begin{cases} u_{\min}, & \text{if} \quad \phi(t) > 0; \\ u_{\max}, & \text{if} \quad \phi(t) < 0; \\ singular, & \text{if} \quad \phi(t) = 0, \quad \forall t \in \left[\tilde{a}, \tilde{b}\right] \subset [a,b]. \end{cases}$$

Finally, the condition (2.6) is replaced by (2.7) and vice-versa. For more details one can read [133]. Note that these contents are used in Sections 5.3.4, 5.3.5, 6.3.4 and 6.3.5.

2.2.3 Sufficient optimality conditions

In this section, we recall well-known sufficient optimality conditions for non-autonomous optimal control problems without time delays, following the approaches used in Chapter 5.2 of [103]. Such theoretical results will be generalized for delayed optimal control problems, in Sections 3.2 and 3.3 of Chapter 3.

Next, we define a state-linear optimal control problem that is a particular case of (OCP) (see Definition 2.1).

Definition 2.8 (See p. 340–341 of [103])**.** *A state-linear optimal control problem* ($\mathrm{OCP_L}$) *is a particular case of* (OCP)*, where*

i) the cost functional is given by

$$\begin{aligned} &g^0\big(x(b)\big) + \int_a^b f^0\big(t, x(t), u(t)\big)dt \\ &= g^0\big(x(b)\big) + \int_a^b f_x^0\big(t, x(t)\big) + f_u^0\big(t, u(t)\big)dt := C_L\big(x(\cdot), u(\cdot)\big); \end{aligned} \tag{2.8}$$

ii) the differential system is linear with respect to the state variable, i.e.,

$$\dot{x}(t) = f\big(t, x(t), u(t)\big) = A(t)x(t) + g\big(t, u(t)\big), \tag{2.9}$$

where $A(t)$ is a real $n \times n$ matrix;

iii) Π *is a closed convex target set.*

In what follows, we define admissible pair for ($\mathrm{OCP_L}$).

Definition 2.9. *We say that* $\big(x(\cdot), u(\cdot)\big)$ *is an* admissible pair *for* ($\mathrm{OCP_L}$) *if it respects the following conditions:*

i) $\big(x(\cdot), u(\cdot)\big) \in W^{1,\infty}\big([a,b], \mathbb{R}^n\big) \times L^{\infty}\big([a,b], \mathbb{R}^m\big)$;

ii) $\big(x(\cdot), u(\cdot)\big)$ *satisfies conditions* (2.2) *and* (2.9), *where* Π *is a closed convex target set;*

iii) $\big(x(t), u(t)\big) \in \mathbb{R}^n \times U$ *for all* $t \in [a,b]$.

The following theorem provides a well-known sufficient optimality condition associated with ($\mathrm{OCP_L}$).

Theorem 2.10 (See Theorem 5 – Chapter 5.2 of [103]). *Consider* ($\mathrm{OCP_L}$) *and assume that*

i) $g^0 \equiv 0$;

ii) functions f_x^0, $\partial_2 f_x^0$, f_u^0, A *and* g *are continuous with respect to all their arguments;*

iii) $f_x^0(t,x)$ *is a convex function in* x *for each fixed* $t \in [a,b]$;

iv) for almost all $t \in [a,b]$, u^* *is a control with response* x^* *that satisfies the* maximality condition

$$\max_{u \in U} H\big(t, x^*(t), u, \eta(t)\big) = H\big(t, x^*(t), u^*(t), \eta(t)\big),$$

where

$$H(t,x,u,\eta) = -\Big(f_x^0(t,x) + f_u^0(t,u)\Big) + \eta\Big(A(t)x + g(t,u)\Big),$$

and $\eta(t)$ *is any non-trivial solution of the* adjoint system

$$\dot{\eta}(t) = \partial_2 f_x^0\big(t, x^*(t)\big) - \eta(t)A(t),$$

satisfying the transversality condition *that ensures that* $\eta(b)^T$ *is an inward normal vector of* Π *at the boundary point* $x^*(b)$.

Then, $\big(x^*(\cdot), u^*(\cdot)\big)$ *is an optimal solution of* ($\mathrm{OCP_L}$) *that leads to the minimal cost* $C_L\big(x^*(\cdot), u^*(\cdot)\big)$.

Remark 2.11 (See p. 342 of [103]). *Note that if* $\Pi = \{x_b\}$, *then the* transversality condition *of Theorem 2.10 is vacuous, because* Π *has a single point. If* $\Pi = \mathbb{R}^n$, *then* $\eta(b) = [0 \quad \cdots \quad 0]_{1\times n}$.

Finally, the following theorem provides a sufficient optimality condition associated with (OCP) (see Definition 2.1).

Theorem 2.12 (See Theorem 7 – Chapter 5.2 of [103]). *Consider* (OCP), *where functions g^0, f^0 and f are of class $\mathscr{C}^1$ with respect to all their arguments. Assume that there is a feedback control*

$$u^*\Big(t, x(t), \eta\big(t, x(t)\big)\Big) \in \mathscr{C}^1\big([a,b] \times \mathbb{R}^{2n}, \mathbb{R}^m\big)$$

such that

$$\begin{aligned}\max_{u \in U} H\Big(t, x(t), u, \eta\big(t, x(t)\big)\Big) &= H\Big(t, x(t), u^*\Big(t, x(t), \eta\big(t, x(t)\big)\Big), \eta\big(t, x(t)\big)\Big) \\ &=: H^0\Big(t, x(t), \eta\big(t, x(t)\big)\Big),\end{aligned}$$

where $H(t,x,u,\eta) = -f^0(t,x,u) + \eta f(t,x,u)$. Furthermore, suppose that

i) the function $S\big(t, x(t)\big) \in \mathscr{C}^2\big([a,b] \times \mathbb{R}^n, \mathbb{R}\big)$ is a solution of the following Hamilton–Jacobi equation:

$$\partial_1 S\big(t, x(t)\big) + H^0\Big(t, x(t), \partial_2 S\big(t, x(t)\big)\Big) = 0$$

with $S\big(b, x(b)\big) = -g^0\big(x(b)\big)$;

ii) the control law

$$u^*\Big(t, x(t), \partial_2 S\big(t, x(t)\big)\Big)$$

determines a response $\tilde{x}(t)$ steering (a, x_a) to (b, Π).

Then,

$$\tilde{u}(\cdot) = u^*\Big(\,\cdot\,, \tilde{x}(\cdot), \partial_2 S\big(\,\cdot\,, \tilde{x}(\cdot)\big)\Big)$$

is an optimal control with respect to the optimal state $\tilde{x}(\cdot)$ of (OCP) *that leads to the minimal cost $C\big(\tilde{x}(\cdot), \tilde{u}(\cdot)\big) = -S(a, x_a)$.*

Note that Theorems 2.10 and 2.12 are used in Sections 3.2 and 3.3, respectively, where we derive new sufficient optimality conditions for optimal control problems with constant time delays in state and control variables.

2.3 Delayed optimal control problems

In this section, we define an optimal control problem with constant time delays in state and control variables. Moreover, for such problem, we recall well-known necessary optimality conditions originally proved in [56].

2.3.1 Statement of the optimal control problem

We begin this section by defining, on a fixed finite time interval, a non-autonomous optimal control problem with constant time delays in state and control variables. Such problem also considers a fixed state on the initial time interval. Note that it is a particular optimal control problem of that that is defined in Section 2 of [56].

Definition 2.13 (See Section 2 of [56]). *Consider that $r \geq 0$ and $s \geq 0$ are constant time delays associated with the state and control variables, respectively. A non-autonomous optimal control problem with constant time delays and with a fixed initial state, on a fixed finite time interval $[a, b]$, is denoted by* ($\mathrm{OCP_D}$) *and consists in*

$$\min \quad C_D\big(x(\cdot), u(\cdot)\big) = g^0\big(x(b)\big) + \int_a^b f^0\big(t, x(t), x(t-r), u(t), u(t-s)\big)dt$$

subject to the delayed differential system

$$\dot{x}(t) = f\big(t, x(t), x(t-r), u(t), u(t-s)\big) \quad \text{for a.e.} \quad t \in [a, b] \tag{2.10}$$

with initial and final conditions

$$\begin{aligned} &x(t) = \varphi(t), \quad t \in I_x \subset \mathbb{R}, \\ &u(t) = \psi(t), \quad t \in [a-s, a[, \\ &x(b) \in \Pi \subseteq \mathbb{R}^n; \end{aligned} \tag{2.11}$$

where

i) the state trajectory is $x(t) \in \mathbb{R}^n$ for all $t \in I_x \cup [a, b]$;

ii) the control is $u(t) \in U \subseteq \mathbb{R}^m$ for all $t \in [a-s, b]$;

iii) $f = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix}^T$.

Next we define admissible pair for ($\mathrm{OCP_D}$).

Definition 2.14 (See Section 2 of [56]). *We say that $\big(x(\cdot), u(\cdot)\big)$ is an* admissible pair *for* ($\mathrm{OCP_D}$) *if it respects the following conditions:*

i) $\big(x(\cdot), u(\cdot)\big) \in W^{1,\infty}\big(I_x \cup [a, b], \mathbb{R}^n\big) \times L^\infty\big([a-s, b], \mathbb{R}^m\big)$;

ii) $\big(x(\cdot), u(\cdot)\big)$ satisfies conditions (2.10) and (2.11);

iii) $\big(x(t), u(t)\big) \in \mathbb{R}^n \times U$ for all $t \in [a, b]$.

2.3.2 Necessary optimality conditions

In what follows, we assume that the time delays r and s respect the following *commensurability assumption*.

Assumption 2.15 (Commensurability – see Assumption 4.1 of [56]). *We consider $r, s \geq 0$ not simultaneously equal to zero and commensurable, that is,*

$$(r, s) \neq (0, 0)$$

and

$$\frac{r}{s} \in \mathbb{Q} \quad \textit{for} \quad s > 0 \quad \textit{or} \quad \frac{s}{r} \in \mathbb{Q} \quad \textit{for} \quad r > 0.$$

Actually, the previous commensurability assumption holds for any couple of rational numbers (r, s) for which at least one number is non-zero (see [56]).

Notation 2.16. *With the purpose to simplify the notation, we define t_τ, t^τ, $t^{\tau_2}_{\tau_1}$ and $I^a_{\tilde{\tau}}$ as follows:*

$$t_\tau = t - \tau, \quad t^\tau = t + \tau, \quad t^{\tau_2}_{\tau_1} = t - \tau_1 + \tau_2 \quad \textit{and} \quad I^a_{\tilde{\tau}} = [a - \tilde{\tau}, a]$$

for time delays $\tau, \tau_1, \tau_2 \in \{r, s\}$, $\tilde{\tau} \in \{r, r + s\}$ and for all $t \in [a, b]$.

In the following theorem, we recall a well-known necessary optimality condition associated with ($\mathrm{OCP_D}$).

Theorem 2.17 (See Theorem 4.2 of [56]). *Consider ($\mathrm{OCP_D}$), where $I_x = I^a_r$ and $\Pi = \mathbb{R}^n$. Assume that functions g^0, f^0 and f are of class $\mathscr{C}^1$ with respect to all their arguments. Furthermore, suppose that $\big(x^*(\cdot), u^*(\cdot)\big)$ is a local minimizer for ($\mathrm{OCP_D}$), satisfying Assumption 2.15. Then, there is a non-zero function $\eta(\cdot)^T \in W^{1,\infty}\big([a, b], \mathbb{R}^n\big)$ that verifies the:*

i) transversality condition

$$\eta(b)^T = \frac{\partial g^0}{\partial x}\big(x^*(b)\big);$$

ii) adjoint system

$$\begin{aligned}\dot{\eta}(t) = &- \partial H_2\big(t, x^*(t), x^*(t_r), u^*(t), u^*(t_s), \eta(t)\big) \\ &- \partial H_3\big(t^r, x^*(t^r), x^*(t), u^*(t^r), u^*(t^r_s), \eta(t^r)\big)\chi_{[a,b-r]}(t)\end{aligned}$$

for almost all $t \in [a, b]$;

iii) maximality condition

$$\begin{aligned}
&\max_{u\in U}\Big\{H\big(t,x^*(t),x^*(t_r),u,u^*(t_s),\eta(t)\big)\\
&\quad+H\big(t^s,x^*(t^s),x^*(t^s_r),u^*(t^s),u,\eta(t^s)\big)\chi_{[a,b-s]}(t)\Big\}\\
&=H\big(t,x^*(t),x^*(t_r),u^*(t),u^*(t_s),\eta(t)\big)\\
&\quad+H\big(t^s,x^*(t^s),x^*(t^s_r),u^*(t^s),u^*(t),\eta(t^s)\big)\chi_{[a,b-s]}(t)
\end{aligned}\tag{2.12}$$

for almost all $t\in[a,b]$*;*

where $H(t,x,y,u,v,\eta)=-f^0(t,x,y,u,v)+\eta f(t,x,y,u,v)$.

The previous result is known as the *Maximum Principle* for delayed optimal control problems, or the *Minimum Principle* if we consider a minimality condition instead of (2.12). Note that Theorem 2.17 is used in the proof of Theorem 5.11 of Section 5.3.4.

2.4 Conclusion

In this chapter we defined several optimal control problems: non-delayed, delayed state-linear and delayed non-linear. For the non-delayed problems, we recalled necessary and sufficient optimality conditions. On the other hand, for optimal control problems with time delays we only recalled necessary optimality conditions, because in the literature there are not, up to our best knowledge, sufficient conditions for this type of problems. With the purpose to answer this open question, in the next chapter we derive sufficient optimality conditions for two types of delayed optimal control problems.

Part II

Original Results

Chapter 3

Sufficient optimality conditions for delayed optimal control problems

In this chapter, we give answer to an open question by proving sufficient optimality conditions for optimal control problems with discrete time delays in state and control variables. In the proof of our main results, we transform delayed optimal control problems to equivalent non-delayed problems, considering the technique proposed by Guinn in [59] and used by Göllmann et al. in [56, 57]. This allows us to use well-known theoretical results, namely Theorems 2.10 and 2.12, that ensure sufficient optimality conditions for non-delayed optimal control problems. We finish by giving examples in order to illustrate the obtained results. These original works are published in [108, 109].

3.1 Introduction

The study of delayed systems, which can be optimized and controlled by a certain control function, has a long history and has been developed by many researchers (see e.g. [7, 8, 14, 19, 43, 52, 56, 60, 91, 121, 122, 123, 124, 135, 167, 196] and references cited therein). Such systems can be called retarded, time-lag, or hereditary processes/optimal control problems. There are many applications of such systems in diverse fields as Biology, Chemistry, Mechanics, Economy and Engineering (see e.g. [8, 43, 57, 81, 91, 152, 167, 196, 197, 198]). Dynamical systems with time delays, in both state and control variables, play an important role in the modelling of real-life phenomena in various fields of applications (see [56, 57]). For instance, in [144] the incubation and pharmacological delays are modelled through the introduction of time delays in both state and control variables. In [159], Silva et al. introduce time delays in the state and control variables for tuberculosis

modelling. They represent the time delay on the diagnosis and commencement of treatment of individuals with active tuberculosis infection and the delays on the treatment of persistent latent individuals, due to clinical and patient reasons.

Delayed linear differential systems have also been investigated, their importance being recognized both from a theoretical and practical points of view. For instance, in [52] Friedman considers linear hereditary processes and apply to them Pontryagin's method, deriving necessary optimality conditions as well as existence and uniqueness results. Analogously, in [135] delayed linear differential equations and optimal control problems involving this kind of systems are studied. Since these first works, many researchers have devoted their attention to linear quadratic optimal control problems with time delays (see e.g. [19, 38, 44, 89, 136]). It turns out that for delayed linear quadratic optimal control problems it is possible to provide an explicit formula for the optimal controls (see [19, 89, 136]).

Delayed optimal control problems with differential systems, which are linear both in state and control variables, have been studied in [19, 32, 38, 44, 89, 92, 93, 102, 134, 136]. In [38, 92, 136], the system is delayed with respect to state and control variables. In [32, 134], the system only considers delays in the state variable. Chyung and Lee derive necessary and sufficient optimality conditions in [32], while Oğuztöreli only proves necessary conditions in [134]. Certain necessary conditions analysed by Chyung and Lee in [32] have been already derived in [62, 140, 141]. However, the system considered in [32] is different from the previously studied hereditary systems, which do not require a initial function of state. In [44], Eller et al. derive a sufficient condition for a control to be optimal for certain problems with time delay. The problems studied by Eller et al. and Khellat in [44] and [89], respectively, consider only one constant lag in the state. The research done by Lee in [102] is different from that of the current chapter (more specifically from that of Section 3.2), because in [102] the aim is to minimize a cost functional which does not consider delays subject to a linear differential system (with respect to state and control variables) and to another constraint. In their differential system, the state variable depends on a constant and fixed delay and the control variable depends on a constant lag, which is not specified a priori. Note that the differential system of the problem considered in [93] is similar to the one of [102]. Although Banks has studied delayed non-linear problems without lags in the control, he has also analysed problems that are linear and delayed with respect to control (see [7]). Later, in 2010, Carlier and Tahraqui investigated optimal control problems with a unique delay in the state (see [23]). In 2012 and 2013, Frederico and Torres devoted their attention to optimal control problems that only contain delays in the state variables and the dependence on the control is linear (see [50, 51]). The most general results on the area of optimal control with delay-differential inclusions in infinite dimensions seem those of Mordukhovich et al. in [121, 122, 123, 124].

Recently, Cacace et al. studied optimal control problems that involve linear differential systems with variable delays only in the control (see [19]). The problems analysed in the current chapter are different from those considered in the mentioned works. In Section 3.2, the optimal control problems involve differential systems that are linear with respect to state, but not with respect to the control. In Section 3.3, we study optimal control problems with non-linear differential systems. Furthermore, in both Sections 3.2 and 3.3, we consider a constant time delay in the state and another one in the control. These two delays are in general not equal.

In [73], Hughes firstly consider variational problems with only one constant lag and derive various necessary and a sufficient optimality conditions for them. The variational problems in [73] can easily be transformed to control problems with only one constant delay (see e.g. [105, p. 53–54]). Hughes also investigates an optimality condition for a control problem with a constant delay, which is the same for state and control. The problems analysed by Chan and Yung in [29] and by Sabbagh in [149] are similar to the first problems studied by Hughes in [73]. Therefore, the problems investigated in [29, 73, 149] are different from the problems studied by us, because in the present chapter the state delay is not necessarily equal to the control delay. The problems considered in [73, 149] are also considered in [137] by Palm and Schmitendorf. For such problems, they derive two conjugate-point conditions, which are not equivalent. Note that their conditions are only necessary and do not give a set of sufficient conditions (see [137]). Recent results include Noether type theorems for problems of the calculus of variations with time delays (see [49, 114, 153]), necessary optimality conditions for quantum (see [51]) and Herglotz variational problems with time delays (see [151, 152]), as well as delayed optimal control problems with integer (see [10, 17, 50]) and non-integer (fractional order) dynamics (see [36, 37]). Applications of such theoretical results are found in Biology and other Natural Sciences, e.g., in tuberculosis (see [159]) and HIV (see [144, 145]).

In [82], Jacobs and Kao investigate delayed problems that consist to minimize a cost functional without delays subject to a differential system defined by a non-linear function with a delay in state and another one in the control. Similar to our cases, these delays do not have to be equal. In contrast, all type of cost functionals considered in this chapter also contain time delays. Therefore, we study here problems that are more general than the one considered in [82]. Jacobs and Kao transform the problem using a Lagrange-multiplier technique and prove a regularity result in the form of a controllability condition, as well as some necessary optimality conditions. Then, in some special restricted cases, they prove existence, uniqueness and sufficient conditions. Such restricted problems consider a differential system which is linear in state and in control variables. Thus, the sufficient conditions of [82] are derived for problems that are less general than ours.

As it is well-known and as Hwang and Bien write in [75], many researchers

have directed their efforts to seek sufficient optimality conditions for control problems with delays (see e.g. [32, 44, 73, 82, 104, 155]). Therefore, it is not a surprise that there are authors that already proved some sufficient optimality conditions for delayed optimal control problems similar but, nevertheless, different from ours. In what respects to research done in [32, 44, 73, 82], we have already seen why they are different. The delayed optimal control problems analysed by Schmitendorf in [155] have a cost functional and a differential system that are more general than ours. However, in [155] the control takes its values in all $\mathbb{R}^m$, while in the present chapter the control values belong to a set $U \subseteq \mathbb{R}^m$, $m \in \mathbb{N}$. In [104], Lee and Yung study a problem that is similar to the one considered in [155], where the control belongs to a subset of $\mathbb{R}^m$, as we consider here. First and second-order sufficient conditions are shown in [104]. Nevertheless, the conditions of [104] are not constructive and practical for the computation of the optimal solution. Indeed, as hypothesis, it is assumed the existence of a symmetric matrix under some conditions, for which is not given a method to calculate its expression. Another similar problem to ours is studied by Bokov in [17], in order to arise a necessary optimality condition in an explicit form. Moreover, a solution to the problem with infinite time horizon is given in [17]. In contrast, in the present chapter we are interested to derive sufficient optimality conditions. In [75], Hwang and Bien prove a sufficient condition for problems involving a differential affine time delay system with the same time delay for the state and the control. The differential systems considered in the present chapter are more general. In 1996, Lee and Yung, considering functions that do not have to be convex, derived various first and second-order sufficient conditions for non-linear optimal control problems with only a constant delay in the state (see [101]). Their class of problems is obviously different from our. In particular, we consider delays for both state and control variables. As in [29, 104], second-order sufficient conditions are shown to be related to the existence of solutions of a Riccati-type matrix differential inequality.

Optimal control problems with multiple delays have also been investigated. In [60], Halanay derive necessary conditions for some optimal control problems with various time lags in state and control variables, using the abstract multiplier rule of Hestenes (see [67]). In [60], all delays related to state are equal to each other and the same happens with the delays associated to the control. Note that the results of [52, 62] are obtained as particular cases of problems considered in [60]. Later, in 1973, a necessary condition is derived for an optimal control problem that involves multiple constant lags only in the control. This delayed dependence occurs both in the cost functional and in the differential system, which is defined by a non-linear function (see [165]). In [64], Haratišvili and Tadumadze prove the existence of an optimal solution and a necessary condition for optimal control systems with multiple variable time lags in the state and multiple variable commensurable time delays in the control. Later, an optimal control problem where

the state variable is solution of an integral equation with multiple delays, both for state and control variables, is studied by Bakke in [6]. Furthermore, necessary conditions and Hamilton–Jacobi equations are derived. In 2006, Basin and Rodriguez-Gonzalez proved a necessary and a sufficient optimality condition for a problem that consists to minimize a quadratic cost functional subject to a linear system with multiple time delays in the control variable (see [9]). In their work, they begin by deriving a necessary condition through Pontryagin's Maximum Principle. Afterwards, sufficiency is proved by verifying if the candidate found, through the Maximum Principle, satisfies the Hamilton–Jacobi–Bellman equation. Although Basin and Rodriguez-Gonzalez consider multiple time delays, the dependence of the state and control in the differential system is linear. In this chapter, the dependence of the control, in the differential systems, is in general non-linear. In 2013, Boccia et al. derived necessary conditions for a free end-time optimal control problem subject to a non-linear differential system with multiple delays in the state (see [14]). The control variable is not influenced by time lags in [14]. Recently, in 2017, Boccia and Vinter obtained necessary conditions for a fixed end-time problem with a constant and unique delay for all variables, as well as free end-time problems without control delays (see [15]).

As Guinn wrote in [59], the classical methods of obtaining necessary conditions for retarded optimal control problems (used, for instance, by Halanay in [60], Haratišvili in [63] and Oğuztöreli in [135]) require complicated and extensive proofs (see e.g. [7, 52, 60, 63, 135]). In 1976, Guinn proposed a method whereby we can reduce some specific time-lag optimal control problems to equivalent and augmented optimal control problems without delays (see [59]). By reducing delayed optimal control problems into non-delayed ones, we can then use well-known theorems applicable for optimal control problems without delays to derive desired optimality conditions for delayed problems (see [59]). In [59], Guinn study specific optimal control problems with a constant delay in state and control variables. These two delays are equal. Later, in 2009, Göllmann et al. studied optimal control problems with a constant delay in state and control variables subject to mixed control-state inequality constraints (see [56]). In that research, the delays do not have to be equal. For technical reasons, the authors need to assume that the ratio between these two time delays is a rational number (see [56]). In [56], the method used by Guinn in [59] is generalized and, consequently, a non-delayed optimal control problem is obtained again. Pontryagin's Minimum Principle, for non-delayed control problems with mixed state-control constraints, is used and first-order necessary optimality conditions are derived for retarded problems (see [56]), as we have already seen in Section 2.3. Furthermore, Göllmann et al. discuss the Euler discretization for the retarded problem and some analytical examples versus correspondent numerical solutions are given. Later, in 2014, Göllmann and Maurer generalized the research mentioned before, by studying optimal control problems with multiple

and constant time delays in state and control, involving mixed state-control inequality constraints (see [57]). Again, necessary optimality conditions are derived (see [57]). Note that the works [56, 57, 59, 60] consider delayed non-linear differential systems.

In Section 3.2 we consider optimal control problems that consist to minimize a delayed non-linear cost functional subject to a delayed differential system that is linear with respect to state, but not with respect to control. Note that the cost functional does not have to be quadratic, but it satisfies some continuity and convexity assumptions. In Section 3.3, we consider optimal control problems that consist to minimize a delayed non-linear cost functional subject to a delayed non-linear differential system. In both Sections 3.2 and 3.3, the delay in the state is the same for the cost functional and for the differential system. The same happens with the time lag of the control variable. To the best of our knowledge, we derive sufficient optimality conditions for this two type of optimal control problems, giving answer to an open question. In order to prove our sufficient optimality conditions, we use the technique proposed by Guinn in [59] and used by Göllmann et al. in [56, 57]. As we have already mentioned before, the technique consists to transform a delayed optimal control problem into an equivalent non-delayed optimal control problem. After doing such transformation, one can apply well-known results for non-delayed optimal control problems and then return to the initial delayed problem. Analogously to Göllmann et al. in [56], we ensure the Commensurability Assumption 2.15 between the, possibly different, delays of state and control variables (see Section 2.3.2). We restrict ourselves to delayed problems with deterministic controls. For the stochastic case, we refer the reader to [49, 53, 55, 81, 98].

This chapter is organised as follows. In Section 3.2.1, we define a state-linear optimal control problem with constant time delays in state and control variables. Then, in Section 3.2.2, we prove a sufficient optimality condition associated with the problem stated in Section 3.2.1. A concrete example is solved in detail in Section 3.2.3, with the purpose to illustrate Theorem 3.3 of Section 3.2.2. In Section 3.3.1, we prove a sufficient optimality condition associated with a non-linear optimal control problem with time lags both in state and control variables (see Definition 2.13). An example that illustrates the obtained theoretical result – Theorem 3.7 of Section 3.3.1 – is given. We end this chapter with some conclusions, in Section 3.4.

3.2 Delayed state-linear optimal control problem

In this section we are interested in state-linear optimal control problems with discrete time delays in state and control variables. Furthermore, we derive a sufficient optimality condition for this type of problems. We finish this section by giving an illustrative example for the proved theoretical results.

All the contents of the current section are published in [108].

3.2.1 Statement of the optimal control problem

Definition 3.1. *Consider that $r \geq 0$ and $s \geq 0$ are the discrete and constant time delays associated with the state and control variables, respectively. We assume that $(r, s) \neq (0, 0)$. A non-autonomous state-linear optimal control problem* ($\mathrm{OCP_{LD}}$) *with time delays and with a fixed initial state, on a fixed finite time interval $[a, b]$, consists in*

$$\min \quad C_{LD}\big(x(\cdot), u(\cdot)\big) = \int_a^b f_x^0\big(t, x(t), x(t-r)\big) + f_u^0\big(t, u(t), u(t-s)\big)dt$$

subject to the delayed differential system

$$\dot{x}(t) = A(t)x(t) + A_D(t)x(t-r) + g\big(t, u(t)\big) + g_D\big(t, u(t-s)\big) \tag{3.1}$$

with the following initial conditions

$$\begin{aligned} x(t) &= \varphi(t), \ t \in [a-r, a], \\ u(t) &= \psi(t), \ t \in [a-s, a[, \end{aligned} \tag{3.2}$$

where

i) the state trajectory is $x(t) \in \mathbb{R}^n$ for each $t \in [a-r, b]$;

ii) the control is $u(t) \in U \subseteq \mathbb{R}^m$ for each $t \in [a-s, b]$;

iii) $A(t)$ and $A_D(t)$ are real $n \times n$ matrices for each $t \in [a, b]$.

Next we define admissible pair for ($\mathrm{OCP_{LD}}$).

Definition 3.2. *We say that $\big(x(\cdot), u(\cdot)\big)$ is an* admissible pair *for* ($\mathrm{OCP_{LD}}$) *if it respects the following conditions:*

i) $\big(x(\cdot), u(\cdot)\big) \in W^{1,\infty}([a-r, b], \mathbb{R}^n) \times L^\infty([a-s, b], \mathbb{R}^m)$;

ii) $\big(x(\cdot), u(\cdot)\big)$ satisfies the conditions (3.1) *and* (3.2)*;*

iii) $\big(x(t), u(t)\big) \in \mathbb{R}^n \times U$ for all $t \in [a, b]$.

3.2.2 Main result

As in Section 2.3.2, in what follows we consider that the time delays r and s respect the Commensurability Assumption 2.15. Moreover, we are going to continue using Notation 2.16.

The following theorem supplies a sufficient optimality condition associated with ($\mathrm{OCP_{LD}}$). Such result generalizes Theorem 2.10 for the delayed state-linear optimal control problem ($\mathrm{OCP_{LD}}$).

Theorem 3.3. *Consider* ($\mathrm{OCP_{LD}}$) *and assume that*

1. *functions* f_x^0, $\partial_2 f_x^0$, $\partial_3 f_x^0$, f_u^0, g, g_D, A *and* A_D *are continuous for all their arguments;*

2. $f_x^0(t,x,x_r)$ *is a convex function in* $(x,x_r)\in\mathbb{R}^{2n}$ *for each* $t\in[a,b]$*;*

3. *for almost all* $t\in[a,b]$, u^* *is a control with response* x^* *that satisfies the* maximality condition

$$\begin{aligned}&\max_{u\in U}\big\{H_D^1\big(t,x^*(t),x^*(t_r),u,u^*(t_s),\eta(t)\big)\\&\quad+H_D^0\big(t^s,x^*(t^s),x^*(t_r^s),u^*(t^s),u,\eta(t^s)\big)\chi_{[a,b-s]}(t)\big\}\\&=H_D^1\big(t,x^*(t),x^*(t_r),u^*(t),u^*(t_s),\eta(t)\big)\\&\quad+H_D^0\big(t^s,x^*(t^s),x^*(t_r^s),u^*(t^s),u^*(t),\eta(t^s)\big)\chi_{[a,b-s]}(t),\end{aligned}\tag{3.3}$$

where

$$\begin{aligned}H_D^p(t,x,y,u,v,\eta)=&-\Big[f_x^0(t,x,y)+f_u^0(t,u,v)\Big]\\&+\eta\Big[A(t)x+A_D(t)y+pg(t,u)+(1-p)g_D(t,v)\Big]\end{aligned}$$

for $p\in\{0,1\}$ *and* $\eta(t)$ *is any non-trivial solution of the* adjoint system

$$\begin{aligned}\dot\eta(t)=&\,\partial_2 f_x^0\big(t,x^*(t),x^*(t_r)\big)+\partial_3 f_x^0\big(t^r,x^*(t^r),x^*(t)\big)\chi_{[a,b-r]}(t)\\&-\eta(t)A(t)-\eta(t^r)A_D(t^r)\chi_{[a,b-r]}(t)\end{aligned}$$

that satisfies the transversality condition $\eta(b)=[0\ \ \cdots\ \ 0]_{1\times n}$.

Then, $\big(x^*(\cdot),u^*(\cdot)\big)$ *is an optimal solution of* ($\mathrm{OCP_{LD}}$) *that leads to the minimal cost* $C_{LD}\big(x^*(\cdot),u^*(\cdot)\big)$.

Proof. We are going to transform the delayed state-linear optimal control problem ($\mathrm{OCP_{LD}}$) into an equivalent non-delayed state-linear optimal control ($\mathrm{OCP_L}$) type problem, using the approach of [56, 59]. Then, we apply Theorem 2.10. Without loss of generality, we assume the first case of Commensurability Assumption 2.15, that is, $\dfrac{r}{s}\in\mathbb{Q}$ for $r>0$ and $s>0$. Consequently, there exist $k,l\in\mathbb{N}$ such that

$$\frac{r}{s}=\frac{k}{l}\Leftrightarrow rl=sk\Leftrightarrow\frac{r}{k}=\frac{s}{l}.$$

Thus, let us divide the interval $[a,b]$ into N subintervals of amplitude

$$h:=\frac{r}{k}=\frac{s}{l}.$$

We can note that

$$r=hk\ \ \text{and}\ \ s=hl.$$

Furthermore, we also assume that

$$a + hN = b \quad \text{and} \quad N > 2k + 1, \tag{3.4}$$

with $N \in \mathbb{N}$.

Remark 3.4. *If $b-a$ is not a multiple of h ($b-a \neq hN$), then we can study* $(\mathrm{OCP_{LD}})$ *for $t \in [a, \tilde{b}]$, where $\tilde{b}$ is the smallest multiple of h, which is greater than b. Thus, we also study* $(\mathrm{OCP_{LD}})$ *for $t \in [a, b]$, because $b < \tilde{b}$.*

For $i = 0, \ldots, N-1$ and for $t \in [a, a+h]$, we define new variables

$$\xi_i(t) = x(t + hi) \quad \text{and} \quad \theta_i(t) = u(t + hi).$$

In Figure 3.1, we can observe a simple scheme for the new state variables.

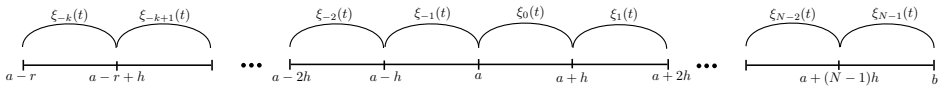


Figure 3.1: Scheme of the new state variables.

The idea is similar for the new control variables. We transform the delayed state-linear problem $(\mathrm{OCP_{LD}})$ into an equivalent non-delayed state-linear problem $(\overline{\mathrm{OCP_L}})$, which consists to minimize the cost functional given by

$$\begin{aligned} &\bar{C}_L\big(\xi(\cdot), \theta(\cdot)\big) \\ &= \int_a^{a+h} \sum_{i=0}^{N-1} \Big[f_x^0\big(t + hi, \xi_i(t), \xi_{i-k}(t)\big) + f_u^0\big(t + hi, \theta_i(t), \theta_{i-l}(t)\big) \Big] dt \end{aligned} \tag{3.5}$$

subject to the non-delayed differential system

$$\dot{\xi}_i(t) = A(t+hi)\xi_i(t) + A_D(t+hi)\xi_{i-k}(t) + g\big(t+hi, \theta_i(t)\big) + g_D\big(t+hi, \theta_{i-l}(t)\big)$$

for $i = 0, \ldots, N-1$ and $t \in [a, a+h]$, and to the initial conditions

$$\begin{aligned} &\xi_i(t) = \varphi(t + hi), \quad i = -k, \ldots, -1 \quad \text{and} \quad t \in [a, a+h]; \\ &\theta_i(t) = \psi(t + hi), \quad i = -l, \ldots, -1 \quad \text{and} \quad t \in [a, a+h[; \\ &\xi_i(a+h) = \xi_{i+1}(a), \quad i = 0, \ldots, N-2. \end{aligned}$$

Consider that

$$\xi = \begin{bmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_{N-1} \end{bmatrix}, \; \xi^- = \begin{bmatrix} \xi_{-k} \\ \xi_{1-k} \\ \vdots \\ \xi_{-1} \end{bmatrix}, \; \theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_{N-1} \end{bmatrix} \text{ and } \theta^- = \begin{bmatrix} \theta_{-l} \\ \theta_{1-l} \\ \vdots \\ \theta_{-1} \end{bmatrix}.$$

Observe that the dimensions of ξ, ξ^-, θ and θ^- are $Nn\times 1$, $kn\times 1$, $Nm\times 1$ and $lm\times 1$, respectively. Note also that ξ and θ represent optimization variables and ξ^- and θ^- not. We know, a priori, the expressions of $\xi^-(t)$, $t\in[a,a+h]$, and $\theta^-(t)$, $t\in[a,a+h[$. Let us write the objective function expressed in (3.5) as a function of the type presented in (2.8):

$$\begin{aligned}
&\sum_{i=0}^{N-1} f_x^0\big(t+hi,\xi_i(t),\xi_{i-k}(t)\big)\\
&= f_x^0\big(t,\xi_0(t),\xi_{-k}(t)\big)+f_x^0\big(t+h,\xi_1(t),\xi_{1-k}(t)\big)\\
&\quad+\cdots+f_x^0\big(t+h(k-1),\xi_{k-1}(t),\xi_{-1}(t)\big)+f_x^0\big(t+hk,\xi_k(t),\xi_0(t)\big)\\
&\quad+\cdots+f_x^0\big(t+h(N-1),\xi_{N-1}(t),\xi_{N-1-k}(t)\big).
\end{aligned}$$

We can simply write $\sum_{i=0}^{N-1} f_x^0\big(t+hi,\xi_i(t),\xi_{i-k}(t)\big)=F_x^0(t,\xi(t))$, because ξ_i and h are known for $i=-k,\ldots,-1$. In a similar way, we can also write $\sum_{i=0}^{N-1} f_u^0\big(t+hi,\theta_i(t),\theta_{i-l}(t)\big)=F_u^0(t,\theta(t))$. Consequently, we have that

$$\begin{aligned}
&\int_a^{a+h}\sum_{i=0}^{N-1}\Big[f_x^0\big(t+hi,\xi_i(t),\xi_{i-k}(t)\big)+f_u^0\big(t+hi,\theta_i(t),\theta_{i-l}(t)\big)\Big]dt\\
&=\int_a^{a+h}\Big[F_x^0\big(t,\xi(t)\big)+F_u^0\big(t,\theta(t)\big)\Big]dt.
\end{aligned}$$

In order to apply Theorem 2.10, we have to write the set of constraints

$$\begin{aligned}
\dot{\xi}_i(t) &= A(t+hi)\xi_i(t)+A_D(t+hi)\xi_{i-k}(t)\\
&\quad+g\big(t+hi,\theta_i(t)\big)+g_D\big(t+hi,\theta_{i-l}(t)\big),\quad i=0,\ldots,N-1,
\end{aligned}\tag{3.6}$$

in the form

$$\dot{\xi}(t)=\tilde{A}(t)\xi(t)+\tilde{G}(t,\theta(t)).\tag{3.7}$$

For $i=0,\ldots,N-1$, consider that $t_i=t+hi$. Thus, we have

$$\begin{aligned}
\begin{bmatrix} A(t_0)\xi_0(t)\\ A(t_1)\xi_1(t)\\ \vdots\\ A(t_{N-1})\xi_{N-1}(t)\end{bmatrix}_{Nn\times 1} &= \begin{bmatrix} A(t_0) & \mathbf{0} & \cdots & \cdots & \mathbf{0}\\ \mathbf{0} & A(t_1) & \mathbf{0} & \cdots & \mathbf{0}\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \mathbf{0}\\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & A(t_{N-1})\end{bmatrix}\times\begin{bmatrix}\xi_0(t)\\ \xi_1(t)\\ \vdots\\ \xi_{N-1}(t)\end{bmatrix}\\
&= M(t)\xi(t)
\end{aligned}$$

and

$$
\begin{aligned}
&\begin{bmatrix} A_D(t_0)\xi_{-k}(t) \\ A_D(t_1)\xi_{1-k}(t) \\ \vdots \\ A_D(t_k)\xi_0(t) \\ \vdots \\ A_D(t_{N-1})\xi_{N-1-k}(t) \end{bmatrix}_{Nn\times 1} \\
= &\begin{bmatrix} A_D(t_0) & \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & A_D(t_1) & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & A_D(t_k) & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \mathbf{0} & A_D(t_{N-1}) \end{bmatrix} \times \begin{bmatrix} \xi_{-k}(t) \\ \xi_{1-k}(t) \\ \vdots \\ \xi_0(t) \\ \vdots \\ \xi_{N-1-k}(t) \end{bmatrix} \\
= &\begin{bmatrix} & & & \mathbf{0}_{kn\times Nn} & & & & \\ A_D(t_k) & \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & A_D(t_{k+1}) & \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & A_D(t_{N-1}) & \mathbf{0} & \cdots & \cdots & \mathbf{0} \end{bmatrix} \times \begin{bmatrix} \xi_0(t) \\ \vdots \\ \xi_{N-1-k}(t) \\ \vdots \\ \xi_{N-1}(t) \end{bmatrix} \\
&+ \begin{bmatrix} A_D(t_0) & \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & A_D(t_1) & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & A_D(t_{k-1}) & \mathbf{0} & \cdots & \mathbf{0} \\ & & & \mathbf{0}_{(N-k)n\times Nn} & & & \end{bmatrix} \times \begin{bmatrix} \xi_{-k}(t) \\ \vdots \\ \xi_{-1}(t) \\ \mathbf{0}_{(N-k)n\times 1} \end{bmatrix} \\
= &\ M_D(t)\xi(t) + M_D^-(t) \begin{bmatrix} \xi^-(t) \\ \mathbf{0}_{(N-k)n\times 1} \end{bmatrix}.
\end{aligned}
$$

Note that $M(t)$, $M_D(t)$ and $M_D^-(t)$ have dimension $Nn \times Nn$. Concluding, we have

$$\tilde{A}(t) = M(t) + M_D(t).$$

Now, we write the sum of the third and fourth terms of (3.6) as a function

of t and $\theta(t)$. Thus,

$$\begin{aligned}
&\begin{bmatrix}
g\big(t_0,\theta_0(t)\big)+g_D\big(t_0,\theta_{-l}(t)\big)\\
g\big(t_1,\theta_1(t)\big)+g_D\big(t_1,\theta_{1-l}(t)\big)\\
\vdots\\
g\big(t_{l-1},\theta_{l-1}(t)\big)+g_D\big(t_{l-1},\theta_{-1}(t)\big)\\
g\big(t_l,\theta_l(t)\big)+g_D\big(t_l,\theta_0(t)\big)\\
\vdots\\
g\big(t_{N-1},\theta_{N-1}(t)\big)+g_D\big(t_{N-1},\theta_{N-1-l}(t)\big)
\end{bmatrix}\\
&=\begin{bmatrix}
g\big(t_0,\theta_0(t)\big)\\
g\big(t_1,\theta_1(t)\big)\\
\vdots\\
g\big(t_{l-1},\theta_{l-1}(t)\big)\\
g\big(t_l,\theta_l(t)\big)+g_D\big(t_l,\theta_0(t)\big)\\
\vdots\\
g\big(t_{N-1},\theta_{N-1}(t)\big)+g_D\big(t_{N-1},\theta_{N-1-l}(t)\big)
\end{bmatrix}+\begin{bmatrix}
g_D\big(t_0,\theta_{-l}(t)\big)\\
g_D\big(t_1,\theta_{1-l}(t)\big)\\
\vdots\\
g_D\big(t_{l-1},\theta_{-1}(t)\big)\\
0\\
\vdots\\
0
\end{bmatrix}\\
&=g_\theta\big(t,\theta(t)\big)+g_{\theta^-}\big(t,\theta^-(t)\big).
\end{aligned}$$

As $\xi^-(t)$ and $\theta^-(t)$ are known, we have that

$$\tilde{G}\big(t,\theta(t)\big)=M_D^-(t)\begin{bmatrix}\xi^-(t)\\ \mathbf{0}_{(N-k)n\times 1}\end{bmatrix}+g_\theta\big(t,\theta(t)\big)+g_{\theta^-}\big(t,\theta^-(t)\big).$$

Therefore, we have the set of constraints (3.6) in form (3.7). To apply Theorem 2.10, we have to ensure that

1. for all $(t,\xi,\theta)\in[a,a+h]\times\mathbb{R}^{Nn+Nm}$, F_x^0, $\partial_2F_x^0$, F_u^0, $\tilde{A}$ and $\tilde{G}$ are continuous;
2. $F_x^0(t,\xi)$ is a convex function in ξ for each fixed $t\in[a,a+h]$;
3. θ^* is a control with response ξ^* that satisfies the *maximality condition*

$$-F_u^0\big(t,\theta^*(t)\big)+\Lambda(t)\tilde{G}\big(t,\theta^*(t)\big)=\max_{\theta\in\tilde{U}}\big[-F_u^0(t,\theta)+\Lambda(t)\tilde{G}(t,\theta)\big]$$

for almost all $t\in[a,a+h]$. Note that $\tilde{U}\subseteq\mathbb{R}^{Nm}$ and $\Lambda(t)$ is any non-trivial solution of the *adjoint system*

$$\dot{\Lambda}(t)=\partial_2F_x^0\big(t,\xi^*(t)\big)-\Lambda(t)\tilde{A}(t)$$

such that $\Lambda^i(a+h)$ is an inward normal vector of the closed convex set

$$\tilde{\Pi}_i=\begin{cases}\{\xi_i^*(a+h)\}, & \text{if } i=0,\dots,N-2\\ \mathbb{R}^n, & \text{if } i=N-1\end{cases}$$

at the boundary point $\xi_i^*(a+h)$ for $i=0,\dots,N-1$.

Thus, $\big(\xi^*(\cdot),\theta^*(\cdot)\big)$ will be an optimal solution of $(\overline{\mathrm{OCP_L}})$ that leads to the minimal cost $\bar{C}_L\big(\xi^*(\cdot),\theta^*(\cdot)\big)$. From now on, we are going to analyse each hypothesis of Theorem 3.3.

1. (a) We have that

$$\begin{aligned} F_x^0\big(t,\xi(t)\big) &= \sum_{i=0}^{N-1} f_x^0\big(t+hi,\xi_i(t),\xi_{i-k}(t)\big) \\ &= \sum_{i=0}^{N-1} f_x^0\big(t+hi,x(t+hi),x(t+h(i-k))\big) \\ &= \sum_{i=0}^{N-1} f_x^0\big(t+hi,x(t+hi),x(t+hi-hk)\big) \\ &= \sum_{i=0}^{N-1} f_x^0\big(t+hi,x(t+hi),x(t+hi-r)\big). \end{aligned}$$

   By hypothesis, function f_x^0 is continuous with respect to all its arguments. Then, F_x^0 is continuous for all $(t,\xi)\in[a,a+h]\times\mathbb{R}^{Nn}$.

   (b) Having in mind that $N>2k+1$ (see (3.4)), that is, $k<N-1-k$, then

$$\begin{aligned} F_x^0\big(t,\xi(t)\big) =& f_x^0\big(t_0,\xi_0(t),\xi_{-k}(t)\big)+f_x^0\big(t_1,\xi_1(t),\xi_{1-k}(t)\big) \\ &+\cdots+f_x^0\big(t_{k-1},\xi_{k-1}(t),\xi_{-1}(t)\big) \\ &+f_x^0\big(t_k,\xi_k(t),\xi_0(t)\big)+f_x^0\big(t_{k+1},\xi_{k+1}(t),\xi_1(t)\big) \\ &+\cdots+f_x^0\big(t_{N-1-k},\xi_{N-1-k}(t),\xi_{N-1-2k}(t)\big) \\ &+\cdots+f_x^0\big(t_{N-1},\xi_{N-1}(t),\xi_{N-1-k}(t)\big). \end{aligned}$$

   So, for $i=0,\dots,N-1-k$, we obtain

$$\begin{aligned} \frac{\partial F_x^0}{\partial\xi_i}\big(t,\xi(t)\big) &= \partial_2 f_x^0\big(t+hi,\xi_i(t),\xi_{i-k}(t)\big) \\ &\quad+\partial_3 f_x^0\big(t+h(k+i),\xi_{k+i}(t),\xi_i(t)\big) \\ &= \partial_2 f_x^0\big(t+hi,x(t+hi),x(t+h(i-k))\big) \\ &\quad+\partial_3 f_x^0\big(t+h(k+i),x(t+h(k+i)),x(t+hi)\big) \\ &= \partial_2 f_x^0\big(t+hi,x(t+hi),x(t+hi-r)\big) \\ &\quad+\partial_3 f_x^0\big(t+hi+r,x(t+hi+r),x(t+hi)\big). \end{aligned}$$

   For $i=0,\dots,N-1-k$ and $t\in[a,a+h]$, we conclude that

$$a\le t+hi\le a+h+h(N-1-k)=b-r.$$

For $i = N-k, \ldots, N-1$ we have

$$\begin{aligned}\frac{\partial F_x^0}{\partial \xi_i}\big(t, \xi(t)\big) &= \partial_2 f_x^0\big(t+hi, \xi_i(t), \xi_{i-k}(t)\big) \\ &= \partial_2 f_x^0\big(t+hi, x(t+hi), x(t+hi-r)\big).\end{aligned}$$

As $i \in \{N-k, \ldots, N-1\}$ and $t \in [a, a+h]$, we obtain

$$a + h(N-k) \leq t + hi \leq a + h + h(N-1) \Leftrightarrow b - r \leq t + hi \leq b.$$

For each $t \in [a, b]$, there exists $j \in \{0, \ldots, N-1\}$ such that

$$a + hj \leq t \leq a + h(j+1) \Leftrightarrow a \leq t - hj \leq a + h.$$

Thus, let us define $t' \in [a, a+h]$ as being $t' = t - hj$. Consequently,

$$\begin{aligned}&\frac{\partial F_x^0}{\partial \xi_j}\big(t', \xi(t')\big) \\ &= \partial_2 f_x^0\big(t'+hj, x(t'+hj), x(t'+hj-r)\big) \\ &\quad + \partial_3 f_x^0\big(t'+hj+r, x(t'+hj+r), x(t'+hj)\big)\chi(j)_{\{0,\ldots,N-1-k\}} \\ &= \partial_2 f_x^0\big(t, x(t), x(t-r)\big) + \partial_3 f_x^0\big(t+r, x(t+r), x(t)\big)\chi(t)_{[a,b-r]}.\end{aligned}$$

Since $\partial_2 f_x^0$ is continuous for all $(t, x, x_r) \in [a, b] \times \mathbb{R}^{2n}$ and function $\partial_3 f_x^0$ is continuous for all $(t, x, x_r) \in [a+r, b] \times \mathbb{R}^{2n}$, then $\dfrac{\partial F_x^0}{\partial \xi}$ is continuous for all $(t, \xi) \in [a, a+h] \times \mathbb{R}^{Nn}$.

(c) We have that

$$\begin{aligned}F_u^0\big(t, \theta(t)\big) &= \sum_{i=0}^{N-1} f_u^0\big(t+hi, \theta_i(t), \theta_{i-l}(t)\big) \\ &= \sum_{i=0}^{N-1} f_u^0\big(t+hi, u(t+hi), u(t+h(i-l))\big) \\ &= \sum_{i=0}^{N-1} f_u^0\big(t+hi, u(t+hi), u(t+hi-hl)\big) \\ &= \sum_{i=0}^{N-1} f_u^0\big(t+hi, u(t+hi), u(t+hi-s)\big).\end{aligned}$$

Function f_u^0 is continuous for all $(t, u, u_s) \in [a, b] \times \mathbb{R}^{2m}$, by hypothesis. Then, F_u^0 is continuous for all $(t, \theta) \in [a, a+h] \times \mathbb{R}^{Nm}$.

(d) We know that $\tilde{A}(t) = M(t) + M_D(t)$. As A and A_D are continuous for all $t \in [a, b]$ and $M(t)$ and $M_D(t)$ are depending on $A(t)$ for $t \in [a, b]$ and on $A_D(t)$ for $t \in [a+r, b]$, then $\tilde{A}$ is continuous for all $t \in [a, a+h]$.

(e) Let us define function $u_s(t)$ by

$$u_s(t) = u(t-s)$$

for all $t \in [a,b]$. We have already defined

$$\tilde{G}\big(t,\theta(t)\big) = M_D^-(t)\begin{bmatrix}\xi^-(t)\\ \mathbf{0}\end{bmatrix} + g_\theta\big(t,\theta(t)\big) + g_{\theta^-}\big(t,\theta^-(t)\big).$$

For $t \in [a, a+r]$, the matrix $M_D^-(t)$ is depending on the matrix $A_D(t)$. As $A_D(t)$ is continuous in the interval $[a,b]$, then

$$M_D^-(t)\begin{bmatrix}\xi^-(t)\\ \mathbf{0}\end{bmatrix}$$

is continuous for all $t \in [a, a+h]$. If, for each $i = 0, \ldots, N-1$, the functions $g\big(t+hi, \theta_i(t)\big)$ and $g_D\big(t+hi, \theta_{i-l}(t)\big)$ are continuous for all $\big(t,\theta_i(t)\big)$, $\big(t,\theta_{i-l}(t)\big) \in [a, a+h] \times \mathbb{R}^m$, respectively; then the function $g_\theta\big(t,\theta(t)\big) + g_{\theta^-}\big(t,\theta^-(t)\big)$ is continuous. We know that $g\big(t+hi,\theta_i(t)\big) = g\big(t+hi, u(t+hi)\big)$ and

$$\begin{aligned} g_D\big(t+hi,\theta_{i-l}(t)\big) &= g_D\big(t+hi, u(t+h(i-l))\big) \\ &= g_D\big(t+hi, u(t+hi-s)\big), \end{aligned}$$

$i = 0, \ldots, N-1$. Moreover, as $g\big(t,u(t)\big)$ and $g_D\big(t,u_s(t)\big)$ are continuous for all $(t,u,u_s) \in [a,b] \times \mathbb{R}^{2m}$, $\tilde{G}$ is continuous for all $(t,\theta) \in [a, a+h] \times \mathbb{R}^{Nm}$.

2. As we know,

$$F_x^0\big(t,\xi(t)\big) = \sum_{i=0}^{N-1} f_x^0\big(t+hi, x(t+hi), x(t+hi-r)\big)$$

for $t \in [a, a+h]$ and f_x^0 is convex in $(x, x_r) \in \mathbb{R}^{2n}$ for each $t \in [a,b]$. Then, F_x^0 is a convex function in ξ for each fixed $t \in [a, a+h]$.

3. If θ^* is a control with response ξ^* that satisfies the *maximality condition*

$$-F_u^0\big(t,\theta^*(t)\big) + \Lambda(t)\tilde{G}\big(t,\theta^*(t)\big) = \max_{\theta \in \tilde{U}} \big[-F_u^0(t,\theta) + \Lambda(t)\tilde{G}(t,\theta)\big]$$

for almost all $t \in [a, a+h]$, then

$$-F_u^0\big(t,\theta^*(t)\big) + \Lambda(t)\tilde{G}\big(t,\theta^*(t)\big) \geq -F_u^0(t,\theta) + \Lambda(t)\tilde{G}(t,\theta) \qquad (3.8)$$

for almost all $t \in [a, a+h]$ and for all admissible $\theta \in \tilde{U}$. If we consider that $\eta(t) = \Lambda^j(t-hj)$, then we have that

$$\Lambda^j(t) = \Lambda^j(t+hj-hj) = \eta(t+hj) \Rightarrow \Lambda^j(t') = \eta(t'+hj) = \eta(t)$$

and

$$\begin{aligned}\Lambda^{j+l}(t) &= \Lambda^{j+l}\big(t + h(j+l) - h(j+l)\big)\\ &= \Lambda^{j+l}\big(t + hj + s - h(j+l)\big)\\ &= \eta(t + hj + s),\end{aligned}$$

which implies $\Lambda^{j+l}(t') = \eta(t' + hj + s) = \eta(t+s)$. As inequality (3.8) is verified for all admissible $\theta \in \tilde{U}$, we can choose an admissible variable $\bar{\theta} \in \tilde{U}$ such that

$$\bar{\theta}_i = \begin{cases} u^*(t' + hi), & i \neq j \\ u, & i = j \end{cases}, \quad i = 0, \dots, N-1,$$

where u is an admissible control for ($\mathrm{OCP_{LD}}$). So, using inequality (3.8) and considering $t'_i = t' + hi$, we have that

$$-F_u^0\big(t', \theta^*(t')\big) + \Lambda(t')\tilde{G}\big(t', \theta^*(t')\big) \geq -F_u^0(t', \bar{\theta}) + \Lambda(t')\tilde{G}(t', \bar{\theta}).$$

The previous inequality is equivalent to

$$\begin{aligned}&\sum_{i=0}^{N-1} \Big\{ - f_u^0\big(t'_i, \theta_i^*(t'), \theta_{i-l}^*(t')\big) + \Lambda^i(t')\big[g\big(t'_i, \theta_i^*(t')\big) + g_D\big(t'_i, \theta_{i-l}^*(t')\big)\big]\Big\}\\ &+ \sum_{i=0}^{k-1} \Lambda^i(t') A_D(t'_i)\xi_{i-k}(t')\\ &\geq \sum_{i=0}^{N-1} \big\{ - f_u^0(t'_i, \bar{\theta}_i, \bar{\theta}_{i-l}) + \Lambda^i(t')[g(t'_i, \bar{\theta}_i) + g_D(t'_i, \bar{\theta}_{i-l})]\big\}\\ &+ \sum_{i=0}^{k-1} \Lambda^i(t') A_D(t'_i)\xi_{i-k}(t').\end{aligned}$$

As the last sums of both sides of previous inequality are equal, we obtain

$$\begin{aligned}&\sum_{i=0}^{N-1} \Big\{ - f_u^0\big(t'_i, \theta_i^*(t'), \theta_{i-l}^*(t')\big)\\ &\qquad + \Lambda^i(t')\big[g\big(t'_i, \theta_i^*(t')\big) + g_D\big(t'_i, \theta_{i-l}^*(t')\big)\big]\Big\} \qquad (3.9)\\ &\geq \sum_{i=0}^{N-1} \big\{ - f_u^0(t'_i, \bar{\theta}_i, \bar{\theta}_{i-l}) + \Lambda^i(t')[g(t'_i, \bar{\theta}_i) + g_D(t'_i, \bar{\theta}_{i-l})]\big\}.\end{aligned}$$

Due to the choice of $\bar{\theta}_i$, $i = 0, \dots, N-1$, some terms of the left-hand side of inequality (3.9) cancel with other terms of the right-hand

side. Let us analyse the sums when we only consider the indexes of set $I = \{0, \dots, N-1\} \backslash \{j, j+l\}$. For the first member, we have

$$\begin{aligned}
&\sum_{i \in I} \Big\{ - f_u^0\big(t_i', \theta_i^*(t'), \theta_{i-l}^*(t')\big) \\
&\qquad + \Lambda^i(t') \big[g\big(t_i', \theta_i^*(t')\big) + g_D\big(t_i', \theta_{i-l}^*(t')\big) \big] \Big\} \\
&= \sum_{i \in I} \Big\{ - f_u^0\big(t_i', u^*(t_i'), u^*(t_i' - s)\big) \\
&\qquad + \Lambda^i(t') \big[g\big(t_i', u^*(t_i')\big) + g_D\big(t_i', u^*(t_i' - s)\big) \big] \Big\}
\end{aligned}$$

while for the second we obtain

$$\begin{aligned}
&\sum_{i \in I} \big\{ - f_u^0(t_i', \bar\theta_i, \bar\theta_{i-l}) + \Lambda^i(t') [g(t_i', \bar\theta_i) + g_D(t_i', \bar\theta_{i-l})] \big\} \\
&= \sum_{i \in I} \Big\{ - f_u^0\big(t_i', u^*(t_i'), u^*(t_i' - s)\big) \\
&\qquad + \Lambda^i(t') \big[g\big(t_i', u^*(t_i')\big) + g_D\big(t_i', u^*(t_i' - s)\big) \big] \Big\}.
\end{aligned}$$

Only the terms associated to the indexes $j, j+l \in \{0, \dots, N-1\}$ are different. Therefore, inequality (3.9) is equivalent to

$$\begin{aligned}
&\sum_{i \in \{j, j+l\}} \Big\{ - f_u^0\big(t_i', \theta_i^*(t'), \theta_{i-l}^*(t')\big) \\
&\qquad + \Lambda^i(t') \big[g\big(t_i', \theta_i^*(t')\big) + g_D\big(t_i', \theta_{i-l}^*(t')\big) \big] \Big\} \\
&\geq \sum_{i \in \{j, j+l\}} \big\{ - f_u^0(t_i', \bar\theta_i, \bar\theta_{i-l}) + \Lambda^i(t') [g(t_i', \bar\theta_i) + g_D(t_i', \bar\theta_{i-l})] \big\}.
\end{aligned}$$

For $i = 0, \dots, N-1$, we know that $\bar\theta_i = u$, if $i = j$. Thus, by the above inequality, it follows that

$$\begin{aligned}
&- f_u^0\big(t' + hj, u^*(t' + hj), u^*(t' + hj - s)\big) \\
&+ \Lambda^j(t') \big[g\big(t' + hj, u^*(t' + hj)\big) + g_D\big(t' + hj, u^*(t' + hj - s)\big) \big] \\
&- f_u^0\big(t' + hj + s, u^*(t' + hj + s), u^*(t' + hj)\big) \chi_{\{0, \dots, N-1-l\}}(j) \\
&+ \Lambda^{j+l}(t') \big[g\big(t' + hj + s, u^*(t' + hj + s)\big) \\
&+ g_D\big(t' + hj + s, u^*(t' + hj)\big) \big] \chi_{\{0, \dots, N-1-l\}}(j) \\
\geq &- f_u^0\big(t' + hj, u, u^*(t' + hj - s)\big) \\
&+ \Lambda^j(t') \big[g(t' + hj, u) + g_D\big(t' + hj, u^*(t' + hj - s)\big) \big] \\
&- f_u^0(t' + hj + s, u^*(t' + hj + s), u) \chi_{\{0, \dots, N-1-l\}}(j) \\
&+ \Lambda^{j+l}(t') \big[g\big(t' + hj + s, u^*(t' + hj + s)\big) \\
&+ g_D(t' + hj + s, u) \big] \chi_{\{0, \dots, N-1-l\}}(j).
\end{aligned}$$

As $t' = t - hj \in [a, a+h]$ and $0 \leq j \leq N-1-l$, then

$$\begin{aligned} 0 \leq hj \leq Nh - h - s \Leftrightarrow\ & a \leq t' + hj \leq a + h + Nh - h - s \\ \Leftrightarrow\ & a \leq t' + hj \leq b - s. \end{aligned}$$

Consequently, we have that

$$\begin{aligned} & - f_u^0\big(t, u^*(t), u^*(t-s)\big) + \Lambda^j(t')\big[g\big(t, u^*(t)\big) + g_D\big(t, u^*(t-s)\big)\big] \\ & - f_u^0\big(t+s, u^*(t+s), u^*(t)\big)\chi_{[a,b-s]}(t) \\ & + \Lambda^{j+l}(t')\big[g\big(t+s, u^*(t+s)\big) + g_D\big(t+s, u^*(t)\big)\big]\chi_{[a,b-s]}(t) \\ \geq & - f_u^0\big(t, u, u^*(t-s)\big) + \Lambda^j(t')\big[g(t,u) + g_D\big(t, u^*(t-s)\big)\big] \\ & - f_u^0(t+s, u^*(t+s), u)\chi_{[a,b-s]}(t) \\ & + \Lambda^{j+l}(t')\big[g\big(t+s, u^*(t+s)\big) + g_D(t+s, u)\big]\chi_{[a,b-s]}(t). \end{aligned}$$

As some terms cancel, we obtain

$$\begin{aligned} & - f_u^0\big(t, u^*(t), u^*(t-s)\big) + \Lambda^j(t')g\big(t, u^*(t)\big) \\ & - f_u^0\big(t+s, u^*(t+s), u^*(t)\big)\chi_{[a,b-s]}(t) \\ & + \Lambda^{j+l}(t')g_D\big(t+s, u^*(t)\big)\chi_{[a,b-s]}(t) \\ \geq & - f_u^0\big(t, u, u^*(t-s)\big) + \Lambda^j(t')g(t,u) \\ & - f_u^0(t+s, u^*(t+s), u)\chi_{[a,b-s]}(t) + \Lambda^{j+l}(t')g_D(t+s,u)\chi_{[a,b-s]}(t). \end{aligned}$$

Using relations $\Lambda^j(t') = \eta(t)$ and $\Lambda^{j+l}(t') = \eta(t+s)$, we have that

$$\begin{aligned} & - f_u^0\big(t, u^*(t), u^*(t-s)\big) + \eta(t)g\big(t, u^*(t)\big) \\ & + \big[- f_u^0\big(t+s, u^*(t+s), u^*(t)\big) + \eta(t+s)g_D\big(t+s, u^*(t)\big)\big]\chi_{[a,b-s]}(t) \\ & \geq -f_u^0\big(t, u, u^*(t-s)\big) + \eta(t)g(t,u) \\ & + \big[- f_u^0(t+s, u^*(t+s), u) + \eta(t+s)g_D(t+s,u)\big]\chi_{[a,b-s]}(t). \end{aligned} \tag{3.10}$$

Attending to the definition of H_D^p, $p \in \{0,1\}$, the inequality (3.10) is equivalent to the *maximality condition* (3.3) of Theorem 3.3. Furthermore, we can not forget that $\Lambda(t)$ is any non-trivial solution of the *adjoint system*

$$\dot{\Lambda}(t) = \partial_2 F_x^0(t, \xi^*(t)) - \Lambda(t)\tilde{A}(t) \tag{3.11}$$

that satisfies the *transversality condition* (see Remark 3.5)

$$\Lambda^{N-1}(a+h) = [0 \quad \cdots \quad 0]_{1\times n}. \tag{3.12}$$

As we know,

$$\tilde{A}(t) = M(t) + M_D(t)$$

and $\Lambda(t) = [\Lambda^0(t) \quad \Lambda^1(t) \quad \cdots \quad \Lambda^{N-1}(t)]$, where $\Lambda^i(t)$ has dimension $1 \times n$ for all $i \in \{0, \ldots, N-1\}$. Consequently, by the adjoint system (3.11), we can write that

$$
\begin{aligned}
&\dot{\Lambda}^i(t)\\
&= \partial_2 f_x^0\big(t + hi, \xi_i^*(t), \xi_{i-k}^*(t)\big)\\
&\quad + \partial_3 f_x^0\big(t + h(i+k), \xi_{k+i}^*(t), \xi_i^*(t)\big)\chi_{\{0,\ldots,N-1-k\}}(i) - \Lambda^i(t)A(t+hi)\\
&\quad - \Lambda^{i+k}(t)A_D\big(t + h(i+k)\big)\chi_{\{0,\ldots,N-1-k\}}(i)\\
&= \partial_2 f_x^0\big(t + hi, x^*(t+hi), x^*(t+hi-hk)\big)\\
&\quad + \partial_3 f_x^0\big(t + hi + hk, x^*(t+hi+hk), x^*(t+hi)\big)\chi_{\{0,\ldots,N-1-k\}}(i)\\
&\quad - \Lambda^i(t)A(t+hi) - \Lambda^{i+k}(t)A_D(t+hi+hk)\chi_{\{0,\ldots,N-1-k\}}(i)\\
&= \partial_2 f_x^0\big(t + hi, x^*(t+hi), x^*(t+hi-r)\big)\\
&\quad + \partial_3 f_x^0\big(t + hi + r, x^*(t+hi+r), x^*(t+hi)\big)\chi_{\{0,\ldots,N-1-k\}}(i)\\
&\quad - \Lambda^i(t)A(t+hi) - \Lambda^{i+k}(t)A_D(t+hi+r)\chi_{\{0,\ldots,N-1-k\}}(i).
\end{aligned}
$$

Furthermore, as $\eta(t) = \Lambda^j(t - hj)$, we conclude that

$$
\begin{aligned}
\dot{\eta}(t) &= \dot{\Lambda}^j(t-hj)\\
&= \partial_2 f_x^0\big(t, x^*(t), x^*(t-r)\big)\\
&\quad + \partial_3 f_x^0\big(t + r, x^*(t+r), x^*(t)\big)\chi_{[a,b-r]}(t)\\
&\quad - \eta(t)A(t) - \eta(t+r)A_D(t+r)\chi_{[a,b-r]}(t).
\end{aligned}
\tag{3.13}
$$

By equation (3.12),

$$
\begin{aligned}
&\Lambda^{N-1}(a+h) = [0 \quad \cdots \quad 0]_{1\times n}\\
\Leftrightarrow\ &\eta(a + h + h(N-1)) = [0 \quad \cdots \quad 0]_{1\times n}\\
\Leftrightarrow\ &\eta(a + hN) = [0 \quad \cdots \quad 0]_{1\times n}.
\end{aligned}
$$

As $a + hN = b$, we obtain the *transversality condition*

$$
\eta(b) = [0 \quad \cdots \quad 0]_{1\times n}. \tag{3.14}
$$

With conditions (3.10), (3.13) and (3.14), we obtain item 3 of Theorem 3.3.

This concludes the proof of Theorem 3.3. □

Remark 3.5. *We can note that: (i) problems* ($\mathrm{OCP_{LD}}$) *and* ($\overline{\mathrm{OCP_L}}$) *are equivalent; (ii) the augmented and non-delayed problem* ($\overline{\mathrm{OCP_L}}$) *is defined for* $t \in [a, a+h]$. *Even more, we can solve* ($\mathrm{OCP_{LD}}$) *by solving* N *subproblems, each one with respect to each subinterval of* $[a, b]$ *with amplitude* h. *Then, we can concatenate the respective* N *optimal solutions in order*

to obtain an optimal solution of $(\mathrm{OCP_{LD}})$. *Thus, we can solve* $(\mathrm{OCP_{LD}})$ *by solving* N *augmented and non-delayed sub-problems* $(\overline{\mathrm{OCP_{L}}i})$ *associated with* $(\mathrm{OCP_{LD}})$, *with* $i = 0, \ldots, N-1$. *For* $i \in \{0, \ldots, N-2\}$, *the* $(i+1)th$ *augmented and non-delayed sub-problem* $(\overline{\mathrm{OCP_{L}}i})$ *consists to minimize*

$$\int_a^{a+h} f_x^0\big(t_i, \xi_i(t), \xi_{i-k}(t)\big) + f_u^0\big(t_i, \theta_i(t), \theta_{i-l}(t)\big)dt$$

subject to

$$\begin{aligned}
\dot{\xi}_i(t) =\ & A(t_i)\xi_i(t) + A_D(t_i)\xi_{i-k}(t) + g\big(t_i, \theta_i(t)\big) + g_D\big(t_i, \theta_{i-l}(t)\big) \\
\xi_i(a) =\ & \begin{cases} \varphi(a), & \text{if } i = 0 \\ \xi_{i-1}(a+h), & \text{if } i = 1, \ldots, N-2 \end{cases} \\
\xi_i(a+h) \in\ & \tilde{\Pi}_i = \big\{\xi_i^*(a+h)\big\}
\end{aligned}$$

for $t \in [a, a+h]$. *Theorem 2.10 can be applied and we can find an optimal pair* $\big(\xi_i^*(\cdot), \theta_i^*(\cdot)\big)$ *in the interval of time* $[a, a+h]$ *that provides an optimal solution* $(x^*(\cdot), u^*(\cdot))$ *in the interval of time* $[a+hi, a+h(i+1)]$. *The set* $\tilde{\Pi}_i$ *has a single point. So,* $\Lambda^i(a+h)$ *is an inward normal vector of* $\tilde{\Pi}_i$ *at the boundary point* $\xi_i^*(a+h)$ *(recall Remark 2.11). The last augmented and non-delayed sub-problem* $(\overline{\mathrm{OCP_{L}}(N-1)})$ *consists to minimize*

$$\int_a^{a+h} f_x^0\big(t_{N-1}, \xi_{N-1}(t), \xi_{N-1-k}(t)\big) + f_u^0\big(t_{N-1}, \theta_{N-1}(t), \theta_{N-1-l}(t)\big)dt$$

subject to

$$\begin{aligned}
\dot{\xi}_{N-1}(t) =\ & A(t_{N-1})\xi_{N-1}(t) + A_D(t_{N-1})\xi_{N-1-k}(t) + g\big(t_{N-1}, \theta_{N-1}(t)\big) \\
& + g_D\big(t_{N-1}, \theta_{N-1-l}(t)\big) \\
\xi_{N-1}(a) =\ & \xi_{N-2}(a+h) \\
\xi_{N-1}(a+h) \in\ & \tilde{\Pi}_{N-1} = \mathbb{R}^n
\end{aligned}$$

for $t \in [a, a+h]$. *Again, Theorem 2.10 can be applied and we can find an optimal pair* $\big(\xi_{N-1}^*(\cdot), \theta_{N-1}^*(\cdot)\big)$ *in interval of time* $[a, a+h]$ *that provides an optimal solution* $\big(x^*(\cdot), u^*(\cdot)\big)$ *in the interval of time* $[a+h(N-1), b]$. *As* $\tilde{\Pi}_{N-1} = \mathbb{R}^n$, *then by Theorem 2.10* $\Lambda^{N-1}(a+h) = [0 \quad \cdots \quad 0]_{1\times n}$.

3.2.3 An illustrative example

In this section we provide an illustrative example for our Theorem 3.3.

Let us consider the delayed state-linear optimal control problem given by

$$\begin{aligned} \min \quad & C_{LD}\big(x(\cdot),u(\cdot)\big) = \int_0^4 x(t) + 100u^2(t)dt \\ \text{s.t.} \quad & \dot{x}(t) = x(t) + x(t-2) - 10u(t-1), \\ & x(t) = 1, \ t \in [-2,0], \\ & u(t) = 0, \ t \in [-1,0[, \end{aligned} \tag{3.15}$$

where $u(t) \in U = \mathbb{R}$ for each $t \in [-1,4]$. Thus, we have that $n = m = 1$, $a = 0$, $b = 4$, $r = 2$, $s = 1$, $f_x^0\big(t,x(t),x(t-2)\big) = x(t)$, $f_u^0\big(t,u(t),u(t-1)\big) = 100u^2(t)$, $A(t) = A_D(t) = 1$, $g\big(t,u(t)\big) = 0$ and $g_D\big(t,u(t-1)\big) = -10u(t-1)$. Note that our functions respect hypothesis 1 and 2 of Theorem 3.3. Let $\bar{u}$ be an admissible control of problem (3.15) and let us maximize function

$$\begin{aligned} & - f_u^0\big(t,u,\bar{u}(t-1)\big) + \eta(t)g(t,u) \\ & + \big[- f_u^0\big(t+1,\bar{u}(t+1),u\big) + \eta(t+1)g_D(t+1,u)\big]\chi_{[0,3]}(t) \\ = & - 100u^2 + \big[- 100\bar{u}^2(t+1) - 10\eta(t+1)u\big]\chi_{[0,3]}(t) \\ = & \begin{cases} -100u^2 - 10\eta(t+1)u - 100\bar{u}^2(t+1), & t \in [0,3] \\ -100u^2, & t \in]3,4] \end{cases} \end{aligned}$$

with respect to $u \in \mathbb{R}$. We obtain

$$u(t) = -\frac{\eta(t+1)}{20}$$

for $t \in [0,3]$ and $u(t) = 0$ for $t \in]3,4]$. Furthermore, we know that $\eta(t)$ is any non-trivial solution of

$$\begin{aligned} \dot{\eta}(t) = & \ \partial_2 f_x^0\big(t,x(t),x(t-2)\big) + \partial_3 f_x^0\big(t+2,x(t+2),x(t)\big)\chi_{[0,2]}(t) \\ & - \eta(t)A(t) - \eta(t+2)A_D(t+2)\chi_{[0,2]}(t) \\ \Leftrightarrow \dot{\eta}(t) = 1 - \eta(t) - \eta(t+2)\chi_{[0,2]}(t) = & \begin{cases} 1 - \eta(t) - \eta(t+2), & t \in [0,2] \\ 1 - \eta(t), & t \in]2,4] \end{cases} \end{aligned}$$

that satisfies the transversality condition $\eta(4) = 0$. The adjoint system is given by

$$\begin{cases} \dot{\eta}(t) = \begin{cases} 1 - \eta(t) - \eta(t+2), & t \in [0,2] \\ 1 - \eta(t), & t \in]2,4] \end{cases} \\ \eta(4) = 0. \end{cases} \tag{3.16}$$

For $t \in]2,4]$, the solution of differential equation

$$\begin{cases} \dot{\eta}(t) = 1 - \eta(t) \\ \eta(4) = 0 \end{cases}$$

is given by $\eta(t) = 1 - e^{4-t}$. Knowing $\eta(t)$, $t \in]2,4]$, and attending to the continuity of function η for all $t \in [0,4]$, we can determine $\eta(t)$ for $t \in [0,2]$ solving the differential equation

$$\begin{cases} \dot{\eta}(t) = 1 - \eta(t) - \eta(t+2) \\ \eta(2) = 1 - e^{4-2} = 1 - e^2 \end{cases}$$

for $t \in [0,2]$. Therefore, we have that $\eta(t) = e^{2-t}\left(t - e^2 - 1\right)$ for $t \in [0,2]$. Consequently, the solution of the adjoint system (3.16) is given by

$$\eta(t) = \begin{cases} e^{2-t}\left(t - e^2 - 1\right), & t \in [0,2] \\ 1 - e^{4-t}, & t \in]2,4]. \end{cases}$$

So, the control is given by

$$u(t) = \frac{1}{20} \begin{cases} 0, & t \in [-1,0[\\ e^{3-t} - e^{1-t}t, & t \in [0,1[\\ e^{3-t} - 1, & t \in [1,3] \\ 0, & t \in]3,4]. \end{cases} \tag{3.17}$$

Knowing the control, we can determine the state by solving the differential equation

$$\begin{cases} \dot{x}(t) = x(t) + x(t-2) - 10u(t-1) \\ x(t) = 1, \ t \in [-2,0]. \end{cases}$$

The state solution $x(t)$ is given by

$$\begin{cases} 1, & t \in [-2,0] \\ -1 + 2e^t, & t \in]0,1] \\ \dfrac{\left(e^2 + 2e^4 - 2e^2 t\right)e^{-t} - 8 + \left(17 - 2e^2\right)e^t}{8}, & t \in]1,2] \\ \dfrac{2e^{4-t} + 4 + \left(-47e^{-2} + 17 - 2e^2 + 16e^{-2}t\right)e^t}{8}, & t \in]2,3] \\ \dfrac{\left(-e^6 + e^4 t\right)e^{-t} + 4 + \left(-51e^{-2} + 24 - 2e^2 + 17e^{-2}t - 2t\right)e^t}{8}, & t \in]3,4]. \end{cases} \tag{3.18}$$

Such analytical expressions can be obtained with the help of a modern computer algebra system. We have used Mathematica. In Figure 3.2, we observe that the numerical solutions for control and state, obtained using AMPL (see [54]) and IPOPT (see [139]), are in agreement with their analytical solutions, given by (3.17) and (3.18), respectively. The numerical solutions were obtained using Euler's forward difference method in AMPL and IPOPT, dividing the interval of time $[0,4]$ into 2000 subintervals. The minimal cost is $\dfrac{23 + e^2 + 34e^4 - 2e^6}{16} \simeq 67.491786$.

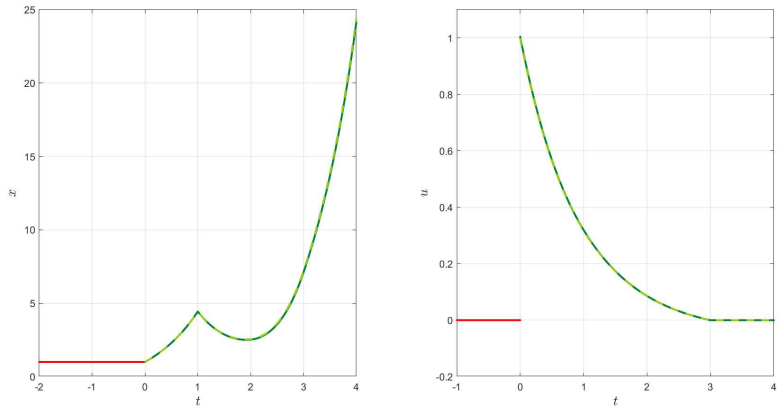


Figure 3.2: Optimal solution: red line – initial data; dark green line – analytical solution; dashed light green line – numerical solution.

3.3 Delayed non-linear optimal control problem

In this section, we are interested in non-linear optimal control problems with discrete time delays in state and control variables. Therefore, we are considering ($\mathrm{OCP_D}$) of Definition 2.13, where $(r,s) \neq (0,0)$. For this kind of problems, we derive a sufficient optimality condition. We finish this section by giving an illustrative example for the proved theoretical results. All the contents of the current section are published in [109].

3.3.1 Main result

As in Sections 2.3.2 and 3.2.2, in what follows we consider that the time delays r and s respect Commensurability Assumption 2.15. Moreover, we continue using Notation 2.16 and the following one.

Notation 3.6. *Let* $x_a = x(a) = \varphi(a)$ *and* $x_r(t) = \big(x(t), x(t-r)\big)$. *Moreover, we define the operators* $[\cdot,\cdot]_r$ *and* $\langle\cdot,\cdot\rangle_r$ *by* $[x,\zeta]_r(t) := \Big(t, x_r(t), \zeta\big(t, x_r(t)\big)\Big)$ *and* $\langle x,\zeta\rangle_r(t) := \Big(t, x_r(t), \zeta\big(t, x(t)\big)\Big)$, *respectively.*

The following theorem provides a sufficient optimality condition associated with ($\mathrm{OCP_D}$) (see Definition 2.13). Such result generalizes Theorem 2.12 for the delayed non-linear optimal control problem ($\mathrm{OCP_D}$).

Theorem 3.7. *Consider* ($\mathrm{OCP_D}$) *and assume that* $I_x = I^a_{r+s}$. *Let the interval* $[a,b]$ *be divided into* $N \in \mathbb{N}$ *subintervals of amplitude* $h = \frac{b-a}{N} > 0$ *and suppose that the functions* g^0, f^0 *and* f *are of class* $\mathscr{C}^1$ *with respect to all their arguments. Assume there exists a* $\mathscr{C}^1\left(\mathbb{R}^{1+3n}, \mathbb{R}^m\right)$ *feedback control*

$u^*\Big(t, x_r(t), \eta\big(t, x_r(t)\big)\Big) = u^*[x,\eta]_r(t)$ *such that*

$$\begin{aligned}
&\max_{u\in U}\Big\{H\Big(t, x_r(t), u, u^*[x,\eta]_r(t_s), \eta\big(t, x_r(t)\big)\Big)\\
&\quad + H\Big(t^s, x_r(t^s), u^*[x,\eta]_r(t^s), u, \eta\big(t^s, x_r(t^s)\big)\Big)\chi_{[a,b-s]}(t)\Big\}\\
&= H\Big(t, x_r(t), u^*[x,\eta]_r(t), u^*[x,\eta]_r(t_s), \eta\big(t, x_r(t)\big)\Big)\\
&\quad + H\Big(t^s, x_r(t^s), u^*[x,\eta]_r(t^s), u^*[x,\eta]_r(t), \eta\big(t^s, x_r(t^s)\big)\Big)\chi_{[a,b-s]}(t)\\
&=: H^0[x,\eta]_r(t) + H^0[x,\eta]_r(t^s)\chi_{[a,b-s]}(t)
\end{aligned} \tag{3.19}$$

for all $t \in [a,b]$, *where*

$$H(t,x,y,u,v,\eta) = -f^0(t,x,y,u,v) + \eta f(t,x,y,u,v).$$

Furthermore, let $I_i = [a+hi, a+h(i+1)]$, $i = 0,\ldots,N-1$, *and suppose that function* $S\big(t,x(t)\big) \in \mathscr{C}^2\left(\mathbb{R}^{1+n},\mathbb{R}\right)$, $t\in[a,b]$, *is a solution of equation*

$$\begin{aligned}
&\partial_1 S\big(t,x(t)\big) + \sum_{i=0}^{N-1}\Big\{ - f^0\big(t, x_r(t), u^*\langle x, \partial_2 S\rangle_r(t), u^*\langle x, \partial_2 S\rangle_r(t_s)\big)\\
&+ \partial_2 S\big(t,x(t)\big) f\big(t, x_r(t), u^*\langle x, \partial_2 S\rangle_r(t), u^*\langle x, \partial_2 S\rangle_r(t_s)\big)\Big\}\chi_{I_i}(t) = 0
\end{aligned} \tag{3.20}$$

with $S\big(b,x(b)\big) = -g^0\big(x(b)\big)$, $x(b)\in\Pi$. *Finally, consider that the control law*

$$u^*\Big(t, x_r(t), \partial_2 S\big(t,x(t)\big)\Big) = u^*\langle x, \partial_2 S\rangle_r(t), \quad t\in[a,b],$$

determines a response $\tilde{x}(t)$ *steering* (a,x_a) *to* (b,Π). *Then,*

$$\tilde{u}(t) = u^*\big(t, \tilde{x}(t), \tilde{x}(t-r), \partial_2 S(t,\tilde{x}(t)\big)$$

is an optimal control of $(\mathrm{OCP_D})$ *that leads to the minimal cost*

$$C_D\big(\tilde{x}(\cdot),\tilde{u}(\cdot)\big) = -S(a,x_a).$$

Proof. We prove Theorem 3.7 as a corollary of Theorem 2.12 by transforming the delayed non-linear optimal control problem $(\mathrm{OCP_D})$ into an equivalent non-linear optimal control problem without delays of type (OCP) (see Definition 2.1). For that, we follow again the approach of [56, 59] used in the proof of Theorem 3.3. Also here, without loss of generality, we assume that $\dfrac{r}{s}\in\mathbb{Q}$ for $r\geq 0$ and $s>0$. Consequently, there exist $k,l\in\mathbb{N}$ such that

$$\frac{r}{s} = \frac{k}{l} \Leftrightarrow rl = sk \Leftrightarrow \frac{r}{k} = \frac{s}{l} =: h.$$

Thus, we also divide the interval $[a,b]$ into $N\in\mathbb{N}$ subintervals of amplitude h. Again, we obtain $r = hk$ and $s = hl$. Also here, we assume that $a+hN = b$ and $N > 2k+1$, with $N\in\mathbb{N}$. Note that Remark 3.4 also holds for $(\mathrm{OCP_D})$.

Again, we define the new variables $\xi_i(t) = x(t+hi)$ and $\theta_i(t) = u(t+hi)$, for $i = 0, \ldots, N-1$ and $t \in [a, a+h]$. The delayed non-linear problem $(\mathrm{OCP_D})$ is transformed into the following equivalent non-linear problem $(\overline{\mathrm{OCP}})$ without delays, which consists to minimize the cost functional given by

$$\begin{aligned}\bar{C}\big(\xi(\cdot),\theta(\cdot)\big) &= g^0\big(\xi_{N-1}(a+h)\big) \\ &\quad + \int_a^{a+h} \sum_{i=0}^{N-1} f^0\big(t+hi, \xi_i(t), \xi_{i-k}(t), \theta_i(t), \theta_{i-l}(t)\big)dt\end{aligned} \tag{3.21}$$

subject to the non-delayed differential system

$$\dot{\xi}_i(t) = f\big(t+hi, \xi_i(t), \xi_{i-k}(t), \theta_i(t), \theta_{i-l}(t)\big) \tag{3.22}$$

for $i = 0, \ldots, N-1$ and $t \in [a, a+h]$, and to the initial conditions

$$\begin{aligned}&\xi_i(t) = \varphi(t+hi), \quad i = -k-l, \ldots, -1 \quad \text{and} \quad t \in [a, a+h]; \\ &\theta_i(t) = \psi(t+hi), \quad i = -l, \ldots, -1 \quad \text{and} \quad t \in [a, a+h[; \\ &\xi_i(a+h) = \xi_{i+1}(a), \quad i = 0, \ldots, N-2.\end{aligned} \tag{3.23}$$

We observe that the cost functional (3.21) depends only on $t \in [a, a+h]$, $\xi(t) = \big[\xi_0(t) \ \cdots \ \xi_{N-1}(t)\big]^T$ and $\theta(t) = \big[\theta_0(t) \ \cdots \ \theta_{N-1}(t)\big]^T$, because

$$\xi^-(t) = [\xi_{-k-l}(t) \ \xi_{1-k-l}(t) \ \cdots \ \xi_{-1}(t)]^T$$

and

$$\theta^-(t) = [\theta_{-l}(t) \ \cdots \ \theta_{-1}(t)]^T$$

are already known. Thus, the integrand function of (3.21) can be written as

$$\sum_{i=0}^{N-1} f^0\big(t+hi, \xi_i(t), \xi_{i-k}(t), \theta_i(t), \theta_{i-l}(t)\big) = F^0\big(t, \xi(t), \theta(t)\big).$$

We can also write

$$g^0\big(\xi_{N-1}(a+h)\big) = G^0\big(\xi(a+h)\big).$$

Note that we are writing G^0 as a function of $\xi(a+h) \in \mathbb{R}^{nN}$ in order to obtain $(\overline{\mathrm{OCP}})$ written in the form used by Theorem 2.12. However, function G^0 depends only on $\xi_{N-1}(a+h) \in \mathbb{R}^n$. Consequently, we have

$$\begin{aligned}&g^0\big(\xi_{N-1}(a+h)\big) + \int_a^{a+h} \sum_{i=0}^{N-1} f^0\big(t+hi, \xi_i(t), \xi_{i-k}(t), \theta_i(t), \theta_{i-l}(t)\big)dt \\ &= G^0\big(\xi(a+h)\big) + \int_a^{a+h} F^0\big(t, \xi(t), \theta(t)\big)dt.\end{aligned}$$

Using similar arguments, the differential system (3.22) can be written as

$$\begin{aligned}\dot{\xi}(t) &= \begin{bmatrix} \dot{\xi}_0(t) \\ \dot{\xi}_1(t) \\ \vdots \\ \dot{\xi}_{N-1}(t) \end{bmatrix} \\ &= \begin{bmatrix} f\big(t, \xi_0(t), \xi_{-k}(t), \theta_0(t), \theta_{-l}(t)\big) \\ f\big(t+h, \xi_1(t), \xi_{1-k}(t), \theta_1(t), \theta_{1-l}(t)\big) \\ \vdots \\ f\big(t+h(N-1), \xi_{N-1}(t), \xi_{N-1-k}(t), \theta_{N-1}(t), \theta_{N-1-l}(t)\big) \end{bmatrix} \\ &= F\big(t, \xi(t), \theta(t)\big), \quad t \in [a, a+h].\end{aligned}$$

In order to apply Theorem 2.12, we consider the initial boundary condition, with respect to variable ξ, given by

$$\begin{aligned}\xi_a = \xi(a) &= \begin{bmatrix} \xi_0(a) & \xi_1(a) & \cdots & \xi_{N-1}(a) \end{bmatrix}^T \\ &= \begin{bmatrix} x_a & \xi_0(a+h) & \cdots & \xi_{N-2}(a+h) \end{bmatrix}^T.\end{aligned}$$

Remark 3.8. *Only the first component of ξ_a is known a priori. The others are determined using the continuity conditions $\xi_i(a+h) = \xi_{i+1}(a)$ of (3.23), $i = 0, \ldots, N-2$, and the fixed value x_a.*

Concluding, ($\overline{\text{OCP}}$) is written in the standard form, as follows:

$$\begin{aligned} \min \;\; & \bar{C}\big(\xi(\cdot), \theta(\cdot)\big) = G^0\big(\xi(a+h)\big) + \int_a^{a+h} F^0\big(t, \xi(t), \theta(t)\big) dt \\ \text{s.t.} \;\; & \dot{\xi}(t) = F\big(t, \xi(t), \theta(t)\big), \; t \in [a, a+h], \\ & \xi(a) = \xi_a = \begin{bmatrix} x_a & \xi_0(a+h) & \cdots & \xi_{N-2}(a+h) \end{bmatrix}^T, \end{aligned}$$

ensuring for all $i \in \{0, \ldots, N-2\}$ the continuity conditions $\xi_i(a+h) = \xi_{i+1}(a)$ of (3.23) and knowing $\xi^-(t)$ for all $t \in [a, a+h]$ and $\theta^-(t)$ for all $t \in [a, a+h[$. Furthermore, we know that

- $\xi(t) \in \mathbb{R}^{nN}$ and $\theta(t) \in \tilde{U} \subseteq \mathbb{R}^{mN}$ for each $t \in [a, a+h]$;
- $\xi(a+h) \in \tilde{\Pi} = \mathbb{R}^{n(N-1)} \times \Pi$;
- functions G^0, F^0 and F are of class $\mathscr{C}^1$ with respect to all their arguments, because g^0, f^0 and f are of class $\mathscr{C}^1$ in all their arguments.

Therefore, we are in condition to apply Theorem 2.12. Firstly, we are going to prove the first part of Theorem 3.7, that is, we show that (3.19) holds. Assume that there exists a feedback control

$$\theta^*\Big(t, \xi(t), \Lambda\big(t, \xi(t)\big)\Big) \in \mathscr{C}^1\big(\mathbb{R}^{1+2nN}, \mathbb{R}^{mN}\big)$$

such that

$$\begin{aligned}\max_{\theta\in\tilde{U}} \bar{H}\Big(t,\xi(t),\theta,\Lambda\big(t,\xi(t)\big)\Big) =& \bar{H}\Big(t,\xi(t),\theta^*\Big(t,\xi(t),\Lambda\big(t,\xi(t)\big)\Big),\Lambda\big(t,\xi(t)\big)\Big)\\ =:& \bar{H}^0\Big(t,\xi(t),\Lambda\big(t,\xi(t)\big)\Big)\end{aligned} \tag{3.24}$$

for all $t\in[a,a+h]$, where $\bar{H}(t,\xi,\theta,\Lambda)=-F^0(t,\xi,\theta)+\Lambda F(t,\xi,\theta)$. In order to write the previous condition with respect to the original variables, we do the following remark.

Remark 3.9. *For each $t\in[a,b]$, there exists $j\in\{0,\dots,N-1\}$ such that*

$$a+hj\le t\le a+h(j+1)\Leftrightarrow a\le t-hj\le a+h.$$

Thus, let us define $t'\in[a,a+h]$ as $t'=t-hj$ and

$$\Lambda^j\big(t,\xi_j(t),\xi_{j-k}(t)\big)=\eta\big(t+hj,x(t+hj),x(t+hj-r)\big).$$

Then, we obtain

$$\begin{aligned}\Lambda^j\big(t',\xi_j(t'),\xi_{j-k}(t')\big) &= \eta\big(t'+hj,x(t'+hj),x(t'+hj-r)\big)\\ &= \eta\big(t,x(t),x(t-r)\big)\end{aligned}$$

and

$$\begin{aligned}&\Lambda^{j+l}\big(t,\xi_{j+l}(t),\xi_{j+l-k}(t)\big)\\ &\quad=\Lambda^{j+l}\big(t+h(j+l)-h(j+l),x(t+hj+hl),x(t+hj+hl-hk)\big)\\ &\quad=\Lambda^{j+l}\big(t+hj+s-h(j+l),x(t+hj+s),x(t+hj+s-r)\big)\\ &\quad=\eta\big(t+hj+s,x(t+hj+s),x(t+hj+s-r)\big),\end{aligned}$$

which implies that

$$\begin{aligned}&\Lambda^{j+l}\big(t',\xi_{j+l}(t'),\xi_{j+l-k}(t')\big)\\ &\quad=\eta\big(t'+hj+s,x(t'+hj+s),x(t'+hj+s-r)\big)\\ &\quad=\eta\big(t+s,x(t+s),x(t+s-r)\big).\end{aligned}$$

As equation (3.24) is verified for all admissible $\theta\in\tilde{U}$, we can choose an admissible control $\bar{\theta}\in\tilde{U}$ such that

$$\bar{\theta}_i=\begin{cases}\theta_i^*\Big(t',\xi_i(t'),\xi_{i-k}(t'),\Lambda^i\big(t',\xi_i(t'),\xi_{i-k}(t')\big)\Big), & i\neq j,\\ \theta_i, & i=j,\end{cases} \tag{3.25}$$

$i=0,\dots,N-1$, where $\theta=[\theta_0\ \ \dots\ \ \theta_{N-1}]^T$ is an admissible control for ($\overline{\text{OCP}}$). From condition (3.24), we can write that

$$\bar{H}\Big(t',\xi(t'),\bar{\theta},\Lambda\big(t',\xi(t')\big)\Big)\le\bar{H}\Big(t',\xi(t'),\theta^{*'},\Lambda\big(t',\xi(t')\big)\Big),$$

where $\theta^{*'} = \theta^*\Big(t', \xi(t'), \Lambda\big(t', \xi(t')\big)\Big)$. From now on, we use the previous notation in order to simplify expressions. So, we have

$$\begin{aligned} & - F^0\big(t', \xi(t'), \bar{\theta}\big) + \Lambda\big(t', \xi(t')\big) F\big(t', \xi(t'), \bar{\theta}\big) \\ \leq & - F^0\Big(t', \xi(t'), \theta^{*'}\Big) + \Lambda\big(t', \xi(t')\big) F\Big(t', \xi(t'), \theta^{*'}\Big), \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \sum_{i=0}^{N-1} \Big\{ - f^0\big(t' + hi, \xi_i(t'), \xi_{i-k}(t'), \bar{\theta}_i, \bar{\theta}_{i-l}\big) \\ & \qquad + \Lambda^i\big(t', \xi_i(t'), \xi_{i-k}(t')\big) f\big(t' + hi, \xi_i(t'), \xi_{i-k}(t'), \bar{\theta}_i, \bar{\theta}_{i-l}\big) \Big\} \\ \leq & \sum_{i=0}^{N-1} \Big\{ - f^0\Big(t' + hi, \xi_i(t'), \xi_{i-k}(t'), \theta_i^{*'}, \theta_{i-l}^{*'}\Big) \\ & \qquad + \Lambda^i\left(t', \xi_i(t'), \xi_{i-k}(t')\right) f\Big(t' + hi, \xi_i(t'), \xi_{i-k}(t'), \theta_i^{*'}, \theta_{i-l}^{*'}\Big) \Big\}. \end{aligned}$$

Considering $I = \{0, \ldots, N-1\} \backslash \{j, j+l\}$ and definition (3.25) for the admissible control $\bar{\theta}$, we obtain that

$$\begin{aligned} & \sum_{i \in I} \Big\{ - f^0\Big(t' + hi, \xi_i(t'), \xi_{i-k}(t'), \theta_i^{*'}, \theta_{i-l}^{*'}\Big) \\ & \qquad + \Lambda^i\left(t', \xi_i(t'), \xi_{i-k}(t')\right) f\Big(t' + hi, \xi_i(t'), \xi_{i-k}(t'), \theta_i^{*'}, \theta_{i-l}^{*'}\Big) \Big\} \\ & \qquad - f^0\Big(t' + hj, \xi_j(t'), \xi_{j-k}(t'), \theta_j, \theta_{j-l}^{*'}\Big) \\ & \qquad + \Lambda^j\left(t', \xi_j(t'), \xi_{j-k}(t')\right) f\Big(t' + hj, \xi_j(t'), \xi_{j-k}(t'), \theta_j, \theta_{j-l}^{*'}\Big) \\ & \qquad + \Big[- f^0\Big(t' + hj + s, \xi_{j+l}(t'), \xi_{j+l-k}(t'), \theta_{j+l}^{*'}, \theta_j\Big) \\ & \qquad\quad + \Lambda^{j+l}\left(t', \xi_{j+l}(t'), \xi_{j+l-k}(t')\right) \\ & \qquad\quad \times f\Big(t' + hj + s, \xi_{j+l}(t'), \xi_{j+l-k}(t'), \theta_{j+l}^{*'}, \theta_j\Big) \Big] \chi_{\{0,\ldots,N-1-l\}}(j) \\ \leq & \sum_{i=0}^{N-1} \Big\{ - f^0\Big(t' + hi, \xi_i(t'), \xi_{i-k}(t'), \theta_i^{*'}, \theta_{i-l}^{*'}\Big) \\ & \qquad + \Lambda^i\left(t', \xi_i(t'), \xi_{i-k}(t')\right) f\Big(t' + hi, \xi_i(t'), \xi_{i-k}(t'), \theta_i^{*'}, \theta_{i-l}^{*'}\Big) \Big\}. \end{aligned}$$

The terms of the first and second members with indexes in set I cancel and

we simply have

$$
\begin{aligned}
&- f^0\left(t'+hj,\xi_j(t'),\xi_{j-k}(t'),\theta_j,\theta^{*'}_{j-l}\right)\\
&+\Lambda^j\left(t',\xi_j(t'),\xi_{j-k}(t')\right)f\left(t'+hj,\xi_j(t'),\xi_{j-k}(t'),\theta_j,\theta^{*'}_{j-l}\right)\\
&+\Bigg[-f^0\left(t'+hj+s,\xi_{j+l}(t'),\xi_{j+l-k}(t'),\theta^{*'}_{j+l},\theta_j\right)\\
&\quad+\Lambda^{j+l}\left(t',\xi_{j+l}(t'),\xi_{j+l-k}(t')\right)\\
&\quad\times f\left(t'+hj+s,\xi_{j+l}(t'),\xi_{j+l-k}(t'),\theta^{*'}_{j+l},\theta_j\right)\Bigg]\chi_{\{0,\ldots,N-1-l\}}(j)\\
\le&-f^0\left(t'+hj,\xi_j(t'),\xi_{j-k}(t'),\theta^{*'}_j,\theta^{*'}_{j-l}\right)\\
&+\Lambda^j\left(t',\xi_j(t'),\xi_{j-k}(t')\right)f(t'+hj,\xi_j(t'),\xi_{j-k}(t'),\theta^{*'}_j,\theta^{*'}_{j-l})\\
&+\Bigg[-f^0\left(t'+hj+s,\xi_{j+l}(t'),\xi_{j+l-k}(t'),\theta^{*'}_{j+l},\theta^{*'}_j\right)\\
&\quad+\Lambda^{j+l}\left(t',\xi_{j+l}(t'),\xi_{j+l-k}(t')\right)\\
&\quad\times f\left(t'+hj+s,\xi_{j+l}(t'),\xi_{j+l-k}(t'),\theta^{*'}_{j+l},\theta^{*'}_j\right)\Bigg]\chi_{\{0,\ldots,N-1-l\}}(j).
\end{aligned}
\tag{3.26}
$$

We can observe that

$$
\begin{aligned}
t'+hj&=t-hj+hj=t;\\
\xi_j(t')&=x(t'+hj)=x(t);\\
\xi_{j-k}(t')&=x(t'+hj-hk)=x(t-r);\\
\xi_{j-l}(t')&=x(t'+hj-hl)=x(t-s);\\
\xi_{j-l-k}(t')&=x(t'+hj-hl-hk)=x(t-s-r);\\
\xi_{j+l}(t')&=x(t'+hj+hl)=x(t+s);\\
\xi_{j+l-k}(t')&=x(t'+hj+hl-hk)=x(t+s-r);
\end{aligned}
$$

$$
\begin{aligned}
\theta^{*'}_j&=\theta^*_j\left(t',\xi_j(t'),\xi_{j-k}(t'),\Lambda^j\left(t',\xi_j(t'),\xi_{j-k}(t')\right)\right)\\
&=u^*\left(t'+hj,x(t),x(t-r),\eta\left(t,x(t),x(t-r)\right)\right)\\
&=u^*[x,\eta]_r(t);\\
\theta^{*'}_{j-l}&=\theta^*_{j-l}\left(t',\xi_{j-l}(t'),\xi_{j-l-k}(t'),\Lambda^{j-l}\left(t',\xi_{j-l}(t'),\xi_{j-l-k}(t')\right)\right)\\
&=u^*\left(t'+hj-hl,x(t-s),x(t-s-r),\eta(t-s,x(t-s),x(t-s-r))\right)\\
&=u^*\left(t_s,x_r(t_s),\eta(t_s,x_r(t_s))\right)=u^*[x,\eta]_r(t_s);
\end{aligned}
$$

$$
\theta^{*'}_{j+l}=\theta^*_{j+l}\left(t',\xi_{j+l}(t'),\xi_{j+l-k}(t'),\Lambda^{j+l}\left(t',\xi_{j+l}(t'),\xi_{j+l-k}(t')\right)\right)
$$

$$= u^*\left(t' + hj + hl, x(t+s), x(t+s-r), \eta(t+s, x(t+s), x(t+s-r))\right)$$
$$= u^*\left(t^s, x_r(t^s), \eta(t^s, x_r(t^s))\right) = u^*[x,\eta]_r(t^s);$$

$\theta_j = u$, where $u \in U$ is an arbitrary admissible control for $(\mathrm{OCP_D})$. Using these relations, we rewrite the first member of inequality (3.26) as

$$\begin{aligned}
&- f^0\left(t' + hj, \xi_j(t'), \xi_{j-k}(t'), \theta_j, \theta^{*'}_{j-l}\right)\\
&+ \Lambda^j\left(t', \xi_j(t'), \xi_{j-k}(t')\right) f\left(t' + hj, \xi_j(t'), \xi_{j-k}(t'), \theta_j, \theta^{*'}_{j-l}\right)\\
&+ \Big[- f^0\left(t' + hj + s, \xi_{j+l}(t'), \xi_{j+l-k}(t'), \theta^{*'}_{j+l}, \theta_j\right)\\
&\qquad + \Lambda^{j+l}\left(t', \xi_{j+l}(t'), \xi_{j+l-k}(t')\right)\\
&\qquad \times f\left(t' + hj + s, \xi_{j+l}(t'), \xi_{j+l-k}(t'), \theta^{*'}_{j+l}, \theta_j\right)\Big]\chi_{\{0,\dots,N-1-l\}}(j)\\
&= - f^0\big(t, x_r(t), u, u^*[x,\eta]_r(t_s)\big) + \eta\big(t, x_r(t)\big) f\big(t, x_r(t), u, u^*[x,\eta]_r(t_s)\big)\\
&\quad + \Big[- f^0\big(t^s, x_r(t^s), u^*[x,\eta]_r(t^s), u\big)\\
&\qquad + \eta\big(t^s, x_r(t^s)\big) f\big(t^s, x_r(t^s), u^*[x,\eta]_r(t^s), u\big)\Big]\chi_{[a,b-s]}(t)\\
&= H\Big(t, x_r(t), u, u^*[x,\eta]_r(t_s), \eta\big(t, x_r(t)\big)\Big)\\
&\quad + H\Big(t^s, x_r(t^s), u^*[x,\eta]_r(t^s), u, \eta\big(t^s, x_r(t^s)\big)\Big)\chi_{[a,b-s]}(t).
\end{aligned}$$

On the other hand, the second member of inequality (3.26) takes the form

$$\begin{aligned}
&- f^0\left(t' + hj, \xi_j(t'), \xi_{j-k}(t'), \theta^{*'}_j, \theta^{*'}_{j-l}\right)\\
&+ \Lambda^j\left(t', \xi_j(t'), \xi_{j-k}(t')\right) f\left(t' + hj, \xi_j(t'), \xi_{j-k}(t'), \theta^{*'}_j, \theta^{*'}_{j-l}\right)\\
&+ \Big[- f^0\left(t' + hj + s, \xi_{j+l}(t'), \xi_{j+l-k}(t'), \theta^{*'}_{j+l}, \theta^{*'}_j\right)\\
&\qquad + \Lambda^{j+l}\left(t', \xi_{j+l}(t'), \xi_{j+l-k}(t')\right)\\
&\qquad \times f\left(t' + hj + s, \xi_{j+l}(t'), \xi_{j+l-k}(t'), \theta^{*'}_{j+l}, \theta^{*'}_j\right)\Big]\chi_{\{0,\dots,N-1-l\}}(j)\\
&= - f^0\big(t, x_r(t), u^*[x,\eta]_r(t), u^*[x,\eta]_r(t_s)\big)\\
&\quad + \eta\big(t, x_r(t)\big) f\big(t, x_r(t), u^*[x,\eta]_r(t), u^*[x,\eta]_r(t_s)\big)\\
&\quad + \Big[- f^0\big(t^s, x_r(t^s), u^*[x,\eta]_r(t^s), u^*[x,\eta]_r(t)\big)\\
&\qquad + \eta\big(t^s, x_r(t^s)\big) f\big(t^s, x_r(t^s), u^*[x,\eta]_r(t^s), u^*[x,\eta]_r(t)\big)\Big]\chi_{[a,b-s]}(t)\\
&= H\Big(t, x_r(t), u^*[x,\eta]_r(t), u^*[x,\eta]_r(t_s), \eta\big(t, x_r(t)\big)\Big)\\
&\quad + H\Big(t^s, x_r(t^s), u^*[x,\eta]_r(t^s), u^*[x,\eta]_r(t), \eta\big(t^s, x_r(t^s)\big)\Big)\chi_{[a,b-s]}(t).
\end{aligned}$$

Therefore, the inequality (3.26) is equivalent to

$$\begin{aligned}
&H\Big(t, x_r(t), u, u^*[x,\eta]_r(t_s), \eta\big(t, x_r(t)\big)\Big) \\
&\quad + H\Big(t^s, x_r(t^s), u^*[x,\eta]_r(t^s), u, \eta\big(t^s, x_r(t^s)\big)\Big)\chi_{[a,b-s]}(t) \\
&\leq H\Big(t, x_r(t), u^*[x,\eta]_r(t), u^*[x,\eta]_r(t_s), \eta\big(t, x_r(t)\big)\Big) \\
&\quad + H\Big(t^s, x_r(t^s), u^*[x,\eta]_r(t^s), u^*[x,\eta]_r(t), \eta\big(t^s, x_r(t^s)\big)\Big)\chi_{[a,b-s]}(t),
\end{aligned}$$

where $u \in U$ is an arbitrary admissible control for $(\mathrm{OCP_D})$. We just proved condition (3.19). Now we proceed by proving equation (3.20). Let us suppose that function $\bar{S}\big(t,\xi(t)\big) \in \mathscr{C}^2\left(\mathbb{R}^{1+nN}, \mathbb{R}\right)$, $t \in [a, a+h]$, is a solution to the Hamilton–Jacobi equation

$$\partial_1 \bar{S}\big(t,\xi(t)\big) + \bar{H}^0\Big(t, \xi(t), \partial_2 \bar{S}\big(t,\xi(t)\big)\Big) = 0 \tag{3.27}$$

with $\bar{S}\big(a+h, \xi(a+h)\big) = -G^0\big(\xi(a+h)\big)$ for $\xi(a+h) \in \tilde{\Pi}$. Now, in order to simplify the notation, we write

- θ^* instead of $\theta^*\Big(t, \xi(t), \partial_2 \bar{S}\big(t,\xi(t)\big)\Big)$;
- θ_i^* instead of $\theta_i^*\Big(t, \xi_i(t), \xi_{i-k}(t), \partial_{i+2} \bar{S}\big(t,\xi(t)\big)\Big)$, for $i = 0, \ldots, N-1$.

Therefore, the Hamilton–Jacobi equation (3.27) is equivalent to

$$\begin{aligned}
&\partial_1 \bar{S}\big(t,\xi(t)\big) + \bar{H}\Big(t, \xi(t), \theta^*, \partial_2 \bar{S}\big(t,\xi(t)\big)\Big) = 0 \\
\Leftrightarrow\ &\partial_1 \bar{S}\big(t,\xi(t)\big) - F^0\big(t,\xi(t),\theta^*\big) + \partial_2 \bar{S}\big(t,\xi(t)\big) F\big(t,\xi(t),\theta^*\big) = 0 \\
\Leftrightarrow\ &\partial_1 \bar{S}\big(t,\xi(t)\big) + \sum_{i=0}^{N-1} \Big\{ -f^0\big(t+hi, \xi_i(t), \xi_{i-k}(t), \theta_i^*, \theta_{i-l}^*\big) \\
&\quad + \partial_{i+2} \bar{S}\big(t,\xi(t)\big) f\big(t+hi, \xi_i(t), \xi_{i-k}(t), \theta_i^*, \theta_{i-l}^*\big) \Big\} = 0.
\end{aligned}$$

For all $t \in [a, a+h]$, one has

$$\begin{aligned}
\bar{S}\big(t,\xi(t)\big) &= \bar{S}\big(t, \xi_0(t), \xi_1(t), \ldots, \xi_{N-1}(t)\big) \\
&= \bar{S}\big(t, x(t), x(t+h), \ldots, x(t+hN-h)\big).
\end{aligned}$$

So, we can simply write $\bar{S}(t,\xi(t))$, for $t \in [a, a+h]$, as a function of t and $x(t)$, for all $t \in [a,b]$:

$$\bar{S}\big(t,\xi(t)\big)\big|_{t\in[a,a+h]} := S\big(t,x(t)\big)\big|_{t\in[a,b]}.$$

We can also observe that

$$\partial_{i+2} \bar{S}\big(t,\xi(t)\big) = \partial_2 S\big(t,x(t)\big)\chi_{I_i}(t),$$

$$
\begin{aligned}
&f^0\big(t+hi,\xi_i(t),\xi_{i-k}(t),\theta_i^*,\theta_{i-l}^*\big)\\
&=f^0\big(t,x_r(t),u^*\langle x,\partial_2 S\rangle_r(t),u^*\langle x,\partial_2 S\rangle_r(t_s)\big)\chi_{I_i}(t),
\end{aligned}
$$

and

$$
\begin{aligned}
&f\big(t+hi,\xi_i(t),\xi_{i-k}(t),\theta_i^*,\theta_{i-l}^*\big)\\
&=f\big(t,x_r(t),u^*\langle x,\partial_2 S\rangle_r(t),u^*\langle x,\partial_2 S\rangle_r(t_s)\big)\chi_{I_i}(t),
\end{aligned}
$$

for $i=0,\ldots,N-1$. Therefore, we obtain

$$
\begin{aligned}
&\partial_1 S\big(t,x(t)\big)+\sum_{i=0}^{N-1}\Big\{-f^0\big(t,x_r(t),u^*\langle x,\partial_2 S\rangle_r(t),u^*\langle x,\partial_2 S\rangle_r(t_s)\big)\\
&+\partial_2 S\big(t,x(t)\big)f\big(t,x_r(t),u^*\langle x,\partial_2 S\rangle_r(t),u^*\langle x,\partial_2 S\rangle_r(t_s)\big)\Big\}\chi_{I_i}(t)=0.
\end{aligned}
$$

Furthermore, we have to ensure that

$$
\begin{aligned}
&\bar{S}\big(a+h,\xi(a+h)\big)=-G^0\big(\xi(a+h)\big)\\
\Leftrightarrow\ &\bar{S}\big(a+h,x(a+h),x(a+2h),\ldots,x(b)\big)=-g^0\big(\xi_{N-1}(a+h)\big)\\
\Leftrightarrow\ &\bar{S}\big(a+h,x(a+h),x(a+2h),\ldots,x(b)\big)=-g^0\big(x(b)\big),
\end{aligned}
$$

which implies that

$$
S\big(b,x(b)\big)=-g^0\big(x(b)\big).
$$

As $\xi(a+h)\in\tilde{\Pi}=\mathbb{R}^{n(N-1)}\times\Pi$, then $\xi_{N-1}(a+h)=x(b)\in\Pi$. Therefore, we obtain equation (3.20) and its conditions $S\big(b,x(b)\big)=-g^0\big(x(b)\big)$, $x(b)\in\Pi$. To finish the proof, let us assume that the control law

$$
\theta_i^*\Big(t,\xi_i(t),\xi_{i-k}(t),\partial_{i+2}\bar{S}\big(t,\xi(t)\big)\Big)=u^*\langle x,\partial_2 S\rangle_r(t)\chi_{I_i}(t)
$$

determines a response $\tilde{\xi}(t)$, $t\in[a,a+h]$, steering $\big(a,\xi_i(a)\big)$ to $(a+h,\tilde{\Pi})$, $i=0,\ldots,N-1$. Such assumption implies that the control law $u^*\langle x,\partial_2 S\rangle_r(t)$ determines a response $\tilde{x}(t)$ steering (a,x_a) to (b,Π), for all $t\in[a,b]$. For $i=0,\ldots,N-1$ and $t\in[a,a+h]$,

$$
\tilde{\theta}_i(t)=\theta_i^*\Big(t,\tilde{\xi}_i(t),\tilde{\xi}_{i-k}(t),\partial_{i+2}\bar{S}\big(t,\tilde{\xi}(t)\big)\Big)
$$

is the ith component of an optimal control $\tilde{\theta}(t)$ that lead us to the minimal cost

$$
\bar{C}\big(\tilde{\xi}(\cdot),\tilde{\theta}(\cdot)\big)=-\bar{S}\big(a,\xi(a)\big)=-\bar{S}\big(a,\xi_0(a),\ldots,\xi_{N-1}(a)\big)=-S(a,x_a).
$$

As $\tilde{\theta}_i(t)=\tilde{u}(t+hi)$, $i=0,\ldots,N-1$ and $t\in[a,a+h]$, then

$$
\tilde{u}(t)=u^*\Big(t,\tilde{x}(t),\tilde{x}(t-r),\partial_2 S\big(t,\tilde{x}(t)\big)\Big),
$$

$t\in[a,b]$, is an optimal control that lead us to the minimal cost

$$
C_D\big(\tilde{x}(\cdot),\tilde{u}(\cdot)\big)=-S(a,x_a).
$$

This completes the proof of Theorem 3.7. □

3.3.2 An illustrative example

Let us consider the following delayed non-linear optimal control problem studied by Göllmann et al. in [56]:

$$\begin{aligned} \min \quad & C_D\big(x(\cdot),u(\cdot)\big) = \int_0^3 x^2(t)+u^2(t)dt, \\ \text{s.t.} \quad & \dot{x}(t) = x(t-1)\ u(t-2), \\ & x(t) = 1, \quad t \in [-1,0], \\ & u(t) = 0, \quad t \in [-2,0], \end{aligned} \tag{3.28}$$

which is a particular case of our delayed non-linear optimal control problem ($\mathrm{OCP_D}$) with $n = m = 1$, $a = 0$, $b = 3$, $r = 1$, $s = 2$, $g^0\big(x(3)\big) = 0$, $f^0(t,x,y,u,v) = x^2+u^2$ and $f(t,x,y,u,v) = yv$. In [56], necessary optimality conditions were proved and applied to (3.28). The following candidate $\big(x^*(\cdot),u^*(\cdot)\big)$ was found:

$$x^*(t) = \begin{cases} 1, & t \in [-1,2], \\ \dfrac{e^{t-2}+e^{4-t}}{e^2+1}, & t \in\ [2,3], \end{cases} \tag{3.29}$$

and

$$u^*(t) = \begin{cases} 0, & t \in [-2,0], \\ \dfrac{e^t-e^{2-t}}{e^2+1}, & t \in\ ]0,1], \\ 0, & t \in\ [1,3]. \end{cases} \tag{3.30}$$

It remains missing in [56], however, a proof that such candidate (3.29)–(3.30) is a solution to the problem. It follows from our sufficient optimality condition that such claim is indeed true.

We denote that $x_0^*(t) = 1$, $t \in [-1,0]$; $x_1^*(t) = 1$, $t \in [0,1]$; $x_2^*(t) = 1$, $t \in [1,2]$; $x_3^*(t) = \frac{e^{t-2}+e^{4-t}}{e^2+1}$, $t \in [2,3]$; $u_0^*(t) = 0$, $t \in [-2,0]$; $u_1^*(t) = \frac{e^t-e^{2-t}}{e^2+1}$, $t \in\ ]0,1]$; $u_2^*(t) = 0$, $t \in [1,2]$ and $u_3^*(t) = 0$, $t \in [2,3]$. Furthermore, the

corresponding adjoint function is given by

$$\eta(t) = \begin{cases} \eta_1(t), & t \in [0,1] \\ \eta_2(t), & t \in [1,2] \\ \eta_3(t), & t \in [2,3] \end{cases}$$

$$= \begin{cases} -2t + 5 + \dfrac{2\big(e^2-1\big)}{\big(e^2+1\big)^2}, & t \in [0,1] \\ -\left(\dfrac{4e^2}{\big(e^2+1\big)^2} + 2\right)t + \dfrac{4\big(e^2-1\big)}{\big(e^2+1\big)^2} + 6 + \dfrac{e^{2t-2} - e^{6-2t}}{\big(e^2+1\big)^2}, & t \in [1,2] \\ \dfrac{2\big(e^{4-t} - e^{t-2}\big)}{e^2+1}, & t \in [2,3]. \end{cases}$$

From now on, we are going to ensure that these functions satisfy the sufficient optimality conditions studied in this section (see Theorem 3.7). So, for $t \in [0,3]$ we intend to find a function $S(t,x)$ that is a solution of equation (3.20) with $S\big(3, x(3)\big) = 0$. As $\eta(t) = \partial_2 S\big(t, x(t)\big)$, we obtain that

$$S(t,x) = \begin{cases} \eta_1(t)x + c_1(t), & t \in [0,1] \\ \eta_2(t)x + c_2(t), & t \in [1,2] \\ \eta_3(t)x + c_3(t), & t \in [2,3], \end{cases}$$

where $c_i(\cdot)$ is a real function of real variable, $i \in \{1,2,3\}$. For $t \in [2,3]$, the equation (3.20) implies that

$$\begin{aligned} & \dot{\eta_3}(t)x^*(t) + \dot{c_3}(t) - \big(x^{*2}(t) + u^{*2}(t)\big) + \eta_3(t)x^*(t-1)u^*(t-2) = 0 \\ \Leftrightarrow\ & \dot{\eta_3}(t)x_3^*(t) + \dot{c_3}(t) - \big(x_3^{*2}(t) + u_3^{*2}(t)\big) + \eta_3(t)x_2^*(t-1)u_1^*(t-2) = 0 \\ \Leftrightarrow\ & -\frac{2\left(e^{4-t} + e^{t-2}\right)}{e^2+1} \times \frac{e^{t-2} + e^{4-t}}{e^2+1} + \dot{c_3}(t) - \left(\frac{e^{t-2} + e^{4-t}}{e^2+1}\right)^2 \\ & + \eta_3(t) \times 1 \times \frac{e^{t-2} - e^{2-(t-2)}}{e^2+1} = 0 \\ \Leftrightarrow\ & \dot{c_3}(t) = \frac{5\big(e^{2t-4} + e^{8-2t}\big) + 2e^2}{\big(e^2+1\big)^2} \end{aligned} \tag{3.31}$$

with $S\big(3, x(3)\big) = c_3(3) = 0$. Solving the differential equation (3.31) with final condition $c_3(3) = 0$, we obtain that

$$c_3(t) = \frac{4e^2(t-3) + 5\big(e^{2t-4} - e^{8-2t}\big)}{2\big(e^2+1\big)^2}.$$

For $t \in [1,2]$, the equation (3.20) implies that

$$\begin{aligned}
&\dot{\eta_2}(t)x^*(t) + \dot{c_2}(t) - \big(x^{*2}(t) + u^{*2}(t)\big) + \eta_2(t)x^*(t-1)u^*(t-2) = 0 \\
\Leftrightarrow\ &\dot{\eta_2}(t)x_2^*(t) + \dot{c_2}(t) - \big(x_2^{*2}(t) + u_2^{*2}(t)\big) + \eta_2(t)x_1^*(t-1)u_0^*(t-2) = 0 \\
\Leftrightarrow\ &-\left(\frac{4e^2}{\big(e^2+1\big)^2} + 2\right) + \frac{2\big(e^{2t-2} + e^{6-2t}\big)}{\big(e^2+1\big)^2} + \dot{c_2}(t) - 1 + \eta_2(t) \times 1 \times 0 = 0 \\
\Leftrightarrow\ &\dot{c_2}(t) = -\frac{2\big(e^{2t-2} + e^{6-2t} - 5e^2\big) - 3\big(e^4+1\big)}{\big(e^2+1\big)^2} \qquad (3.32)
\end{aligned}$$

with $\eta_2(2)x_2^*(2) + c_2(2) = \eta_3(2)x_3^*(2) + c_3(2)$. Therefore, the previous condition is equivalent to

$$c_2(2) = c_3(2) = \frac{5\big(1-e^4\big) - 4e^2}{2\big(e^2+1\big)^2}. \qquad (3.33)$$

Solving the differential equation (3.32) with the condition (3.33), we have that

$$c_2(t) = \frac{2t\big(3e^4 + 10e^2 + 3\big) + 2\big(e^{6-2t} - e^{2t-2}\big) - 17e^4 - 44e^2 - 7}{2\big(e^2+1\big)^2}.$$

For $t \in [0,1]$, the equation (3.20) implies that

$$\begin{aligned}
&\dot{\eta_1}(t)x^*(t) + \dot{c_1}(t) - \big(x^{*2}(t) + u^{*2}(t)\big) + \eta_1(t)x^*(t-1)u^*(t-2) = 0 \\
\Leftrightarrow\ &\dot{\eta_1}(t)x_1^*(t) + \dot{c_1}(t) - \big(x_1^{*2}(t) + u_1^{*2}(t)\big) + \eta_1(t)x_0^*(t-1)u_0^*(t-2) = 0 \\
\Leftrightarrow\ &-2 + \dot{c_1}(t) - 1 - \left(\frac{e^t - e^{2-t}}{e^2+1}\right)^2 + \eta_1(t) \times 1 \times 0 = 0 \\
\Leftrightarrow\ &\dot{c_1}(t) = \frac{e^{4-2t} + e^{2t} + 3e^4 + 4e^2 + 3}{\big(e^2+1\big)^2} \qquad (3.34)
\end{aligned}$$

with $\eta_1(1)x_1^*(1) + c_1(1) = \eta_2(1)x_2^*(1) + c_2(1)$. Therefore, the previous condition is equivalent to

$$c_1(1) = c_2(1) = \frac{-9e^4 - 24e^2 - 3}{2\big(e^2+1\big)^2}. \qquad (3.35)$$

Solving the differential equation (3.34) with the condition (3.35), we obtain that

$$c_1(t) = \frac{2t\big(3e^4 + 4e^2 + 3\big) + e^{2t} - e^{4-2t} - 15e^4 - 32e^2 - 9}{2\big(e^2+1\big)^2}.$$

Concluding, the previous computations show the following result.

Proposition 3.10. *Function*

$$S(t,x) = \begin{cases} \eta_1(t)x + c_1(t), & t \in [0,1], \\ \eta_2(t)x + c_2(t), & t \in [1,2], \\ \eta_3(t)x + c_3(t), & t \in [2,3], \end{cases}$$

with

$$\eta_1(t) = -2t + 5 + \frac{2\big(e^2-1\big)}{\big(e^2+1\big)^2},$$
$$\eta_2(t) = -\left(\frac{4e^2}{\big(e^2+1\big)^2} + 2\right)t + \frac{4\big(e^2-1\big)}{\big(e^2+1\big)^2} + 6 + \frac{e^{2t-2} - e^{6-2t}}{\big(e^2+1\big)^2},$$
$$\eta_3(t) = \frac{2\big(e^{4-t} - e^{t-2}\big)}{e^2+1},$$

and

$$c_1(t) = \frac{2t\big(3e^4+4e^2+3\big) + e^{2t} - e^{4-2t} - 15e^4 - 32e^2 - 9}{2\big(e^2+1\big)^2},$$
$$c_2(t) = \frac{2t\big(3e^4+10e^2+3\big) + 2\big(e^{6-2t} - e^{2t-2}\big) - 17e^4 - 44e^2 - 7}{2\big(e^2+1\big)^2},$$
$$c_3(t) = \frac{4e^2(t-3) + 5\big(e^{2t-4} - e^{8-2t}\big)}{2\big(e^2+1\big)^2},$$

is solution of the Hamilton–Jacobi equation (3.20) *with* $S\big(3, x^*(3)\big) = 0$.

3.4 Conclusion

To the best of our knowledge, in this chapter we give answer to an open question, by proving sufficient optimality conditions for control problems with constant time delays in both state and control variables. The proof is based on the transformation of delayed optimal control problems into equivalent and augmented non-delayed ones, following the approach proposed in [59] and used in [56]. Analogously to [56], we ensure the Commensurability Assumption 2.15 between the, possibly different, state and control delays. Examples are provided with the purpose to illustrate the usefulness of obtained sufficient optimality conditions.

In the next chapter, we are going to do a brief explanation about an infectious disease that has caused a lot of deaths worldwide – cholera. We propose several mathematical models to translate the transmission dynamics of cholera, using different types of treatment and prevention measures. Later, such models are going to be incorporated in optimal control problems (see Sections 5.2.4, 5.3.4 and 6.3.4).

Chapter 4

Cholera mathematical models

We begin this chapter by explaining an infectious disease that remains a global threat to public health and an indicator of inequity and lack of social development – cholera (see [106, 107, 192]). The number of cholera cases reported by World Health Organization (WHO) has continued to be high over the last few years. During 2017, 1227391 cases were notified from 34 countries, including 5654 deaths (see [192]). Moreover, we give a general idea about what has already been done to understand the dynamics of cholera through the study of mathematical models. Next, we propose and explain several models that can translate the spread of cholera and that consider different types of treatment or prevention measures.

4.1 Introduction

Cholera is an acute diarrhoeal infectious disease caused by infection of the intestine with the bacterium *Vibrio cholerae*, which lives in an aquatic organism. There are 200 serogroups of the bacterium *Vibrio cholerae*, but only two of them (O1 and O139) are responsible for the cholera disease (see [35, 90]). They pass through and survive the gastric acid barrier of the stomach. Then, they penetrate the mucus lining that coats the intestinal epithelial (see [35, 142]). They colonize the intestine, producing enterotoxin which stimulates water and electrolyte secretion by the endothelial cells of the small intestine (see [35]). Cholera is a disease of poverty and closely linked to poor sanitation and a lack of clean drinking water (see [191]), remaining a global threat to public health, as we have mentioned before. The ingestion of contaminated food or water can cause cholera outbreaks, as proved by John Snow in 1854 (see [157]). Nevertheless, there are other ways of spreading. Susceptible individuals can also become infected if they contact with infectious individuals. If these individuals are at an increased risk of infection, they can transmit the disease to other people who live with them and are involved in food preparation or use water storage containers (see

e.g. [157]). An individual can be infective with or without symptoms which can appear from a few hours to 5 days, after infection. However, symptoms typically appear in 2–3 days (see [27]). Some symptoms are vomiting, leg cramps and copious, painless and watery diarrhoea. It is very important that infective individuals can get treatment as soon as possible, because without it they become dehydrated, suffering from acidosis and circulatory collapse. Even worse, this situation can lead to death within 12 to 24 hours (see [127, 157]). Some studies and experiments suggest that a recovered individual can be immune to the disease during a period of 3 to 10 years. On the other hand, recent researches suggest that immunity can be lost after a period of weeks to months (see [120, 157]). Diseases involving diarrhoea are the major cause of child mortality in developing countries, because the access to clean drinking water and sanitation is difficult (see [13]). Moreover, Sun et al. write in [168] that this disease has generated a great threat to human society and caused enormous morbidity and mortality with weak surveillance systems. Thus, it is very important to study mathematical models of the cholera spread in order to know how to curtail it.

Several mathematical models for the dynamics of cholera transmission have been studied since, at least, 1979 (see e.g. [20, 22, 33, 65, 72, 83, 120, 126, 127, 129, 130, 138, 157, 179] and references cited therein). In [120], the authors propose a SIR (Susceptible–Infectious–Recovered) type model. Such model considers two classes for the bacterial concentration (less-infectious and hyper-infectious) and two classes for the infective individuals (asymptomatic and symptomatic). The authors compare a cost-effective balance of multiple intervention methods of two endemic populations, using optimal control theory, parameter sensitivity analysis and numerical simulations. In [179], Wang and Modnak also consider a SIR type model with a class for the *Vibrio cholerae* concentration in the environment. The model incorporates three control measures: vaccination, therapeutic treatment and water sanitation. The stability analysis of equilibrium points is done when the controls are given by constant values. They also study a more general cholera model with time-dependent controls, proving existence of solution to an optimal control problem and deriving necessary optimality conditions based on Pontryagin's Maximum Principle. The authors of [127] incorporate in a SIR type model public health educational campaigns, vaccination, quarantine and treatment (as control strategies) with the purpose to curtail the disease. The model also considers a class for the bacterial concentration. The education-induced, vaccination-induced and treatment-induced reproduction numbers, as well as the combined reproduction number, are compared with the basic reproduction number to assess the possible community benefits of these strategies. The stability analysis of the equilibria is performed using a Lyapunov functional approach. In [157], a SIR type model with distributed delays is proposed. It incorporates hyperinfectivity (where infectivity varies with the time since the pathogen was shed) and temporary immunity. The

basic reproduction number is computed and it plays an important role to know if the disease dies out or not. Numerical simulations are carried out in order to illustrate important details of the unique endemic equilibrium's stability. In [178], Wang and Liao present a SIR type model with a class for the *Vibrio cholerae* concentration in the contaminated water. It is an unified deterministic model for cholera, because it considers a general incidence rate and a general formulation of the pathogen concentration. The basic reproduction number is computed and conditions are derived for the existence of the disease-free and endemic equilibrium points. The local asymptotic and global stability analysis of the equilibrium points are studied. The authors show that different models can be studied in a single framework, using three representative cholera models presented in [33, 65, 125]. A mathematical model that considers public health educational campaigns, vaccination, sanitation and treatment (as control strategies) is formulated in [42]. The reproduction number for the cases with single and combined controls is determined and compared. The authors conclude that, when one considers a single control measure, treatment yields the best results, followed by education campaigns, sanitation and vaccination; cf. the numerical simulations of [42]. Nevertheless, the more control strategies are considered, better results can be obtained. Furthermore, the authors perform a sensitivity analysis on the key parameters that drive the disease dynamics in order to find their relative importance to cholera's spread and prevalence. In [13], a SIR type model with a class for the bacterial concentration in the environment is proposed. Such model incorporates media coverage. The existence and stability of the equilibria is analysed. Numerical simulations suggest that the number of infections decreases faster, when media coverage is very efficient. So, media alert and awareness campaigns are crucial for controlling the spread of cholera. In [168], a SIR type mathematical model for cholera transmission is used to characterize the cholera spread in China. With the purpose of avoiding cholera outbreaks in China, the researchers suggest to increase the immunization coverage rate and to make efforts for improving environmental management, mainly for drinking water (see [168]).

The use of quarantine for controlling epidemic diseases has always been controversial, because such strategy raises ethical, socio-economic and political issues, requiring a careful balance between public interest and individual rights (see [171]). Quarantine was adopted as a mean of separating persons, animals and goods that may have been exposed to a contagious disease. Since the fourteenth century, quarantine has been the cornerstone of a coordinated disease-control strategy, including isolation, sanitary cordons, bills of health issued to ships, fumigation, disinfection and regulation of groups of persons who were believed to be responsible for spreading of the infection (see [116, 171]). The WHO does not recommend quarantine measures and embargoes on the movement of people and goods for cholera. However, cholera is still on the list of quarantinable diseases of the USA National

Archives and Records Administration (see [26]). Furthermore, cholera is one of the international quarantine infectious diseases, as stipulated by the International Health Regulations (see [35]).

Several cholera outbreaks have occurred since 2007, namely in Angola, Haiti, Zimbabwe and Yemen (see [4, 157, 188, 193]). The consequences of a humanitarian crisis, such as the disruption of water and sanitation systems or the displacement of populations to inadequate and overcrowded camps, can increase the risk of cholera transmission (see [192]).

As we have already mentioned before, optimal control theory is a branch of Mathematics developed to find optimal ways to control a dynamical system (see [28, 47, 140]). There are few papers that apply optimal control to cholera models (see e.g. [120]). With the purpose to enrich the mathematical research applied to cholera, later we are going to propose several cholera optimal control problems.

This chapter is organised as follows. In Section 4.2, we propose and explain two mathematical models to translate the dynamics of cholera transmission. Such models incorporate treatment through quarantine. In Section 4.3, we formulate and explain other cholera mathematical model that considers vaccination and a more general treatment. We end this chapter with some conclusions in Section 4.4.

4.2 Cholera mathematical models with quarantine

In this section, we present two mathematical models for the transmission of cholera whose treatment consists in isolating infective individuals who are also submitted to an appropriate medication. We say that these individuals are in quarantine.

For both proposed models, we consider a SIQRB (Susceptible–Infectious–Quarantined–Recovered–Bacterial) type mathematical model. Such models incorporate a class for the bacterial concentration in the environment with respect to dynamics of cholera. The total human population is divided into four classes: S (susceptible), I (infective with symptoms), Q (in treatment through quarantine) and R (recovered). Note that $N(t)$ gives the total human population at time $t \geq 0$:

$$N(t) = S(t) + I(t) + Q(t) + R(t).$$

Furthermore, we consider a class B that reflects the bacterial concentration in the environment, i.e., the water.

Remark 4.1. *From now on, sometimes we are going to write simply "*infective individuals*" instead of "*infective individuals with symptoms*". Nevertheless, in the context of proposed models of Chapters 4, 5 and 6, we are going to consider* infective individuals with symptoms *all the time.*

We assume that there is a constant recruitment rate $\Lambda > 0$ into the susceptible class S and a constant natural death rate $\mu > 0$ for all time $t \geq 0$ under study. Susceptible individuals can become infected with cholera by ingestion of bacteria from the environment at rate $\frac{\beta B(t)}{\kappa + B(t)} \geq 0$ that is dependent on time $t \geq 0$. Note that $\beta > 0$ is the ingestion rate of the bacteria through contaminated sources, $\kappa > 0$ is the half saturation constant of the bacteria population and $\frac{B(t)}{\kappa + B(t)}$ is the likeliness of an infective individual to have the disease with symptoms, given a contact with contaminated sources (see [127]). Any recovered individual can lose the immunity at rate $\omega_1 \geq 0$ and therefore becomes susceptible again. Moreover, it is assumed that infective individuals are subject to quarantine during the treatment period. During this time they are isolated and subject to a proper medication at rate $\delta \geq 0$. The quarantined individuals can recover at rate $\varepsilon \geq 0$. The disease-related death rates associated with the individuals that are infective and in quarantine are $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$, respectively. In class R people are not ill and, therefore, the disease-related mortality is not present. Note that the individuals in classes I, Q and R can also die at natural death rate $\mu > 0$. In the environment, bacteria can not survive, being $d > 0$ their mortality rate. Nevertheless, each infective individual contributes to the increase of the bacterial concentration at rate $\eta > 0$. Indeed, within the body of the infective, they reproduce and this is the cause of the illness. Then, the bacteria are released in the open environment. A similar phenomenon could occur for the quarantined individuals. They are still subject to the disease, but as they are isolated and treated in the hospitals, it is assumed that measures are taken so that they can not propagate the infection. In particular, they are prevented from fouling the water with new bacteria coming from the dejections of their bodies.

In the first model, we consider a time delay, $\tau \geq 0$, that is related with the passage of individuals from class S to class I. This time delay represents the time between the instant in which an individual becomes infected and the instant in which he begins to show symptoms. The introduction of this delay is done with the goal to better approximate the reality (see Section 5.3.5: *Delayed SIB sub-model*). The symptoms of cholera can appear from a few hours to 5 days after infection. Nevertheless, they typically appear in 2–3 days (see [27]). Thus, the first proposed model is translated by:

$$\begin{cases} \dot{S}(t) = \Lambda - \dfrac{\beta B(t) S(t)}{\kappa + B(t)} + \omega_1 R(t) - \mu S(t), \\ \dot{I}(t) = \dfrac{\beta B(t-\tau) S(t-\tau)}{\kappa + B(t-\tau)} - (\delta + \alpha_1 + \mu) I(t), \\ \dot{Q}(t) = \delta I(t) - (\varepsilon + \alpha_2 + \mu) Q(t), \\ \dot{R}(t) = \varepsilon Q(t) - (\omega_1 + \mu) R(t), \\ \dot{B}(t) = \eta I(t) - dB(t). \end{cases} \tag{4.1}$$

Note that the respective non-delayed model is easily obtained, when $\tau = 0$. In Figure 4.1, model (4.1) is presented in a schematic way.

In the second model we do not consider time delays ($\tau = 0$) and we also assume that, to become infected, a healthy individual must intake bacteria from the environment. Consequently, these bacteria are removed from the aquatic medium. In other words, we suppose that there is an uptake of bacteria from the water by healthy individuals during the infection process. This feature, absent in model (4.1) with $\tau = 0$, must be incorporated in the model, to have a meaningful formulation. Thus, the second proposed model is translated by:

$$\begin{cases} \dot{S}(t) = \Lambda - \dfrac{\beta B(t) S(t)}{\kappa + B(t)} + \omega_1 R(t) - \mu S(t), \\ \dot{I}(t) = \dfrac{\beta B(t) S(t)}{\kappa + B(t)} - (\delta + \alpha_1 + \mu) I(t), \\ \dot{Q}(t) = \delta I(t) - (\varepsilon + \alpha_2 + \mu) Q(t), \\ \dot{R}(t) = \varepsilon Q(t) - (\omega_1 + \mu) R(t), \\ \dot{B}(t) = \eta I(t) - dB(t) - \dfrac{\beta B(t) S(t)}{\kappa + B(t)}. \end{cases} \tag{4.2}$$

Note that model (4.1) with $\tau = 0$ is improved by the previous model, since the removal of the intake bacteria from the environment by susceptible individuals is not contemplated for model (4.1) with $\tau = 0$. In Figure 4.2, model (4.2) is presented in a schematic way.

4.3 Cholera mathematical model with vaccination

In this section, we add a vaccination class to model (4.1), considering $\tau = 0$ and different kinds of treatment for cholera. Thus, for the dynamics of cholera transmission we propose a SITRVB (Susceptible–Infectious–Treated–Recovered–Vaccinated–Bacterial) type mathematical model. The

total human population is divided into five classes: S (susceptible), I (infective with symptoms), T (in treatment), R (recovered) and V (vaccinated). Again, we consider a class B that reflects the bacterial concentration in the environment – water. Note that $N(t)$ gives the total human population at time $t \geq 0$:

$$N(t) = S(t) + I(t) + T(t) + R(t) + V(t).$$

Susceptible individuals can be vaccinated at rate $\varphi \geq 0$. Any vaccinated individual can die at natural mortality rate $\mu > 0$ and can lose the immunity at rate $\omega_2 \geq 0$, becoming susceptible again. Different types of treatment for cholera infective individuals are considered based on [186, 192]. These assumptions are translated into the following mathematical model:

$$\begin{cases} \dot{S}(t) = \Lambda - \dfrac{\beta B(t) S(t)}{\kappa + B(t)} + \omega_1 R(t) + \omega_2 V(t) - (\varphi + \mu) S(t), \\ \dot{I}(t) = \dfrac{\beta B(t) S(t)}{\kappa + B(t)} - (\delta + \alpha_1 + \mu) I(t), \\ \dot{T}(t) = \delta I(t) - (\varepsilon + \alpha_2 + \mu) T(t), \\ \dot{R}(t) = \varepsilon T(t) - (\omega_1 + \mu) R(t), \\ \dot{V}(t) = \varphi S(t) - (\omega_2 + \mu) V(t), \\ \dot{B}(t) = \eta I(t) - d B(t). \end{cases} \tag{4.3}$$

In Figure 4.3, model (4.3) is presented in a schematic way.

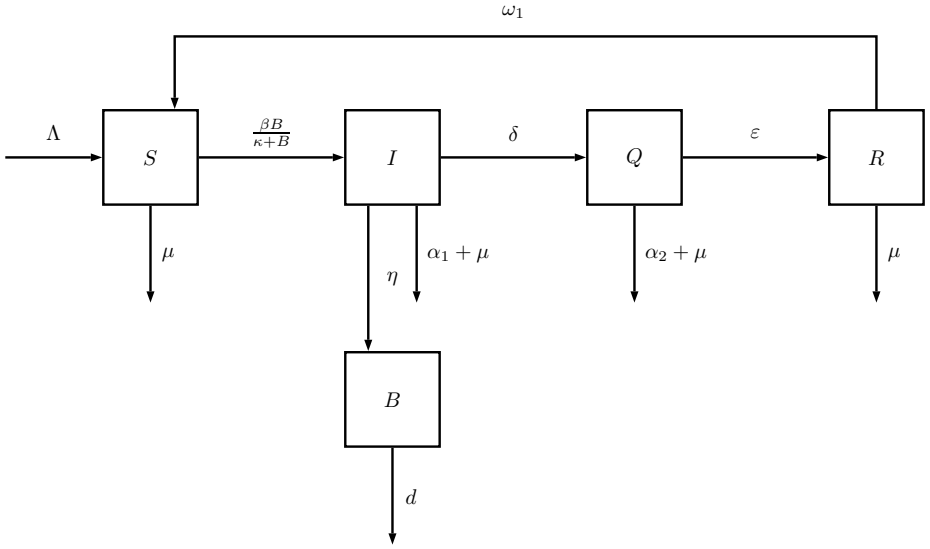


Figure 4.1: Diagram of the dynamical model (4.1).

A brief description of all used parameters is given in Table 4.1.

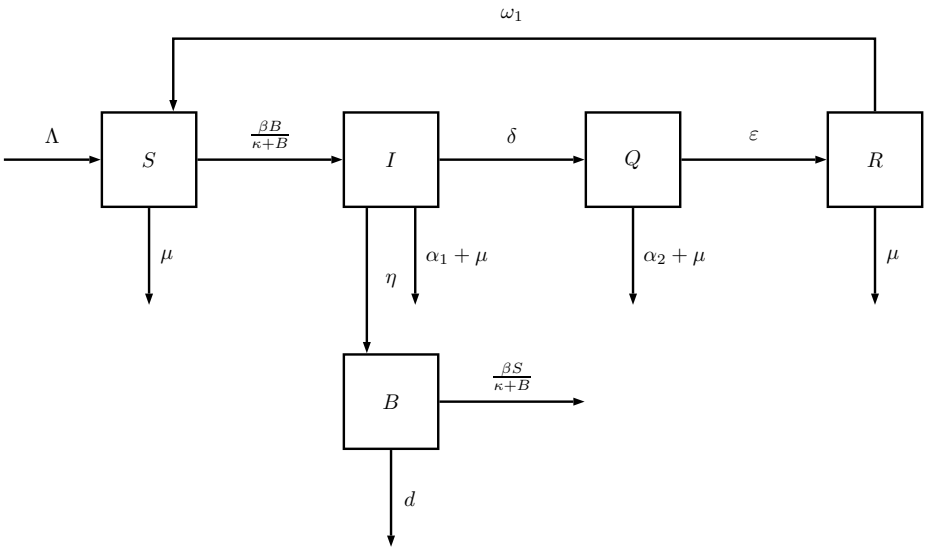


Figure 4.2: Diagram of the dynamical model (4.2).

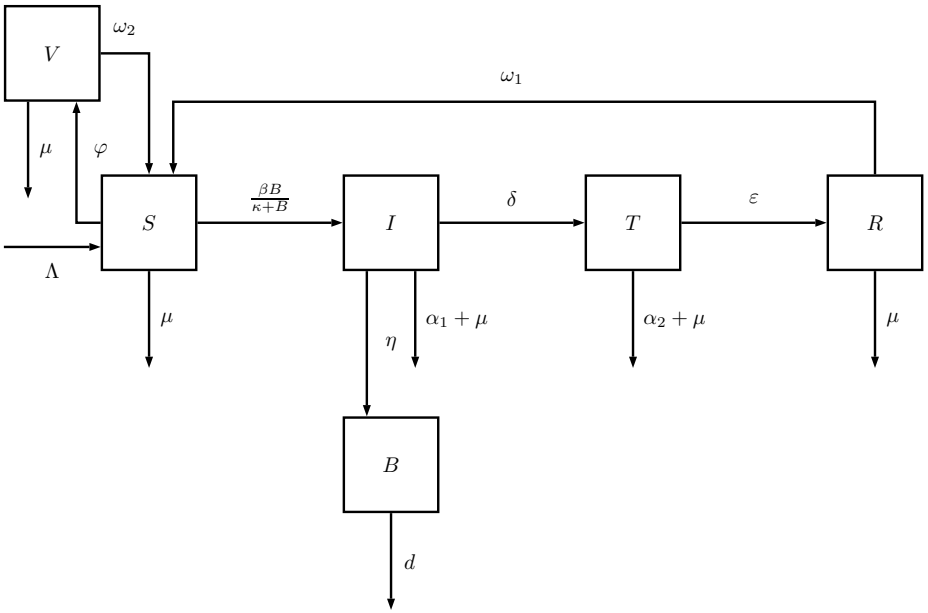


Figure 4.3: Diagram of the dynamical model (4.3).

4.4 Conclusion

In this chapter, we give some explanations about cholera transmission and a state of the art on mathematical models for this disease. After, new mathematical models for cholera were proposed.

In the next two chapters, we analyse mathematically each of the new

Parameter	Description
Λ	Recruitment rate
μ	Natural death rate
β	Ingestion rate
κ	Half saturation constant
ω_1	Immunity waning rate
ω_2	Efficacy vaccination waning rate
φ	Vaccination rate
δ	Quarantine rate
ε	Recovery rate
α_1	Death rate (infective)
α_2	Death rate (in quarantine/treatment)
η	Shedding rate (from infective)
d	Bacteria death rate
τ	Time delay
t_f	Final time
S_0	Susceptible individuals for $t \in [-\tau, 0]$
I_0	Infective individuals for $t \in [-\tau, 0]$
Q_0	Individuals in quarantine for $t \in [-\tau, 0]$
T_0	Individuals in treatment for $t \in [-\tau, 0]$
R_0	Recovered individuals for $t \in [-\tau, 0]$
V_0	Vaccinated individuals for $t \in [-\tau, 0]$
B_0	Bacterial concentration for $t \in [-\tau, 0]$
W	Weight constant for the treatment cost

Table 4.1: Description of parameters.

models, by determining their basic reproduction numbers and equilibrium points. The stability study of each equilibrium point is also carried out. For some models we are going to formulate corresponding optimal control problems and derive the respective necessary optimality conditions. Moreover, we are going to fit well-known cholera outbreaks to the models and then to apply optimal control theory in order to obtain measures that could improve the consequences of these cholera's outbreaks. It is really important to know and understand better such control strategies, because they can be a useful tool for health authorities and policy makers, when similar outbreaks occur.

Chapter 5

Optimal control of cholera outbreak in Haiti

In this chapter we consider and study two types of cholera mathematical models with quarantine. The second one is obtained from the first by adding a time delay that represents the time that a susceptible individual, who got infected, takes to have symptoms of the infection by the bacterium *Vibrio cholerae*. We prove that both models are biologically meaningful. Moreover, we determine the equilibrium points and the basic reproduction number. The local asymptotic stability of such points is also analysed. For both, it is assumed that infective individuals are subject to quarantine during the treatment period. We also propose and analyse non-delayed and delayed optimal control problems, where the control function represents the fraction of infective individuals that will be submitted to treatment in quarantine until complete recovery. The goal is to find the treatment strategy through quarantine that minimizes the number of infective individuals and the bacterial concentration, as well as the cost of interventions associated with quarantine. Finally, numerical simulations associated with cholera outbreak that occurred in the Department of Artibonite – Haiti, in 2010, are carried out. We show that the delayed cholera model fits better the Haiti cholera outbreak, than the non-delayed. Considering the data of the cholera outbreak in Haiti, we solve, numerically, several optimal control problems and propose solutions for outbreak control and eradication. All the contents of Section 5.2 are published in [106].

5.1 Introduction

Mathematical models have been developed and studied in order to understand the dynamics of cholera transmission, mostly focusing on the epidemic that occurred in Haiti, 2010–2011 (see [4]). The first cases were reported in the Department of Artibonite, on 14th October 2010. The disease prop-

agated along the Artibonite river and reached several departments. Only within one month, all departments had reported cases in rural areas and places without good conditions of public health (see [193]). In this chapter, we study cholera mathematical models and corresponding optimal control problems, with the purpose to simulate what happened in Haiti and to obtain solutions for the outbreak control and eradication. Such simulations illustrate the usefulness of the models and their analysis. We use the data of the cholera outbreak that occurred in the Department of Artibonite – Haiti, from 1st November 2010 to 1st May 2011 (see [193]). Many studies have been developed with the purpose to find and evaluate measures to contain the cholera spread. Nevertheless, it was not possible to obtain yet solutions in real time that can stop the cholera epidemics (see [4]).

This chapter is organised as follows. In Section 5.2, we consider a non-delayed SIQRB type mathematical model for cholera with treatment through quarantine. The model is shown to be both epidemiologically and mathematically well posed, because every solution of the considered model with initial conditions in a certain meaningful set remains in that set for all time (see Section 5.2.1). The existence of unique disease-free and endemic equilibrium points is proved and the basic reproduction number is computed (see Section 5.2.2). Then, we study the local asymptotic stability of these equilibrium points, in Section 5.2.3. A non-delayed optimal control problem is proposed and analysed, whose goal is to obtain a successful treatment through quarantine (see Section 5.2.4). The respective necessary optimality conditions are derived, according to Pontryagin's Minimum Principle (see [140]). In Section 5.2.5, we provide numerical simulations for the cholera outbreak in the Department of Artibonite – Haiti, from 1st November 2010 until 1st May 2011 (see [193]). More precisely, we show that the number of infective individuals decreases significantly and that the bacterial concentration is a strictly decreasing function, when our control strategy is applied. Furthermore, we provide the quarantine strategy (through the extremal solution for the control) for the minimization of the number of infective individuals and the bacterial concentration, as well as the costs associated with the quarantine. As the symptoms of cholera can not appear immediately after the infection (see [27]), we propose in Section 5.3 an improvement of the work done in Section 5.2, by considering and analysing a delayed SIQRB model based on that of Section 5.2. More precisely, we introduce a discrete time delay that represents the time between the instant at which an individual becomes infected and the instant at which he begins to have symptoms of cholera. The delayed model is analysed, proving the non-negativity of the solutions for non-negative initial conditions (see Section 5.3.1). The equilibrium points and the basic reproduction number are the same of those obtained in Section 5.2.2. The local asymptotic stability of the equilibrium points is analysed for non-negative time delays, in Section 5.3.3. Concretely, the stability analysis of the endemic equilibrium is carried out as function

of the ingestion rate of the bacteria through contaminated sources β. In Section 5.3.4, we formulate and analyse an optimal control problem with a non-negative state delay and with linear or quadratic cost functionals with respect to control variable. The control function and the delayed optimal control problem have, respectively, the same meaning and the same goal of those considered in Section 5.2.4. We also apply the Minimum Principle for delayed optimal control problems (see [56, 57]), deriving the respective necessary optimality conditions. In Section 5.3.5, we consider again the cholera outbreak that occurred in the Department of Artibonite – Haiti, improving the numerical simulations done in Section 5.2.5 by considering a positive time delay, treatment and recovery. We also consider different non-delayed and delayed optimal control problems and compute extremal solutions using discretization and non-linear programming methods. We interpret the numerical solutions from an epidemiological point of view. We finish this chapter with some conclusions, in Section 5.4.

We believe that the work of Sections 5.2 and 5.3 is of great significance, because it provides an approach to cholera with big positive impact on the number of infective individuals and on the bacterial concentration. This is well illustrated with the real data of the cholera outbreak that occured in the Department of Artibonite – Haiti, from 1st November 2010 to 1st May 2011 (see [193]).

5.2 Non-delayed model with quarantine

In this section we consider model (4.1) with $\tau = 0$, given by:

$$\begin{cases} \dot{S}(t) = \Lambda - \dfrac{\beta B(t) S(t)}{\kappa + B(t)} + \omega_1 R(t) - \mu S(t), \\ \dot{I}(t) = \dfrac{\beta B(t) S(t)}{\kappa + B(t)} - (\delta + \alpha_1 + \mu) I(t), \\ \dot{Q}(t) = \delta I(t) - (\varepsilon + \alpha_2 + \mu) Q(t), \\ \dot{R}(t) = \varepsilon Q(t) - (\omega_1 + \mu) R(t), \\ \dot{B}(t) = \eta I(t) - d B(t). \end{cases} \tag{5.1}$$

Moreover, throughout this section, we assume that the initial conditions of system (5.1) are non-negative:

$$S(0) \geq 0\,, \quad I(0) \geq 0\,, \quad Q(0) \geq 0\,, \quad R(0) \geq 0\,, \quad B(0) \geq 0\,. \tag{5.2}$$

Here, S_0, I_0, Q_0, R_0, B_0 and N_0 denote $S(0)$, $I(0)$, $Q(0)$, $R(0)$, $B(0)$ and $S(0) + I(0) + Q(0) + R(0)$, respectively, because $\tau = 0$ (see Table 4.1). In this section, we also consider some notations in order to simplify the writing.

Namely, a_1, a_2 and a_3 are defined as follows:

$$a_1 = \delta + \alpha_1 + \mu, \quad a_2 = \varepsilon + \alpha_2 + \mu \text{ and } a_3 = \omega_1 + \mu. \tag{5.3}$$

5.2.1 Non-negativity and boundedness of solutions

Our first lemma shows that model (5.1)–(5.2) is biologically meaningful.

Lemma 5.1. *The solutions* $\big(S(t), I(t), Q(t), R(t), B(t)\big)$ *of system* (5.1) *are non-negative for all* $t \geq 0$ *with non-negative initial conditions* (5.2).

Proof. We have

$$\begin{cases} \left.\dfrac{dS(t)}{dt}\right|_{\xi(S)} = \Lambda + \omega_1 R(t) > 0\,, \\ \left.\dfrac{dI(t)}{dt}\right|_{\xi(I)} = \dfrac{\beta B(t) S(t)}{\kappa + B(t)} \geq 0\,, \\ \left.\dfrac{dQ(t)}{dt}\right|_{\xi(Q)} = \delta I(t) \geq 0\,, \\ \left.\dfrac{dR(t)}{dt}\right|_{\xi(R)} = \varepsilon Q(t) \geq 0\,, \\ \left.\dfrac{dB(t)}{dt}\right|_{\xi(B)} = \eta I(t) \geq 0\,, \end{cases}$$

where $\xi(\upsilon) = \big\{\upsilon(t) = 0 \text{ and } S(\cdot), I(\cdot), Q(\cdot), R(\cdot), B(\cdot) \in \mathscr{C}\left(\mathbb{R}_0^+, \mathbb{R}_0^+\right)\big\}$ and $\upsilon \in \{S, I, Q, R, B\}$. So, due to Lemma 1.42, we can conclude that any solution of system (5.1) is such that $\big(S(t), I(t), Q(t), R(t), B(t)\big) \in \left(\mathbb{R}_0^+\right)^5$ for all time $t \geq 0$. This concludes the proof. □

Next Lemma 5.2 shows that it is sufficient to consider the dynamics of the flow generated by (5.1)–(5.2) in a certain region Ω.

Lemma 5.2. *Let*

$$\Omega_H = \left\{(S, I, Q, R) \in \left(\mathbb{R}_0^+\right)^4 \,|\, 0 \leq S(t) + I(t) + Q(t) + R(t) \leq \frac{\Lambda}{\mu}\right\} \tag{5.4}$$

and

$$\Omega_B = \left\{B \in \mathbb{R}_0^+ \,|\, 0 \leq B(t) \leq \frac{\Lambda\eta}{\mu d}\right\}. \tag{5.5}$$

Define

$$\Omega = \Omega_H \times \Omega_B. \tag{5.6}$$

If $N(0) \leq \frac{\Lambda}{\mu}$ *and* $B(0) \leq \frac{\Lambda\eta}{\mu d}$, *then the region* Ω *is positively invariant for model* (5.1) *with non-negative initial conditions* (5.2) *in* $\left(\mathbb{R}_0^+\right)^5$.

Proof. Let us split system (5.1) into two parts: the human population, i.e., $S(t)$, $I(t)$, $Q(t)$ and $R(t)$, and the pathogen population, i.e., $B(t)$. Adding the first four equations of system (5.1) gives

$$\begin{aligned}\dot{N}(t) &= \dot{S}(t) + \dot{I}(t) + \dot{Q}(t) + \dot{R}(t) \\ &= \Lambda - \mu N(t) - \alpha_1 I(t) - \alpha_2 Q(t) \leq \Lambda - \mu N(t)\,.\end{aligned}$$

Assuming that $N(0) \leq \frac{\Lambda}{\mu}$, we conclude that $N(t) \leq \frac{\Lambda}{\mu}$. For this reason, (5.4) defines the biologically feasible region for the human population. For the pathogen population, it follows that

$$\dot{B}(t) = \eta I(t) - dB(t) \leq \eta\frac{\Lambda}{\mu} - dB(t)\,.$$

If $B(0) \leq \frac{\Lambda\eta}{\mu d}$, then $B(t) \leq \frac{\Lambda\eta}{\mu d}$ and, in agreement, (5.5) defines the biologically feasible region for the pathogen population. From (5.4) and (5.5), we know that $N(t)$ and $B(t)$ are bounded for all $t \geq 0$. Therefore, every solution of system (5.1) with initial condition in Ω remains in Ω. This concludes the proof. □

In region Ω defined by (5.6), our model is epidemiologically and mathematically well posed, in the sense of [69]. In other words, every solution of model (5.1), with initial conditions in Ω, remains in Ω for all $t \geq 0$.

5.2.2 Equilibrium points and the basic reproduction number

The disease-free equilibrium (DFE) of model (5.1) is given by

$$E^0 = \left(S^0, I^0, Q^0, R^0, B^0\right) = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0\right). \tag{5.7}$$

Next, following the approach of [127, 176] (see Section 1.5), we compute the basic reproduction number R_0.

Proposition 5.3 (Basic reproduction number of (5.1))**.** *The basic reproduction number of model* (5.1) *is given by*

$$R_0 = \frac{\beta\Lambda\eta}{\mu\kappa d(\delta + \alpha_1 + \mu)}. \tag{5.8}$$

Proof. Consider that $\mathscr{F}_i(t)$ is the rate of appearance of new infections in the compartment associated with index i, $\mathscr{V}_i^+(t)$ is the rate of transfer of "individuals" into the compartment associated with index i by all other means and $\mathscr{V}_i^-(t)$ is the rate of transfer of "individuals" out of compartment associated with index i. In this way, recalling (5.3), the matrices $\mathscr{F}(t)$, $\mathscr{V}^+(t)$

and $\mathscr{V}^-(t)$ associated with model (5.1) are given by

$$\begin{cases} \mathscr{F}(t) = \begin{bmatrix} 0 & \dfrac{\beta B(t)S(t)}{\kappa + B(t)} & 0 & 0 & 0 \end{bmatrix}^T, \\ \mathscr{V}^+(t) = \begin{bmatrix} \Lambda + \omega_1 R(t) & 0 & \delta I(t) & \varepsilon Q(t) & \eta I(t) \end{bmatrix}^T, \\ \mathscr{V}^-(t) = \begin{bmatrix} \dfrac{\beta B(t)S(t)}{\kappa + B(t)} + \mu S(t) & a_1 I(t) & a_2 Q(t) & a_3 R(t) & dB(t) \end{bmatrix}^T. \end{cases}$$

Therefore, considering $\mathscr{V}(t) = \mathscr{V}^-(t) - \mathscr{V}^+(t)$, we have that

$$\begin{bmatrix} \dot{S}(t) & \dot{I}(t) & \dot{Q}(t) & \dot{R}(t) & \dot{B}(t) \end{bmatrix}^T = \mathscr{F}(t) - \mathscr{V}(t).$$

The Jacobian matrices of $\mathscr{F}(t)$ and of $\mathscr{V}(t)$ are, respectively, given by

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \dfrac{\beta B(t)}{\kappa + B(t)} & 0 & 0 & 0 & \dfrac{\beta \kappa S(t)}{\big(\kappa + B(t)\big)^2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$V = \begin{bmatrix} \dfrac{\beta B(t)}{\kappa + B(t)} + \mu & 0 & 0 & -\omega_1 & \dfrac{\beta \kappa S(t)}{\big(\kappa + B(t)\big)^2} \\ 0 & a_1 & 0 & 0 & 0 \\ 0 & -\delta & a_2 & 0 & 0 \\ 0 & 0 & -\varepsilon & a_3 & 0 \\ 0 & -\eta & 0 & 0 & d \end{bmatrix}.$$

At the disease-free equilibrium E^0 (5.7), we obtain the matrices F_0 and V_0 given by

$$F_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{\beta\Lambda}{\mu\kappa} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad V_0 = \begin{bmatrix} \mu & 0 & 0 & -\omega_1 & \dfrac{\beta\Lambda}{\mu\kappa} \\ 0 & a_1 & 0 & 0 & 0 \\ 0 & -\delta & a_2 & 0 & 0 \\ 0 & 0 & -\varepsilon & a_3 & 0 \\ 0 & -\eta & 0 & 0 & d \end{bmatrix}.$$

The basic reproduction number of model (5.1) is then given by

$$R_0 = \rho\left(F_0 V_0^{-1}\right) = \frac{\beta\Lambda\eta}{\mu\kappa d a_1} = \frac{\beta\Lambda\eta}{\mu\kappa d(\delta + \alpha_1 + \mu)},$$

which is obtained with the help of the computer algebra system Maple. This concludes the proof. □

Next we prove the existence of an endemic equilibrium, when the basic reproduction number R_0, given by (5.8), is greater than one.

Proposition 5.4 (Endemic equilibrium)**.** *Assume that* λ^*, δ, ε, $\omega_1 > 0$. *If* $R_0 > 1$, *then model* (5.1) *has an endemic equilibrium given by*

$$\begin{aligned} E^* &= (S^*, I^*, Q^*, R^*, B^*) \\ &= \left(\frac{\Lambda a_1 a_2 a_3}{D}, \frac{\Lambda a_2 a_3 \lambda^*}{D}, \frac{\Lambda \delta a_3 \lambda^*}{D}, \frac{\Lambda \delta \varepsilon \lambda^*}{D}, \frac{\Lambda \eta a_2 a_3 \lambda^*}{Dd} \right), \end{aligned} \tag{5.9}$$

where we use notation (5.3) *and define* D *and* λ^* *as*

$$D = a_1 a_2 a_3 (\lambda^* + \mu) - \delta \varepsilon \omega_1 \lambda^* \quad \textit{and} \quad \lambda^* = \frac{\beta B^*}{\kappa + B^*}. \tag{5.10}$$

Proof. In order to exist disease, the rate of infection must satisfy the inequality $\frac{\beta B(t)}{\kappa + B(t)} > 0$. Considering that $E^* = (S^*, I^*, Q^*, R^*, B^*)$ is an endemic equilibrium of (5.1), let us define λ^* to be the rate of infection in the presence of disease, that is,

$$\lambda^* = \frac{\beta B^*}{\kappa + B^*}.$$

Using (5.3), considering $D = a_1 a_2 a_3 (\lambda^* + \mu) - \delta \varepsilon \omega_1 \lambda^*$ and setting the left-hand side of the equations of (5.1) equal to zero, we obtain the endemic equilibrium (5.9). Thus, we can compute λ^*:

$$\begin{aligned} \lambda^* = \frac{\beta B^*}{\kappa + B^*} &= \frac{\beta \Lambda \eta a_2 a_3 \lambda^*}{\kappa D d + \Lambda \eta a_2 a_3 \lambda^*} \\ \Leftrightarrow \lambda^* \left(1 - \frac{\beta \Lambda \eta a_2 a_3}{\kappa D d + \Lambda \eta a_2 a_3 \lambda^*} \right) &= 0 \\ \Leftrightarrow \lambda^* \left(\frac{\kappa D d + \Lambda \eta a_2 a_3 \lambda^* - \beta \Lambda \eta a_2 a_3}{\kappa D d + \Lambda \eta a_2 a_3 \lambda^*} \right) &= 0. \end{aligned}$$

The solution $\lambda^* = 0$ does not make sense in this context. Therefore, we only consider the solution of $\kappa D d + \Lambda \eta a_2 a_3 \lambda^* - \beta \Lambda \eta a_2 a_3 = 0$. We have,

$$\begin{aligned} & \kappa D d + \Lambda \eta a_2 a_3 \lambda^* - \beta \Lambda \eta a_2 a_3 = 0 \\ \Leftrightarrow\ & \kappa \big(a_1 a_2 a_3 (\lambda^* + \mu) - \delta \varepsilon \omega_1 \lambda^* \big) d + \Lambda \eta a_2 a_3 \lambda^* - \beta \Lambda \eta a_2 a_3 = 0 \\ \Leftrightarrow\ & \big(\kappa (a_1 a_2 a_3 - \delta \varepsilon \omega_1) d + \Lambda \eta a_2 a_3 \big) \lambda^* = -\kappa a_1 a_2 a_3 \mu d + \beta \Lambda \eta a_2 a_3 \\ \Leftrightarrow\ & \lambda^* = \frac{a_2 a_3 (\beta \Lambda \eta - \mu \kappa d a_1)}{\kappa (a_1 a_2 a_3 - \delta \varepsilon \omega_1) d + \Lambda \eta a_2 a_3} = \frac{\mu \kappa d a_1 a_2 a_3 (R_0 - 1)}{\kappa d (a_1 a_2 a_3 - \delta \varepsilon \omega_1) + \Lambda \eta a_2 a_3}. \end{aligned}$$

Note that $a_1 a_2 a_3 - \delta \varepsilon \omega_1 = (\delta + \alpha_1 + \mu)(\varepsilon + \alpha_2 + \mu)(\omega_1 + \mu) - \delta \varepsilon \omega_1 > 0$, because $\alpha_1, \alpha_2 \geq 0$ and $\mu > 0$. Furthermore, since κ, d, Λ, $\eta > 0$, we have that $\mu \kappa d a_1 a_2 a_3$ and $\kappa (a_1 a_2 a_3 - \delta \varepsilon \omega_1) d + \Lambda \eta a_2 a_3$ are positive. Concluding, if $R_0 > 1$, then $\lambda^* > 0$ and, consequently, model (5.1) has an endemic equilibrium given by (5.9). This concludes the proof. □

5.2.3 Stability analysis

We begin this section by analysing the local asymptotic stability of the disease-free equilibrium E^0 of model (5.1), given by (5.7).

Theorem 5.5 (Local asymptotic stability of (5.7))**.** *The disease-free equilibrium E^0 of model* (5.1) *is locally asymptotic stable, if $R_0 < 1$.*

Proof. The characteristic polynomial associated with the linearised system of model (5.1) is given by

$$p(\chi) = \det(F_0 - V_0 - \chi I_5).$$

In order to compute the roots of polynomial p, we have that

$$\begin{vmatrix} -\mu-\chi & 0 & 0 & \omega_1 & -\dfrac{\beta\Lambda}{\mu\kappa} \\ 0 & -a_1-\chi & 0 & 0 & \dfrac{\beta\Lambda}{\mu\kappa} \\ 0 & \delta & -a_2-\chi & 0 & 0 \\ 0 & 0 & \varepsilon & -a_3-\chi & 0 \\ 0 & \eta & 0 & 0 & -d-\chi \end{vmatrix} = 0,$$

that is,

$$\chi = -\mu \vee \chi = -a_2 \vee \chi = -a_3 \vee \tilde{p}(\chi) = \chi^2 + (a_1+d)\chi + a_1 d - \frac{\beta\Lambda\eta}{\mu\kappa} = 0.$$

By Routh–Hurwitz Criterion (see Theorem 1.27), if all coefficients of polynomial $\tilde{p}(\chi)$ have the same signal, then the roots of $\tilde{p}(\chi)$ have negative real part and, consequently, the DFE E^0 is locally asymptotic stable. The coefficients of $\tilde{p}(\chi)$ are $\tilde{p}_2 = 1 > 0$, $\tilde{p}_1 = a_1 + d > 0$ and $\tilde{p}_0 = a_1 d - \frac{\beta\Lambda\eta}{\mu\kappa}$. Note that

$$a_1 d - \frac{\beta\Lambda\eta}{\mu\kappa} > 0 \Leftrightarrow \beta\Lambda\eta < \mu\kappa d a_1 \Leftrightarrow \frac{\beta\Lambda\eta}{\mu\kappa d a_1} < 1 \Leftrightarrow R_0 < 1.$$

Therefore, DFE (5.7) is locally asymptotic stable, if $R_0 < 1$. This concludes the proof. □

With respect to model (5.1), we are going to study the local asymptotic stability of its endemic equilibrium E^* (see (5.9)) and, moreover, the instability of its disease-free equilibrium E^0 (see (5.7)) for $R_0 > 1$. Our proof is based on the *Center Manifold Theory* (see [24]), as described in Theorem 1.25.

Theorem 5.6 (Instability of (5.7) and local asymptotic stability of (5.9))**.** *The equilibrium points E^0 and E^* of model* (5.1) *(see* (5.7) *and* (5.9)*) are, respectively, unstable and locally asymptotic stable for $R_0 > 1$.*

Proof. To apply the method described in Theorem 1.25, we consider a change of variables. Let

$$X = (x_1, x_2, x_3, x_4, x_5) = (S, I, Q, R, B). \tag{5.11}$$

Consequently, we have that the total number of individuals is $N = \sum_{i=1}^{4} x_i$. Thus, model (5.1) can be written as follows:

$$\begin{cases} \dot{x}_1(t) = f_1\big(X(t)\big) = \Lambda - \dfrac{\beta x_1(t)x_5(t)}{\kappa + x_5(t)} + \omega_1 x_4(t) - \mu x_1(t) \\ \dot{x}_2(t) = f_2\big(X(t)\big) = \dfrac{\beta x_1(t)x_5(t)}{\kappa + x_5(t)} - (\delta + \alpha_1 + \mu)x_2(t) \\ \dot{x}_3(t) = f_3\big(X(t)\big) = \delta x_2(t) - (\varepsilon + \alpha_2 + \mu)x_3(t) \\ \dot{x}_4(t) = f_4\big(X(t)\big) = \varepsilon x_3(t) - (\omega_1 + \mu)x_4(t) \\ \dot{x}_5(t) = f_5\big(X(t)\big) = \eta x_2(t) - d x_5(t). \end{cases} \tag{5.12}$$

Choosing β^* as bifurcation parameter and solving for β from $R_0 = 1$, we obtain that

$$\beta^* = \frac{\mu\kappa d(\delta + \alpha_1 + \mu)}{\Lambda\eta}.$$

Considering $\beta = \beta^*$, the Jacobian of system (5.12) evaluated at E^0 is given by

$$J_0^* = \begin{bmatrix} -\mu & 0 & 0 & \omega_1 & -\dfrac{a_1 d}{\eta} \\ 0 & -a_1 & 0 & 0 & \dfrac{a_1 d}{\eta} \\ 0 & \delta & -a_2 & 0 & 0 \\ 0 & 0 & \varepsilon & -a_3 & 0 \\ 0 & \eta & 0 & 0 & -d \end{bmatrix}.$$

The eigenvalues of J_0^* are $-d - a_1$, $-a_2$, $-a_3$, $-\mu$ and 0. We conclude that zero is a simple eigenvalue of J_0^* and all other eigenvalues of J_0^* have negative real parts. Therefore, the *Center Manifold Theory* (see [24]) can be applied to study the dynamics of (5.12) near $\beta = \beta^*$. Theorem 1.25 is used to show the local asymptotic stability of the endemic equilibrium point of (5.12), for β near β^*. The Jacobian J_0^* has a non-negative right eigenvector w and a left eigenvector v associated with the zero eigenvalue. With respect to w, we have that

$$\begin{aligned} & J_0^* w = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ \Leftrightarrow\ & J_0^* \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ \Leftrightarrow\ & w = \left[\left(\frac{\delta\varepsilon\omega_1}{a_2 a_3} - a_1 \right) \frac{1}{\mu} \quad 1 \quad \frac{\delta}{a_2} \quad \frac{\delta\varepsilon}{a_2 a_3} \quad \frac{\eta}{d} \right]^T w_2, \end{aligned}$$

where w_2 is an arbitrary constant. So, we can choose $w_2 = 1$ and, consequently, we obtain that

$$w = \left[\left(\frac{\delta\varepsilon\omega_1}{a_2a_3} - a_1\right)\frac{1}{\mu} \qquad 1 \qquad \frac{\delta}{a_2} \qquad \frac{\delta\varepsilon}{a_2a_3} \qquad \frac{\eta}{d}\right]^T.$$

One can observe that

$$\begin{aligned}\left(\frac{\delta\varepsilon\omega_1}{a_2a_3} - a_1\right)\frac{1}{\mu} &= \frac{\delta\varepsilon\omega_1 - a_1a_2a_3}{a_2a_3\mu} \\ &= \frac{\delta\varepsilon\omega_1 - (\delta+\alpha_1+\mu)(\varepsilon+\alpha_2+\mu)(\omega_1+\mu)}{a_2a_3\mu} < 0,\end{aligned}$$

because $\delta, \varepsilon, \omega_1, \alpha_1, \alpha_2 \geq 0$ and $\mu > 0$. Nevertheless, attending to Remark 1.26, as E^0 (see (5.7)) is a non-negative equilibrium point of interest and the first component of E^0 is positive, then the first component of w does not need to be positive. Clearly, other components of w are non-negative. With respect to v, we have that

$$\begin{aligned}vJ_0^* = \begin{bmatrix}0 & 0 & 0 & 0 & 0\end{bmatrix} &\Leftrightarrow \begin{bmatrix}v_1 & v_2 & v_3 & v_4 & v_5\end{bmatrix} J_0^* = \begin{bmatrix}0 & 0 & 0 & 0 & 0\end{bmatrix} \\ \Leftrightarrow v = \left[0 \qquad 1 \qquad 0 \qquad 0 \qquad \frac{a_1}{\eta}\right] &v_2,\end{aligned}$$

where v_2 is an arbitrary constant. So, we can choose $v_2 = 1$ and, consequently, we obtain that

$$v = \left[0 \qquad 1 \qquad 0 \qquad 0 \qquad \frac{a_1}{\eta}\right].$$

Remember that f_l represents the right-hand side of the lth equation of system (5.12) and x_l is the state variable whose derivative is given by the lth equation, for $l = 1, \ldots, 5$. The local stability near the bifurcation point $\beta = \beta^*$ is determined by the signs of two associated constants a and b defined by

$$a = \sum_{i,j,k=1}^{5} v_k w_i w_j \left[\frac{\partial^2 f_k}{\partial x_i \partial x_j}\left(E^0\right)\right]_{\beta=\beta^*}$$

and

$$b = \sum_{i,k=1}^{5} v_k w_i \left[\frac{\partial^2 f_k}{\partial x_i \partial \phi}\left(E^0\right)\right]_{\beta=\beta^*},$$

with $\phi = \beta - \beta^*$. As $v_1 = v_3 = v_4 = 0$, the non-zero partial derivatives at the disease free equilibrium E^0 are given by

$$\left[\frac{\partial^2 f_2}{\partial x_1 \partial x_5}\left(E^0\right)\right]_{\beta=\beta^*} = \left[\frac{\partial^2 f_2}{\partial x_5 \partial x_1}\left(E^0\right)\right]_{\beta=\beta^*} = \frac{\beta^*}{\kappa}$$

and

$$\left[\frac{\partial^2 f_2}{\partial x_5^2}\left(E^0\right)\right]_{\beta=\beta^*} = -\frac{2\beta^*\Lambda}{\mu\kappa^2}.$$

Therefore, the constant a is given by

$$\begin{aligned} a &= \frac{2\beta^*\eta}{\mu\kappa d}\left(\frac{\delta\varepsilon\omega_1 - a_1a_2a_3}{a_2a_3} - \frac{\Lambda\eta}{\kappa d}\right)v_2w_2^2 \\ &= \frac{2\beta^*\eta}{\mu\kappa d}\left(\frac{\delta\varepsilon\omega_1 - a_1a_2a_3}{a_2a_3} - \frac{\Lambda\eta}{\kappa d}\right) < 0. \end{aligned}$$

Furthermore, we have that

$$\begin{aligned} b =& \sum_{i=1}^{5}\left(v_2w_i\left[\frac{\partial^2 f_2}{\partial x_i\partial\phi}\left(E^0\right)\right]_{\beta=\beta^*} + v_5w_i\left[\frac{\partial^2 f_5}{\partial x_i\partial\phi}\left(E^0\right)\right]_{\beta=\beta^*}\right) \\ =& \sum_{i=1}^{5} v_2w_i\left[\frac{\partial}{\partial x_i}\left(\frac{x_1x_5}{\kappa+x_5}\right)\left(E^0\right)\right]_{\beta=\beta^*} \\ =& \frac{v_2w_5\Lambda}{\mu\kappa} \\ =& \frac{\Lambda\eta}{\mu\kappa d}v_2w_2 \\ =& \frac{\Lambda\eta}{\mu\kappa d} > 0. \end{aligned}$$

Thus, as

$$\begin{cases} a < 0 \\ b > 0 \\ \phi = \beta - \beta^* = \dfrac{\mu\kappa d a_1}{\Lambda\eta}(R_0 - 1) > 0 \end{cases} \Leftrightarrow \begin{cases} a < 0 \\ b > 0 \\ R_0 > 1, \end{cases}$$

we conclude from Theorem 1.25 that the equilibrium points E^0 and E^* of (5.1) (see (5.7) and (5.9)) are, respectively, unstable and locally asymptotic stable, for a value of the basic reproduction number such that $R_0 > 1$. This concludes the proof. □

5.2.4 Non-delayed optimal control problem

So far, we have proposed a mathematical model and we have showed to be both mathematically and epidemiologically well posed for the reality under investigation. These investigations give a model to study and understand a certain reality, but do not allow us to interfere and manipulate it. In this section, we introduce a control that allow us to decide how many individuals move to quarantine. Naturally, the question is then to know how to choose such control in an optimal way. For that, we use the theory of optimal

control that is a branch of Mathematics developed to find optimal ways to control a dynamic system (see Chapter 2 and [28, 47, 140]). There are few papers that apply optimal control to cholera models (see e.g. [120]).

Formulation of a non-delayed optimal control problem

Here, we propose and analyse an optimal control problem applied to cholera dynamics described by model (5.1). We add to model (5.1) a control function $u(\cdot)$ that represents the fraction of infective individuals I that are submitted to treatment in quarantine until complete recovery. Given the meaning of the control u, it is natural that the control takes values in the closed set $[0,1]$: $u \equiv 0$ means no infective individual is put under quarantine and $u \equiv 1$ means all infective people are put under quarantine. Only values of u on the interval $[0,1]$ make sense. The model with control is given by the following system of non-linear ordinary differential equations:

$$\begin{cases} \dot{x}_1(t) = \Lambda - \dfrac{\beta x_1(t) x_5(t)}{\kappa + x_5(t)} + \omega_1 x_4(t) - \mu x_1(t), \\ \dot{x}_2(t) = \dfrac{\beta x_1(t) x_5(t)}{\kappa + x_5(t)} - \big(\delta u(t) + \alpha_1 + \mu\big) x_2(t), \\ \dot{x}_3(t) = \delta u(t) x_2(t) - (\varepsilon + \alpha_2 + \mu) x_3(t), \\ \dot{x}_4(t) = \varepsilon x_3(t) - (\omega_1 + \mu) x_4(t), \\ \dot{x}_5(t) = \eta x_2(t) - d x_5(t), \end{cases} \tag{5.13}$$

with initial conditions given by (5.2). The set $\mathscr{X}$ of admissible trajectories is given by

$$\mathscr{X} = \big\{ X(\cdot) \in W^{1,1}\big([0,t_f];\mathbb{R}^5\big) \,|\, (5.2) \text{ and } (5.13) \text{ are satisfied} \big\}$$

with X defined in (5.11) and the admissible control set $\mathscr{U}$ is given by

$$\mathscr{U} = \big\{ u(\cdot) \in L^\infty\big([0,t_f];\mathbb{R}\big) \,|\, 0 \leq u(t) \leq 1, \, \forall\, t \in [0,t_f] \big\}.$$

We consider the objective functional

$$J\big(X(\cdot),u(\cdot)\big) = \int_0^{t_f} \big(x_2(t) + x_5(t) + W u^2(t)\big)\, dt\,, \tag{5.14}$$

where the positive constant W is a weight constant that represents the cost of the interventions associated with the control u, that is, associated with the treatment of infective individuals keeping them in quarantine during all the treatment period. Our aim is to minimize the number of infective individuals and the bacterial concentration, as well as the cost of interventions associated with the control treatment through quarantine. The optimal control problem consists of determining the vector function $X^\diamond(\cdot) =$

$\big(x_1^\diamond(\cdot), x_2^\diamond(\cdot), x_3^\diamond(\cdot), x_4^\diamond(\cdot), x_5^\diamond(\cdot)\big) \in \mathscr{X}$ associated with an admissible control $u^\diamond(\cdot) \in \mathscr{U}$, on the time interval $[0, t_f]$, minimizing the cost functional (5.14), i.e.,

$$J\big(X^\diamond(\cdot), u^\diamond(\cdot)\big) = \min_{(X(\cdot),u(\cdot))\in\mathscr{X}\times\mathscr{U}} J\big(X(\cdot), u(\cdot)\big). \tag{5.15}$$

Note that $\big(x_1^\diamond(\cdot), x_2^\diamond(\cdot), x_3^\diamond(\cdot), x_4^\diamond(\cdot), x_5^\diamond(\cdot)\big) = \big(S^\diamond(\cdot), I^\diamond(\cdot), Q^\diamond(\cdot), R^\diamond(\cdot), B^\diamond(\cdot)\big)$.

Necessary optimality conditions: Pontryagin's Minimum Principle

Before deriving necessary optimality conditions for optimal control problem (5.15), it is important to note that the existence of an optimal control $u^\diamond(\cdot)$ comes from the convexity of the cost functional (5.14) with respect to the controls and the regularity of system (5.13) (see e.g. [28, 47]).

Remark 5.7. *In optimal control theory and in its many applications, it is standard to consider objective functionals with integrands that are convex with respect to the control variables (see e.g. [110]). Such convexity easily ensures the existence and the regularity of solution to the problem (see e.g. [172]) as well as good performance of numerical methods (see e.g. [39]). In our case, we considered a quadratic expression of the control in order to indicate non-linear costs potentially arising at high treatment levels, as proposed in [120].*

The following theorem provides the necessary optimality conditions associated with optimal control problem (5.15) and it ensures the existence of a unique solution (see [140]).

Theorem 5.8. *Optimal control problem* (5.15) *with fixed final time* $t_f \in \mathbb{R}^+$ *admits a unique optimal state* $X^\diamond = \big(x_1^\diamond(\cdot), x_2^\diamond(\cdot), x_3^\diamond(\cdot), x_4^\diamond(\cdot), x_5^\diamond(\cdot)\big) \in \mathscr{X}$ *associated with an optimal control* $u^\diamond(\cdot) \in \mathscr{U}$. *Moreover, there is an adjoint function* $\lambda^\diamond = \big(\lambda_1^\diamond, \lambda_2^\diamond, \lambda_3^\diamond, \lambda_4^\diamond, \lambda_5^\diamond\big) : [0, t_f] \to \mathbb{R}^5$ *such that*

$$\begin{cases} \dot\lambda_1^\diamond(t) = \left(\dfrac{\beta x_5^\diamond(t)}{\kappa + x_5^\diamond(t)} + \mu\right)\lambda_1^\diamond(t) - \dfrac{\beta x_5^\diamond(t)}{\kappa + x_5^\diamond(t)}\lambda_2^\diamond(t)\,, \\ \dot\lambda_2^\diamond(t) = -1 + \big(\delta u^\diamond(t) + \alpha_1 + \mu\big)\lambda_2^\diamond(t) - \delta u^\diamond(t)\lambda_3^\diamond(t) - \eta\lambda_5^\diamond(t)\,, \\ \dot\lambda_3^\diamond(t) = (\varepsilon + \alpha_2 + \mu)\,\lambda_3^\diamond(t) - \varepsilon\lambda_4^\diamond(t)\,, \\ \dot\lambda_4^\diamond(t) = -\omega_1\lambda_1^\diamond(t) + (\omega_1 + \mu)\,\lambda_4^\diamond(t)\,, \\ \dot\lambda_5^\diamond(t) = -1 + \dfrac{\beta\kappa x_1^\diamond(t)}{\big(\kappa + x_5^\diamond(t)\big)^2}\big(\lambda_1^\diamond(t) - \lambda_2^\diamond(t)\big) + d\lambda_5^\diamond(t)\,, \end{cases} \tag{5.16}$$

with transversality conditions

$$\lambda_i^\diamond(t_f) = 0, \quad i = 1, \ldots, 5 \tag{5.17}$$

for almost all $t \in [0, t_f]$. *Furthermore,*

$$u^\diamond(t) = \min\left\{\max\left\{0, \frac{\delta x_2^\diamond(t)}{2W}\big(\lambda_2^\diamond(t) - \lambda_3^\diamond(t)\big)\right\}, 1\right\} \tag{5.18}$$

for almost all $t \in [0, t_f]$.

Proof. Existence of an optimal solution $X^\diamond = (x_1^\diamond, x_2^\diamond, x_3^\diamond, x_4^\diamond, x_5^\diamond)$ associated with an optimal control $u^\diamond$ comes from the convexity of the integrand of the cost function J with respect to the control u and the Lipschitz property of the state system with respect to the state variables $(x_1, x_2, x_3, x_4, x_5)$ (see e.g. [28, 47]).

The necessary optimality conditions for an optimal solution of (5.15) are given by Pontryagin's Minimum Principle (see Theorem 2.5). The Hamiltonian function is defined by

$$\begin{aligned} H(X, u, \lambda) = {} & x_2 + x_5 + Wu^2 + \lambda_1\left(\Lambda - \frac{\beta x_1 x_5}{\kappa + x_5} + \omega_1 x_4 - \mu x_1\right) \\ & + \lambda_2\left(\frac{\beta x_1 x_5}{\kappa + x_5} - (\delta u + \alpha_1 + \mu)x_2\right) \\ & + \lambda_3\big(\delta u x_2 - (\varepsilon + \alpha_2 + \mu)x_3\big) + \lambda_4\big(\varepsilon x_3 - (\omega_1 + \mu)x_4\big) \\ & + \lambda_5\big(\eta x_2 - d x_5\big). \end{aligned} \tag{5.19}$$

Let us suppose that $\big(X^\diamond(\cdot), u^\diamond(\cdot)\big) \in \mathscr{X} \times \mathscr{U}$ is an optimal solution of (5.15) with fixed final time $t_f \in \mathbb{R}^+$. Then, according to Pontryagin's Minimum Principle (see [140] and Section 2.2.2), there is an adjoint function $\lambda^\diamond = \big(\lambda_1^\diamond, \lambda_2^\diamond, \lambda_3^\diamond, \lambda_4^\diamond, \lambda_5^\diamond\big) : [0, t_f] \to \mathbb{R}^5$, $\lambda^\diamond(\cdot) \in W^{1,1}\big([0, t_f]; \mathbb{R}^5\big)$, that satisfies, for almost all $t \in [0, t_f]$, the

1) transversality conditions:

$$\lambda_i^\diamond(t_f) = 0, \quad i = 1, \ldots, 5, \tag{5.20}$$

in view of the free terminal state $X(t_f)$;

2) adjoint system:

$$\dot{\lambda}_i^\diamond(t) = -\frac{\partial H}{\partial x_i}\big(X^\diamond(t), u^\diamond(t), \lambda^\diamond(t)\big), \quad i = 1, \ldots, 5; \tag{5.21}$$

3) minimality condition:

$$\min_{0 \le u \le 1} H\big(X^\diamond(t), u, \lambda^\diamond(t)\big) = H\big(X^\diamond(t), u^\diamond(t), \lambda^\diamond(t)\big). \tag{5.22}$$

So, conditions (5.17) are derived from transversality conditions (5.20). Moreover, system (5.16) is obtained from adjoint system (5.21). Let us evaluate

the minimality condition (5.22). To this end, we consider the so-called free (unconstrained) control u_f defined by the equation

$$\frac{\partial H}{\partial u}\big(X(t), u_f(t), \lambda(t)\big) = 2Wu_f(t) + \delta x_2(t)\big(\lambda_3(t) - \lambda_2(t)\big) = 0,$$

which yieds

$$u_f(t) = u_f\big(X(t), \lambda(t)\big) = \frac{\delta x_2(t)}{2W}\big(\lambda_2(t) - \lambda_3(t)\big).$$

Then, the constrained control $u(t) \in [0,1]$ minimizing the Hamiltonian (5.19) is given by the projection of u_f onto $[0,1]$, which gives the following control law:

$$\begin{aligned} u^\diamond(t) &= Proj_{[0,1]}\Big(u_f\big(X^\diamond(t), \lambda^\diamond(t)\big)\Big) \\ &= \min\left\{\max\left\{0, \frac{\delta x_2^\diamond(t)}{2W}\big(\lambda_2^\diamond(t) - \lambda_3^\diamond(t)\big)\right\}, 1\right\}. \end{aligned}$$

Concluding, optimal control (5.18) comes from the minimality condition (5.22) of Pontryagin's Minimum Principle (see e.g. [140] and Section 2.2.2).

For small final time t_f, the optimal control given by (5.18) is unique due to the boundedness of the state and adjoint function and the Lipschitz property of systems (5.13) and (5.16). Uniqueness extends to any t_f due to the fact that our problem is autonomous (see [161] and references cited therein). This completes the proof. □

5.2.5 Numerical simulations

After the theoretical study done in Section 5.2.4, we are going to provide numerical simulations associated with the cholera outbreak that occurred in Haiti in 2010. Such simulations are going to show how we can manipulate and improve the reality.

We start by fitting the cholera epidemic that occurred in the Department of Artibonite – Haiti, from 1st November 2010 to 1st May 2011 (see [193]), through a sub-model of (5.1) (see *Non-delayed SIB sub-model* below). Then, we illustrate the local stability of the endemic equilibrium for model (5.1). Finally, we solve numerically the optimal control problem proposed and studied in Section 5.2.4, when applied to Haiti case.

For all numerical simulations of this section, the values of Λ, μ, η, d, S_0, I_0, Q_0, R_0, B_0 and N_0 are fixed in Table 5.1. Consequently, we have that:

$$N_0 = 7450, \quad \frac{\Lambda}{\mu} \simeq 2.97 \times N_0, \quad B_0 = 2.75 \times 10^5 \quad \text{and} \quad \frac{\Lambda\eta}{\mu d} \simeq 6.71 \times 10^5.$$

Therefore, the initial values of the human population and bacterial concentration belong to (5.4) and (5.5), respectively:

$$N_0 = S_0 + I_0 + Q_0 + R_0 \in \left[0, \frac{\Lambda}{\mu}\right] \quad \text{and} \quad B_0 \in \left[0, \frac{\Lambda\eta}{\mu d}\right].$$

Therefore, $(S_0, I_0, Q_0, R_0, B_0) \in \Omega = \Omega_H \times \Omega_B$. This implies that all the following numerical solutions (S, I, Q, R, B) belong to the positively invariant set $\Omega = \Omega_H \times \Omega_B$ (see Lemmas 5.1 and 5.2). We use real data provided by World Health Organization (WHO) in [193].

Parameter	**Value**	**Unity**	**Reference**
Λ	$(24.4N_0)/365000$	person day^{-1}	[76]
μ	2.2493×10^{-5}	day^{-1}	[78]
β	0.8	day^{-1}	[22]
κ	10^6	cell/ml	[150]
ω_1	0.4/365	day^{-1}	[120]
δ	0.05	day^{-1}	Assumed
ε	0.2	day^{-1}	[127]
α_1	0.015	day^{-1}	[127]
α_2	0.0001	day^{-1}	[127]
η	10	cell/ml day^{-1} person^{-1}	[22]
d	0.33	day^{-1}	[22]
S_0	5750	person	Assumed
I_0	1700	person	[193]
Q_0	0	person	Assumed
R_0	0	person	Assumed
N_0	7450	person	–
B_0	275000	cell/ml	Assumed
t_f	182	days	[193]
W	1000	Adimensional	Assumed

Table 5.1: Parameter values and initial conditions for optimal control problem (5.15).

Non-delayed SIB sub-model

The existing data of the cholera outbreak that occurred in the Department of Artibonite – Haiti does not include quarantine and, consequently, recovered individuals (see [193]). Then, in order to approximate better the real data, we choose $\delta = \varepsilon = \omega_1 = \alpha_2 = Q(0) = R(0) = 0$. Consequently, we obtain a sub-model of (5.1) given by

$$\begin{cases} \dot{S}(t) = \Lambda - \dfrac{\beta B(t) S(t)}{\kappa + B(t)} - \mu S(t), \\ \dot{I}(t) = \dfrac{\beta B(t) S(t)}{\kappa + B(t)} - (\alpha_1 + \mu) I(t), \\ \dot{B}(t) = \eta I(t) - d B(t). \end{cases} \tag{5.23}$$

By considering all the other parameter values from Table 5.1, the submodel (5.23) approximates well the cholera outbreak in the Department of Artibonite – Haiti: see Figure 5.1a. In this situation, the basic reproduction number (5.8) is

$$R_0 = 35.7306$$

and the endemic equilibrium (5.9) is

$$E^* = (S^*, I^*, B^*) = (620.2829,\ 32.2234,\ 976.4658).$$

This numerical simulation, denoted by (NS1), was obtained with the help of the integration routines in Matlab.

Local stability of the endemic equilibrium associated with the non-delayed SIQRB model

For the parameter values in Table 5.1, we have that the basic reproduction number (5.8) is

$$R_0 = 8.2550$$

and the endemic equilibrium (5.9) is

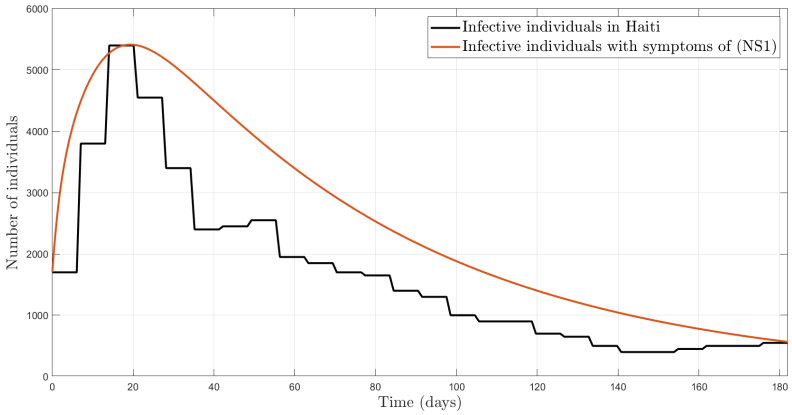
$$E^* = (2684.3930,\ 27.2540,\ 6.8093,\ 1217.7101,\ 825.8793).$$

In Figure 5.2, we can observe agreement between the state trajectories predicted by model (5.1) and the analysis of the local asymptotic stability of the endemic equilibrium E^* done in Section 5.2.3, considering $t_f = 2.5 \times 10^4$ days and all the other values of Table 5.1. This conclusion was also obtained with the help of the integration routines in Matlab.

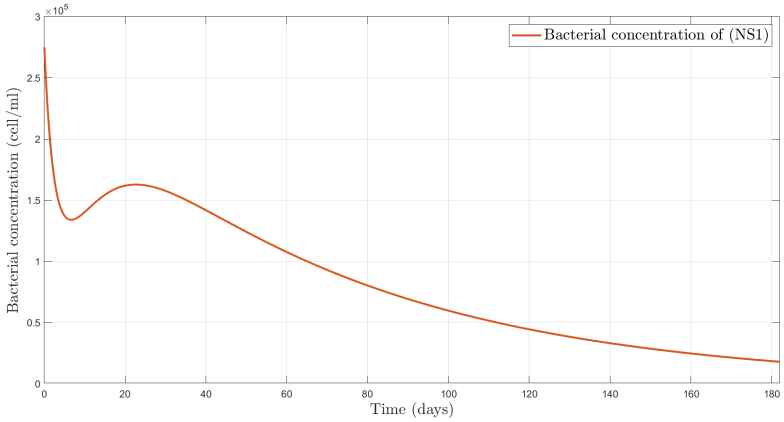
Numerical solutions of the non-delayed SIQRB control model

We now solve numerically the optimal control problem proposed in Section 5.2.4 for $W = 1000$ (see [180]), considering the parameter values and initial conditions of Table 5.1.

The optimal control takes the maximum value for $t \in [0, 87.36]$ days. For $t \in]87.36, 182]$, the optimal control is a strictly decreasing function and at the final time we have $u^\diamond(182) \simeq 0.00159$ (see Figure 5.3). At the end of approximately 88 days, the number of infective individuals is approximately 86. At the final time ($t_f = 182$ days), the number of infective individuals associated with the optimal control is, approximately, 23. Note that the curve of infective individuals associated with optimal control strategy is represented in Figure 5.3. One can observe that the strategy associated with the control $u^\diamond$ allows an important decrease on the number of infective individuals as well as on the concentration of bacteria, by comparing Figures 5.1 and 5.3. The maximum value of the number of infective individuals also decreases significantly when the control strategy is applied. The



(a) Infective individuals with symptoms of (NS1) versus real data from the cholera outbreak in the Department of Artibonite – Haiti, from 1st November 2010 to 1st May 2011.



(b) Bacterial concentration of (NS1).

Figure 5.1: State trajectories $I(t)$ and $B(t)$ for all $t \in [0, 182]$, predicted by model (5.1), assuming that $\delta = \varepsilon = \omega_1 = \alpha_2 = 0$ and all the other values of Table 5.1.

optimal control implies a significant transfer of individuals to the recovered class. These numerical simulations were obtained through ACADO Toolkit – Toolkit for Automatic Control and Dynamic Optimization (see [70, 71]). For our numerical computations, we used $N = 100$ grid points and the tolerance was set to $tol = 1 \times 10^{-6}$.

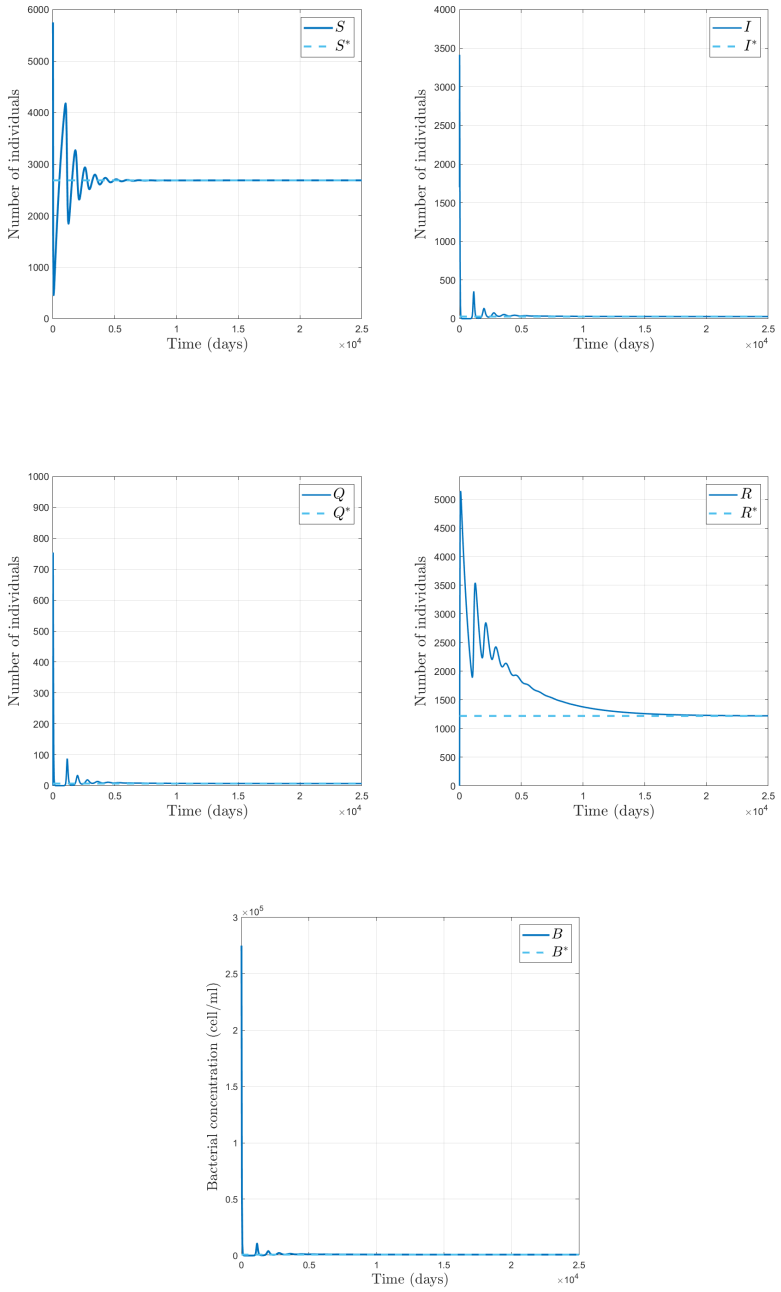


Figure 5.2: State trajectories of model (5.1) versus endemic equilibrium (5.9) for $t_f = 2.5 \times 10^4$ days and considering all the other values of Table 5.1.

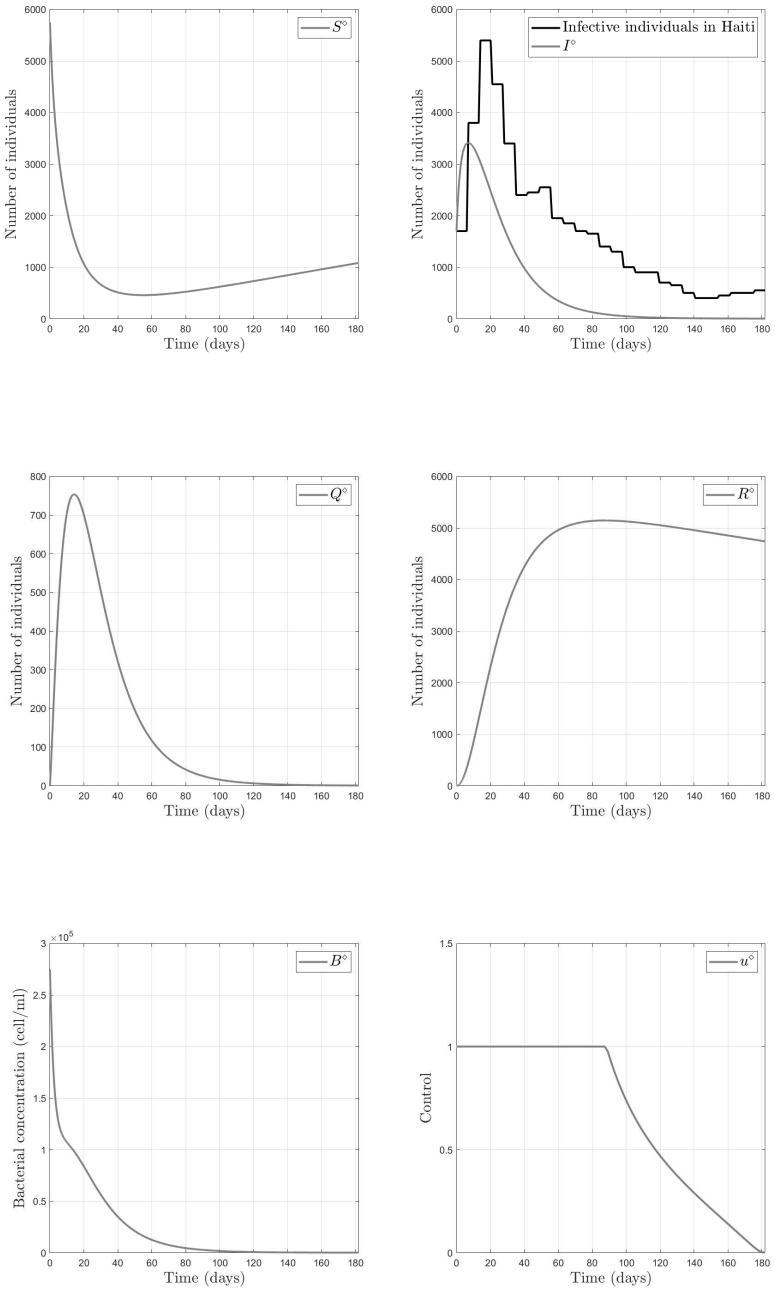


Figure 5.3: Optimal state trajectories $S^\diamond(t)$, $I^\diamond(t)$ (versus real data from the cholera outbreak in the Department of Artibonite – Haiti, from 1st November 2010 to 1st May 2011), $Q^\diamond(t)$, $R^\diamond(t)$ and $B^\diamond(t)$ and optimal control $u^\diamond(t)$ (satisfying the control law (5.18)) associated with optimal control problem (5.15) for all $t \in [0, 182]$, using the values of Table 5.1.

5.3 Delayed model with quarantine

Here we consider model (4.1) (with a time delay $\tau \geq 0$) that generalizes model (5.1). As in Section 5.2, we assume throughout this section that the initial conditions of system (4.1) are non-negative:

$$\forall t \in [-\tau, 0]\,,\ S(t) \geq 0\,,\ I(t) \geq 0\,,\ Q(t) \geq 0\,,\ R(t) \geq 0\,,\ B(t) \geq 0. \quad (5.24)$$

Here, S_0, I_0, Q_0, R_0, B_0 and N_0 denote $S(t)$, $I(t)$, $Q(t)$, $R(t)$, $B(t)$ and $S(t) + I(t) + Q(t) + R(t)$ for $t \in [-\tau, 0]$, respectively (see Table 4.1). Again, with the purpose to simplify the writing we are going to continue using notations (5.3) and (5.10). Furthermore, in this section we also need to consider more notations. Namely, ρ, $\bar{D}$, A and $\tilde{A}$ are defined as follows:

i) $\rho = \Lambda\eta a_2 a_3 + \kappa d(a_1 a_2 a_3 - \delta\varepsilon\omega_1)$;

ii) $\bar{D} = a_1 a_2 a_3 \mu + \beta(a_1 a_2 a_3 - \delta\varepsilon\omega_1)$;

iii) $A = a_1 a_2 a_3$;

iv) $\tilde{A} = a_1 a_2 a_3 - \delta\varepsilon\omega_1$.

5.3.1 Non-negativity of solutions

In this section, we prove that the delayed model (4.1) subject to (5.24) makes sense from the biological point of view, since the solutions of (4.1) are non-negative under non-negative initial conditions (5.24).

Lemma 5.9. *The solutions* $\big(S(t), I(t), Q(t), R(t), B(t)\big)$ *of* (4.1) *are non-negative, for all* $t \geq -\tau$, *with non-negative initial conditions* (5.24).

Proof. We have

$$\begin{cases} \left.\dfrac{dS(t)}{dt}\right|_{\xi(S)} = \Lambda + \omega_1 R(t) > 0\,, \\ \left.\dfrac{dI(t)}{dt}\right|_{\xi(I)} = \dfrac{\beta B(t-\tau) S(t-\tau)}{\kappa + B(t-\tau)} \geq 0\,, \\ \left.\dfrac{dQ(t)}{dt}\right|_{\xi(Q)} = \delta I(t) \geq 0\,, \\ \left.\dfrac{dR(t)}{dt}\right|_{\xi(R)} = \varepsilon Q(t) \geq 0\,, \\ \left.\dfrac{dB(t)}{dt}\right|_{\xi(B)} = \eta I(t) \geq 0\,, \end{cases}$$

where $\xi(\upsilon) = \big\{\upsilon(t) = 0$ and $S(\cdot), I(\cdot), Q(\cdot), R(\cdot), B(\cdot) \in \mathscr{C}\left([-\tau, +\infty[, \mathbb{R}_0^+\right)\big\}$ and $\upsilon \in \{S, I, Q, R, B\}$. Therefore, due to Lemma 1.42, any solution of

system (4.1) is such that $\big(S(t), I(t), Q(t), R(t), B(t)\big) \in \big(\mathbb{R}_0^+\big)^5$ for all $t \geq -\tau$. This concludes the proof. □

5.3.2 Equilibrium points and the basic reproduction number

From Section 5.2.2, we know that model (4.1) has a DFE given by (5.7) and, recalling notation (5.3), the basic reproduction number has the following expression (see (5.8))

$$R_0 = \frac{\beta\Lambda\eta}{\mu\kappa d a_1}.$$

Moreover, when $R_0 > 1$, there is an endemic equilibrium E^* given by (5.9). Let us write E^* in a different way, having in mind that

$$\lambda^* = \frac{a_1a_2a_3\mu\kappa d(R_0-1)}{\kappa d(a_1a_2a_3 - \delta\varepsilon\omega_1) + \Lambda\eta a_2a_3} = \frac{a_1a_2a_3\mu\kappa d(R_0-1)}{\rho}$$

(see the proof of Proposition 5.4). For that we begin by finding an equivalent expression for $D = a_1a_2a_3\left(\lambda^* + \mu\right) - \delta\varepsilon\omega_1\lambda^*$ (see (5.10)):

$$\begin{aligned}
& a_1a_2a_3(\lambda^*+\mu) - \delta\varepsilon\omega_1\lambda^* = \lambda^*\left(a_1a_2a_3 - \delta\varepsilon\omega_1\right) + a_1a_2a_3\mu \\
= \;& \lambda^*\tilde{A} + a_1a_2a_3\mu = \frac{\mu\kappa d a_1a_2a_3(R_0-1)\tilde{A}}{\rho} + a_1a_2a_3\mu \\
= \;& \frac{\dfrac{\beta\Lambda\eta}{R_0}a_2a_3(R_0-1)\tilde{A}}{\rho} + a_1a_2a_3\mu = \frac{\beta\Lambda\eta a_2a_3(R_0-1)\tilde{A}}{R_0\rho} + a_1a_2a_3\mu \\
= \;& \frac{\beta\Lambda\eta a_2a_3(R_0-1)\tilde{A} + a_1a_2a_3\mu R_0\rho}{R_0\rho} \\
= \;& \frac{R_0\mu\kappa d a_1a_2a_3(R_0-1)\tilde{A} + a_1a_2a_3\mu R_0\rho}{R_0\rho} \\
\underset{R_0\neq 0}{=} \;& \frac{a_1a_2a_3\mu\Big(\kappa d(R_0-1)\tilde{A} + \rho\Big)}{\rho} \\
= \;& \frac{a_1a_2a_3\mu\left(\kappa d R_0\tilde{A} - \kappa d\tilde{A} + \Lambda\eta a_2a_3 + \kappa d\tilde{A}\right)}{\rho} \\
= \;& \frac{a_1a_2a_3\mu\left(\kappa d R_0\tilde{A} + \Lambda\eta a_2a_3\right)}{\rho}.
\end{aligned}$$

Moreover, we have that

$$\begin{aligned}
\frac{\lambda^*}{D} &= \frac{a_1a_2a_3\mu\kappa d(R_0-1)}{\rho} \times \frac{\rho}{a_1a_2a_3\mu\left(\kappa d R_0\tilde{A} + \Lambda\eta a_2a_3\right)} \\
&\underset{a_1,a_2,a_3,\mu,\rho\neq 0}{=} \frac{\kappa d(R_0-1)}{\kappa d R_0\tilde{A} + \Lambda\eta a_2a_3}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\kappa d(R_0-1)}{\frac{\beta\kappa d R_0\tilde{A}}{\beta}+\frac{R_0\mu\kappa d a_1}{\beta}\times a_2a_3}\\
&\underset{\kappa,d\neq 0}{=} \frac{\beta(R_0-1)}{R_0\left(\beta\tilde{A}+a_1a_2a_3\mu\right)}\\
&= \frac{\beta(R_0-1)}{R_0\bar{D}}.
\end{aligned}
$$

So, with respect to the first component of E^* given by (5.9), we obtain

$$
\begin{aligned}
S^* &= \frac{\Lambda a_1a_2a_3}{D}=\Lambda a_1a_2a_3\times\frac{\rho}{a_1a_2a_3\mu\left(\kappa d R_0\tilde{A}+\Lambda\eta a_2a_3\right)}\\
&\underset{a_1,a_2,a_3\neq 0}{=} \frac{\Lambda\rho}{\mu\left(\kappa d\times\frac{\beta\Lambda\eta}{\mu\kappa d a_1}\times\tilde{A}+\Lambda\eta a_2a_3\right)}\\
&\underset{\Lambda,\kappa,d\neq 0}{=} \frac{\rho}{\mu\left(\frac{\beta\eta}{\mu a_1}\times\tilde{A}+\eta a_2a_3\right)}\\
&\underset{\mu\neq 0}{=} \frac{\rho}{\frac{\beta\eta\tilde{A}}{a_1}+\frac{\eta a_1a_2a_3\mu}{a_1}}\\
&= \frac{a_1\rho}{\eta\left(\beta\tilde{A}+a_1a_2a_3\mu\right)}\\
&= \frac{a_1\rho}{\eta\bar{D}}.
\end{aligned}
$$

Furthermore, having in mind that $\frac{\lambda^*}{D}=\frac{\beta(R_0-1)}{R_0\bar{D}}$ and the second, third, fourth and fifth components of E^* given by (5.9), we have that

i) $I^*=\frac{\Lambda a_2a_3\lambda^*}{D}=\frac{\beta\Lambda a_2a_3(R_0-1)}{R_0\bar{D}}$;

ii) $Q^*=\frac{\Lambda\delta a_3\lambda^*}{D}=\frac{\beta\Lambda a_3\delta(R_0-1)}{R_0\bar{D}}$;

iii) $R^*=\frac{\Lambda\delta\varepsilon\lambda^*}{D}=\frac{\beta\Lambda\delta\varepsilon(R_0-1)}{R_0\bar{D}}$;

iv) $B^*=\frac{\Lambda\eta a_2a_3\lambda^*}{Dd}=\frac{\beta\Lambda\eta a_2a_3(R_0-1)}{R_0\bar{D}d}$.

Concluding, equilibrium point E^* given by (5.9) can be represented, in an equivalent way, by

$$
E^*=\left(\frac{a_1\rho}{\eta\bar{D}},\frac{\beta\Lambda a_2a_3\bar{R}_0}{R_0\bar{D}},\frac{\beta\Lambda a_3\delta\bar{R}_0}{R_0\bar{D}},\frac{\beta\Lambda\delta\varepsilon\bar{R}_0}{R_0\bar{D}},\frac{\beta\Lambda\eta a_2a_3\bar{R}_0}{R_0\bar{D}d}\right); \qquad (5.25)
$$

where $\bar{R}_0 = R_0 - 1$.

5.3.3 Stability analysis

We proceed with the linearisation of model (4.1), which allows to derive some important results needed in the stability study of the equilibria. With this purpose, we consider again notation (5.11):

$$X = (x_1, x_2, x_3, x_4, x_5) = (S, I, Q, R, B).$$

Then, we can write system (4.1) in the following way:

$$\dot{x}(t) = f\big(x(t), x(t-\tau)\big),$$

where $x(t) = \big(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t)\big)$. Let $\bar{E} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5)$ be an arbitrary equilibrium point of (4.1) and let us consider the following change of variables:

$$z_i(t) = x_i(t) - \bar{x}_i, \quad i = 1, \ldots, 5.$$

Thus, the linearised system of (4.1) is given by

$$\dot{z} = \left.\frac{\partial f}{\partial x}\right|_{\bar{E}} z + \left.\frac{\partial f}{\partial x_\tau}\right|_{\bar{E}} z_\tau,$$

where $z(t) = \big(z_1(t), z_2(t), z_3(t), z_4(t), z_5(t)\big)$, $z_\tau(t) = z(t-\tau)$ and, moreover, $x_\tau(t) = x(t-\tau)$. Furthermore, we have

$$A_0 := \left.\frac{\partial f}{\partial x}\right|_{\bar{E}} = \begin{bmatrix} -\bar{\lambda} - \mu & 0 & 0 & \omega_1 & -C \\ 0 & -a_1 & 0 & 0 & 0 \\ 0 & \delta & -a_2 & 0 & 0 \\ 0 & 0 & \varepsilon & -a_3 & 0 \\ 0 & \eta & 0 & 0 & -d \end{bmatrix}$$

and

$$A_1 := \left.\frac{\partial f}{\partial x_\tau}\right|_{\bar{E}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \bar{\lambda} & 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\bar{\lambda} = \dfrac{\beta \bar{x}_5}{\kappa + \bar{x}_5}$ and $C = \dfrac{\beta \kappa \bar{x}_1}{(\kappa + \bar{x}_5)^2}$. So, the characteristic polynomial associated with the linearised system of model (4.1) is given by

$$p_\tau(\chi) = \det(\chi I_5 - A_0 - e^{-\tau\chi} A_1) = p_1(\chi) + e^{-\tau\chi} p_2(\chi), \tag{5.26}$$

where

$$p_1(\chi) = (\chi + a_1)(\chi + a_2)(\chi + a_3)(\chi + d)(\chi + \bar{\lambda} + \mu)$$

and

$$\begin{aligned} p_2(\chi) = &- \eta C\chi^3 - \eta C(a_2+a_3+\mu)\chi^2 \\ &- \Big(\eta C(a_2a_3+a_2\mu+a_3\mu) + \delta\varepsilon\omega_1\bar{\lambda}\Big)\chi - \eta C a_2a_3\mu - \delta\varepsilon\omega_1 d\bar{\lambda}. \end{aligned}$$

For more details about the previous study see Section 1.4.

In order to study the stability of the equilibria, we are going to follow the approach of [34] corrected in [16] (see Theorem 1.39). We shall prove now that the conditions $i)$, $ii)$, $iv)$ and $v)$ of Theorem 1.39 are satisfied for an arbitrary equilibrium point

$$\bar{E} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5) \in \left(\mathbb{R}_0^+\right)^5$$

of model (4.1). The polynomial p_1 has only real zeros ($\chi = -a_1$ or $\chi = -a_2$ or $\chi = -a_3$ or $\chi = -d$ or $\chi = -\bar{\lambda} - \mu$). Thus, the polynomials p_1 and p_2 can not have common imaginary zeros so that condition $i)$ is satisfied. In order to fulfil the hypothesis of condition $ii)$, we are going to compute $p_1(yi)$ and $p_2(yi)$. We have

$$\begin{aligned} p_1(yi) = &\big[a_1(\bar{\lambda}+\mu) - y^2\big]\big[a_2a_3d - (a_2+a_3+d)y^2\big] \\ &- (a_1+\bar{\lambda}+\mu)\big[a_2a_3 + (a_2+a_3)d - y^2\big]y^2 \\ &+ \Big\{\big[a_1(\bar{\lambda}+\mu) - y^2\big]\big[a_2a_3 + (a_2+a_3)d - y^2\big] \\ &+ (a_1+\bar{\lambda}+\mu)\big[a_2a_3d - (a_2+a_3+d)y^2\big]\Big\}yi \\ = &\ \overline{p_1(-yi)} \end{aligned}$$

and obtain

$$\begin{aligned} p_2(yi) = &\ \eta C(a_2+a_3+\mu)y^2 - \eta C a_2a_3\mu - \delta\varepsilon\omega_1\bar{\lambda}d \\ &+ \Big\{\eta C y^3 - \big[\eta C(a_2a_3+a_2\mu+a_3\mu) + \delta\varepsilon\omega_1\bar{\lambda}\big]y\Big\}i \\ = &\ \overline{p_2(-yi)}. \end{aligned}$$

Therefore, condition $ii)$ is satisfied. As the degree of polynomial p_1 (equal to 5) is bigger than the degree of p_2 (equal to 3), then the condition $iv)$ given by

$$\lim_{|\lambda|\to\infty,\ \Re(\lambda)\geq 0} \sup\left\{\left|\frac{p_2(\lambda)}{p_1(\lambda)}\right|\right\} < 1$$

is obviously satisfied. Furthermore, the function defined by

$$F(y) = |p_1(yi)|^2 - |p_2(yi)|^2$$

is a polynomial with degree equal to ten. Thus, the function F has at most a finite number (ten) of real zeros. Concluding, the condition $v)$ is also verified.

Disease-free equilibrium

Now, we are going to study the stability of the disease-free equilibrium E^0 of delayed model (4.1), given by (5.7).

Theorem 5.10 (Stability of (5.7))**.** *Assume that $R_0 \neq 1$. If $a_1 d < 1$, then there exists $\tau^* \in \mathbb{R}_0^+$ such that:*

- *there is at most a finite number of stability switches, when $\tau \in [0, \tau^*]$;*
- *instability occurs, when $\tau \in]\tau^*, +\infty[$.*

For all $\tau \geq 0$, if $a_1 d \geq 1$, then the DFE (5.7) is:

- *locally asymptotic stable, when $R_0 < 1$;*
- *unstable, when $R_0 > 1$.*

Proof. In order to study the stability of the DFE (5.7), we follow the approach of [16, 34]. We already know that the conditions *i*), *ii*), *iv*) and *v*) of Theorem 1.39 are satisfied for the DFE (5.7). Thus, we analyse the condition *iii*) and compute the zeros of the polynomial F for the DFE (5.7). Computing $p_1(0) + p_2(0)$, we obtain

$$\begin{aligned} p_1(0) + p_2(0) &= a_1 a_2 a_3 d\mu - \eta C a_2 a_3 \mu = a_1 a_2 a_3 d\mu - a_1 d R_0 a_2 a_3 \mu \\ &= a_1 a_2 a_3 d\mu (1 - R_0). \end{aligned}$$

Concluding, $p_1(0) + p_2(0) \neq 0 \Leftrightarrow R_0 \neq 1$, because $a_1, a_2, a_3, d, \mu > 0$. Therefore, the condition *iii*) is verified for the DFE (5.7) if and only if $R_0 \neq 1$.

As the conditions *i*)–*v*) are verified with respect to the DFE (5.7) if and only if $R_0 \neq 1$ holds, the stability of the DFE (5.7) depends on the roots of the polynomial F, according to Theorem 1.39. Solving $F(y) = 0$, we get

$$\begin{aligned} &y = \pm a_2 i \vee y = \pm a_3 i \vee y = \pm \mu i \vee y = \pm \frac{\sqrt{2}}{2} \sqrt{-a_1^2 - d^2 + \sqrt{(a_1^2 - d^2)^2 + 4}} \\ &\vee y = \pm \frac{\sqrt{2}}{2} \left(\sqrt{a_1^2 + d^2 + \sqrt{(a_1^2 - d^2)^2 + 4}} \right) i. \end{aligned}$$

If $-a_1^2 - d^2 + \sqrt{(a_1^2 - d^2)^2 + 4} > 0$, which is equivalent to

$$\begin{aligned} &\sqrt{(a_1^2 - d^2)^2 + 4} > a_1^2 + d^2 \underset{a_1^2 + d^2 > 0}{\Leftrightarrow} (a_1^2 - d^2)^2 + 4 > (a_1^2 + d^2)^2 \\ &\Leftrightarrow (a_1 d)^2 < 1 \underset{a_1, d > 0}{\Leftrightarrow} 0 < a_1 d < 1, \end{aligned}$$

then the polynomial F has at least one positive root, which is simple. According to Theorem 1.39, when $a_1 d < 1$, we can state that there is $\tau^* > 0$ such that

- at most a finite number of stability switches may occur, if $\tau \in [0, \tau^*]$;
- instability occurs, if $\tau \in]\tau^*, +\infty[$.

On the other hand, if $a_1 d \geq 1$, then the polynomial F has not positive roots. In this case, the stability/instability is determined by the stability/instability that occurs when $\tau = 0$, according to item (a) of Theorem 1.39. When $\tau = 0$, one has the model studied in Section 5.2. Therefore, when $a_1 d \geq 1$ the DFE (5.7) is

- locally asymptotic stable if $R_0 < 1$;
- unstable if $R_0 > 1$;

for all $\tau \geq 0$ (see Theorems 5.5 and 5.6). This concludes the proof. □

Endemic equilibrium

For the endemic equilibrium point (5.25), we have that

$$C^* = \frac{\beta\kappa S^*}{(\kappa + B^*)^2} = \frac{\beta\Lambda\Big(R_0\kappa d(a_1a_2a_3 - \delta\varepsilon\omega_1) + \Lambda\eta a_2a_3\Big)}{\mu\kappa R_0^2\rho},$$

which implies

$$\begin{aligned}
\eta C^* &= \frac{\mu\kappa d a_1 R_0\Big(R_0\kappa d(a_1a_2a_3 - \delta\varepsilon\omega_1) + \Lambda\eta a_2a_3\Big)}{\mu\kappa R_0^2\rho} \\
&\underset{\mu,\kappa,R_0\neq 0}{=} \frac{a_1 d}{R_0\rho}\Big(R_0\kappa d(a_1a_2a_3 - \delta\varepsilon\omega_1) + \Lambda\eta a_2a_3\Big) \\
&= \frac{a_1 d}{R_0\rho}\Big(R_0(\rho - \Lambda\eta a_2a_3) + \Lambda\eta a_2a_3\Big) = \frac{a_1 d}{R_0\rho}\Big(R_0\rho + \Lambda\eta a_2a_3(1 - R_0)\Big) \\
&\underset{R_0,\rho\neq 0}{=} a_1 d + \frac{\Lambda\eta a_1a_2a_3 d(1 - R_0)}{R_0\rho} = a_1 d + \frac{\Lambda\eta a_1^2a_2a_3 d^2\mu\kappa(1 - R_0)}{\beta\Lambda\eta\rho} \\
&\underset{\Lambda,\eta\neq 0}{=} a_1 d + \frac{(a_1 d)^2 a_2a_3\mu\kappa(1 - R_0)}{\beta\rho} = a_1 d\left(1 + \frac{a_1a_2a_3\mu\kappa d(1 - R_0)}{\beta\rho}\right).
\end{aligned}$$

Again, we have to study the condition *iii*) of Theorem 1.39 and the roots of F with respect to the endemic equilibrium (5.25). Computing $p_1(0) + p_2(0)$ for E^*, we obtain

$$\begin{aligned}
p_1(0) + p_2(0) &= -\frac{a_2a_3}{\kappa R_0^2\rho}\big(\beta\Lambda\eta - R_0^2\mu\kappa d a_1\big)\big(R_0\kappa d(a_1a_2a_3 - \delta\varepsilon\omega_1) + \Lambda\eta a_2a_3\big) \\
&= -\frac{a_2a_3}{\kappa R_0^2\rho}\big(R_0\mu\kappa d a_1 - R_0^2\mu\kappa d a_1\big) \\
&\quad \times\big(R_0\kappa d(a_1a_2a_3 - \delta\varepsilon\omega_1) + \Lambda\eta a_2a_3\big)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{R_0\mu\kappa d a_1a_2a_3}{\kappa R_0^2\rho}\big(1-R_0\big)\big(R_0\kappa d(a_1a_2a_3-\delta\varepsilon\omega_1)+\Lambda\eta a_2a_3\big)\\
&\underset{R_0,\kappa\neq 0}{=} -\frac{\mu d a_1a_2a_3}{R_0\rho}\big(1-R_0\big)\big(R_0\kappa d(a_1a_2a_3-\delta\varepsilon\omega_1)+\Lambda\eta a_2a_3\big).
\end{aligned}
$$

As $\mu d a_1a_2a_3>0$, $R_0\rho>0$ and $R_0\kappa d(a_1a_2a_3-\delta\varepsilon\omega_1)+\Lambda\eta a_2a_3>0$, then $p_1(0)+p_2(0)\neq 0$ with respect to E^* if and only if $R_0\neq 1$. Now, we are going to write function F as a function of E^* in order to obtain its roots. We obtain

$$
F_{end}(y)=F(y)|_{E^*}=c_0+c_2y^2+c_4y^4+c_6y^6+c_8y^8+c_{10}y^{10},
$$

where c_0, c_2, c_4, c_6, c_8 and c_{10} are real coefficients given by

$$
\begin{aligned}
c_0 &= \left(\frac{A\mu}{\beta\rho}\right)^2 d^3\kappa\left(A\mu+\beta\tilde{A}\right)(R_0-1)\\
&\quad\times\left\{\frac{\beta^2\Lambda\eta}{\mu}\left(a_2a_3+\frac{\delta\varepsilon\omega_1}{a_1}\right)+\rho+\kappa d(A\mu-2\beta\delta\varepsilon\omega_1)\right\};\\
c_2 &= \left\{(a_2a_3\mu)^2+\frac{2a_1(a_2a_3)^3\mu^2\kappa d(R_0-1)}{\rho}\right\}\left(a_1^2+d^2\right)\\
&\quad+\left\{\frac{(a_1d)^2a_2a_3\mu\kappa(R_0-1)}{\beta\rho}\right\}\left\{\frac{a_2a_3\mu\kappa(R_0-1)}{\beta\rho}\Big[A^2\left(\beta^2-d^2\right)\right.\\
&\quad+(a_1d)^2\left(\beta^2-\mu^2\right)\left(a_2^2+a_3^2\right)\\
&\quad+2\beta\delta\varepsilon\omega_1(a_2a_3+a_2\mu+a_3\mu-a_2d-a_3d-\mu d)a_1d\\
&\quad\left.+\beta^2\left[(a_2a_3d)^2-(\delta\varepsilon\omega_1)^2\right]\Big]\right.\\
&\quad-2\beta\delta\varepsilon\omega_1(a_2a_3+a_2\mu+a_3\mu-a_2d-a_3d-\mu d)\\
&\quad+2a_1d\Big[(a_2a_3)^2+(a_2\mu)^2+(a_3\mu)^2+\beta\mu\left(a_2^2+a_3^2\right)\Big]\Big\};\\
c_4 &= \frac{(a_1d)^2a_2a_3\mu\kappa(R_0-1)}{\beta\rho}\left\{\frac{a_1d\left(a_2^2+a_3^2+\mu^2\right)}{\beta\rho}\left(\beta\rho+\beta\kappa d\tilde{A}+A\mu\kappa d\right)\right.\\
&\quad\left.+2a_1\mu d\beta+\frac{2\kappa d\delta\varepsilon\omega_1\left(A\mu+\beta\tilde{A}\right)}{\rho}\right\}+(a_2a_3)^2\left(a_1^2+d^2\right)+(a_1d\lambda^*)^2\\
&\quad+\left\{(a_1a_2)^2+(a_1a_3)^2+(a_2a_3)^2+(a_2d)^2+(a_3d)^2\right\}(\lambda^*+\mu)^2;\\
c_6 &= \frac{(a_1d)^3a_2a_3\mu\kappa(R_0-1)}{(\beta\rho)^2}\left(\beta\rho+\beta\kappa d\tilde{A}+A\mu\kappa d\right)+a_2^2\left(a_1^2+d^2\right)\\
&\quad+a_3^2\left(a_1^2+a_2^2+d^2\right)+(\lambda^*+\mu)^2\left(a_1^2+a_2^2+a_3^2+d^2\right);\\
c_8 &= a_1^2+a_2^2+a_3^2+d^2+(\lambda^*+\mu)^2;\\
c_{10} &= 1;
\end{aligned}
$$

where $\lambda^* = \dfrac{a_1 a_2 a_3 \mu\kappa d(R_0 - 1)}{\rho}$. The study of the stability of E^* depends on the roots of the polynomial F_{end}. It is not easy to obtain their analytical expressions, but we can note that if $R_0 > 1$, then the coefficients c_4, c_6, c_8 and c_{10} are all non-negative for any admissible parameters. Nevertheless, the sign of the coefficients c_0 and c_2 is yet an open question. If all coefficients would be positive, then the polynomial F_{end} would not have positive roots by Descartes' Rule of Signs (see Theorem 1.29). Thus, the stability/instability would be determined by the stability/instability that occurs when $\tau = 0$, according to item (a) of Theorem 1.39. In this way, we would obtain the stability result expressed in Theorem 5.6 for the endemic equilibrium (5.25) of the delayed model (4.1). Though c_0 and c_2 are given by complicated expressions, we can derive some conclusions about their signs by studying them as a function of the ingestion rate $\beta \in]0, 5]$ and fixing all parameters to the values of Table 5.3. Thus, for the existence of an infectious disease we have to assume that $R_0 > 1$, which is equivalent to $\beta > 1.103245 \times 10^{-1}$. Analysing the signs of the coefficients c_0 and c_2 as functions of $\beta \in]0, 5]$, we obtain

$$c_0 = 0 \Leftrightarrow \beta \simeq 2.286124 \times 10^{-5} \vee \beta \simeq 6.533173 \times 10^{-2} \vee \beta \simeq 1.103245 \times 10^{-1}$$

and

$$c_2 = 0 \Leftrightarrow \beta \simeq 1.103243 \times 10^{-1}.$$

In Figures 5.4a and 5.4b, we analyse the signs of the coefficients c_0 and c_2 for $\beta \in]0, 5]$. Thus, we conclude that

$$c_0 > 0 \Leftrightarrow \beta \in]2.286124 \times 10^{-5}, 6.533173 \times 10^{-2}[\cup]1.103245 \times 10^{-1}, 5]$$

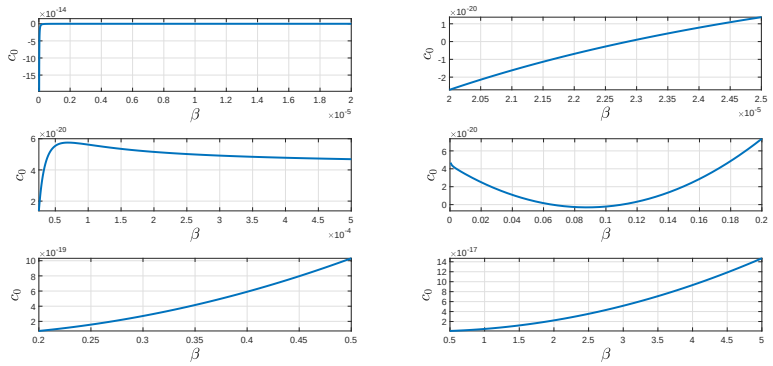
and

$$c_2 > 0 \Leftrightarrow \beta \in]1.103243 \times 10^{-1}, 5].$$

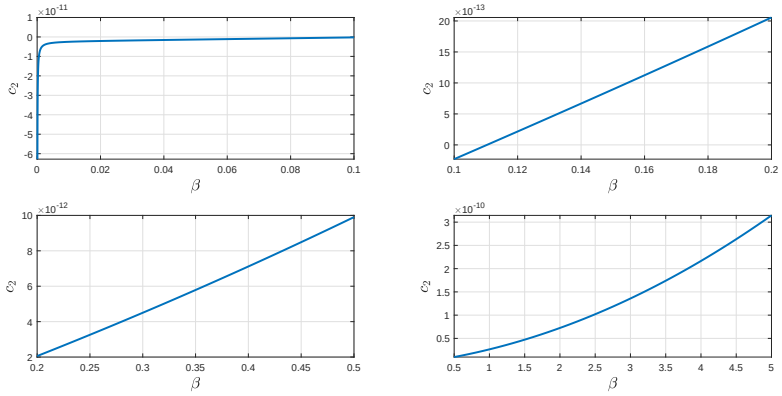
Let us consider that $1.2 \times 10^{-1} \leq \beta \leq 5$ and the other parameters are fixed to the values of Table 5.3. Hence, a disease occurs in view of $R_0 > 1$. In this case, F_{end} has only positive coefficients. So, by Descartes' Rule of Signs (see Theorem 1.29), we can state that the polynomial F_{end} has no positive roots. Therefore, using Theorem 1.39, we can conclude that the stability/instability for $\tau = 0$ remains for all $\tau \geq 0$. As $R_0 > 1$, we can state that, for $\beta \in [1.2 \times 10^{-1}, 5]$, the endemic equilibrium obtained with the values of Table 5.3 is locally asymptotic stable (see Theorem 5.6).

5.3.4 Delayed optimal control problem

We begin this section by formulating an optimal control problem associated with the delayed SIQRB model (4.1). Then, we derive the respective necessary optimality conditions, following the Minimum Principle for



(a) Coefficient c_0.



(b) Coefficient c_2.

Figure 5.4: Coefficients c_0 and c_2 in function of $\beta \in]0, 5]$.

delayed optimal control problems (see Theorem 2.17). Throughout the current section, we consider again notation (5.11): $X = (x_1, x_2, x_3, x_4, x_5) = (S, I, Q, R, B)$.

Formulation of a delayed optimal control problem

We propose a delayed optimal control problem similar to the one formulated in Section 5.2.4. Again, we add to model (4.1) a control function $u(\cdot)$ that represents the fraction of infective individuals, I, that are submitted to treatment in quarantine until complete recovery. Only values of u on the closed interval $[0, 1]$ make sense. If $u \equiv 0$, then no infective individual get treatment through quarantine. On the other hand, if $u \equiv 1$, then all infective people are put under quarantine all the time. The controlled model is given by the following system of delayed non-linear ordinary differential equations:

$$\begin{cases} \dot{x}_1(t) = \Lambda - \dfrac{\beta x_1(t) x_5(t)}{\kappa + x_5(t)} + \omega_1 x_4(t) - \mu x_1(t), \\ \dot{x}_2(t) = \dfrac{\beta x_1(t-\tau) x_5(t-\tau)}{\kappa + x_5(t-\tau)} - \big(\delta u(t) + \alpha_1 + \mu\big) x_2(t), \\ \dot{x}_3(t) = \delta u(t) x_2(t) - (\varepsilon + \alpha_2 + \mu) x_3(t), \\ \dot{x}_4(t) = \varepsilon x_3(t) - (\omega_1 + \mu) x_4(t), \\ \dot{x}_5(t) = \eta x_2(t) - d x_5(t), \end{cases} \tag{5.27}$$

with initial conditions given by (5.24). Our aim is to minimize the number of infective individuals and the bacterial concentration, as well as the cost of interventions associated with the control treatment through quarantine. Thus, we consider the following objective functional:

$$J_q\big(X(\cdot), u(\cdot)\big) = \int_0^{t_f} \big(x_2(t) + x_5(t) + W u^q(t)\big) dt\,, \quad q \in \{1, 2\}, \tag{5.28}$$

where $t_f > 0$ is the final time and the positive constant W is a measure of the cost of the interventions associated with the control u, that is, associated with the treatment of infective individuals keeping them in quarantine during all the treatment period. The set $\mathscr{X}$ of admissible trajectories is given by

$$\mathscr{X} = \big\{X(\cdot) \in W^{1,1}\big([0, t_f]; \mathbb{R}^5\big) \,|\, (5.24) \text{ and } (5.27) \text{ are satisfied}\big\}$$

with X defined in (5.11) and the admissible control set $\mathscr{U}$ is given by

$$\mathscr{U} = \big\{u(\cdot) \in L^q\big([0, t_f]; \mathbb{R}\big) \,|\, 0 \leq u(t) \leq 1,\, \forall\, t \in [0, t_f]\,\big\}.$$

The optimal control problem consists of determining the vector function $X^\diamond(\cdot) = \big(S^\diamond(\cdot), I^\diamond(\cdot), Q^\diamond(\cdot), R^\diamond(\cdot), B^\diamond(\cdot)\big) \in \mathscr{X}$ associated with an admissible

control $u^\diamond(\cdot) \in \mathscr{U}$ on the time interval $[0, t_f]$, minimizing the cost functional (5.28), i.e.,

$$J_q\big(X^\diamond(\cdot), u^\diamond(\cdot)\big) = \min_{(X(\cdot),u(\cdot))\in\mathscr{X}\times\mathscr{U}} J_q\big(X(\cdot), u(\cdot)\big), \quad q \in \{1, 2\}. \tag{5.29}$$

Necessary optimality conditions: Minimum Principle

The following theorem provides the necessary optimality conditions associated with optimal control problem (5.29).

Theorem 5.11. *Assume that* $X^\diamond = \big(x_1^\diamond(\cdot), x_2^\diamond(\cdot), x_3^\diamond(\cdot), x_4^\diamond(\cdot), x_5^\diamond(\cdot)\big) \in \mathscr{X}$ *is a local optimal state associated with a local optimal control* $u^\diamond(\cdot) \in \mathscr{U}$ *of optimal control problem* (5.29) *with fixed final time* $t_f \in \mathbb{R}^+$. *Then, there is an adjoint function* $\lambda^\diamond = \big(\lambda_1^\diamond, \lambda_2^\diamond, \lambda_3^\diamond, \lambda_4^\diamond, \lambda_5^\diamond\big) : [0, t_f] \to \mathbb{R}^5$ *that satisfies the adjoint system*

$$\begin{cases} \dot{\lambda}_1^\diamond(t) = \dfrac{\beta x_5^\diamond(t)}{\kappa + x_5^\diamond(t)}\Big(\lambda_1^\diamond(t) - \lambda_2^\diamond(t+\tau)\chi_{[0,t_f-\tau]}(t)\Big) + \mu\lambda_1^\diamond(t), \\ \dot{\lambda}_2^\diamond(t) = -1 + \lambda_2^\diamond(t)\Big(\delta u^\diamond(t) + \alpha_1 + \mu\Big) - \delta\lambda_3^\diamond(t)u^\diamond(t) - \eta\lambda_5^\diamond(t), \\ \dot{\lambda}_3^\diamond(t) = \Big(\varepsilon + \alpha_2 + \mu\Big)\lambda_3^\diamond(t) - \varepsilon\lambda_4^\diamond(t), \\ \dot{\lambda}_4^\diamond(t) = -\omega_1\lambda_1^\diamond(t) + \Big(\omega_1 + \mu\Big)\lambda_4^\diamond(t), \\ \dot{\lambda}_5^\diamond(t) = -1 + \dfrac{\beta\kappa x_1^\diamond(t)}{\big(\kappa + x_5^\diamond(t)\big)^2}\Big(\lambda_1^\diamond(t) - \lambda_2^\diamond(t+\tau)\chi_{[0,t_f-\tau]}(t)\Big) \\ \qquad\quad + d\lambda_5^\diamond(t), \end{cases} \tag{5.30}$$

with transversality conditions

$$\lambda_i^\diamond(t_f) = 0, \quad i = 1, \ldots, 5$$

for almost all $t \in [0, t_f]$. *Moreover, when* $q = 1$ *and* $q = 2$ *the control law is given, respectively, by*

$$u^\diamond(t) = \begin{cases} 1, & \text{if} \quad \phi(t) < 0; \\ 0, & \text{if} \quad \phi(t) > 0; \\ \text{singular}, & \text{if} \quad \phi(t) = 0 \ \forall t \in I_s \subset [0, t_f]; \end{cases} \tag{5.31}$$

and

$$u^\diamond(t) = \min\left\{\max\left\{0, \frac{\delta x_2^\diamond(t)}{2W}\big(\lambda_2^\diamond(t) - \lambda_3^\diamond(t)\big)\right\}, 1\right\}; \tag{5.32}$$

where ϕ *is the switching function defined by*

$$\phi(t) = W + \delta x_2^\diamond(t)\big(\lambda_3^\diamond(t) - \lambda_2^\diamond(t)\big) \tag{5.33}$$

for almost all $t \in [0, t_f]$.

Proof. The necessary optimality conditions for a local optimal solution of (5.29) are given by the Minimum Principle for delayed optimal control problems (see Theorem 2.17). Let us denote the delayed state variables in the following way:

i) $S(t-\tau) = x_1(t-\tau) = y_1(t)$;

ii) $B(t-\tau) = x_5(t-\tau) = y_5(t)$.

The Hamiltonian function is defined by

$$\begin{aligned} H(X, y_1, y_5, u, \lambda) = {} & x_2 + x_5 + Wu^q \\ & + \lambda_1 \left(\Lambda - \frac{\beta x_1 x_5}{\kappa + x_5} + \omega_1 x_4 - \mu x_1 \right) \\ & + \lambda_2 \left(\frac{\beta y_1 y_5}{\kappa + y_5} - (\delta u + \alpha_1 + \mu) x_2 \right) \\ & + \lambda_3 \big(\delta u x_2 - (\varepsilon + \alpha_2 + \mu) x_3 \big) \\ & + \lambda_4 \big(\varepsilon x_3 - (\omega_1 + \mu) x_4 \big) + \lambda_5 (\eta x_2 - d x_5), \end{aligned} \tag{5.34}$$

where $q \in \{1, 2\}$. Let us suppose that the pair $\big(X^\diamond(\cdot), u^\diamond(\cdot)\big) \in \mathscr{X} \times \mathscr{U}$ is a local optimal solution of (5.29) with fixed final time $t_f \in \mathbb{R}^+$. Then, according to the Minimum Principle for delayed optimal control problems (for more details see [56] and Section 2.3.2), there is an adjoint function $\lambda^\diamond = \big(\lambda_1^\diamond, \lambda_2^\diamond, \lambda_3^\diamond, \lambda_4^\diamond, \lambda_5^\diamond\big) : [0, t_f] \to \mathbb{R}^5$, $\lambda^\diamond(\cdot) \in W^{1,1}\big([0, t_f]; \mathbb{R}^5\big)$, that satisfies, for almost all $t \in [0, t_f]$, the

1) transversality conditions:

$$\lambda_i^\diamond(t_f) = 0, \quad i = 1, \ldots, 5, \tag{5.35}$$

in view of the free terminal state $X(t_f)$;

2) adjoint system:

$$\begin{cases} \dot{\lambda}_i^\diamond(t) = -\dfrac{\partial H}{\partial x_i}[t] - \dfrac{\partial H}{\partial y_i}[t+\tau]\chi_{[0, t_f - \tau]}(t), \quad i = 1, 5, \\ \dot{\lambda}_i^\diamond(t) = -\dfrac{\partial H}{\partial x_i}[t], \quad i = 2, 3, 4; \end{cases} \tag{5.36}$$

3) minimality condition:

$$\begin{aligned} & \min_{0 \le u \le 1} H\big(X^\diamond(t), y_1^\diamond(t), y_5^\diamond(t), u, \lambda^\diamond(t)\big) \\ & = H\big(X^\diamond(t), y_1^\diamond(t), y_5^\diamond(t), u^\diamond(t), \lambda^\diamond(t)\big). \end{aligned} \tag{5.37}$$

Note that when we write $H[t]$, we mean that the arguments of H are applied in t, that is, $H[t] = H\big(X^\diamond(t), y_1^\diamond(t), y_5^\diamond(t), u^\diamond(t), \lambda^\diamond(t)\big)$. Due to previous point 1), we can conclude that conditions $\lambda_i^\diamond(t_f) = 0, \quad i = 1, \ldots, 5$, are derived from transversality conditions (5.35). Moreover, system (5.30) is obtained from adjoint system (5.36). Now, let us evaluate the minimality condition (5.37) for $q = 1$. The Hamiltonian (5.34) is linear in the control variable. Hence, the minimizer control is determined by the sign of the switching function

$$\phi(t) = \frac{\partial H}{\partial u}[t] = W + \delta x_2^\diamond(t)\big(\lambda_3^\diamond(t) - \lambda_2^\diamond(t)\big)$$

(see (5.33)) as follows:

$$u^\diamond(t) = \begin{cases} 1, & \text{if} \quad \phi(t) < 0; \\ 0, & \text{if} \quad \phi(t) > 0; \\ \text{singular}, & \text{if} \quad \phi(t) = 0 \ \forall t \in I_s \subset [0, t_f]. \end{cases}$$

For more details, see Section 2.2.2. If the switching function has only finitely many isolated zeros in an interval $I_b \subset [0, t_f]$, then the control $u^\diamond$ is called *bang-bang* on I_b. The case of a *singular control*, where $\phi(t) = 0$ on $I_s \subset [0, t_f]$, will not be further discussed here, since in our computations we never encountered *singular controls*. For $q = 2$ we consider the so-called free (unconstrained) control u_f defined by the equation

$$\frac{\partial H}{\partial u}\big(X(t), y_1(t), y_5(t), u_f(t), \lambda(t)\big) = 2W u_f(t) + \delta x_2(t)\big(\lambda_3(t) - \lambda_2(t)\big) = 0,$$

which yieds

$$u_f(t) = u_f\big(X(t), \lambda(t)\big) = \frac{\delta x_2(t)}{2W}\big(\lambda_2(t) - \lambda_3(t)\big).$$

Then, the constrained control $u(t) \in [0, 1]$ minimizing the Hamiltonian (5.34) is given by the projection of u_f onto $[0, 1]$, which gives the following control law:

$$\begin{aligned} u^\diamond(t) &= Proj_{[0,1]}\Big(u_f\big(X^\diamond(t), \lambda^\diamond(t)\big)\Big) \\ &= \min\left\{\max\left\{0, \frac{\delta x_2^\diamond(t)}{2W}\big(\lambda_2^\diamond(t) - \lambda_3^\diamond(t)\big)\right\}, 1\right\}. \end{aligned}$$

Consequently, optimal controls (5.31) and (5.32) comes from the minimality condition (5.37). This concludes the proof. □

5.3.5 Numerical simulations

After the theoretical study done in Section 5.3.4, we are going to provide again numerical simulations for the cholera outbreak that occurred in the

Department of Artibonite – Haiti, from 1st November 2010 to 1st May 2011 (see [193]). We start this section by showing that model (4.1) with a positive time delay translates better the cholera outbreak in Haiti, than model (5.1).

We end this section with the study of several numerical solutions of optimal control problem (5.29) varying the value of the delay ($\tau = 0$ or $\tau = 3.3$) and the type of control (linear or quadratic). These studies are done with the purpose to obtain control strategies that could stop the spread of the considered outbreak in Haiti.

Delayed SIB sub-model

The cholera outbreak that occurred in the Department of Artibonite – Haiti, from 1st November 2010 to 1st May 2011 (see [193]), is approximated through a numerical simulation of model (5.1) in Section 5.2.5: *Non-delayed SIB sub-model* (see Figure 5.1a). Such simulation, which does not consider treatment and recovery ($\omega_1 = \delta = \varepsilon = \alpha_2 = Q(0) = R(0) = 0$), is improved here and denominated by (NS1). Moreover, let us call (NS2) to the numerical simulation that improves (NS1), presented next. Both for (NS1) and (NS2), we assume the values of Table 5.3 for the parameters Λ, μ, κ, η, d, S_0, I_0, B_0 and t_f. Furthermore, we also assume, for both, that $\omega_1 = \delta = \varepsilon = \alpha_2 = 0$ and $Q(t) = R(t) = 0$ for all $t \in [-\tau, 0]$. With such choice, we obtain $Q(t) = R(t) = 0$ for all $t \in [-\tau, t_f]$. Consequently, (NS1) and (NS2) are related to the sub-model of (4.1) given by

$$\begin{cases} \dot{S}(t) = \Lambda - \dfrac{\beta B(t) S(t)}{\kappa + B(t)} - \mu S(t), \\ \dot{I}(t) = \dfrac{\beta B(t-\tau) S(t-\tau)}{\kappa + B(t-\tau)} - (\alpha_1 + \mu) I(t), \\ \dot{B}(t) = \eta I(t) - d B(t). \end{cases} \tag{5.38}$$

We assume that $\tau = 0$, $\beta = 0.8$ and $\alpha_1 = 0.015$ for (NS1) and that $\tau = 3.3$, $\beta = 0.5$ and $\alpha_1 = 0.024$ for (NS2). The numerical simulation (NS2) approximates better the cholera outbreak in the Department of Artibonite – Haiti than (NS1): see Figure 5.5b. The average of relative error per day of (NS2) decreases more than half with respect to (NS1). The numerical simulations (NS1) and (NS2) were obtained with the help of the integration routines in Matlab.

Remark 5.12. *It is important to note that the values of the parameters β and α_1 are different for (NS1) and (NS2), because for each numerical simulation, without or with delay, we consider the parameter values that better fit the outbreak in Haiti. So, here (NS1) is a motivation for the numerical simulations done in the current section.*

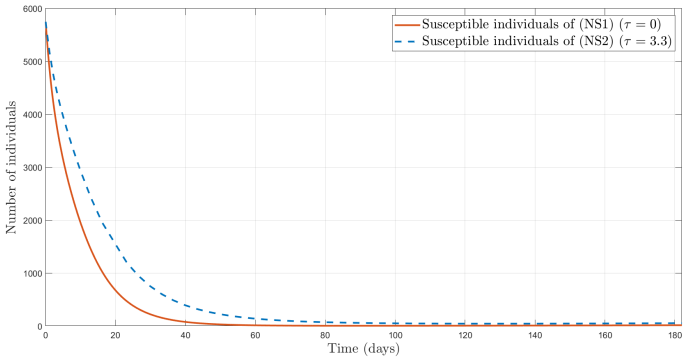
In Table 5.2, we can compare the basic reproduction number (5.8) and the endemic equilibrium (5.9) for numerical simulations (NS1) and (NS2). Furthermore, Figures 5.5a and 5.5c give, respectively, the solutions of S and B for (NS1) and (NS2). We know that the endemic equilibrium of (NS1) is locally asymptotic stable, because the corresponding reproduction number is given by $R_0 = 35.730565 > 1$ (see Theorem 5.6). One can conclude that the endemic equilibrium of (NS2) is locally asymptotic stable too, by doing numerical simulations similar to those of Figure 5.5 for a sufficiently large time interval.

	R_0	S^*	I^*	Q^*	R^*	B^*
(NS1)	35.730565	620.282930	32.223372	0	0	976.465804
(NS2)	13.965093	1586.409227	19.246308	0	0	583.221449

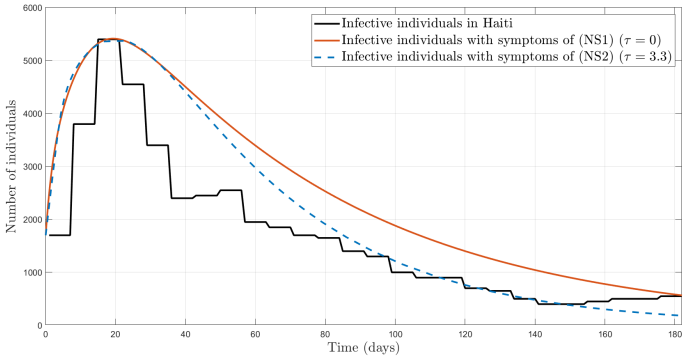
Table 5.2: Some results associated with numerical simulations (NS1) and (NS2).

Parameter	**Value**	**Unity**	**Reference**
Λ	$(24.4N_0)/365000$	person day^{-1}	[76]
μ	2.2493×10^{-5}	day^{-1}	[78]
β	0.5	day^{-1}	Assumed
κ	10^6	cell/ml	[150]
ω_1	0.4/365	day^{-1}	[120]
δ	0.05	day^{-1}	Assumed
ε	0.2	day^{-1}	[127]
α_1	0.024	day^{-1}	Assumed
α_2	0.0001	day^{-1}	[127]
η	10	cell/ml day^{-1} person^{-1}	[22]
d	0.33	day^{-1}	[22]
τ	3.3	days	[27]
S_0	5750	person	Assumed
I_0	1700	person	[193]
Q_0	0	person	Assumed
R_0	0	person	Assumed
N_0	7450	person	–
B_0	275000	cell/ml	Assumed
t_f	182	days	[193]
W	1000	Adimensional	Assumed

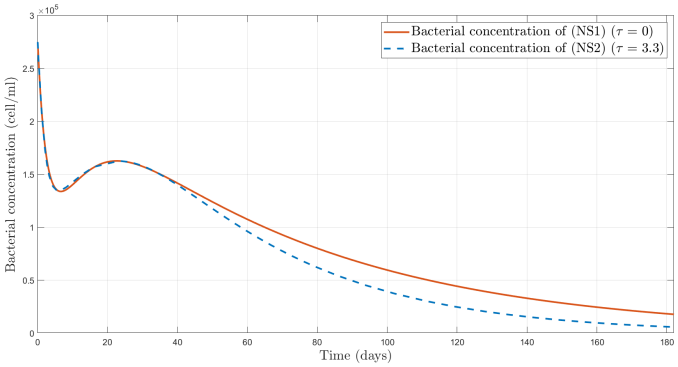
Table 5.3: Parameter values and initial conditions for delayed optimal control problem (5.29).



(a) Susceptible individuals of (NS1) and (NS2).



(b) Infective individuals with symptoms of (NS1) and (NS2) versus real data from the cholera outbreak in the Department of Artibonite – Haiti, from 1st November 2010 to 1st May 2011.



(c) Bacterial concentration of (NS1) and (NS2).

Figure 5.5: State trajectories $S(t)$, $I(t)$ and $B(t)$ for all $t \in [0, 182]$, predicted by models (4.1) (dashed blue lines) and (5.1) (solid orange lines), assuming that $\delta = \varepsilon = \omega_1 = \alpha_2 = 0$. To obtain solid orange and dashed blue lines, we use all the other values of Tables 5.1 and 5.3, respectively.

Numerical solutions of the delayed SIQRB control model

With the purpose to obtain control strategies for the Haiti cholera outbreak mentioned previously, we consider here that there is treatment through quarantine ($\omega_1,\ \delta,\ \varepsilon,\ \alpha_2 \in \mathbb{R}^+$) and that control measures are taken. The absence of a control measure means that we are considering that $u(t) = 1$ for all time $t \in [0, t_f]$, i.e., all infective individuals are moved to quarantine all the time. Sometimes, to treat all infective individuals, it is not necessary to consider $u(t) = 1$ for all $t \in [0, t_f]$. In this way, it is possible to treat infective people and decrease the respective costs. To illustrate this, we are going to solve numerically the optimal control problem (5.29), considering several cases. For solving it, we apply the discretization methods developed in [56, 57]. The resulting large-scale non-linear programming problem (NLP) can be conveniently formulated using the Applied Modeling Programming Language – AMPL (see [48, 54]), which can be linked to several efficient optimization solvers. We use the Interior-Point optimization solver – IPOPT, developed by Wächter and Biegler (see [177]). As integration method we implement either Euler's method or the trapezoidal rule. For our numerical computations, we use the trapezoidal rule with $t_f = 182$ days and $N = 30 \times t_f = 5460$ grid points. The tolerance for IPOPT is set to $tol = 10^{-10}$. In the following, we shall compare non-delayed solutions ($\tau = 0$) with delayed solutions ($\tau = 3.3$), considering four cases:

1) Case 1: $\tau = 0$ and $q = 1$;

2) Case 2: $\tau = 3.3$ and $q = 1$;

3) Case 3: $\tau = 0$ and $q = 2$;

4) Case 4: $\tau = 3.3$ and $q = 2$.

For these computations, we use the other parameters values and the initial conditions of Table 5.3.

Case 1 ($\tau = 0$ and $q = 1$). The obtained cost functional value and initial value of the adjoint function are, approximately, given by

$$J_1\big(X^\diamond(\cdot), u^\diamond(\cdot)\big) \simeq 3.64958 \times 10^6$$

and

$$\lambda^\diamond(0) \simeq (443.90,\ 538.42,\ 15.226,\ 17.362,\ 3.6544).$$

We also obtain the following *bang-bang control* with only one switch:

$$u^\diamond(t) = \begin{cases} 1, & \text{if} \quad 0 \le t < t_s^\diamond, \\ 0, & \text{if} \quad t_s^\diamond \le t \le t_f, \end{cases} \tag{5.39}$$

with the switching time $t_s^\diamond \simeq 126.4$ days (see the solid dark green curve in the right plot of Figure 5.6a). Here, the *bang-bang control* induces an optimization problem, where the switching time t_s is the only optimization variable. Therefore, the cost reduces to $\tilde{J}_1(t_s) = J_1\big(X(\cdot), u(\cdot)\big)$. We check numerically that the second derivative of $\tilde{J}_1(t_s^\diamond)$ is positive and get $\ddot{\tilde{J}}_1(t_s^\diamond) \simeq 64.2$. Moreover, the switching function satisfies the strict *bang-bang property* (see the right plot of Figure 5.6b):

$$\phi(t) < 0 \quad \text{for} \quad 0 \leq t < t_s^\diamond, \quad \dot{\phi}(t_s^\diamond) > 0, \quad \phi(t) > 0 \quad \text{for} \quad t_s^\diamond < t \leq t_f.$$

Then, it follows from Theorem 7.10 of [133] that the *bang-bang control* (5.39) provides a local minimum.

Case 2 ($\tau = 3.3$ and $q = 1$). The obtained cost functional value and initial value of the adjoint function are, approximately, given by

$$J_1\big(X^\diamond(\cdot), u^\diamond(\cdot)\big) \simeq 4.71343 \times 10^6$$

and

$$\lambda^\diamond(0) \simeq (439.10,\ 489.51,\ 16.594,\ 18.779,\ 3.3692).$$

The control has the *bang-bang* structure (5.39) with only one switch at $t = t_s^\diamond \simeq 125.5$ days (see the solid light green curve of the left plot of Figure 5.6a). The switching time for $\tau = 3.3$ ($t_s^\diamond = 125.5$ days) is smaller than that for $\tau = 0$ ($t_s^\diamond = 126.4$ days). Concluding, in Cases 1 and 2 we have to treat through quarantine all infective individuals in the first 126.4 and 125.5 days, respectively. After this, there is no more treatment. However, the quarantined time is similar for Cases 1 and 2.

To our knowledge, no sufficient conditions for *bang-bang controls* with $\tau > 0$ and $q = 1$ are available in the literature. But the left plot of the Figure 5.6b shows that the computed control satisfies the necessary conditions and, hence, is an extremal solution.

Remark 5.13. *With respect to Cases 1 and 2, we present in both plots of Figure 5.6b the function $\frac{\phi}{W}$ instead of ϕ, because the values obtained by ϕ are much bigger than those achieved by control $u^\diamond$. Thus, using the function $\frac{\phi}{W}$, it is possible to compare the signal of ϕ with the behaviour of the linear control $u^\diamond$ in the same plot, either for $\tau = 0$ or $\tau = 3.3$.*

Figure 5.7 shows the comparison of delayed ($\tau = 3.3$) and non-delayed ($\tau = 0$) state trajectories $S^\diamond$, $I^\diamond$, $Q^\diamond$, $R^\diamond$ and $B^\diamond$ for optimal control problem (5.29) with $q = 1$. The delayed and non-delayed trajectories are represented, respectively, by solid light green and dashed dark green curves. We see a significant difference in the delayed and non-delayed state trajectories. Although the number of infective individuals is not represented by a strictly decreasing function for Cases 1 and 2, there are improvements for both cases,

when we compare them with the situation depicted in Figure 5.5b. The number of infective individuals for $\tau = 3.3$ is much higher than that for $\tau = 0$ up to time $t = 90$ days and, moreover, the maximum value of infective individuals in the delayed case is bigger than in the non-delayed case (see top right plot of Figure 5.7). Because of this, the number of susceptible individuals is slightly smaller for $\tau = 3.3$ than that for $\tau = 0$, as one can observe in the top left plot of Figure 5.7. Consequently, when $\tau = 3.3$, more infective individuals have to be quarantined with the purpose to recover later. Actually, for $\tau = 3.3$, the number of quarantined and recovered individuals is significantly larger than in the non-delayed case (see middle plots of Figure 5.7). When we consider a control treatment through quarantine, we improve the situation translated by Figure 5.5c, because the bacterial concentration is represented here by a strictly decreasing function for both Cases 1 and 2 (see bottom plot of Figure 5.7). Despite of this, the solution data shows that the decrease is faster in the time interval $[3.5, 80]$ for Case 1.

Case 3 ($\tau = 0$ and $q = 2$). The obtained cost functional value and initial value of the adjoint function are, approximately, given by

$$J_2\big(X^\diamond(\cdot), u^\diamond(\cdot)\big) \simeq 3.64246 \times 10^6$$

and

$$\lambda^\diamond(0) \simeq (444.26,\ 538.27,\ 15.236,\ 17.374,\ 3.6514).$$

These values are very close to the ones obtained for Case 1 ($\tau = 0$ and $q = 1$). The control is given by

$$u^\diamond(t) = \begin{cases} 1, & \text{if} \quad 0 \leq t < t_s^\diamond, \\ u_f(t), & \text{if} \quad t_s^\diamond \leq t \leq t_f. \end{cases} \tag{5.40}$$

The control has a boundary arc with exit time $t_s^\diamond \simeq 99.9$ days (see the dashed light blue curve in the right plot of Figure 5.6a). The control is *continuous* at t_s, which also comes from the fact that the strict Legendre–Clebsch condition holds and the Hamiltonian (5.34) is regular, i.e., admits an unique minimum. To check second-order sufficient conditions (SSC) for this solution, one would have to show that an associated matrix Riccati equation has a bounded solution on $[t_s, t_f]$ and satisfies a certain boundary condition at $t = t_f$ (see [119, 133, 154]). However, we refrain here from performing this cumbersome test.

Case 4 ($\tau = 3.3$ and $q = 2$). The obtained cost functional value and initial value of the adjoint function are, approximately, given by

$$J_2\big(X^\diamond(\cdot), u^\diamond(\cdot)\big) \simeq 4.70657 \times 10^6$$

and

$$\lambda^\diamond(0) \simeq (439.35,\ 489.42,\ 16.578,\ 18.765,\ 3.3675).$$

We can note that these values are very similar to those for Case 2 ($\tau = 3.3$ and $q = 1$). As in the non-delayed problem, the control is *continuous* and has the control structure (5.40) with a slightly different exit point $t_s^\diamond \simeq 100.5$ days of the boundary (see the dashed red curve of the left plot of Figure 5.6a). Second-order sufficient conditions (SSC) can probably be derived by applying Guinn's transformation technique (see [59]) and the matrix Riccati type results in [119, 154, 133]. We remark again that this will lead to a numerically difficult test.

Concluding, in Cases 3 and 4, we have to treat all infective individuals in the first 99.9 and 100.5 days, respectively. After this moment, we are going to decrease the fraction of infective individuals who are being moved to quarantine until the final time t_f. In both quadratic cases (Cases 3 and 4) the exit time values are similar. As the extremal state trajectories for $q = 2$ are very similar to those for $q = 1$ (Figure 5.7), we do not display them here.

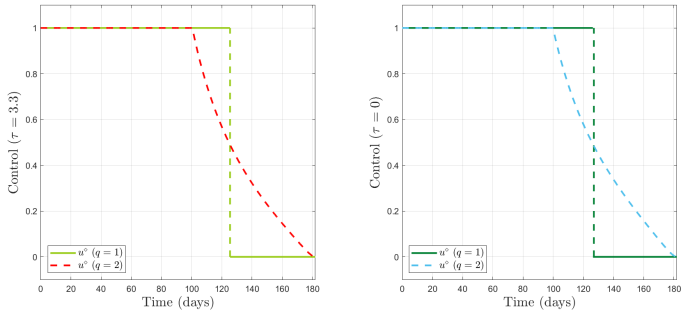
Figures 5.6a–5.7 summarize our numerical findings in the four cases. Figure 5.6a displays the linear ($q = 1$) and quadratic ($q = 2$) controls either to the delayed (left side) or to the non-delayed (right side) SIQRB model. The main message of Figures 5.6b and 5.6c is that the controls satisfy the control laws (5.31) for $q = 1$ and (5.32) for $q = 2$. Hence, the necessary conditions are satisfied and thus we have found the *extremals*. Only in case $\tau = 0$ and $q = 1$ we could verify sufficient conditions.

The cost functional values of the non-delayed cases (Cases 1 and 3) are smaller than those of the delayed cases (Cases 2 and 4), maybe because there are more infective individuals when we use $\tau = 3.3$. Nevertheless, it is more realistic to consider $\tau > 0$ in the context of the infectious disease in study – cholera (see [27]). Although we have studied numerical simulations either to linear or to quadratic control, in the biomedical framework it is more appropriate to use a linear control since the cost is directly proportional to the dosage control. Moreover, in real life it is easier to apply the linear control measure than the quadratic (see [99, 100]).

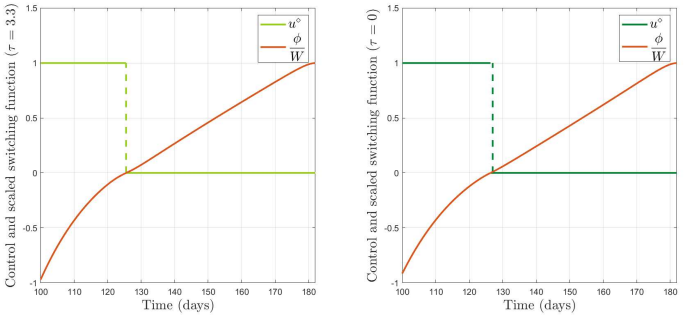
Furthermore, we can also observe that the endemic equilibrium (5.25) of the delayed model (4.1) is locally asymptotic stable, when we consider all the values of Table 5.3 (see Figure 5.8).

5.4 Conclusion

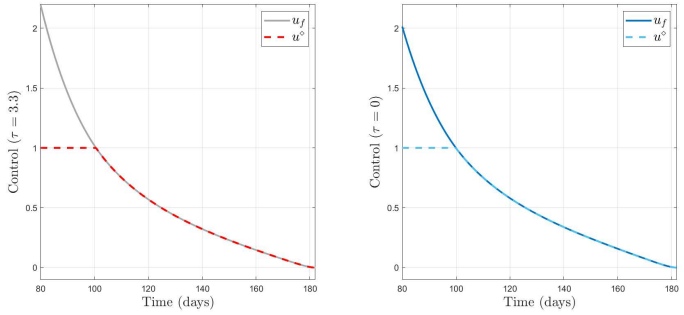
SIR (Susceptible–Infectious–Recovered) type models and optimal control theory provide powerful tools to describe and control infectious disease dynamics (see [96, 115, 162]). In this chapter, we analysed analytically and numerically two SIQRB type models for the dynamics of cholera transmission. These models differ from the other cholera mathematical models in the literature, because they assume that infective individuals subject to treatment stay in quarantine during that period. While the first one is non-delayed (see



(a) Control $u^\diamond$ for Cases 1 (solid dark green curve in the right column), 2 (solid light green curve in the left column), 3 (dashed light blue curve in the right column) and 4 (dashed red curve in the left column).



(b) Zoom into $u^\diamond$ and $\frac{\phi}{W}$ associated with (5.29) for $q = 1$, satisfying the control law (5.31), either for $\tau = 0$ (right column) or for $\tau = 3.3$ (left column).



(c) Zoom into u_f and $u^\diamond$ associated with (5.29) for $q = 2$, satisfying the control law (5.32), either for $\tau = 0$ (right column) or for $\tau = 3.3$ (left column).

Figure 5.6: Control $u^\diamond(t)$, scaled switching function $\frac{\phi(t)}{W}$ and free control $u_f(t)$ associated with optimal control problem (5.29) for all $t \in [0, 182]$, either when $\tau = 0$ or when $\tau = 3.3$, using all the other values of Table 5.3.

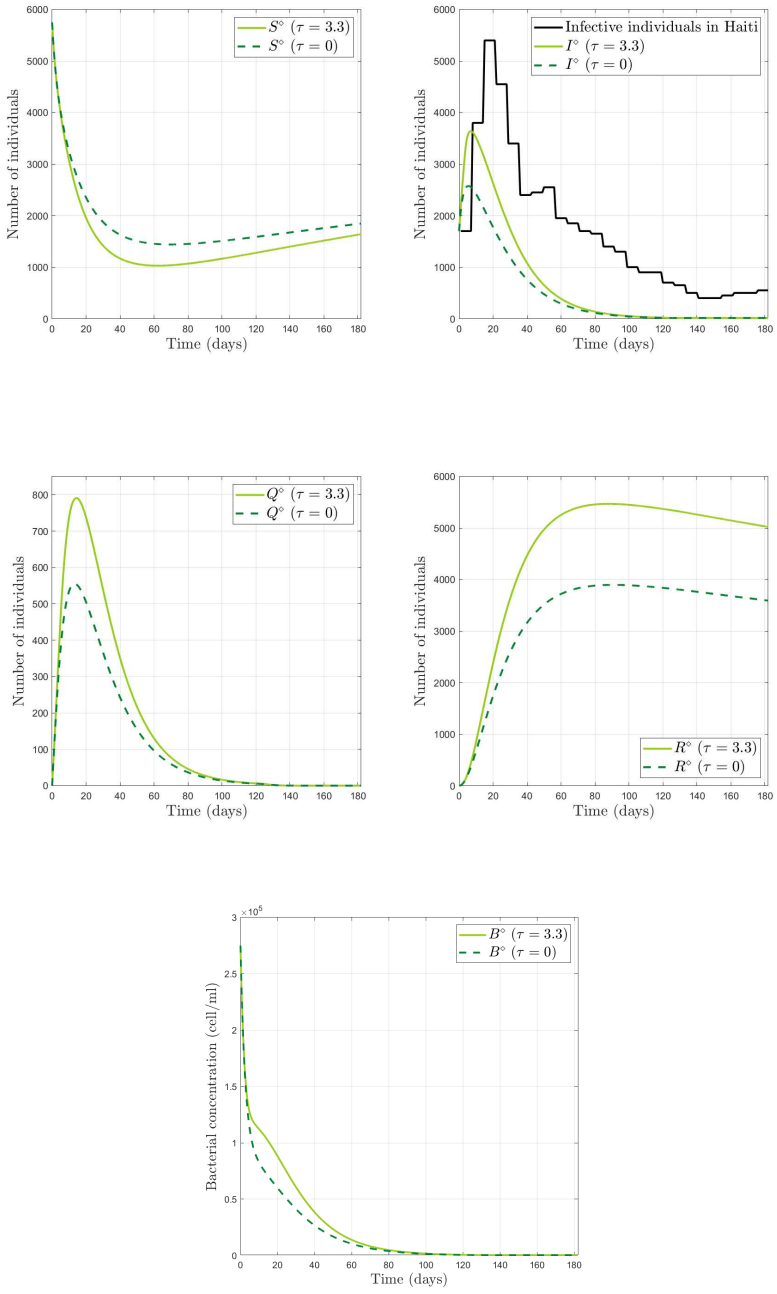


Figure 5.7: State trajectories $S^\diamond(t)$, $I^\diamond(t)$ (versus real data from the cholera outbreak in the Department of Artibonite – Haiti, from 1st November 2010 to 1st May 2011), $Q^\diamond(t)$, $R^\diamond(t)$ and $B^\diamond(t)$ associated with optimal control problem (5.29) for all $t \in [0, 182]$ and $q = 1$, either when $\tau = 0$ (dashed dark green curves) or when $\tau = 3.3$ (solid light green curves), using all the other values of Table 5.3.

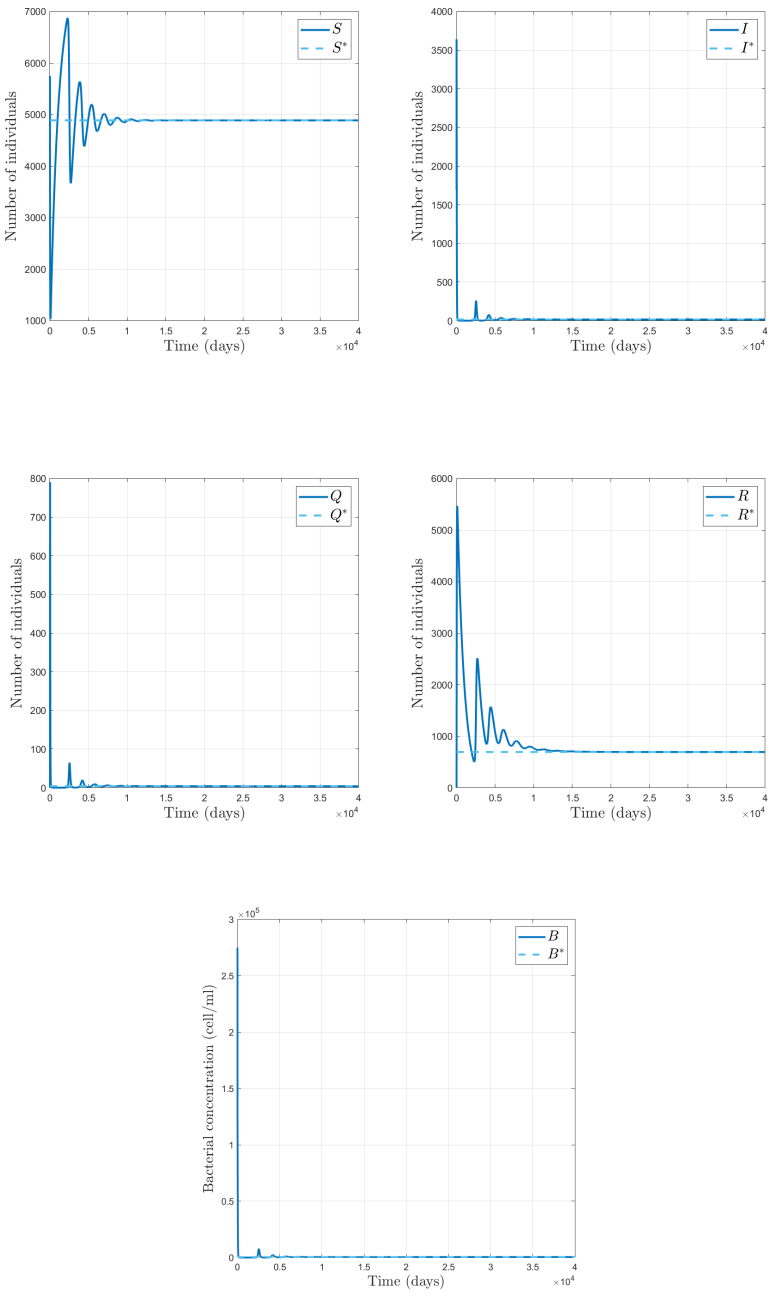


Figure 5.8: State trajectories of model (4.1) versus endemic equilibrium (5.9) for $t_f = 4 \times 10^4$ days and considering all the other values of Table 5.3.

Section 5.2), the second considers a time delay (see Section 5.3). The model of Section 5.2 is a particular case of that of Section 5.3. The considered

delay represents the time that a susceptible individual, who got infected, takes to have symptoms of the infection by the bacterium *Vibrio cholerae*. The delayed model is more realistic and describes better the reality, since the symptoms of cholera disease can appear from a few hours to 5 days after infection. Usually, the symptoms appear in 2–3 days after infection (see [27]).

For both models we proved that their solutions are non-negative, if non-negative initial conditions are considered. The equilibrium points and the basic reproduction number were computed for the non-delayed model and are the same for the delayed one. In Section 5.2, we studied the local asymptotic stability for the equilibrium points, considering arbitrary parameters. In Section 5.3, we also analysed the local asymptotic stability for the disease-free equilibrium, considering arbitrary parameters. With respect to the delayed model, the local asymptotic stability associated with the endemic equilibrium was carried out in function of the ingestion rate of the bacteria through contaminated sources β. So, we could conclude that the ingestion rate of the bacteria through contaminated sources β has an important influence on the stability of the endemic equilibrium.

Using the two proposed cholera models, we tried to simulate the cholera outbreak in the Department of Artibonite – Haiti, from 1st November 2010 to 1st May 2011 [193]. For these numerical simulations, we did not consider treatment and, consequently, recovery. The numerical simulations associated with such epidemic, that were developed in Section 5.3.5, improved those done in Section 5.2.5. Thus, we showed that the delayed model fits better this cholera outbreak.

For both models we add the same control function with the purpose to find the best way of using quarantine with the less possible cost and, simultaneously, to minimize the number of infective individuals and the bacterial concentration. Consequently, we proposed two optimal control problems (without and with a time delay) which are analysed analytically. While in Section 5.2, we only consider a quadratic cost functional with respect to control, in Section 5.3 we consider a quadratic and a linear cost functional. Necessary optimality conditions were applied to all considered optimal control problems. Sufficient optimality conditions were only applied to the non-delayed optimal control problem with linear control in the cost functional, in Section 5.3.

Let us call Case 0 to the numerical simulation of the non-delayed SIQRB control model done in Section 5.2.5. So, Case 0 shows that after approximately three months (87.36 days) the optimal strategy implies a gradual reduction of the fraction of infective individuals that stay in quarantine. To be precise, by introducing the optimal strategy through quarantine, as a way of systematizing treatment, one reduces the 1700 infective individuals at $t = 0$ days (reported by WHO in [193]) to just 86 infective individuals at the end of 87.36 days, approximately. On the other hand, $I(88) \simeq 76$ indi-

viduals for Cases 1 and 3 of Section 5.3.5. For Cases 2 and 4 of Section 5.3.5, $I(88) \simeq 91$ individuals. So, with respect to the number of infective individuals at the end of 88 days, approximately, we can conclude that Cases 1 and 3 attained the best values.

Since quarantine implies a big economic, social and individual effort, it is important to know the instant of time from which the infective individuals may leave quarantine without compromising the minimization of the number of infective individuals and the bacterial concentration. The switching time $t_s^\diamond$ (in days) and the number of infective individuals at the final time ($t_f = 182$ days) for all Cases 0, 1, 2, 3 and 4 are presented in Table 5.4. Although the switching times attained by Cases 1 and 2 are the biggest, the *bang-bang* structure (5.39) of Cases 1 and 2 are the easiest to implement. Furthermore, by Table 5.4 one can conclude that the smallest number of infective individuals at the end is obtained for Case 4.

	$t_s^\diamond$ (days)	$I(182)$ (number of individuals)
Case 0	$\simeq 87.36$ days	$\simeq 23$ individuals
Case 1	$\simeq 126.37$ days	$\simeq 14$ individuals
Case 2	$\simeq 125.53$ days	$\simeq 13$ individuals
Case 3	$\simeq 99.87$ days	$\simeq 12$ individuals
Case 4	$\simeq 100.50$ days	$\simeq 11$ individuals

Table 5.4: Switching time and the number of infective individuals at final time ($t_f = 182$ days) for Cases 0, 1, 2, 3 and 4.

In the next chapter, we are going to study different mathematical models with purpose to fit a cholera epidemic that is known as the biggest cholera outbreak in the human history (see [170]). It has occurred in Yemen since October 2016 (see [182, 186]). We are going to study it from 27th April 2017 to 15th April 2018 (see [189]). Moreover, we intend to propose strategies to eradicate such type of outbreaks.

Chapter 6

Optimal control of cholera outbreak in Yemen

In this chapter, we analyse two mathematical models for the transmission dynamics of some strains of the bacterium *Vibrio cholerae*, responsible for the cholera disease in humans. We show that they are epidemiologically and mathematically well posed. For both models, we also prove the existence and uniqueness of disease-free and endemic equilibrium points; we determine the basic reproduction number and we study the local asymptotic stability of equilibria. The biggest cholera outbreak of world's history began in October 2016, in Yemen. Between 27th April 2017 and 15th April 2018 there were 2275 deaths due to this epidemic. A vaccination campaign began on 6th May 2018 and ended on 15th May 2018. We show that the first proposed model, that considers vaccination, is able to describe well this outbreak. Moreover, we prove that the number of infective individuals would have been much lower provided the vaccination campaign had begun earlier. A control function is introduced into the second model, representing the distribution of chlorine water tablets (CWT) for water purification. An optimal control problem is then proposed and analysed, where the goal is to determine the fraction of susceptible individuals that should have access to CWT in order to minimize the number of infective individuals and bacteria concentration, as well as the costs associated with the distribution of CWT. Finally, we consider the real data of the cholera outbreak in Yemen, from 27 April 2017 to 15 April 2018. Using optimal control results, we show, numerically, that the distribution of CWT could have stopped, in a fast way, the worst cholera outbreak that ever occurred in human history. Due to the critical situation of Yemen, we also simulate the case where only a small percentage of susceptible individuals has access to CWT and obtain an extremal control that decreases, substantially, the maximum number of infective individuals attained at the outbreak. All the contents of Section 6.2 are published in [107].

6.1 Introduction

Recently, the biggest outbreak of cholera in the history of the world has occurred in Yemen (see [170]). The epidemic began in October 2016 and in February–March 2017 was in decline. However, on 27th April 2017 the epidemic returned. This happened ten days after Sana'a's sewer system had stopped working. Problems in infrastructures, health, water and sanitation systems in Yemen allowed the fast spread of the disease (see [182]). Between 27th April 2017 and 1st July 2018 there were 1115378 suspected cases reported and 2310 deaths due to cholera (see [190]).

In [130], Nishiura et al. study mathematically this outbreak, trying to forecast the cholera epidemic in Yemen by explicitly addressing the reporting delay and ascertainment bias (see also [129]). The Yemen outbreak data available in the website of World Health Organization (WHO) is also fitted by He et al. in [66], considering a mathematical model based on differential equations. Their model considers five classes: S (susceptible individuals), I (infectious individuals), R (recovered individuals), B (concentration of the bacterium in the environment) and M (availability of medical resources in the country). Such model translates the interaction between the human hosts and the pathogenic bacteria, under the impact of limited medical resources. The results obtained in [66] support that improvement of the public health system and strategic implementation of control measures with respect to time and location can facilitate the prevention and intervention related to this disease in Yemen.

A vaccine for cholera is currently available, but poor sanitation and the lack of access to vaccines promote the spread of the disease (see [4]). According to WHO recommendations, the first-ever oral vaccination campaign against cholera had been launched on 6th May 2018 in Yemen and it was concluded on 15th May 2018 (see [187]), due to the lack of national governmental authorization to do the vaccination (see [143]). Aid workers say that one reason of the delayed of the campaign vaccination is due to some senior Houthi officials who objected to vaccination. This campaign coincided with the rainy season and some health workers fear that this could spread the disease (see [143]). This campaign just covered four districts in Aden, which were at a high risk of fast spread of the disease, and just 350000 individuals (including pregnant women) were vaccinated (see [143, 170]). It is important to note that the vaccinated individuals represent, approximately, 1.21% of the total population, since the total population of Yemen, in 2018, is 28915284 (see [194]). Lorenzo Pizzoli, WHO's cholera expert, said that the campaign hoped to cover at least four million people in areas at risk (corresponding, approximately, to 14% of total population) and Michael Ryan, WHO's Assistant Director-General, revealed that they were negotiating with Yemen health authorities in order to vaccinate people from all high risks zones (see [143]). The International Coordinating Group on Vaccine Provi-

sion had planned one million cholera vaccines for Yemen in July 2017, but WHO and Yemen local authorities decided to postpone it and the doses were diverted to South Sudan. WHO and GAVI, the Vaccine Alliance, affirmed that the largest cholera vaccination is now being carried out in five countries (Kenya, Malawi, South Sudan, Uganda and Zambia). It is expected that this campaign targets more than two million people across Africa (see [143]).

This chapter is organised as follows. In Section 6.2, we analyse a cholera mathematical model that considers vaccination. In Section 6.3, we study other cholera model that is an improvement of the one analysed in Section 5.2. For both models, we analyse the non-negativity and boundedness of the solutions, proving in this way that the models are mathematically well posed and they have biological meaning (see Sections 6.2.1 and 6.3.1). Then, we prove the existence and uniqueness of the disease-free and endemic equilibrium points and compute the basic reproduction number for both models (see Sections 6.2.2 and 6.3.2). The sensitivity of the basic reproduction number with respect to all parameters of the first model is analysed in Section 6.2.3. Moreover, the stability analysis of equilibria is carried out for both models (see Sections 6.2.4 and 6.3.3). With respect to the model with vaccination, we show numerically that it admits a sub-model which fits well the cholera outbreak in Yemen, between 27th April 2017 and 15th April 2018 (see Section 6.2.5). Note that the considered sub-model does not consider vaccination. Then, we also illustrate the impact of vaccination of susceptible individuals in Yemen. The numerical results support the importance of vaccination to prevent cholera spread. With respect to the second model, we formulate and analyse an optimal control problem in Section 6.3.4, deriving the respective necessary optimality conditions according to Pontryagin's Minimum Principle (see [140]). The goal of such problem is to minimize the new infections and bacterial concentration, through the distribution of the CWT, as well as the costs associated with this measure. Finally, Section 6.3.5 is devoted to numerical simulations and a case study in Yemen, associated with the second considered model. The concluding Section 6.4 discusses the impact of vaccination and CWT distribution on the control of the cholera outbreak in Yemen.

6.2 Model with vaccination

In this section, we consider model (4.3). Throughout this section, we assume that the initial conditions of system (4.3) are non-negative:

$$S(0) \geq 0\,,\ I(0) \geq 0\,,\ T(0) \geq 0\,,\ R(0) \geq 0\,,\ V(0) \geq 0\,,\ B(0) \geq 0. \qquad (6.1)$$

The constants S_0, I_0, T_0, R_0, V_0, B_0 and N_0 denote $S(0)$, $I(0)$, $T(0)$, $R(0)$, $V(0)$, $B(0)$ and $S(0)+I(0)+T(0)+R(0)+V(0)$, respectively, since model (4.3) does not consider time delays: $\tau = 0$ (see Table 4.1). In this

section we are going to continue using notation (5.3) and moreover, we also need to do more considerations. Namely, a_0 and a_4 are defined as follows:

$$a_0 = \varphi + \mu \quad \text{and} \quad a_4 = \omega_2 + \mu. \tag{6.2}$$

6.2.1 Non-negativity and boundedness of solutions

Our first lemma shows that the considered model (4.3) subject to (6.1) is biologically meaningful.

Lemma 6.1. *For all $t \geq 0$ the solutions $\big(S(t), I(t), T(t), R(t), V(t), B(t)\big)$ of system (4.3) are non-negative with non-negative initial conditions (6.1) in $\big(\mathbb{R}_0^+\big)^6$.*

Proof. We have

$$\begin{cases} \left.\dfrac{dS(t)}{dt}\right|_{\xi(S)} &= \Lambda + \omega_1 R(t) + \omega_2 V(t) > 0\,, \\ \left.\dfrac{dI(t)}{dt}\right|_{\xi(I)} &= \dfrac{\beta B(t)}{\kappa + B(t)} S(t) \geq 0\,, \\ \left.\dfrac{dT(t)}{dt}\right|_{\xi(T)} &= \delta I(t) \geq 0\,, \\ \left.\dfrac{dR(t)}{dt}\right|_{\xi(R)} &= \varepsilon T(t) \geq 0\,, \\ \left.\dfrac{dV(t)}{dt}\right|_{\xi(V)} &= \varphi S(t) \geq 0\,, \\ \left.\dfrac{dB(t)}{dt}\right|_{\xi(B)} &= \eta I(t) \geq 0\,, \end{cases}$$

where $\xi(\upsilon) = \big\{\upsilon(t) = 0 \text{ and } S(\cdot), I(\cdot), T(\cdot), R(\cdot), V(\cdot), B(\cdot) \in \mathscr{C}\left(\mathbb{R}_0^+, \mathbb{R}_0^+\right)\big\}$ and $\upsilon \in \{S, I, T, R, V, B\}$. Therefore, due to Lemma 1.42, any solution of system (4.3) is such that $\big(S(t), I(t), T(t), R(t), V(t), B(t)\big) \in \big(\mathbb{R}_0^+\big)^6$ for all $t \geq 0$. □

Lemma 6.2 shows that it is enough to consider the dynamics of the flow generated by (4.3) and (6.1) in a certain region Ω_V.

Lemma 6.2. *Let*

$$\Omega_{H_V} = \left\{ \big(S, I, T, R, V\big) \in \big(\mathbb{R}_0^+\big)^5 \,|\, 0 \leq N(t) \leq \frac{\Lambda}{\mu} \right\} \tag{6.3}$$

and

$$\Omega_{B_V} = \left\{ B \in \mathbb{R}_0^+ \,|\, 0 \leq B(t) \leq \frac{\Lambda\eta}{\mu d} \right\}, \tag{6.4}$$

where $N(t) = S(t) + I(t) + T(t) + R(t) + V(t)$. *Define*

$$\Omega_V = \Omega_{H_V} \times \Omega_{B_V}. \tag{6.5}$$

If $N(0) \leq \dfrac{\Lambda}{\mu}$ *and* $B(0) \leq \dfrac{\Lambda\eta}{\mu d}$, *then the region* Ω_V *is positively invariant for model* (4.3) *with non-negative initial conditions* (6.1) *in* $\left(\mathbb{R}_0^+\right)^6$.

Proof. Let us split system (4.3) into two parts: the human population, i.e., $S(t)$, $I(t)$, $T(t)$, $R(t)$ and $V(t)$, and the pathogen population, i.e., $B(t)$. Adding the first five equations of system (4.3) gives

$$\begin{aligned}\dot{N}(t) &= \dot{S}(t) + \dot{I}(t) + \dot{T}(t) + \dot{R}(t) + \dot{V}(t)\\ &= \Lambda - \mu N(t) - \alpha_1 I(t) - \alpha_2 T(t) \leq \Lambda - \mu N(t)\,.\end{aligned}$$

Assuming that $N(0) \leq \dfrac{\Lambda}{\mu}$, we conclude that $N(t) \leq \dfrac{\Lambda}{\mu}$. For this reason, (6.3) defines the biologically feasible region for the human population. As it is proved in Lemma 5.2, the region (6.4) defines the biologically feasible region for the pathogen population. From (6.3) and (6.4), we know that $N(t)$ and $B(t)$ are bounded for all $t \geq 0$. Therefore, every solution of system (4.3) with initial conditions in Ω_V remains in Ω_V for all $t \geq 0$. In other words, in region Ω_V defined by (6.5), our model is epidemiologically and mathematically well posed, in the sense of [69]. □

6.2.2 Equilibrium points and the basic reproduction number

Recalling notation (6.2), the disease-free equilibrium point (DFE) of model (4.3) is given by

$$\begin{aligned}E^0 &= \left(S^0, I^0, T^0, R^0, V^0, B^0\right)\\ &= \left(\frac{\Lambda a_4}{a_0 a_4 - \varphi\omega_2}, 0, 0, 0, \frac{\Lambda\varphi}{a_0 a_4 - \varphi\omega_2}, 0\right).\end{aligned} \tag{6.6}$$

Remark 6.3. *Note that one has* $a_0 a_4 - \varphi\omega_2 = (\varphi + \mu)(\omega_2 + \mu) - \varphi\omega_2 > 0$, *because* $\mu > 0$.

Next, following the approach of [127, 176] (see Section (1.5)) and recalling notations (5.3) and (6.2), we compute the basic reproduction number R_0.

Proposition 6.4 (Basic reproduction number of (4.3))**.** *The basic reproduction number of model* (4.3) *is given by*

$$R_0 = \frac{\beta\Lambda\eta(\omega_2 + \mu)}{\left((\varphi + \mu)(\omega_2 + \mu) - \varphi\omega_2\right)\kappa d(\delta + \alpha_1 + \mu)}. \tag{6.7}$$

Proof. Consider that $\mathscr{F}_i(t)$ is the rate of appearance of new infections in the compartment associated with index i, $\mathscr{V}_i^+(t)$ is the rate of transfer of "individuals" into the compartment associated with index i by all other means and $\mathscr{V}_i^-(t)$ is the rate of transfer of "individuals" out of compartment associated with index i. In this way, recalling notations (5.3) and (6.2), the matrices $\mathscr{F}(t)$, $\mathscr{V}^+(t)$ and $\mathscr{V}^-(t)$, associated with model (4.3), are given by

$$\begin{cases} \mathscr{F}(t) = \begin{bmatrix} 0 & \lambda(t) & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\ \mathscr{V}^+(t) = \begin{bmatrix} \Lambda + \omega_1 R(t) + \omega_2 V(t) & 0 & \delta I(t) & \varepsilon T(t) & \varphi S(t) & \eta I(t) \end{bmatrix}^T, \\ \mathscr{V}^-(t) = \begin{bmatrix} \lambda(t) + a_0 S(t) & a_1 I(t) & a_2 T(t) & a_3 R(t) & a_4 V(t) & dB(t) \end{bmatrix}^T, \end{cases}$$

where $\lambda(t) = \frac{\beta B(t) S(t)}{\kappa + B(t)}$. Therefore, by considering $\mathscr{V}(t) = \mathscr{V}^-(t) - \mathscr{V}^+(t)$, we have that

$$\begin{bmatrix} \dot{S}(t) & \dot{I}(t) & \dot{T}(t) & \dot{R}(t) & \dot{V}(t) & \dot{B}(t) \end{bmatrix}^T = \mathscr{F}(t) - \mathscr{V}(t).$$

The Jacobian matrices of $\mathscr{F}(t)$ and of $\mathscr{V}(t)$ are, respectively, given by

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{\beta B(t)}{\kappa + B(t)} & 0 & 0 & 0 & 0 & \dfrac{\beta \kappa S(t)}{\big(\kappa + B(t)\big)^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$V = \begin{bmatrix} \dfrac{\beta B(t)}{\kappa + B(t)} + a_0 & 0 & 0 & -\omega_1 & -\omega_2 & \dfrac{\beta \kappa S(t)}{\big(\kappa + B(t)\big)^2} \\ 0 & a_1 & 0 & 0 & 0 & 0 \\ 0 & -\delta & a_2 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon & a_3 & 0 & 0 \\ -\varphi & 0 & 0 & 0 & a_4 & 0 \\ 0 & -\eta & 0 & 0 & 0 & d \end{bmatrix}.$$

At the disease-free equilibrium E^0 defined by (6.6), we obtain the matrices F_0 and V_0 given by

$$F_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{\beta \Lambda a_4}{(a_0 a_4 - \varphi \omega_2)\kappa} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$V_0 = \begin{bmatrix} a_0 & 0 & 0 & -\omega_1 & -\omega_2 & \dfrac{\beta\Lambda a_4}{(a_0a_4-\varphi\omega_2)\kappa} \\ 0 & a_1 & 0 & 0 & 0 & 0 \\ 0 & -\delta & a_2 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon & a_3 & 0 & 0 \\ -\varphi & 0 & 0 & 0 & a_4 & 0 \\ 0 & -\eta & 0 & 0 & 0 & d \end{bmatrix}.$$

The basic reproduction number of model (4.3) is then given by

$$\begin{aligned} R_0 = \rho\left(F_0V_0^{-1}\right) &= \frac{\beta\Lambda\eta a_4}{(a_0a_4-\varphi\omega_2)\kappa d a_1} \\ &= \frac{\beta\Lambda\eta(\omega_2+\mu)}{\big((\varphi+\mu)(\omega_2+\mu)-\varphi\omega_2\big)\kappa d(\delta+\alpha_1+\mu)}, \end{aligned}$$

which is easily obtained with the help of the computer algebra system Maple. This concludes the proof. □

Now we prove the existence of an endemic equilibrium, when R_0, given by (6.7), is greater than one.

Proposition 6.5 (Endemic equilibrium). *Assume that* $\delta,\ \varepsilon,\ \omega_1,\ \omega_2,\ \varphi > 0$. *If the basic reproduction number* (6.7) *is such that* $R_0 > 1$, *then model* (4.3) *has an endemic equilibrium given by*

$$E^* = (S^*, I^*, T^*, R^*, V^*, B^*), \tag{6.8}$$

where

$$\begin{cases} S^* = \dfrac{a_1a_4\{\kappa d(a_1a_2a_3-\delta\varepsilon\omega_1)+\Lambda\eta a_2a_3\}}{\eta\tilde{D}}, \\ I^* = \dfrac{a_2a_3\{\beta\Lambda\eta a_4-(a_0a_4-\varphi\omega_2)\kappa d a_1\}}{\eta\tilde{D}}, \\ T^* = \dfrac{a_3\delta\{\beta\Lambda\eta a_4-(a_0a_4-\varphi\omega_2)\kappa d a_1\}}{\eta\tilde{D}}, \\ R^* = \dfrac{\delta\varepsilon\{\beta\Lambda\eta a_4-(a_0a_4-\varphi\omega_2)\kappa d a_1\}}{\eta\tilde{D}}, \\ V^* = \dfrac{a_1\varphi\{\kappa d(a_1a_2a_3-\delta\varepsilon\omega_1)+\Lambda\eta a_2a_3\}}{\eta\tilde{D}}, \\ B^* = \dfrac{a_2a_3\{\beta\Lambda\eta a_4-(a_0a_4-\varphi\omega_2)\kappa d a_1\}}{d\tilde{D}} \end{cases}$$

and $\tilde{D} = a_1a_2a_3(a_0a_4-\varphi\omega_2)+\beta a_4(a_1a_2a_3-\delta\varepsilon\omega_1)$.

Proof. We note that

1) $a_1 = \delta+\alpha_1+\mu > 0$, because $\alpha_1 \geq 0$ and $\delta, \mu > 0$;

2) $a_2 = \varepsilon + \alpha_2 + \mu > 0$, because $\alpha_2 \geq 0$ and $\varepsilon, \mu > 0$;

3) $a_3 = \omega_1 + \mu > 0$, because $\omega_1, \mu > 0$;

4) $a_4 = \omega_2 + \mu > 0$, because $\omega_2, \mu > 0$;

5) $\beta, \kappa, d, \delta, \varepsilon, \varphi > 0$;

6) $a_0 a_4 - \varphi\omega_2 > 0$ (see Remark 6.3);

7) $a_1 a_2 a_3 - \delta\varepsilon\omega_1 = (\delta + \alpha_1 + \mu)(\varepsilon + \alpha_2 + \mu)(\omega_1 + \mu) - \delta\varepsilon\omega_1 > 0$, because $\alpha_1, \alpha_2 \geq 0$ and $\mu > 0$;

8) $\Lambda\eta a_2 a_3 > 0$, because $\Lambda, \eta, a_2, a_3 > 0$.

With the above inequalities, we conclude that $\tilde{D} > 0$ and, consequently, that $S^* > 0$ and $V^* \geq 0$. The basic reproduction number is given by $\dfrac{\beta\Lambda\eta a_4}{(a_0 a_4 - \varphi\omega_2)\kappa d a_1}$. Thus, it follows that

$$
\begin{aligned}
& \beta\Lambda\eta a_4 = R_0(a_0 a_4 - \varphi\omega_2)\kappa d a_1 \\
\Leftrightarrow\ & \beta\Lambda\eta a_4 - (a_0 a_4 - \varphi\omega_2)\kappa d a_1 = R_0(a_0 a_4 - \varphi\omega_2)\kappa d a_1 - (a_0 a_4 - \varphi\omega_2)\kappa d a_1 \\
\Leftrightarrow\ & \beta\Lambda\eta a_4 - (a_0 a_4 - \varphi\omega_2)\kappa d a_1 = (a_0 a_4 - \varphi\omega_2)\kappa d a_1 (R_0 - 1).
\end{aligned}
$$

Therefore, we have that

$$
\begin{cases}
I^* = \dfrac{a_1 a_2 a_3 \kappa d (a_0 a_4 - \varphi\omega_2)(R_0 - 1)}{\eta\tilde{D}}, \\
T^* = \dfrac{a_1 a_3 \kappa d \delta (a_0 a_4 - \varphi\omega_2)(R_0 - 1)}{\eta\tilde{D}}, \\
R^* = \dfrac{a_1 \kappa d \delta\varepsilon (a_0 a_4 - \varphi\omega_2)(R_0 - 1)}{\eta\tilde{D}}, \\
B^* = \dfrac{a_1 a_2 a_3 \kappa (a_0 a_4 - \varphi\omega_2)(R_0 - 1)}{\tilde{D}}.
\end{cases}
$$

In order to obtain an endemic equilibrium, we have to ensure that $I^*, B^* > 0$. Thus, we obtain $I^*, B^* > 0$ if and only if $R_0 - 1 > 0 \Leftrightarrow R_0 > 1$. In this case $(R_0 > 1)$, we also have that $T^*, R^* > 0$. □

6.2.3 Sensitivity of the basic reproduction number

In this section, we are going to study the sensitivity of R_0 with respect to all parameters p of model (4.3), computing the respective normalized forward sensitive indexes $\Upsilon_p^{R_0}$, given in Definition 6.6. They are presented in Table 6.1.

Definition 6.6 (See [30, 95, 161]). *The normalized forward sensitivity index of a variable z that depends differentiably on a parameter p is defined by*

$$\Upsilon_p^z = \frac{\partial z}{\partial p} \times \frac{p}{|z|}.$$

Remark 6.7. *When a parameter p is one of the most sensitive parameters with respect to a variable z, then we have $\Upsilon_p^z = \pm 1$. If $\Upsilon_p^z = 1$, then an increase (decrease) of p by $\gamma\%$ provokes an increase (decrease) of z by $\gamma\%$. On the other hand, if $\Upsilon_p^z = -1$, then an increase (decrease) of p by $\gamma\%$ provokes a decrease (increase) of z by $\gamma\%$ (see [161]).*

Parameter p	$\Upsilon_p^{R_0}$
Λ	1
μ	$\mu\left(\frac{1}{a_4} - \frac{1}{a_1} - \frac{\varphi+\omega_2+2\mu}{a_0 a_4 - \varphi\omega_2}\right)$
β	1
κ	-1
ω_1	0
ω_2	$\frac{\varphi\omega_2}{a_4(\varphi+\omega_2+\mu)}$
φ	$-\frac{\varphi}{\varphi+\omega_2+\mu}$
δ	$-\frac{\delta}{a_1}$
ε	0
α_1	$-\frac{\alpha_1}{a_1}$
α_2	0
η	1
d	-1

Table 6.1: The normalized forward sensitivity indexes $\Upsilon_p^{R_0}$ with respect to all parameters of model (4.3).

6.2.4 Stability analysis

Now we prove the local asymptotic stability of the disease-free equilibrium E^0 of model (4.3), given by (6.6).

Theorem 6.8 (Local asymptotic stability of (6.6)). *The disease-free equilibrium E^0 of model (4.3) is locally asymptotic stable, if $R_0 < 1$.*

Proof. The characteristic polynomial associated with the linearised system of model (4.3) is given by

$$p_V(\chi) = \det(F_0 - V_0 - \chi I_6).$$

In order to compute the roots of the polynomial p_V, we have that

$$\begin{vmatrix} -a_0-\chi & 0 & 0 & \omega_1 & \omega_2 & -\dfrac{\beta\Lambda a_4}{(a_0a_4-\varphi\omega_2)\kappa} \\ 0 & -a_1-\chi & 0 & 0 & 0 & \dfrac{\beta\Lambda a_4}{(a_0a_4-\varphi\omega_2)\kappa} \\ 0 & \delta & -a_2-\chi & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & -a_3-\chi & 0 & 0 \\ \varphi & 0 & 0 & 0 & -a_4-\chi & 0 \\ 0 & \eta & 0 & 0 & 0 & -d-\chi \end{vmatrix} = 0,$$

that is,

$$\begin{aligned} &\chi^2+(a_0+a_4)\chi+(a_0a_4-\varphi\omega_2)=0 \\ &\vee\, \chi^2+(a_1+d)\chi+a_1d-\frac{\beta\Lambda\eta a_4}{(a_0a_4-\varphi\omega_2)\kappa}=0 \vee \chi=-a_2 \vee \chi=-a_3. \end{aligned}$$

As the coefficients of polynomial $\chi^2+(a_0+a_4)\chi+(a_0a_4-\varphi\omega_2)$ have the same sign (see Remark 6.3), then it follows from Routh–Hurwitz Criterion that their roots have negative real part (see Theorem 1.27). Furthermore, using similar arguments, the roots of the polynomial

$$\chi^2+(a_1+d)\chi+a_1d-\frac{\beta\Lambda\eta a_4}{(a_0a_4-\varphi\omega_2)\kappa}$$

have negative real part if and only if

$$a_1d-\frac{\beta\Lambda\eta a_4}{(a_0a_4-\varphi\omega_2)\kappa}>0 \Leftrightarrow R_0<1.$$

Therefore, the DFE (6.6) is locally asymptotic stable, if $R_0<1$. □

With respect to model (4.3), we are going to study the local asymptotic stability of its endemic equilibrium E^* (see (6.8)) and, moreover, the instability of its disease-free equilibrium E^0 (see (6.6)) for $R_0>1$. Our proof is based on the *Center Manifold Theory* (see [24]), as described in Theorem 1.25.

Theorem 6.9 (Instability of (6.6) and local asymptotic stability of (6.8))**.** *The equilibrium points E^0 and E^* of model* (4.3) *(see* (6.6) *and* (6.8)*) are, respectively, unstable and locally asymptotic stable for $R_0>1$.*

Proof. In order to apply the method described in Theorem 1.25, we are going to do the following change of variables. Let us consider that

$$X=(x_1,x_2,x_3,x_4,x_5,x_6)=(S,I,T,R,V,B).$$

So, the total number of individuals is given by $N = \sum_{i=1}^{5} x_i$. Thus, we can write model (4.3) as follows:

$$\begin{cases} \dot{x}_1(t) = f_1\big(X(t)\big) = \Lambda - \dfrac{\beta x_1(t)x_6(t)}{\kappa + x_6(t)} + \omega_1 x_4(t) + \omega_2 x_5(t) \\ \qquad\qquad\qquad\qquad - a_0 x_1(t), \\ \dot{x}_2(t) = f_2\big(X(t)\big) = \dfrac{\beta x_1(t)x_6(t)}{\kappa + x_6(t)} - a_1 x_2(t), \\ \dot{x}_3(t) = f_3\big(X(t)\big) = \delta x_2(t) - a_2 x_3(t), \\ \dot{x}_4(t) = f_4\big(X(t)\big) = \varepsilon x_3(t) - a_3 x_4(t), \\ \dot{x}_5(t) = f_5\big(X(t)\big) = \varphi x_1(t) - a_4 x_5(t), \\ \dot{x}_6(t) = f_6\big(X(t)\big) = \eta x_2(t) - d x_6(t). \end{cases} \tag{6.9}$$

Choosing β^* as bifurcation parameter and solving for β, from $R_0 = 1$ we have that

$$\beta^* = \frac{(a_0 a_4 - \varphi\omega_2)\kappa d a_1}{\Lambda\eta a_4}.$$

Considering $\beta = \beta^*$, the Jacobian of system (6.9) evaluated at E^0 is given by

$$J_0^* = \begin{bmatrix} -a_0 & 0 & 0 & \omega_1 & \omega_2 & -\dfrac{a_1 d}{\eta} \\ 0 & -a_1 & 0 & 0 & 0 & \dfrac{a_1 d}{\eta} \\ 0 & \delta & -a_2 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & -a_3 & 0 & 0 \\ \varphi & 0 & 0 & 0 & -a_4 & 0 \\ 0 & \eta & 0 & 0 & 0 & -d \end{bmatrix}.$$

The eigenvalues of J_0^* are obtained solving the equation $\det(J_0^* - \chi I_6) = 0$. Thus, we have that

$$\begin{aligned} \det(J_0^* - \chi I_6) = 0 \Leftrightarrow \chi = 0 \ \vee \ \chi = -a_1 - d \ \vee \ \chi = -a_2 \ \vee \ \chi = -a_3 \\ \vee \ \chi = -\frac{1}{2}\left(a_0 + a_4 \pm \sqrt{(a_0 - a_4)^2 + 4\varphi\omega_2}\right). \end{aligned}$$

Note that the eigenvalue $\chi = -\frac{1}{2}\left(a_0 + a_4 - \sqrt{(a_0 - a_4)^2 + 4\varphi\omega_2}\right)$ is a negative real number, because

$$\begin{aligned} & -\frac{1}{2}\left(a_0 + a_4 - \sqrt{(a_0 - a_4)^2 + 4\varphi\omega_2}\right) \\ = \quad & -\frac{1}{2}\left(\varphi + \mu + \omega_2 + \mu - \sqrt{(\varphi + \mu - \omega_2 - \mu)^2 + 4\varphi\omega_2}\right) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\left(\varphi+\omega_2+2\mu-\sqrt{\varphi^2-2\varphi\omega_2+\omega_2^2+4\varphi\omega_2}\right)\\
&= -\frac{1}{2}\left(\varphi+\omega_2+2\mu-\sqrt{(\varphi+\omega_2)^2}\right)\\
&\underset{\varphi+\omega_2\geq 0}{=} -\frac{1}{2}\big(\varphi+\omega_2+2\mu-(\varphi+\omega_2)\big)\\
&= -\mu<0.
\end{aligned}
$$

Therefore, we can conclude that a simple eigenvalue of J_0^* is zero, while all other eigenvalues of J_0^* have negative real part. So, the *Center Manifold Theory* (see [24]) can be applied to study the dynamics of (6.9) near $\beta=\beta^*$. Theorem 1.25 is used to show the local asymptotic stability of the endemic equilibrium point of (6.9), for β near β^*. The Jacobian J_0^* has a non-negative right eigenvector w and a left eigenvector v associated with the zero eigenvalue. With respect to w we have that

$$
\begin{aligned}
&J_0^* w=\begin{bmatrix}0&0&0&0&0&0\end{bmatrix}^T\\
\Leftrightarrow\ &J_0^*\begin{bmatrix}w_1&w_2&w_3&w_4&w_5&w_6\end{bmatrix}^T=\begin{bmatrix}0&0&0&0&0&0\end{bmatrix}^T\\
\Leftrightarrow\ &w=\begin{bmatrix}-\dfrac{a_4(a_1a_2a_3-\delta\varepsilon\omega_1)}{a_2a_3(a_0a_4-\varphi\omega_2)}&1&\dfrac{\delta}{a_2}&\dfrac{\delta\varepsilon}{a_2a_3}&\dfrac{\varphi}{a_4}&\dfrac{\eta}{d}\end{bmatrix}^T w_2,
\end{aligned}
$$

where w_2 is an arbitrary constant. So, we can choose $w_2=1$ and, consequently, we obtain that

$$
w=\begin{bmatrix}-\dfrac{a_4(a_1a_2a_3-\delta\varepsilon\omega_1)}{a_2a_3(a_0a_4-\varphi\omega_2)}&1&\dfrac{\delta}{a_2}&\dfrac{\delta\varepsilon}{a_2a_3}&\dfrac{\varphi}{a_4}&\dfrac{\eta}{d}\end{bmatrix}^T.
$$

Recalling Remark 6.3 and the beginning of the proof of Proposition 6.5, one can observe that

$$
-\frac{a_4(a_1a_2a_3-\delta\varepsilon\omega_1)}{a_2a_3(a_0a_4-\varphi\omega_2)}<0.
$$

Nevertheless, attending to Remark 1.26, as E^0 (see (6.6)) is a non-negative equilibrium point of interest and the first component of E^0 is positive, then the first component of w does not need to be positive. Clearly, other components of w are non-negative. With respect to v, we have that

$$
\begin{aligned}
&vJ_0^*=\begin{bmatrix}0&0&0&0&0&0\end{bmatrix}\\
\Leftrightarrow\ &\begin{bmatrix}v_1&v_2&v_3&v_4&v_5&v_6\end{bmatrix}J_0^*=\begin{bmatrix}0&0&0&0&0&0\end{bmatrix}\\
\Leftrightarrow\ &v=\begin{bmatrix}0&1&0&0&0&\dfrac{a_1}{\eta}\end{bmatrix}v_2,
\end{aligned}
$$

where v_2 is an arbitrary constant. So, we can choose $v_2=1$ and, consequently, we obtain that

$$
v=\begin{bmatrix}0&1&0&0&0&\dfrac{a_1}{\eta}\end{bmatrix}.
$$

Recall that f_l represents the right-hand side of the lth equation of system (6.9) and x_l is the state variable whose derivative is given by the lth equation, $l = 1, \ldots, 6$. The local stability near the bifurcation point $\beta = \beta^*$ is determined by the signs of two associated constants a and b defined by

$$a = \sum_{i,j,k=1}^{6} w_i w_j v_k \left[\frac{\partial^2 f_k}{\partial x_i \partial x_j} \left(E^0 \right) \right]_{\beta=\beta^*}$$

and

$$b = \sum_{i,k=1}^{6} w_i v_k \left[\frac{\partial^2 f_k}{\partial x_i \partial \phi} \left(E^0 \right) \right]_{\beta=\beta^*}$$

with $\phi = \beta - \beta^*$. As $v_1 = v_3 = v_4 = v_5 = 0$, we only have to consider the following non-zero partial derivatives at the disease free equilibrium E^0:

$$\left[\frac{\partial^2 f_2}{\partial x_1 \partial x_6} \left(E^0 \right) \right]_{\beta=\beta^*} = \left[\frac{\partial^2 f_2}{\partial x_6 \partial x_1} \left(E^0 \right) \right]_{\beta=\beta^*} = \frac{\beta^*}{\kappa}$$

and

$$\left[\frac{\partial^2 f_2}{\partial x_6^2} \left(E^0 \right) \right]_{\beta=\beta^*} = -\frac{2\beta^* \Lambda a_4}{a_0 a_4 - \varphi \omega_2}.$$

Therefore, the constant a is

$$\begin{aligned} a =& -\frac{2\beta^* \eta a_4}{d(a_0 a_4 - \varphi\omega_2)} \left(\frac{a_1 a_2 a_3 - \delta\varepsilon\omega_1}{a_2 a_3 \kappa} + \frac{\Lambda\eta}{d} \right) v_2 w_2^2 \\ =& -\frac{2\beta^* \eta a_4}{d(a_0 a_4 - \varphi\omega_2)} \left(\frac{a_1 a_2 a_3 - \delta\varepsilon\omega_1}{a_2 a_3 \kappa} + \frac{\Lambda\eta}{d} \right) < 0. \end{aligned}$$

Furthermore, we have that

$$\begin{aligned} b =& v_2 w_6 \left[\frac{\partial^2 f}{\partial x_6 \partial \phi} \left(E^0 \right) \right]_{\beta=\beta^*} = \frac{\Lambda \eta a_4}{\kappa d(a_0 a_4 - \varphi\omega_2)} v_2 w_2 \\ =& \frac{\Lambda \eta a_4}{\kappa d(a_0 a_4 - \varphi\omega_2)} > 0. \end{aligned}$$

Thus, as

$$\begin{cases} a < 0 \\ b > 0 \\ \phi = \beta - \beta^* = \dfrac{a_1 \kappa d(a_0 a_4 - \varphi\omega_2)}{\Lambda \eta a_4}(R_0 - 1) > 0 \end{cases} \Leftrightarrow \begin{cases} a < 0 \\ b > 0 \\ R_0 > 1, \end{cases}$$

we conclude from Theorem 1.25 that the equilibrium points E^0 and E^* of (4.3) (see (6.6) and (6.8)) are, respectively, unstable and locally asymptotic stable for a value of the basic reproduction number such that $R_0 > 1$. This concludes the proof. □

6.2.5 Numerical simulations

In this section, we simulate the worst cholera outbreak that ever occurred in human history. It happened in Yemen since October 2016. We are going to study it from 27th April 2017 to 15th April 2018 (see [189]). As the first-ever oral cholera vaccination campaign had been launched only on 6th May 2018 and was concluded on 15th May 2018 (see [187]), to describe such reality of Yemen, a numerical simulation of our model is carried out with $\varphi = \omega_2 = V(0) = 0$ (in absence of vaccination) and with all the other values as in Table 6.2. We also simulate an hypothetical situation that includes vaccination from the beginning of the outbreak, considering in that case all parameter values of Table 6.2. Let us denote the numerical simulation without and with vaccination by (NS) and (NSV), respectively. The curves of infective individuals for (NS) and (NSV) can be observed in Figure 6.1, respectively in solid light blue line and in dashed dark blue line. Our results allow us to state that if a vaccination campaign had been considered earlier in time, the number of infective individuals would have been significantly lower. Furthermore, the basic reproduction number of the simulation without vaccination is $R_0 \simeq 6.132305 > 1$ and the one with vaccination is $R_0 \simeq 0.753969 < 1$. This means that if vaccination had been considered from the beginning of the outbreak, then the spread of cholera would have been extinguished. Consequently, there would not have been so many deaths. Note that the decrease of R_0 with the introduction of a vaccination campaign is expected, because

$$\Upsilon_{\varphi}^{R_0} = -\frac{\varphi}{\varphi + \omega_2 + \mu} \simeq -0.877050 < 0.$$

Concluding, vaccination campaigns would have been very important on the control and eradication of this cholera outbreak. Furthermore, for (NS), we obtain an endemic equilibrium point given by

$$(S^*, I^*, T^*, R^*, V^*, B^*),$$

where

$$\begin{cases} S^* = 2.943350 \times 10^7, \\ I^* = 1.035599 \times 10^5, \\ T^* = 5.954131 \times 10^5, \\ R^* = 1.070992 \times 10^8, \\ V^* = 0, \\ B^* = 3.138180 \times 10^6 \end{cases}$$

and for (NSV) we have a disease-free equilibrium point given by

$$\left(S^0, I^0, T^0, R^0, V^0, B^0\right) = (1.689119 \times 10^7,\ 0,\ 0,\ 0,\ 1.204910 \times 10^8,\ 0).$$

Note that the previous figures correspond to the equilibrium points for the parameter values of Table 6.2, which can be obtained numerically for a final time of approximately 1370 years. We also call attention to the fact that the recruitment rate Λ of Yemen is big and this leads to a huge growth of the population.

For all numerical simulations of this section, the values of Λ, μ, η, d, S_0, I_0, T_0, R_0, V_0, B_0 and N_0 are fixed in Table 6.2. Consequently, we have that:

$$N_0 = 28250420, \quad \frac{\Lambda}{\mu} \simeq 4.86 \times N_0, \quad B_0 = 2.75 \times 10^5 \quad \text{and} \quad \frac{\Lambda\eta}{\mu d} \simeq 4.16 \times 10^9.$$

Therefore, the initial values of the human population and bacterial concentration belong to (6.3) and (6.4), respectively:

$$N_0 = S_0 + I_0 + T_0 + R_0 + V_0 \in \left[0, \frac{\Lambda}{\mu}\right] \quad \text{and} \quad B_0 \in \left[0, \frac{\Lambda\eta}{\mu d}\right].$$

Therefore, $(S_0, I_0, T_0, R_0, V_0, B_0) \in \Omega_V = \Omega_{H_V} \times \Omega_{B_V}$. This implies that all the following numerical solutions (S, I, T, R, V, B) belong to the positively invariant set $\Omega_V = \Omega_{H_V} \times \Omega_{B_V}$ (see Lemmas 6.1 and 6.2). We use real data provided by WHO in [189]. All the numerical simulations were obtained with the help of integration routines in Matlab.

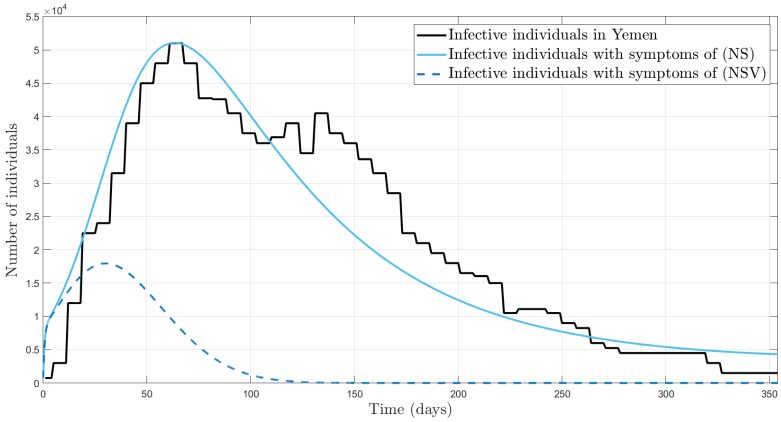


Figure 6.1: State trajectory $I(t)$ for all $t \in [0, 354]$, predicted by model (4.3), assuming that $\varphi = \omega_2 = V(0) = 0$ and all the other values of Table 6.2 – solid light blue line; state trajectory $I(t)$ for all $t \in [0, 354]$, predicted by model (4.3), assuming the values of Table 6.2 – dashed dark blue line; real data from the cholera outbreak in Yemen, from 27th April 2017 to 15th April 2018 (available in [189]) – solid black line.

Parameter	Value	Unity	Reference
Λ	$(28.4N_0)/365000$	person day^{-1}	[77]
μ	1.6×10^{-5}	day^{-1}	[79]
β	0.01694	day^{-1}	Assumed
κ	10^7	cell/ml	Assumed
ω_1	0.4/365	day^{-1}	[120]
ω_2	1/1460	day^{-1}	[31]
φ	5/1000	day^{-1}	Assumed
δ	1.15	day^{-1}	Assumed
ε	0.2	day^{-1}	[127]
α_1	6×10^{-6}	day^{-1}	[79, 188]
α_2	3×10^{-6}	day^{-1}	Assumed
η	10	cell/ml day^{-1} person^{-1}	[22]
d	0.33	day^{-1}	[22]
S_0	28249670	person	[194]
I_0	750	person	[189]
T_0	0	person	Assumed
R_0	0	person	Assumed
V_0	0	person	[187]
N_0	28250420	person	–
B_0	275000	cell/ml	Assumed

Table 6.2: Parameter values and initial conditions for the SITRVB model (4.3).

6.3 Model with chlorine water tablets supply

In this section, we consider model (4.2) and we assume that the initial conditions of system (4.2) are non-negative:

$$S(0) \geq 0\,, \quad I(0) \geq 0\,, \quad Q(0) \geq 0\,, \quad R(0) \geq 0\,, \quad B(0) \geq 0\,. \tag{6.10}$$

Let S_0, I_0, Q_0, R_0, B_0 and N_0 denote $S(0)$, $I(0)$, $Q(0)$, $R(0)$, $B(0)$ and $S(0)+I(0)+Q(0)+R(0)$, respectively, because $\tau = 0$ (see Table 4.1). Here, we are going to continue using notation (5.3).

6.3.1 Non-negativity and boundedness of solutions

As we assume the ecologically meaningful non-negative initial conditions for the populations, the solutions of model (4.2) remain non-negative for all time. This result is contained in Lemma 5.1 and translates in this situation without any change. The solutions not only remain in the positive cone, but are also bounded and the positively invariant set Ω is the same already found in Lemma 5.2 (see (5.4)–(5.6)). The proof is essentially the same, with only

a change in the derivation of the upper bound for the bacteria population, which is obtained by dropping the last term of the last equation in (4.2) to obtain the same bound:

$$\dot{B}(t) \leq \eta I(t) - dB(t) \leq \eta \frac{\Lambda}{\mu} - dB(t)\,.$$

6.3.2 Equilibrium points and the basic reproduction number

The only possible equilibria of model (4.2) are the disease-free point and coexistence, or the endemic equilibrium, as in Section 5.2. However, we will repeat here the analysis in some detail as it entails some relevant differences.

For the disease-free equilibrium (DFE), we find

$$E^0 = (S^0, I^0, Q^0, R^0, B^0) = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0\right). \tag{6.11}$$

The basic reproduction number R_0 can then be evaluated, following [127, 176] (see Section 1.5).

Proposition 6.10 (Basic reproduction number of (4.2))**.** *The basic reproduction number of model* (4.2) *is*

$$R_0 = \frac{\beta\Lambda\eta}{(\beta\Lambda + \mu\kappa d)(\delta + \alpha_1 + \mu)}. \tag{6.12}$$

Proof. Let $\mathscr{F}_i(t)$ be the rate at which new infections appear in the ith compartment and $\mathscr{V}_i^+(t)$ be the "individuals" transfer rate into the ith compartment by all other ways. Similarly, let $\mathscr{V}_i^-(t)$ denote the "individuals" transfer rate out of the ith compartment. Considering that $\mathscr{V}(t) = \mathscr{V}^-(t) - \mathscr{V}^+(t)$, we obtain the following equality:

$$\begin{bmatrix}\dot{S}(t) & \dot{I}(t) & \dot{Q}(t) & \dot{R}(t) & \dot{B}(t)\end{bmatrix}^T = \mathscr{F}(t) - \mathscr{V}(t).$$

Using again notation (5.3): $a_1 = \delta + \alpha_1 + \mu$, $a_2 = \varepsilon + \alpha_2 + \mu$ and $a_3 = \omega_1 + \mu$; we specifically find

$$\begin{cases} \mathscr{F}(t) = \begin{bmatrix}0 & \varpi(t) & 0 & 0 & 0\end{bmatrix}^T, \\ \mathscr{V}^+(t) = \begin{bmatrix}\Lambda + \omega_1 R(t) & 0 & \delta I(t) & \varepsilon Q(t) & \eta I(t)\end{bmatrix}^T, \\ \mathscr{V}^-(t) = \begin{bmatrix}\varpi(t) + \mu S(t) & a_1 I(t) & a_2 Q(t) & a_3 R(t) & dB(t) + \varpi(t)\end{bmatrix}^T, \end{cases}$$

where again $\varpi(t) = \lambda(t)S(t) = \frac{\beta B(t) S(t)}{\kappa + B(t)}$. The Jacobian matrices of $\mathscr{F}(t)$

and of $\mathscr{V}(t)$ are, respectively, given by:

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \dfrac{\beta B(t)}{\kappa + B(t)} & 0 & 0 & 0 & \dfrac{\beta\kappa S(t)}{\big(\kappa + B(t)\big)^2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$V = \begin{bmatrix} \dfrac{\beta B(t)}{\kappa + B(t)} + \mu & 0 & 0 & -\omega_1 & \dfrac{\beta\kappa S(t)}{\big(\kappa + B(t)\big)^2} \\ 0 & a_1 & 0 & 0 & 0 \\ 0 & -\delta & a_2 & 0 & 0 \\ 0 & 0 & -\varepsilon & a_3 & 0 \\ \dfrac{\beta B(t)}{\kappa + B(t)} & -\eta & 0 & 0 & d + \dfrac{\beta\kappa S(t)}{\big(\kappa + B(t)\big)^2} \end{bmatrix}.$$

Evaluating these matrices at the disease-free equilibrium E^0 (see (6.11)), we find

$$F_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{\beta\Lambda}{\mu\kappa} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad V_0 = \begin{bmatrix} \mu & 0 & 0 & -\omega_1 & \dfrac{\beta\Lambda}{\mu\kappa} \\ 0 & a_1 & 0 & 0 & 0 \\ 0 & -\delta & a_2 & 0 & 0 \\ 0 & 0 & -\varepsilon & a_3 & 0 \\ 0 & -\eta & 0 & 0 & d + \dfrac{\beta\Lambda}{\mu\kappa} \end{bmatrix}.$$

Recalling notation (5.3), the basic reproduction number of model (4.2) is then given by

$$R_0 = \rho\left(F_0 V_0^{-1}\right) = \frac{\beta\Lambda\eta}{(\beta\Lambda + \mu\kappa d)a_1} = \frac{\beta\Lambda\eta}{(\beta\Lambda + \mu\kappa d)(\delta + \alpha_1 + \mu)},$$

which is easily obtained with the help of the computer algebra system Maple. This concludes the proof. □

Next we prove the existence of an endemic equilibrium, when R_0, given by (6.12), is greater than one.

Proposition 6.11 (Endemic equilibrium)**.** *Recalling notation* (5.3) *and assuming that* λ^*, δ, ε, $\omega_1 > 0$*; whenever* $R_0 > 1$*, model* (4.2) *has the endemic equilibrium*

$$\begin{aligned} E^* &= (S^*, I^*, Q^*, R^*, B^*) \\ &= \left(\frac{\Lambda a_1 a_2 a_3}{D}, \frac{\Lambda a_2 a_3 \lambda^*}{D}, \frac{\Lambda\delta a_3\lambda^*}{D}, \frac{\Lambda\delta\varepsilon\lambda^*}{D}, \frac{\Lambda a_2 a_3(\eta - a_1)\lambda^*}{Dd}\right), \end{aligned} \tag{6.13}$$

which is feasible if

$$\eta > a_1 = \delta + \alpha_1 + \mu; \tag{6.14}$$

where

$$D = a_1 a_2 a_3(\lambda^* + \mu) - \delta\varepsilon\omega_1\lambda^* \quad \textit{and} \quad \lambda^* = \frac{\beta B^*}{\kappa + B^*}. \tag{6.15}$$

Proof. For this equilibrium to be feasible, the transmission rate must be strictly positive:

$$\lambda^* = \frac{\beta B^*}{\kappa + B^*} > 0.$$

Solving in turn the second, third and fourth equilibrium equation of (4.2), we find

$$S^* = \frac{a_1}{\lambda^*} I^*, \quad I^* = \frac{a_2}{\delta} Q^*, \quad Q^* = \frac{a_3}{\varepsilon} R^*. \tag{6.16}$$

Then, we obtain

$$S^* = \frac{a_1}{\lambda^*} \times \frac{a_2}{\delta} Q^* = \frac{a_1 a_2}{\lambda^* \delta} \times \frac{a_3}{\varepsilon} R^* = \frac{a_1 a_2 a_3}{\lambda^* \delta \varepsilon} R^*.$$

Substituting the last evaluated value of S^* into the first equilibrium equation, we then obtain $\Lambda\lambda^*\delta\varepsilon - DR^* = 0$, which gives the fourth component of (5.9). Furthermore, by back substitution in (6.16), we also determine the first three components of (5.9). Finally, the fifth equilibrium equation provides the value of B^*, which must be non-negative to be feasible, giving thus (6.14). Note that

$$a_1 a_2 a_3 - \delta\varepsilon\omega_1 = (\delta + \alpha_1 + \mu)(\varepsilon + \alpha_2 + \mu)(\omega_1 + \mu) - \delta\varepsilon\omega_1 > 0, \tag{6.17}$$

because $\alpha_1, \alpha_2 \geq 0$ and $\mu > 0$. Consequently, as we are considering that $\lambda^* > 0$, we can conclude that

$$D = a_1 a_2 a_3 \left(\lambda^* + \mu\right) - \delta\varepsilon\omega_1\lambda^* = (a_1 a_2 a_3 - \delta\varepsilon\omega_1)\lambda^* + a_1 a_2 a_3 \mu > 0.$$

Now, from $\lambda^* = \beta B^*(\kappa + B^*)^{-1}$, substituting the value of B^* and rearranging, we obtain

$$\Big\{\big[\Lambda(\eta - a_1) + \kappa d a_1\big] a_2 a_3 - \kappa d \delta\varepsilon\omega_1\Big\}\lambda^* = \big[\beta\Lambda\eta - (\beta\Lambda + \mu\kappa d)a_1\big] a_2 a_3.$$

It follows that

$$\begin{aligned}\lambda^* &= \frac{\left[\dfrac{\beta\Lambda\eta}{(\beta\Lambda + \mu\kappa d)a_1} - 1\right](\beta\Lambda + \mu\kappa d)a_1 a_2 a_3}{\big[\Lambda(\eta - a_1) + \kappa d a_1\big] a_2 a_3 - \kappa d\delta\varepsilon\omega_1} \\ &= \frac{(R_0 - 1)(\beta\Lambda + \mu\kappa d)a_1 a_2 a_3}{\Lambda(\eta - a_1)a_2 a_3 + \kappa d(a_1 a_2 a_3 - \delta\varepsilon\omega_1)}.\end{aligned}$$

As we are assuming (6.14), then the above value of λ^* is positive if and only if $R_0 > 1$, because all the other coefficients in the above expression of λ^* are positive. In such case, model (4.2) has the endemic equilibrium (6.13). This concludes the proof. □

Remark 6.12. *Note that for model* (4.2) *the feasibility result for endemic equilibrium* (6.13) *differs from the corresponding one in Section 5.2 (see Proposition 5.4). Here for the epidemics to subsist, it is necessary that the rate at which the bacteria are spread by the infective individuals must exceed the combined rates at which infective individuals leave their compartment. It means that* η *must be larger than the sum of the rate at which individuals are quarantined and of the disease-related* α_1 *and the natural* μ *mortality rates:* $\eta > \delta + \alpha_1 + \mu$.

6.3.3 Stability analysis

In this section, we analyse the local asymptotic stability of the equilibria of model (4.2). The basic reproduction number proves to be instrumental in the local asymptotic stability issue of the DFE E^0, given by (6.11), as it is shown in the next result.

Theorem 6.13 (Local asymptotic stability of (6.11))**.** *The disease-free equilibrium* E^0 *of model* (4.2) *is locally asymptotic stable, if* $R_0 < 1$.

Proof. Let the characteristic equation of (4.2) evaluated at the DFE (6.11) given by

$$p(\chi) = \det(F_0 - V_0 - \chi I_5) = 0.$$

It is equivalent to

$$\chi = -\mu \vee \chi = -a_2 \vee \chi = -a_3 \vee \tilde{p}(\chi) = (\chi + a_1)\left(\chi + d + \frac{\beta\Lambda}{\mu\kappa}\right) - \frac{\beta\Lambda\eta}{\mu\kappa} = 0.$$

So, we obtained three explicit negative eigenvalues: $-\mu$, $-a_2$ and $-a_3$; and a quadratic equation in χ given by

$$\tilde{p}(\chi) = \chi^2 + \left(a_1 + d + \frac{\beta\Lambda}{\mu\kappa}\right)\chi + a_1\left(d + \frac{\beta\Lambda}{\mu\kappa}\right) - \frac{\beta\Lambda\eta}{\mu\kappa} = 0.$$

By Routh–Hurwitz Criterion (see Theorem 1.27), all coefficients of polynomial $\tilde{p}$ have the same signal if and only if the roots of $\tilde{p}$ have negative real part. Moreover, if all roots of $\tilde{p}$ have negative real part, then DFE (6.11) is locally asymptotic stable. The coefficients of $\tilde{p}$ are $\tilde{p}_2 = 1 > 0$, $\tilde{p}_1 = a_1 + d + \frac{\beta\Lambda}{\mu\kappa} > 0$ and $\tilde{p}_0 = a_1\left(d + \frac{\beta\Lambda}{\mu\kappa}\right) - \frac{\beta\Lambda\eta}{\mu\kappa}$. Note that

$$a_1\left(d + \frac{\beta\Lambda}{\mu\kappa}\right) - \frac{\beta\Lambda\eta}{\mu\kappa} > 0 \Leftrightarrow a_1(\mu\kappa d + \beta\Lambda) - \beta\Lambda\eta > 0$$

$$\Leftrightarrow \beta\Lambda\eta < a_1(\mu\kappa d + \beta\Lambda) \Leftrightarrow \frac{\beta\Lambda\eta}{a_1(\mu\kappa d + \beta\Lambda)} < 1 \Leftrightarrow R_0 < 1.$$

So, we can conclude that DFE (6.11) is locally asymptotic stable, if $R_0 < 1$. This concludes the proof. □

Remark 6.14. *Comparing the local asymptotic stability condition for DFE* (6.11) *($R_0 < 1$) and the feasibility condition for endemic equilibrium* (6.13) *($\eta > a_1$), it is easily seen that for $R_0 = 1$ a transcritical bifurcation occurs for which the endemic equilibrium* (6.13) *emanates from the DFE* (6.11).

With respect to model (4.2), we are going to study the local asymptotic stability of its endemic equilibrium E^* (see (6.13)) and, moreover, the instability of its disease-free equilibrium E^0 (see (6.11)) for $R_0 > 1$. Our proof is based on the *Center Manifold Theory* (see [24]), as described in Theorem 1.25. Although the final result coincides with the one obtained in Theorem 5.6, there are some details that change. Thus, we present also its proof, following the same steps and considerations.

Theorem 6.15 (Instability of (6.11) and local asymptotic stability of (6.13))**.** *The equilibrium points E^0 and E^* of model* (4.2) *(see* (6.11) *and* (6.13)*) are, respectively, unstable and locally asymptotic stable for $R_0 > 1$.*

Proof. To apply the method described in Theorem 1.25, we consider a change of variables. Let us consider that $X = (x_1, x_2, x_3, x_4, x_5) = (S, I, Q, R, B)$. Consequently, we have that the total number of individuals is $N = \sum_{i=1}^{4} x_i$. Thus, model (4.2) can be written as follows:

$$\begin{cases} \dot{x}_1(t) = f_1\big(X(t)\big) = \Lambda - \dfrac{\beta x_1(t)x_5(t)}{\kappa + x_5(t)} + \omega_1 x_4(t) - \mu x_1(t) \\ \dot{x}_2(t) = f_2\big(X(t)\big) = \dfrac{\beta x_1(t)x_5(t)}{\kappa + x_5(t)} - (\delta + \alpha_1 + \mu)x_2(t) \\ \dot{x}_3(t) = f_3\big(X(t)\big) = \delta x_2(t) - (\varepsilon + \alpha_2 + \mu)x_3(t) \\ \dot{x}_4(t) = f_4\big(X(t)\big) = \varepsilon x_3(t) - (\omega_1 + \mu)x_4(t) \\ \dot{x}_5(t) = f_5\big(X(t)\big) = \eta x_2(t) - d x_5(t) - \dfrac{\beta x_1(t)x_5(t)}{\kappa + x_5(t)}. \end{cases} \tag{6.18}$$

Choosing β^* as bifurcation parameter and solving for β from $R_0 = 1$, we obtain that

$$\beta^* = \frac{\mu\kappa d(\delta + \alpha_1 + \mu)}{\Lambda\big[\eta - (\delta + \alpha_1 + \mu)\big]} = \frac{\mu\kappa d a_1}{\Lambda(\eta - a_1)},$$

which is positive in view of (6.14). At β^*, the Jacobian of (6.18) evaluated at E^0 becomes

$$J_0^* = \begin{bmatrix} -\mu & 0 & 0 & \omega_1 & -\dfrac{a_1 d}{\eta - a_1} \\ 0 & -a_1 & 0 & 0 & \dfrac{a_1 d}{\eta - a_1} \\ 0 & \delta & -a_2 & 0 & 0 \\ 0 & 0 & \varepsilon & -a_3 & 0 \\ 0 & \eta & 0 & 0 & -d\left(1 + \dfrac{a_1}{\eta - a_1}\right) \end{bmatrix}.$$

Its eigenvalues are $-a_1 - d\eta(\eta - a_1)^{-1}$, $-a_2$, $-a_3$, $-\mu$ and 0. Thus, zero is a simple eigenvalue of J_0^* and, recalling (6.14), all the other eigenvalues have negative real parts. The *Center Manifold Theory* (see [24]) can thus be employed to assess the behaviour of (6.18) near $\beta = \beta^*$. The tool for studying the local asymptotic stability property of endemic equilibrium (6.13) for β near β^* is provided by Theorem 1.25. Note that the Jacobian J_0^* has a non-negative right eigenvector w and a left eigenvector v associated with the zero eigenvalue. With respect to w, we have that

$$\begin{aligned} & J_0^* w = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ \Leftrightarrow\ & J_0^* \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ \Leftrightarrow\ & w = \begin{bmatrix} \left(\dfrac{\delta\varepsilon\omega_1}{a_2 a_3} - a_1 \right) \dfrac{1}{\mu} & 1 & \dfrac{\delta}{a_2} & \dfrac{\delta\varepsilon}{a_2 a_3} & \dfrac{\eta - a_1}{d} \end{bmatrix}^T w_2, \end{aligned}$$

where w_2 is an arbitrary constant. So, we can choose $w_2 = 1$ and, consequently, we obtain that

$$w = \begin{bmatrix} \left(\dfrac{\delta\varepsilon\omega_1}{a_2 a_3} - a_1 \right) \dfrac{1}{\mu} & 1 & \dfrac{\delta}{a_2} & \dfrac{\delta\varepsilon}{a_2 a_3} & \dfrac{\eta - a_1}{d} \end{bmatrix}^T.$$

As we have seen in the proof of Theorem 5.6,

$$\left(\frac{\delta\varepsilon\omega_1}{a_2 a_3} - a_1 \right) \frac{1}{\mu} < 0.$$

Again, attending to Remark 1.26, as E^0 (see (6.11)) is a non-negative equilibrium point of interest and the first component of E^0 is positive, then the first component of w does not need to be positive. Assuming the feasibility condition (6.14), other components of w are non-negative. With respect to v, we have that

$$\begin{aligned} & v J_0^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{bmatrix} J_0^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \Leftrightarrow\ & v = \begin{bmatrix} 0 & 1 & 0 & 0 & \dfrac{a_1}{\eta} \end{bmatrix} v_2, \end{aligned}$$

where v_2 is an arbitrary constant. So, we can choose $v_2 = 1$ and, consequently, we obtain that

$$v = \begin{bmatrix} 0 & 1 & 0 & 0 & \dfrac{a_1}{\eta} \end{bmatrix}.$$

In view of the fact that $v_1 = v_3 = v_4 = 0$, the only non-vanishing derivatives,

in the above expressions, are

$$\begin{cases} \left[\dfrac{\partial^2 f_2}{\partial x_1 \partial x_5}\left(E^0\right)\right]_{\beta=\beta^*} = \left[\dfrac{\partial^2 f_2}{\partial x_5 \partial x_1}\left(E^0\right)\right]_{\beta=\beta^*} = \dfrac{\beta^*}{\kappa}, \\ \left[\dfrac{\partial^2 f_2}{\partial x_5^2}\left(E^0\right)\right]_{\beta=\beta^*} = -\dfrac{2\beta^*\Lambda}{\mu\kappa^2}, \\ \left[\dfrac{\partial^2 f_5}{\partial x_1 \partial x_5}\left(E^0\right)\right]_{\beta=\beta^*} = \left[\dfrac{\partial^2 f_5}{\partial x_5 \partial x_1}\left(E^0\right)\right]_{\beta=\beta^*} = -\dfrac{\beta^*}{\kappa}, \\ \left[\dfrac{\partial^2 f_5}{\partial x_5^2}\left(E^0\right)\right]_{\beta=\beta^*} = \dfrac{2\beta^*\Lambda}{\mu\kappa^2}. \end{cases}$$

Let us assume that $\phi = \beta - \beta^*$. Therefore, recalling (6.14) and (6.17), for constants a and b, we find

$$a = -\frac{2\beta^*(\eta - a_1)^2}{a_2 a_3 \mu\eta(\kappa d)^2}\left\{(a_1 a_2 a_3 - \delta\varepsilon\omega_1)\kappa d + \Lambda(\eta - a_1)a_2 a_3\right\} < 0$$

and

$$\begin{aligned} b &= \sum_{i=1}^{5}\left(v_2 w_i\left[\frac{\partial^2 f_2}{\partial x_i \partial\phi}\left(E^0\right)\right]_{\beta=\beta^*} + v_5 w_i\left[\frac{\partial^2 f_5}{\partial x_i \partial\phi}\left(E^0\right)\right]_{\beta=\beta^*}\right) \\ &= v_2 w_5\left[\frac{\partial}{\partial x_5}\left(\frac{x_1 x_5}{\kappa + x_5}\right)\left(E^0\right)\right]_{\beta=\beta^*} - v_5 w_5\left[\frac{\partial}{\partial x_5}\left(\frac{x_1 x_5}{\kappa + x_5}\right)\left(E^0\right)\right]_{\beta=\beta^*} \\ &= \frac{\Lambda}{\mu\kappa} w_5(v_2 - v_5) = \frac{\Lambda}{\mu\kappa} \times \frac{\eta - a_1}{d} \times \left(1 - \frac{a_1}{\eta}\right) \\ &= \frac{\Lambda(\eta - a_1)^2}{\mu\kappa d\eta} > 0, \end{aligned}$$

respectively. Thus, since $\eta > a_1$, as

$$\begin{cases} a < 0 \\ b > 0 \\ \phi = \beta - \beta^* = \dfrac{a_1(\beta\Lambda + \mu\kappa d)}{\Lambda(\eta - a_1)}(R_0 - 1) > 0 \end{cases} \Leftrightarrow \begin{cases} a < 0 \\ b > 0 \\ R_0 > 1, \end{cases}$$

we conclude from Theorem 1.25 that the equilibrium points E^0 and E^* of (4.2) (see (6.11) and (6.13)) are, respectively, unstable and locally asymptotic stable for a value of the basic reproduction number such that $R_0 > 1$. This concludes the proof. □

6.3.4 Non-delayed optimal control problem

In this section, we define an optimal control problem associated with model (4.2) with the purpose to curtail the spread of the epidemic. Furthermore, we write the respective necessary optimality conditions, following Pontryagin's Minimum Principle (see [140] and Section 2.2.2). We keep on using notation (5.11): $X = (x_1, x_2, x_3, x_4, x_5) = (S, I, Q, R, B)$.

Formulation of a non-delayed optimal control problem

Cholera transmission is linked to inadequate access to clean water and sanitation facilities. The distribution of chlorine water tablets (CWT) for water purification is one of the possible strategies to improve the quality of the water and to control cholera outbreaks. So, we introduce into model (4.2) a control function $u(\cdot)$ that represents the fraction of susceptible individuals who has access to CWT for water purification (see [80]). Therefore, they are protected from infection. This control measure is such that $u(t) \in [0, u_{\max}]$ for all $t \in [0, t_f]$, where $t_f > 0$ is the final time and $0 \leq u_{\max} < 1$. If $u \equiv 0$, then nobody receives those chlorine water tablets, that is, there is no control measure. Note that it makes no sense, from a practical point of view, to consider the case $u \equiv 1$, since it means that there are no new recruitments in the class of infective individuals I. The objective is to find the control strategy through the use of CWT that minimizes the number of infective individuals and the bacterial concentration, as well as the cost of interventions associated with CWT. CWT are effervescent tablets that kill micro-organisms in water to prevent cholera, typhoid, dysentery and other water borne diseases. There are different sizes of CWT and each tablet size is formulated to treat a specific volume of water, ranging from 1 litre to 2500 litres. The model with control is then given by the following system of non-linear ordinary differential equations:

$$\begin{cases} \dot{x}_1(t) = \Lambda - \dfrac{\beta x_1(t) x_5(t)}{\kappa + x_5(t)}\Big(1 - u(t)\Big) + \omega_1 x_4(t) - \mu x_1(t), \\ \dot{x}_2(t) = \dfrac{\beta x_1(t) x_5(t)}{\kappa + x_5(t)}\Big(1 - u(t)\Big) - (\delta + \alpha_1 + \mu) x_2(t), \\ \dot{x}_3(t) = \delta x_2(t) - (\varepsilon + \alpha_2 + \mu) x_3(t), \\ \dot{x}_4(t) = \varepsilon x_3(t) - (\omega_1 + \mu) x_4(t), \\ \dot{x}_5(t) = \eta x_2(t) - d x_5(t) - \dfrac{\beta x_1(t) x_5(t)}{\kappa + x_5(t)}, \end{cases} \tag{6.19}$$

together with initial conditions (6.10). The set $\mathscr{X}$ of admissible trajectories and the admissible control set $\mathscr{U}$ are, respectively, given by

$$\mathscr{X} = \big\{ X(\cdot) \in W^{1,1}\big([0, t_f]; \mathbb{R}^5\big) \,|\, (6.10) \text{ and } (6.19) \text{ are satisfied} \big\}$$

and

$$\mathscr{U} = \big\{ u(\cdot) \in L^1\big([0, t_f]; \mathbb{R}\big) \,|\, 0 \leq u(t) \leq u_{\max},\, \forall\, t \in [0, t_f] \,\big\},$$

where $0 \leq u_{\max} < 1$. The functional to be minimized is represented by

$$J\big(X(\cdot), u(\cdot)\big) = \int_0^{t_f} \big(x_2(t) + x_5(t) + u(t)\big) dt. \tag{6.20}$$

Clearly, one would like to eradicate the epidemic at the least possible cost. The optimal control problem consists of determining the vector function $X^\diamond(\cdot) = \big(S^\diamond(\cdot), I^\diamond(\cdot), Q^\diamond(\cdot), R^\diamond(\cdot), B^\diamond(\cdot)\big) \in \mathscr{X}$ associated with an admissible control $u^\diamond(\cdot) \in \mathscr{U}$, on the time interval $[0, t_f]$, that provides the minimal value to the cost functional (6.20), i.e.,

$$J\big(X^\diamond(\cdot), u^\diamond(\cdot)\big) = \min_{(X(\cdot),u(\cdot))\in\mathscr{X}\times\mathscr{U}} J\big(X(\cdot), u(\cdot)\big). \tag{6.21}$$

Necessary optimality conditions: Pontryagin's Minimum Principle

The following theorem provides the necessary optimality conditions associated with optimal control problem (6.21).

Theorem 6.16. *Assume that $X^\diamond = \big(x_1^\diamond(\cdot), x_2^\diamond(\cdot), x_3^\diamond(\cdot), x_4^\diamond(\cdot), x_5^\diamond(\cdot)\big) \in \mathscr{X}$ is an optimal state associated with an optimal control $u^\diamond(\cdot) \in \mathscr{U}$ of optimal control problem (6.21) with fixed final time $t_f \in \mathbb{R}^+$. Then, there is an adjoint function $\lambda^\diamond = \big(\lambda_1^\diamond, \lambda_2^\diamond, \lambda_3^\diamond, \lambda_4^\diamond, \lambda_5^\diamond\big) : [0, t_f] \to \mathbb{R}^5$ that satisfies the adjoint system*

$$\begin{cases} \dot\lambda_1^\diamond(t) = \dfrac{\beta x_5^\diamond(t)}{\kappa + x_5^\diamond(t)}\Big(\big[\lambda_1^\diamond(t) - \lambda_2^\diamond(t)\big]\big[1 - u(t)\big] + \lambda_5^\diamond(t)\Big) + \mu\lambda_1^\diamond(t), \\ \dot\lambda_2^\diamond(t) = -1 + \Big(\delta + \alpha_1 + \mu\Big)\lambda_2^\diamond(t) - \delta\lambda_3^\diamond(t) - \eta\lambda_5^\diamond(t), \\ \dot\lambda_3^\diamond(t) = \Big(\varepsilon + \alpha_2 + \mu\Big)\lambda_3^\diamond(t) - \varepsilon\lambda_4^\diamond(t), \\ \dot\lambda_4^\diamond(t) = -\omega_1\lambda_1^\diamond(t) + \Big(\omega_1 + \mu\Big)\lambda_4^\diamond(t), \\ \dot\lambda_5^\diamond(t) = -1 + \dfrac{\beta\kappa x_1^\diamond(t)}{(\kappa + x_5^\diamond(t))^2}\Big(\big[\lambda_1^\diamond(t) - \lambda_2^\diamond(t)\big]\big[1 - u(t)\big] + \lambda_5^\diamond(t)\Big) \\ \qquad\quad + d\lambda_5^\diamond(t), \end{cases} \tag{6.22}$$

with transversality conditions

$$\lambda_i^\diamond(t_f) = 0, \quad i = 1, \ldots, 5 \tag{6.23}$$

for almost all $t \in [0, t_f]$. Moreover, the control law is characterized by

$$u^\diamond(t) = \begin{cases} u_{\max}, & \text{if} \quad \phi(t) < 0; \\ 0, & \text{if} \quad \phi(t) > 0; \\ \text{singular}, & \text{if} \quad \phi(t) = 0 \text{ on } I_s \subset [0, t_f]; \end{cases} \tag{6.24}$$

where ϕ is the switching function defined by

$$\phi(t) = 1 + \frac{\beta x_1^\diamond(t) x_5^\diamond(t)}{\kappa + x_5^\diamond(t)}\Big(\lambda_1^\diamond(t) - \lambda_2^\diamond(t)\Big) \tag{6.25}$$

for almost all $t \in [0, t_f]$.

Proof. The necessary optimality conditions for an optimal solution of (6.21) are given by Pontryagin's Minimum Principle (see Theorem 2.5). The Hamiltonian function is defined by

$$\begin{aligned} H(X,u,\lambda) = {} & x_2 + x_5 + u + \lambda_1 \left(\Lambda - \frac{\beta x_1 x_5}{\kappa + x_5}\big(1-u\big) + \omega_1 x_4 - \mu x_1 \right) \\ & + \lambda_2 \left(\frac{\beta x_1 x_5}{\kappa + x_5}\big(1-u\big) - (\delta + \alpha_1 + \mu) x_2 \right) \\ & + \lambda_3\big(\delta x_2 - (\varepsilon + \alpha_2 + \mu) x_3\big) + \lambda_4\big(\varepsilon x_3 - (\omega_1 + \mu) x_4\big) \\ & + \lambda_5 \left(\eta x_2 - d x_5 - \frac{\beta x_1 x_5}{\kappa + x_5} \right). \end{aligned} \tag{6.26}$$

Let us suppose that $\big(X^\diamond(\cdot), u^\diamond(\cdot)\big) \in \mathscr{X} \times \mathscr{U}$ is an optimal solution of (6.21) with fixed final time $t_f \in \mathbb{R}^+$. Then, there is an adjoint function $\lambda^\diamond = \big(\lambda_1^\diamond, \lambda_2^\diamond, \lambda_3^\diamond, \lambda_4^\diamond, \lambda_5^\diamond\big) : [0, t_f] \to \mathbb{R}^5$, $\lambda^\diamond(\cdot) \in W^{1,1}\big([0,t_f];\mathbb{R}^5\big)$, that satisfies, for almost all $t \in [0, t_f]$, the

1) transversality conditions:

$$\lambda_i^\diamond(t_f) = 0, \quad i = 1, \ldots, 5, \tag{6.27}$$

in view of the free terminal state $X(t_f)$;

2) adjoint system:

$$\dot{\lambda}_i^\diamond(t) = -\frac{\partial H}{\partial x_i}\big(X^\diamond(t), u^\diamond(t), \lambda^\diamond(t)\big), \quad i = 1, \ldots, 5; \tag{6.28}$$

3) minimality condition:

$$\min_{0 \le u \le u_{\max}} H\big(X^\diamond(t), u, \lambda^\diamond(t)\big) = H\big(X^\diamond(t), u^\diamond(t), \lambda^\diamond(t)\big), \tag{6.29}$$

where $0 \le u_{\max} < 1$.

So, conditions (6.23) are derived from transversality conditions (6.27). Moreover, system (6.22) is obtained from adjoint system (6.28). Let us evaluate the minimality condition (6.29). The Hamiltonian (6.26) is linear in the control variable. Hence, the minimizer control is determined by the sign of the switching function

$$\phi(t) = \frac{\partial H}{\partial u}\big(X^\diamond(t), u, \lambda^\diamond(t)\big) = 1 + \frac{\beta x_1^\diamond(t) x_5^\diamond(t)}{\kappa + x_5^\diamond(t)}\Big(\lambda_1^\diamond(t) - \lambda_2^\diamond(t)\Big)$$

(see (6.25)) as follows:

$$u^\diamond(t) = \begin{cases} u_{\max}, & \text{if} \quad \phi(t) < 0; \\ 0, & \text{if} \quad \phi(t) > 0; \\ \text{singular}, & \text{if} \quad \phi(t) = 0 \text{ on } I_s \subset [0, t_f]. \end{cases}$$

For more details see Section 2.2.2. If the switching function has only finitely many isolated zeros in an interval $I_b \subset [0, t_f]$, then the control $u^\diamond$ is called *bang-bang* on I_b. The case of a *singular control*, where $\phi(t) = 0$ on $I_s \subset [0, t_f]$, will not be further discussed here, since in our computations we never encountered *singular controls*. This concludes the proof. □

6.3.5 Numerical simulations

In this section, we show that the control measure defined in Section 6.3.4 could have stopped more quickly the worst cholera outbreak that ever occurred in human history, which began in Yemen in October 2016. We consider the real data of the number of infective individuals in Yemen, from 27th April 2017 to 15th April 2018 (see [189]), represented in Figure 6.2. In this period, the maximum number of infective individuals was 51000.

In order to better simulate real life situations, where there is a lack of resources needed to distribute CWT for water purification, we consider three situations:

- low resources ($u_{\max} = 0.20$);
- enough resources (two cases considered: $u_{\max} = 0.55$ and $u_{\max} = 0.90$);
- abundance of resources ($u_{\max} = 0.95$);

finding the interval of time needed to stop the outbreak in Yemen.

Firstly, we consider low resources for CWT distribution ($u_{\max} = 0.20$). It means that only a small percentage (20%) of susceptible individuals has access to the CWT. In second place, we consider enough resources to decrease the outbreak (cases $u_{\max} = 0.55$ and $u_{\max} = 0.90$). Finally, we present numerical simulations with abundance of resources: $u_{\max} = 0.95$. In this case almost all susceptible population has access to pure water.

In order to solve numerically optimal control problem (6.21) for the three cases mentioned before, we use IPOPT – AMPL, as in Section 5.3.5: *Numerical solutions of the delayed SIQRB control model.* We implement Euler's method as integration method and the tolerance is set to $tol = 10^{-8}$. Moreover, we use $N = 100 \times t_f$ grid points in all cases. For all numerical simulations of this section, the values of Λ, μ, η, d, S_0, I_0, Q_0, R_0, B_0 and N_0 are fixed in Table 6.3. Consequently, we have that:

$$N_0 = 28250420, \quad \frac{\Lambda}{\mu} \simeq 4.86 \times N_0, \quad B_0 = 2.75 \times 10^5 \quad \text{and} \quad \frac{\Lambda\eta}{\mu d} \simeq 4.16 \times 10^9.$$

Therefore, the initial values of the human population and bacterial concentration belong to (5.4) and (5.5), respectively:

$$N_0 = S_0 + I_0 + Q_0 + R_0 \in \left[0, \frac{\Lambda}{\mu}\right] \quad \text{and} \quad B_0 \in \left[0, \frac{\Lambda\eta}{\mu d}\right].$$

Therefore, $(S_0, I_0, Q_0, R_0, B_0) \in \Omega = \Omega_H \times \Omega_B$. This implies that all the following numerical solutions (S, I, Q, R, B) belong to the positively invariant set $\Omega = \Omega_H \times \Omega_B$ (see Section 6.3.1 and Lemmas 5.1 and 5.2).

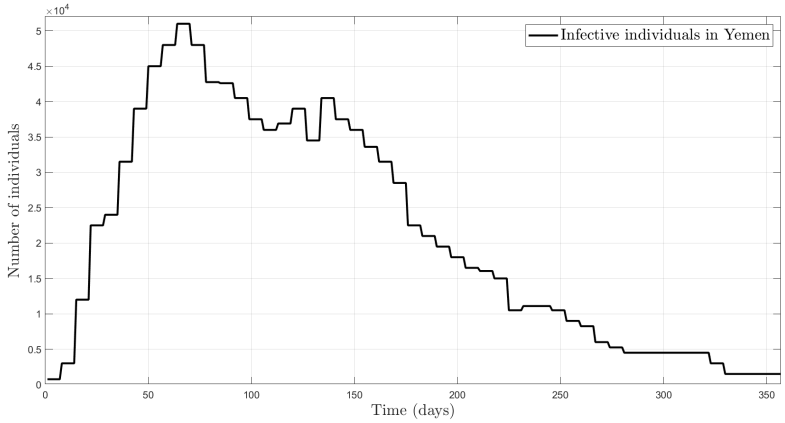


Figure 6.2: Real data from the cholera outbreak in Yemen, from 27th April 2017 to 15th April 2018 (see [189]).

Parameter	**Value**	**Unity**	**Reference**
Λ	$(28.4N_0)/365000$	person day^{-1}	[77]
μ	1.6×10^{-5}	day^{-1}	[79]
β	0.01891	day^{-1}	Assumed
κ	10^7	cell/ml	Assumed
ω_1	$0.4/365$	day^{-1}	[120]
δ	1.15	day^{-1}	Assumed
ε	0.2	day^{-1}	[127]
α_1	6×10^{-6}	day^{-1}	[79, 188]
α_2	3×10^{-6}	day^{-1}	Assumed
η	10	cell/ml day^{-1} person^{-1}	[22]
d	0.33	day^{-1}	[22]
S_0	28249670	person	[194]
I_0	750	person	[188]
Q_0	0	person	Assumed
R_0	0	person	Assumed
N_0	28250420	person	–
B_0	275000	cell/ml	Assumed

Table 6.3: Parameter values and initial conditions for optimal control problem (6.21).

Extremal solution in case of low resources

We start by assuming that $u_{\max} = 0.20$, that is, the maximum percentage of susceptible individuals that have access to the CWT is 20%. As we consider that $t_f = 354$ days, then the number of grid points is $N = 100 \times t_f = 35400$. The numerical simulations for the control are in agreement with Theorem 6.16. Note that, from (6.23)–(6.25), we know that $\phi(t_f) = 1$ and $u(t_f) = 0$, which explains why the values of the control u decrease to zero at $t_f = 354$ (see bottom right plot of Figure 6.3). However, all resources are being used during almost all the time period considered in the simulation (354 days). The state trajectories associated with the extremal control are plotted in Figure 6.3. From this last figure, we observe that although B is a strictly decreasing function, I is not. Although the resources are insufficient to eradicate the disease, in the considered time interval, the distribution of CWT to 20% of the susceptible population is enough to improve the real situation represented in Figure 6.2, decreasing significantly the maximum number of infective individuals. Note that the real maximum number of infective individuals was 51000 and the one associated with $u_{\max} = 0.20$ is approximately equal to 7431, an important improvement.

Extremal solution in case of sufficient resources

As we may deduce from previous numerical simulation, we need to consider larger values for $u_{\max}$ to curtail the spread of the epidemic more quickly and in a better way. Now, let us take $u_{\max} = 0.55$ and $u_{\max} = 0.90$. In the first case, a little bit more than half of susceptible individuals receives CWT (55%). In the second one, only 10% of the susceptible population does not have access to CWT ($u_{\max} = 0.90$). Moreover, when $u_{\max} = 0.55$ and $u_{\max} = 0.90$, we consider that $t_f = 100$ days and $t_f = 70$ days, respectively. So, the number of grid points is $N = 100 \times t_f = 10000$ and $N = 100 \times t_f = 7000$, when $u_{\max} = 0.55$ and $u_{\max} = 0.90$, respectively.

Even considering these larger values for $u_{\max}$, the solution of infective individuals does not become a strictly decreasing function: neither when $u_{\max} = 0.55$ nor when $u_{\max} = 0.90$ (see the curves of I in Figures 6.4 and 6.5). Nevertheless, the maximum value of infective individuals decreases significantly with respect to the one obtained for $u_{\max} = 0.20$ (7431). Here this value is approximately equal to 3749 and 942 for $u_{\max} = 0.55$ and $u_{\max} = 0.90$, respectively.

The extremal control is bang-bang for both values of $u_{\max}$. We need approximately 93 days to solve the epidemic, when $u_{\max} = 0.55$. Thus, at the end of approximately three months, the supply of CWT to susceptible population can be discontinued, because the control decreases to zero (see left plot of Figure 6.7). As we expected, one needs less time to curtail the spread of the epidemic when we consider $u_{\max} = 0.90$: at the end of approximately

48 days, the control decreases to zero and the disease is eradicated (see left plot of Figure 6.8).

Pontryagin's Minimum Principle is a first order necessary optimality condition. Therefore, the control law given by (6.24) is just an extremal of optimal control problem (6.21). However, a stronger condition, the so-called *strict bang-bang property* (for more details see Section 2.2.2) is also satisfied for $u_{\max} = 0.55$ and $u_{\max} = 0.90$, that is, the *bang-bang control* and the switching function match the following switching conditions:

$$\begin{cases} \phi_{55}(t) < 0, & \text{if } t \in \left[0, t_s^{55}\right[; \\ \dot{\phi}_{55}\left(t_s^{55}\right) > 0; \\ \phi_{55}(t) > 0, & \text{if } t \in \left]t_s^{55}, 100\right]; \end{cases}$$

and

$$\begin{cases} \phi_{90}(t) < 0, & \text{if } t \in \left[0, t_s^{90}\right[; \\ \dot{\phi}_{90}\left(t_s^{90}\right) > 0; \\ \phi_{90}(t) > 0, & \text{if } t \in \left]t_s^{90}, 70\right]; \end{cases}$$

where t_s^p and ϕ_p denote, respectively, the switching time and switching function ϕ for $u_{\max} = \frac{p}{100}$ (see Figures 6.7 and 6.8). Moreover, the respective minimum costs are given by

$$J_{55} \simeq 1.630731 \times 10^6 \quad \text{and} \quad J_{90} \simeq 8.402090 \times 10^5,$$

where J_p is the value of functional (6.20) corresponding to the problem with $u_{\max} = \frac{p}{100}$.

Extremal solution in case of abundance of resources

Now, we consider $u_{\max} = 0.95$, that is, 95% of susceptible population has access to CWT for water purification. This corresponds to a situation where there is abundance of resources. Furthermore, we consider that $t_f = 70$ days and, consequently, the number of grid points is $N = 100 \times t_f = 7000$. In this case, the numerical solution for the number of infective individuals I is a strictly decreasing function (see right plot of Figure 6.6). In this situation, there is a timely and effective distribution of CWT, which avoids the increase of the number of infective individuals. Consequently, it is possible to achieve a low maximum value of infective individuals equal to $I_0 = 750$. For $0.95 < u_{\max} < 1$ we are not able to obtain a feasible candidate for optimal control problem (6.21), considering the values of Table 6.3.

When $u_{\max} = 0.95$, we only need to distribute CWT in the first 44 days (see left plot of Figure 6.9). The minimum cost (6.20) takes the value

$$J_{95} \simeq 7.849481 \times 10^5$$

and the extremal control is also bang-bang for $u_{\max} = 0.95$. As in cases $u_{\max} = 0.55$ and $u_{\max} = 0.90$, the bang-bang control and the switching function match the switching condition (6.24) and satisfy the *strict bang-bang property* with respect to Pontryagin's Minimum Principle:

$$\begin{cases} \phi_{95}(t) < 0, & \text{if } t \in \left[0, t_s^{95}\right[; \\ \dot{\phi}_{95}\left(t_s^{95}\right) > 0; \\ \phi_{95}(t) > 0, & \text{if } t \in \left]t_s^{95}, 70\right]; \end{cases}$$

where t_s^{95} and ϕ_{95} denote, respectively, the switching time and switching function ϕ for problem with $u_{\max} = \frac{95}{100}$ (see Figure 6.9).

6.4 Conclusion

In Section 6.2, we proposed and analysed, analytically and numerically, a SITRVB model for the dynamics of cholera transmission. In order to fit the biggest cholera outbreak worldwide, which has occurred very recently in Yemen, we provided a numerical simulation that does not consider vaccination. Indeed, this measure of prevention did not exist in Yemen from 27th April 2017 to 15th April 2018. Simulations of the SITRVB mathematical model (with and without vaccination) show that the introduction of a vaccination campaign since the beginning of the epidemic in Yemen could have changed the situation substantially. Namely, with this measure it could be obtained the case $R_0 < 1$, where the disease extinguishes naturally. We trust that the work of Section 6.2 is of great significance, because it supplies a mathematical model for cholera that is deeply studied and allows to obtain important conclusions about the relevance of vaccination campaigns in cholera outbreaks. Actually, we believe that the absence of this type of prevention measures in Yemen was one of the responsible for provoking the biggest cholera outbreak in world's history, killing 2310 individuals between 27th April 2017 and 1st July 2018 (see [190]). Therefore, our research motivates and fortifies the importance of vaccination in cholera epidemics.

In Section 6.3, we considered a SIQRB model that is an improvement of that considered in Section 5.2, because it is assumed that a healthy individual must intake bacteria from the environment to become infected and, by doing so, these bacteria are removed from the aquatic medium. In contrast to what happened in Section 5.2.2, the feasibility of the endemic equilibrium depends on the rate, η, at which the bacteria are spread by the infective individuals. Moreover, this rate, η, must exceed the combined rates at which infective individuals leave their compartment, i.e., must be larger than the sum of the rates at which individuals are quarantined and die either naturally or due to disease. The conditions for the local stability of the endemic equilibrium also differ from the ones obtained in Section 5.2.3.

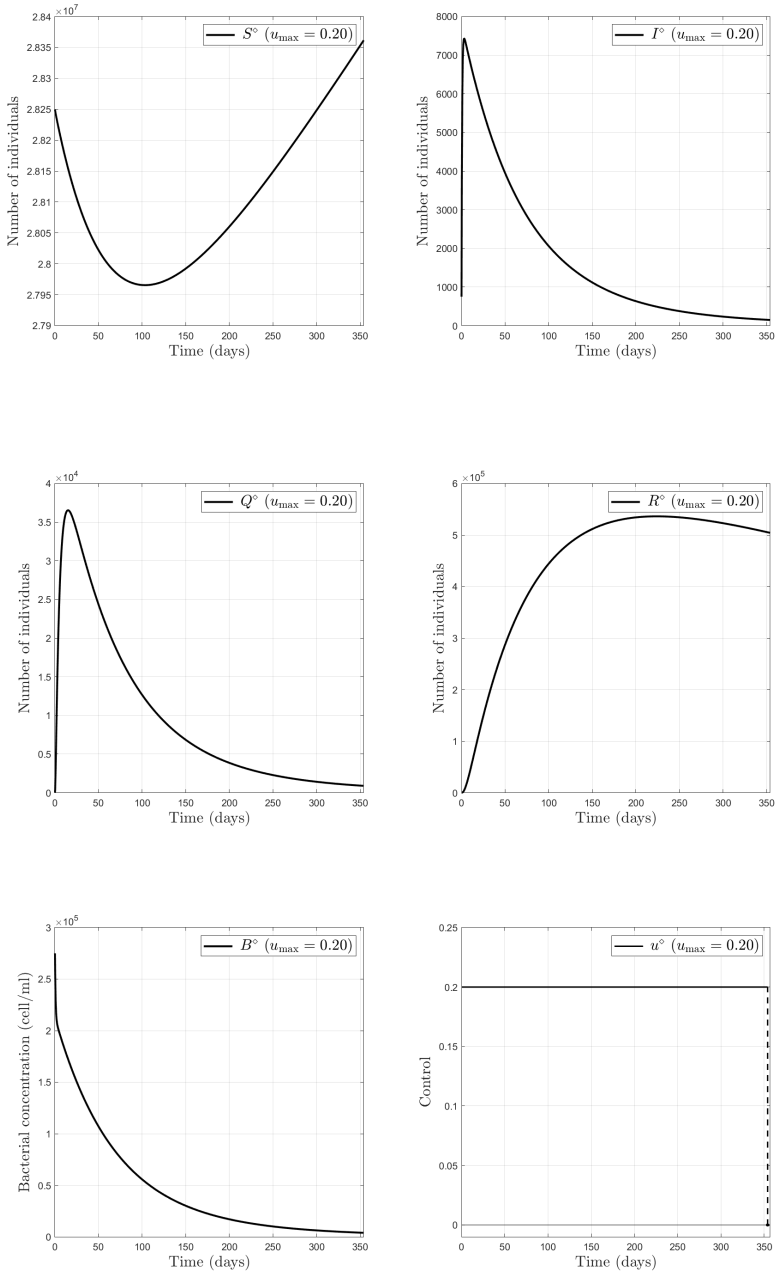


Figure 6.3: Extremal state trajectories $S^\diamond(t)$, $I^\diamond(t)$, $Q^\diamond(t)$, $R^\diamond(t)$ and $B^\diamond(t)$ and extremal control $u^\diamond(t)$ (satisfying the control law (6.24)) associated with optimal control problem (6.21) for all $t \in [0, 354]$ and $u_{\max} = 0.20$, using all the other values of Table 6.3.

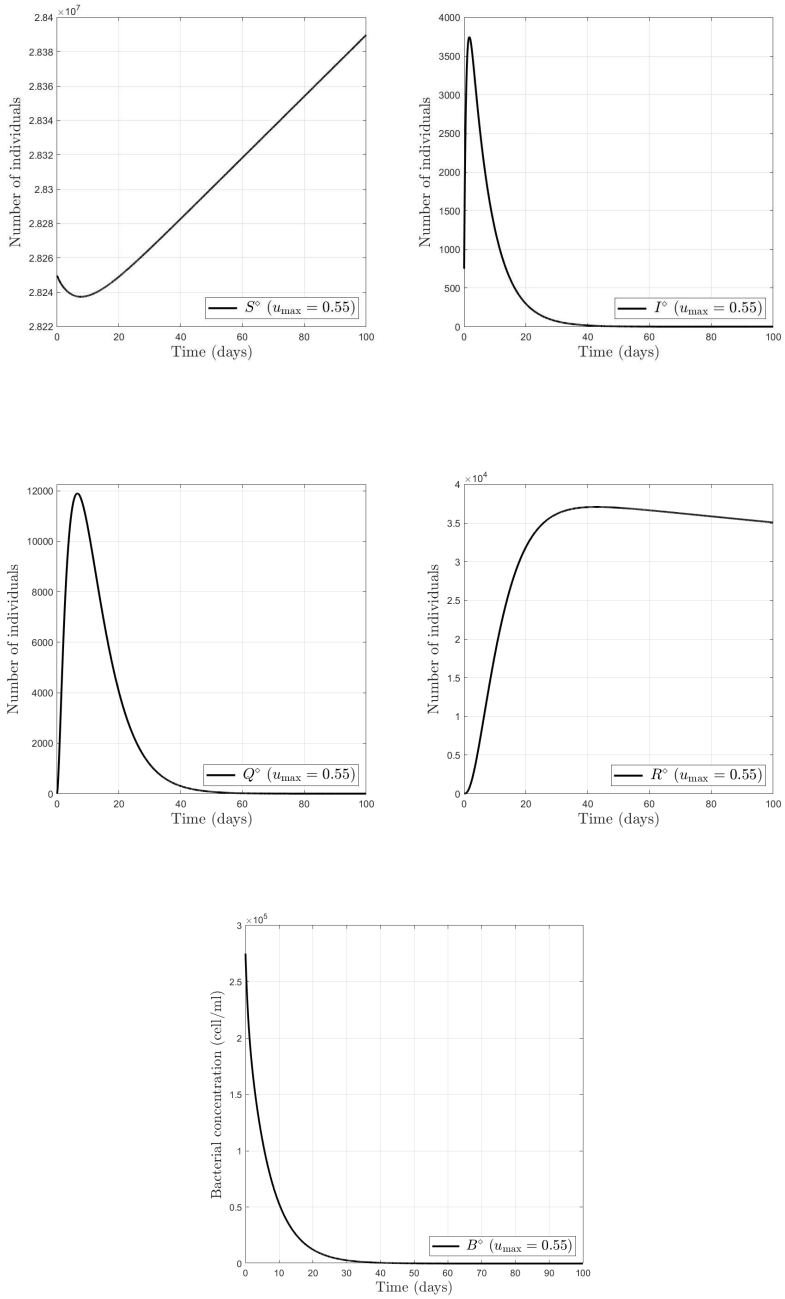


Figure 6.4: Extremal state trajectories $S^\diamond(t)$, $I^\diamond(t)$, $Q^\diamond(t)$, $R^\diamond(t)$ and $B^\diamond(t)$ associated with optimal control problem (6.21) for all $t \in [0, 100]$ and $u_{\max} = 0.55$, using all the other values of Table 6.3.

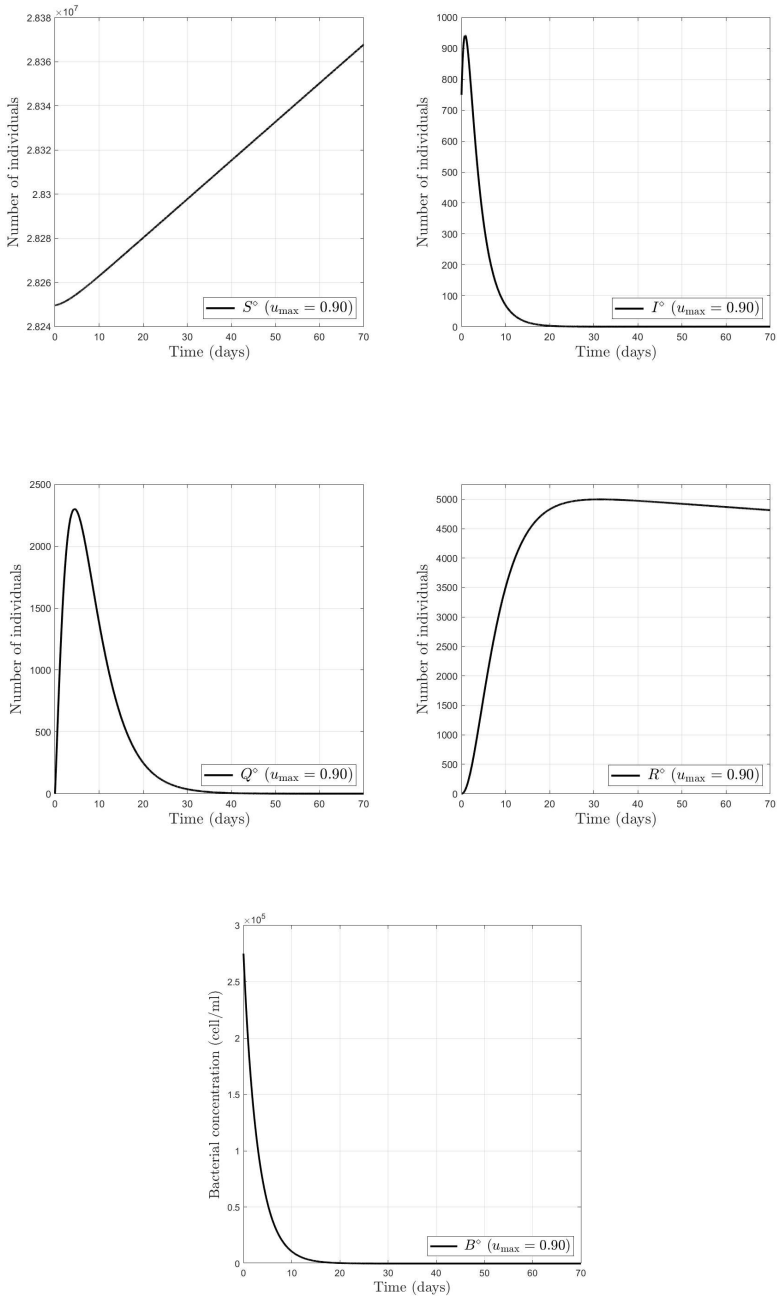


Figure 6.5: Extremal state trajectories $S^\diamond(t)$, $I^\diamond(t)$, $Q^\diamond(t)$, $R^\diamond(t)$ and $B^\diamond(t)$ associated with optimal control problem (6.21) for all $t \in [0, 70]$ and $u_{\max} = 0.90$, using all the other values of Table 6.3.

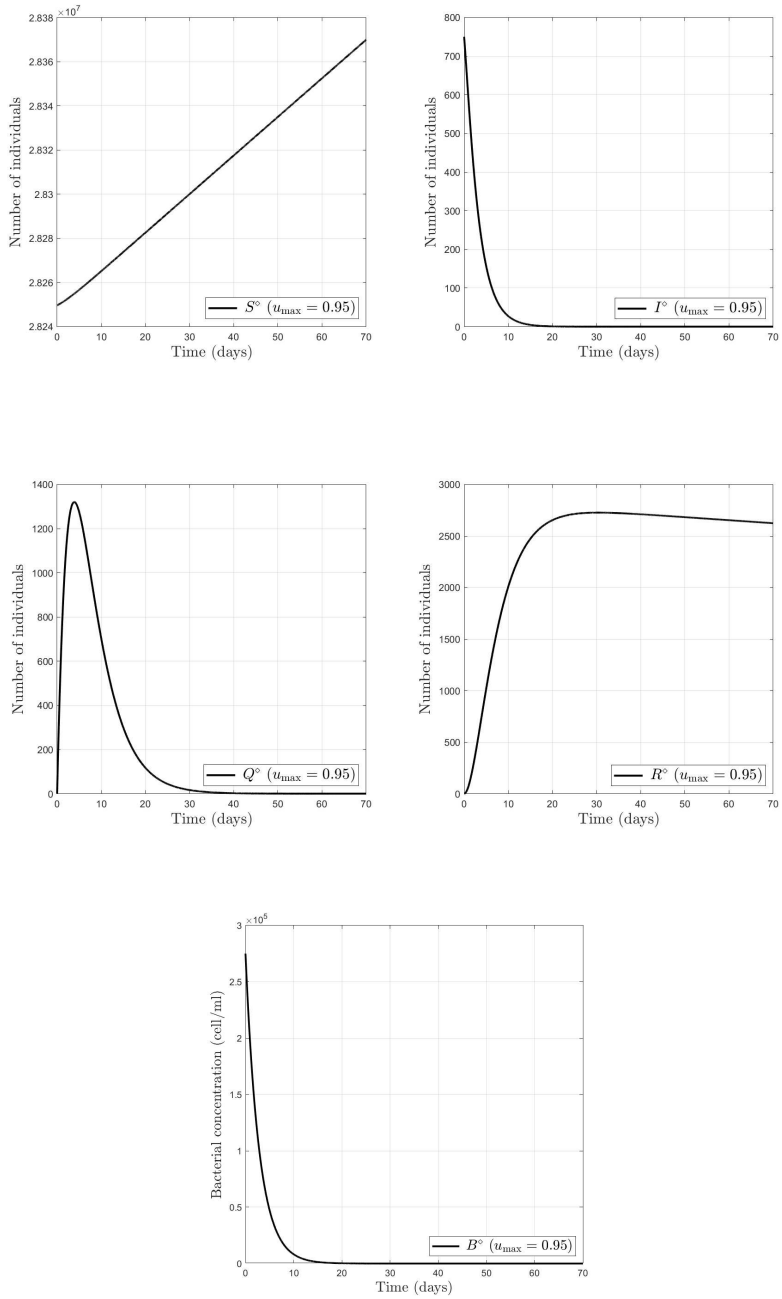


Figure 6.6: Extremal state trajectories $S^\diamond(t)$, $I^\diamond(t)$, $Q^\diamond(t)$, $R^\diamond(t)$ and $B^\diamond(t)$ associated with optimal control problem (6.21) for all $t \in [0, 70]$ and $u_{\max} = 0.95$, using all the other values of Table 6.3.

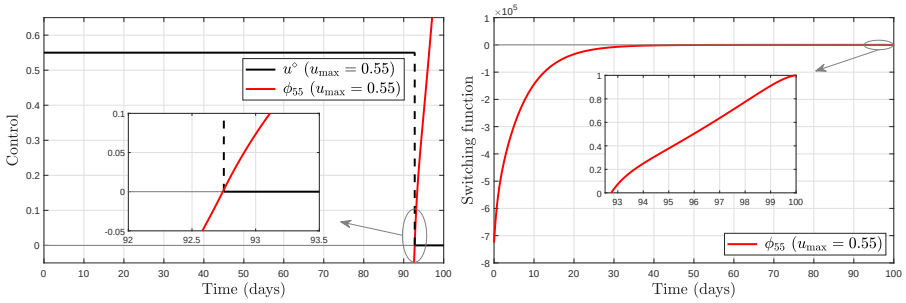


Figure 6.7: Extremal control $u^\diamond(t)$ and switching function $\phi_{55}(t)$ associated with optimal control problem (6.21) for all $t \in [0, 100]$, satisfying the control law (6.24), when we consider $u_{\max} = 0.55$ and all the other values of Table 6.3.

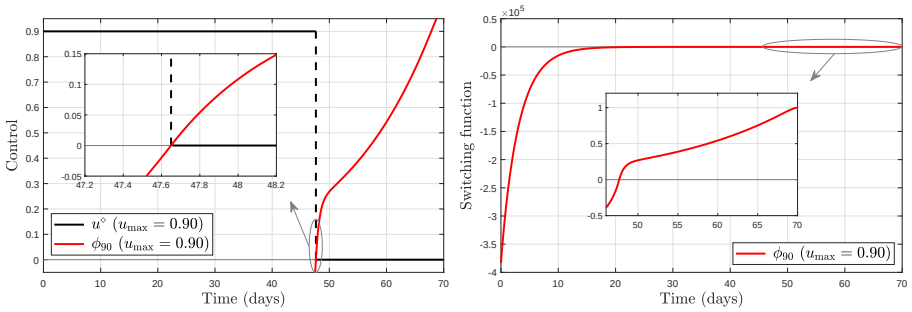


Figure 6.8: Extremal control $u^\diamond(t)$ and switching function $\phi_{90}(t)$ associated with optimal control problem (6.21) for all $t \in [0, 70]$, satisfying the control law (6.24), when we consider $u_{\max} = 0.90$ and all the other values of Table 6.3.

Furthermore, we proposed and analysed an optimal control problem, where the control function represents the fraction of susceptible population who receives chlorine water tablets (CWT) for water purification. The objective of such optimal control problem is to minimize the number of infective individuals and the environmental bacterial concentration, as well as the cost associated with the distribution of CWT. The extremal solution has been characterized both analytically and numerically. The extremal bang-bang controls satisfy the so-called strict bang-bang property with respect to

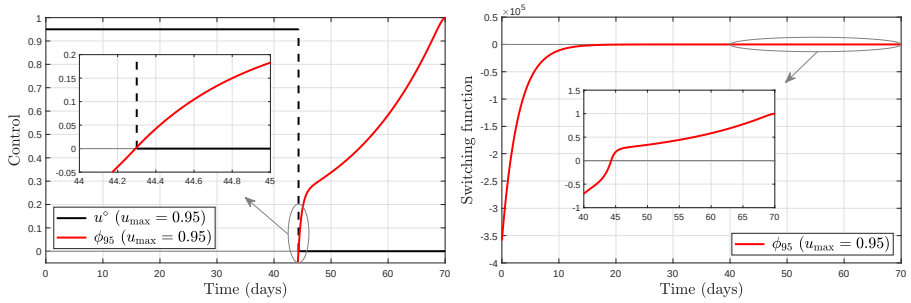


Figure 6.9: Extremal control $u^{\diamond}(t)$ and switching function $\phi_{95}(t)$ associated with optimal control problem (6.21) for all $t \in [0, 70]$, satisfying the control law (6.24), when we consider $u_{\max} = 0.95$ and all the other values of Table 6.3.

Pontryagin's Minimum Principle. Thus, the proposed strategies for the distribution of CWT represent suitable means for containing cholera outbreaks, in different scenarios and periods of time. This is supported by the current situation (March 2019) in Mozambique. There the Portuguese army purified around 4000 litres of water per day using chlorine with the purpose to fight the cholera epidemic caused by the passage of cyclone Idai, in March 2019 (see [84]).

Conclusion and future work

With this PhD thesis we gave answer to an open question, by proving sufficient optimality conditions for two types of optimal control problems with constant time delays in both state and control variables. Usually, these time delays are not equal and we ensured the Commensurability Assumption between them, as Göllmann et al. in [56]. In the first type of problems, the differential system is linear with respect to state function and in the second it is, in general, non-linear. The proof was based on the transformation of delayed optimal control problems into equivalent and augmented non-delayed ones, following the approach proposed in [59] and used in [56]. In this way, we were able to apply well-known sufficient optimality conditions for non-delayed optimal control problems, recalled in Chapter 2 of this thesis. Then, we returned to the initial delayed optimal control problems, obtaining sufficient optimality conditions for such type of problems for the first time in the literature. Furthermore, we solved examples with the purpose to illustrate the usefulness of obtained conditions.

Secondly, we gave an overview of an infectious disease that still continues to cause a high number of deaths worldwide. This disease is cholera. Moreover, we also provided a state of the art of mathematical studies that have been carried out to understand the spread of such disease and to suggest some treatment/prevention measures to stop its transmission. Based on some of them, we formulate new cholera mathematical models and new optimal control problems, with and without time delays, that consider these new models. We studied, from a theoretical point of view, both models and optimal control problems. Then, we provided some numerical simulations of both to fit real outbreaks and to propose control measures that allowed an improvement of what reality was.

As one can see in Section 5.3.4, in the mathematical analysis of the delayed optimal control problem (5.29), we only obtained necessary optimality conditions. It remains missing the study of sufficient optimality conditions for (5.29), using the ones studied in Sections 3.2.2 and 3.3.1. It is important to highlight that:

i) the sufficient conditions, derived in Section 3.2.2, were obtained for delayed optimal control problems which consider a state-linear differential

system and it is difficult to translate the dynamics of cholera transmission through a state-linear model;

ii) the sufficient conditions, derived in Section 3.3.1, applied to delayed optimal control problem (5.29), imply very cumbersome calculations.

Consequently, future work consists to fill the gap, by deriving sufficient optimality conditions for (5.29) and other delayed optimal control problems associated with the mathematical study of infectious diseases.

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Index