"CONTAS DE CABEÇA" - A BOOK OF MATHEMATICAL RIDDLES EDITED BY THE PORTUGUESE FOOTBALL FEDERATION

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A situação é matemática
E há a problemática
Que é a arte de lidar bem o problema[1]

—Dilema (Jorge Palma)

In March 2018, a book was published that brought together the Portuguese Football Federation (FPF, the publisher) and the Portuguese Mathematical Society (SPM, a partner): *Contas de Cabeça – 50 Desafios Matemáticos de Futebol* (fig. 1). In this book, we present several math problems and riddles using football as the common theme. Many of them are classical puzzles from well-known authors in mathematics such as Bolt, Dudeney, Loyd, Perelman, Gardner or Tahan, but now presented in a new and appealing context: football – the passion that excites almost the entire World. In this paper, we will show, for instance, how we transformed a Gardner problem about goldfishes into a problem of goals. The math essence of the problem is the same but we hope that this new approach will be more appealing and attractive to younger generations of students.

Many times, we hear "math is everywhere". If that is true, with a little imagination and work, we can always find new contexts to modernize old interesting problems.

This book, as the subtitle indicates, is a set of 50 math riddles/puzzles related to football. They are diverse enigmas that have football as the background and seek to mobilize math skills to be solved. The choice of this theme lays in the universal interest of football in our society, especially among students, who in general have little interest in Math (a subject that

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[1]The situation is mathematical || And there's the problematical || Which is the art of dealing well with the problem.

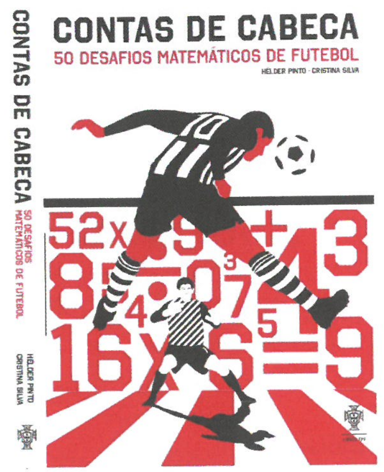


Figure 1: Frontpage of the book *Contas de Cabeça* (FPF, 2018).

is, many times, considered "the seven headed hydra"). The purpose is not showing practical applications of Math to football but getting some attention for Math through the generalized interest in football (fig. 2).

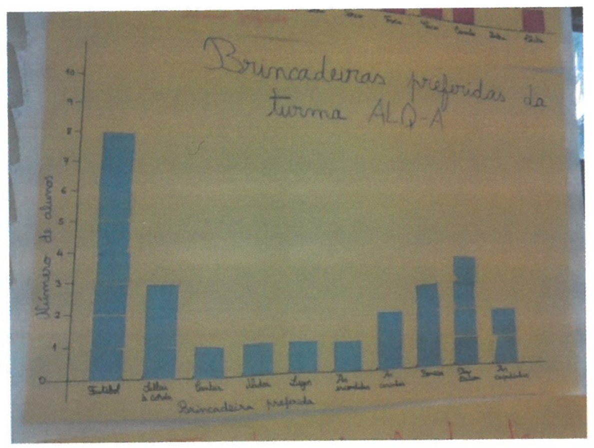


Figure 2: Favourite games from a first school year class in V. N. Gaia (the highest column is football).

We choose to adapt some math puzzles to football in a variety of situations. We tried to make the problem statements not too "far-fetched", making a great effort to have texts understandable to almost everyone, even to those who do not have any formal knowledge of Math. In order to understand the problems, the readers just need to be familiar with the terminology used

in football. Furthermore, we tried to make the problems plausible and not seem to come out of a typical school book.

The riddles presented in this book have different origins:

- some are published as they were originally written by their authors;
- many of them are adapted from classical riddles that use other themes and contexts;
- some are completely new.

The authors of classic riddles in this book include names as remarkable as Brian Bolt, Henry Dudeney, Martin Gardner, Sam Loyd, Mariano Mataix, Adrián Paenza, Yakov Perelman, Dennis Shasha, Malba Tahan and José Paulo Viana. This book presents, in some riddles, fictional situations but in others uses both the Portuguese reality and the international context. For example:

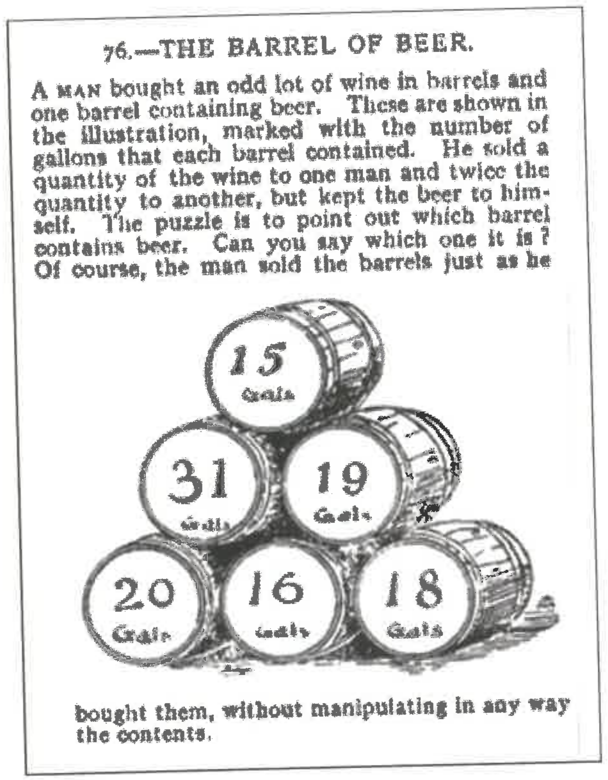
- Portuguese football tournaments (Championship and Cup);
- international football tournaments (Euro and CAN);
- well-known players such as Ronaldo, Zidane, Di Maria, Figo, Eusébio and Pepe.

The readers need different levels of knowledge to get the answers, from the most basic to some more advanced contents – the questions are simple; however the solution is not always easy to obtain. The problems were chosen in order to allow any reader to make attempts solving it without using formal math knowledge, but only logical reasoning and systematic organization. The idea is to show the strength of mathematics in solving problems that would often be very hard to solve without math tools. The answers included in the book provide an insight into the potential of mathematical reasoning in solving problems, highlighting that formal mathematics is a strong and useful tool. The solution methods can range from the simple organization of ideas to the use of school curricula contents such as equations, geometry or combinatory.

In the next pages, several math riddles will be presented as well as their adaptation to the theme of football.

The barrels – The sale of players

Henry Dudeney is one of the most classical authors in the creation of math riddles. One of the many problems with barrels that he created is shown below.


76.—THE BARREL OF BEER.

A MAN bought an odd lot of wine in barrels and one barrel containing beer. These are shown in the illustration, marked with the number of gallons that each barrel contained. He sold a quantity of the wine to one man and twice the quantity to another, but kept the beer to himself. The puzzle is to point out which barrel contains beer. Can you say which one it is? Of course, the man sold the barrels just as he bought them, without manipulating in any way the contents.

Figure 3: Dudeney's original problem (1917).

Today, the math content of this problem remains very interesting, because it is possible to solve it with modern school content: the criteria of divisibility by 3. In our book, we transformed this problem into a problem of selling football players:

A club has placed six players on sale for the values listed in the table below.

Player	*Price (million euros)*
Antonic	*15*
Brunic	*31*
Carlovic	*19*
Dinic	*20*
Eugenic	*16*
Fabic	*18*

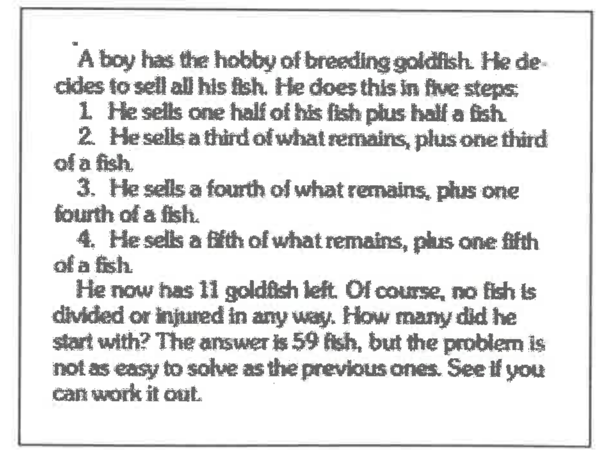
However, this team only managed to sell five of these players to two different clubs. One of the clubs paid twice more money than the other.

Which player was not sold by the club?

The original problem used a theme (barrels) that was common in its time but that is now outdated. However, the math contents are still interesting, and we could always give it "new clothing".

The goldfishes – The scorers

Another well-known author in recreational mathematics is Martin Gardner. One example of his countless math riddles is the following.


A boy has the hobby of breeding goldfish. He decides to sell all his fish. He does this in five steps:

1. He sells one half of his fish plus half a fish.
2. He sells a third of what remains, plus one third of a fish.
3. He sells a fourth of what remains, plus one fourth of a fish.
4. He sells a fifth of what remains, plus one fifth of a fish.

He now has 11 goldfish left. Of course, no fish is divided or injured in any way. How many did he start with? The answer is 59 fish, but the problem is not as easy to solve as the previous ones. See if you can work it out.

Figure 4: Gardner's original problem (1978).

The *wow* factor of this problem is how could we have a fraction (a half, a third, ...) of a fish without injuring any of them. The mathematical content of this problem is still interesting today, but there are not many current young students who have the hobby of breeding fish... So, this is how we transformed a Gardner problem about goldfishes into a problem of goals:

At the end of a championship, the goals scored by a football team were distributed as follows:

- *Archimedes scored half of the team's goals, plus* half a goal!
- *Euclid scored a third of the remaining goals, plus a* third of a goal!
- *Thales scored a fourth of the remaining goals, plus a* fourth of a goal!
- *Descartes scored a fifth of the remaining goals, plus a* fifth of a goal!
- *Eleven other players scored the team's remaining goals (one goal each).*

How many goals has this football team scored?

How many goals has each one of the top four players scored?

Notice that the mathematical essence of the problem is the same and the *wow* factor remained: how could we have a half or a third of a goal? We modernized this problem in order to be more appealing to students using a theme that is well known by the majority of the Portuguese population: football.

The handshake problem - The tournament

Another classic problem in math is the *"Handshake problem"* that has several versions, but it is generally presented like this: how many different handshakes are possible in a room with n persons? Students study this kind of problem in Combinatory (in the context of Probabilities) and we decided to create a new version of this problem using the theme of football:

At the end of a children's tournament – where each team has played with all of the other teams just once –- 66 football matches were played in total.

How many teams played in this tournament?

The 3rd club

As championships get near to the end, calculations go to the top of football analysis. Many times, supporters have to calculate the possibilities of their favourite team being a champion (or relegated) depending on the results of several games of their opponents. For instance, *"My team will be the champion if we win every last match and our rival draws, at least, one of*

Figure 5: Young footballers.

his games until the end" – this is the type of reasoning that every fan is used to doing almost every year...

In the book, we present a riddle similar to this kind of problem:

The championship table before the last day of games is presented below (every team will have one more match to play: 3 points for a victory and 1 point for a draw).

Club	*Points*
1. Clubenense	*60*
2. Vitorense	*59*
3. Sportense	*57*
...	...

In the newspaper, the following is stated: "It's a guarantee that Sportense will end the championship with fewer points than the future champion". *What will happen in the last day of games so that the newspaper can make such a statement?*

CAN 2000

We present another riddle where the final table of a group in an international competition is shown, and the reader needs to find out the results of every match played within this group.

The 2000 African Cup of Nations of Football (known by the French acronym CAN: Coupe d'Afrique des Nations*) was a joint organization between Ghana* *and Nigeria. The classification in group A was quite peculiar and can be seen in the following table (GF – goals for; GA – goals against).*

Group A	*GF*	*GA*	*Points*
1. Cameroon	*4*	*2*	*4*
2. Ghana	*3*	*3*	*4*
3. Ivory Coast	*3*	*4*	*4*
4. Togo	*2*	*3*	*4*

What is the result of the six games played in this group knowing that there have been goals scored in all games and that the first two teams have a draw with each other? (remember that every team played against all the others only once)

The tennis tournament - The Portuguese Cup

Paenza's riddle on a tennis tournament was the inspiration for our riddle on the Portuguese Cup. The idea is finding the total number of matches to be played in a knockout competition until finding the winner while considering a fixed number of teams. In the official Portuguese Cup regulations, the following is stated:

> *«The Portuguese Cup is played by elimination matches, and the loser of each game is eliminated, until the two finalists are found, in obedience of the following rules:*
>
> *a) The rounds are played in a single match [...];*
>
> *b) The semi-finals are played in two legs;*
>
> *...*
>
> *e) The winner in each round qualifies for the next one [...]»*

Knowing that in 2013-14 the Portuguese Cup was played by 156 clubs, how many matches were played?

In these kinds of knockout competitions, a way of approaching the problem is to think backwards, starting with the final game:

- final (2 clubs): 1 ($= 2^0$) match;
- semi-finals (4 clubs): 2 ($= 2^1$) matches;
- quarter-finals (8 clubs): 4 ($= 2^2$) matches;
- round of 16: 8 ($= 2^3$) matches;

- round of 32: 16 ($= 2^4$) matches;
- round of 64 : 32 ($= 2^5$) matches;
- round of 128: 64 ($= 2^6$) matches;
- round of 256: 128 ($= 2^7$) matches;
- ...

In the end, we just have to add up the number of matches in each round. In Paenza's riddle there are 128 participants, so the answer is: $64+32+16+8+4+2+1 = 127$ matches. But frequently, this kind of tournament does not start with a number of teams that is a power of 2 – that is the situation in our Portuguese Cup with 156 clubs. A usual solution is to observe the power of 2 just before – in this case, 128 ($= 2^7$) – and organize a preliminary round with the number of matches corresponding to the difference: $156128 =$ 28 matches played by 56 clubs (usually with the less valued ones in the competition). After that, the rounds go nicely with the power of 2 matches until the final. In our riddle, we cannot forget that the semi-finals are played in two legs, and thus, it would be like this: $28 + 64 + 32 + 16 + 8 + 4 + (2 + 2) + 1 = 157$ matches.

In reality, the 2013-14 Portuguese Cup went like this:

Round	Clubs	Matches
I	123 clubs from 3rd and 4th levels	44 matches (35 free teams)
II	79 clubs from round I + 17 clubs from 2nd level League	48 matches
III	48 clubs from round II + 16 clubs from 1st level League	32
IV	32 clubs from round III	16
V	16 clubs from round IV	8
VI (quarter)	8 clubs from round V	4
VII (semi)	4 clubs from round VI	2+2 (two legs)
VIII (Final)	2 clubs from round VII	1
		Total: 157

By now, we have two different ways to get to the answer of 157 matches: the real one (that we need to dig for in a football archive) and a mathematical one (that anyone can try... by organizing the counting). Nevertheless, Mathematics always seeks an elegant and simple solution. Here a sharp analysis on the similarity of numbers should push us to try a different approach...

In these kinds of knockout competitions, each team loses one and only one game, with an exception: the winner. So, in Paenza's 128 participants tournament, we must have 127 matches. And back to our 156 teams Cup, we should have 155 but, as the semi-finals has two legs, we have two extra matches and arrive at the total of 157 matches.

This is a good example of a riddle based on real data, known by any football fan, which allows a kind of exhaustive approach that can be attempted by several readers. Nevertheless, it can also be solved with a completely different idea, that turns out to be simpler and synthetic. The straightforwardness of math reasoning does not always rely on calculations. Actually it should start with a good organization of ideas, establishing relations and looking for patterns that allow us to elaborate a solution strategy, preferentially with little calculation effort.

The four 4's

A very amusing riddle that kept us entertained as university students is the four 4's riddle: how to write natural numbers through numeric expressions using as digits just four 4's and any combination of symbols. One of Tahan's riddles in his *"The Man Who Counted"* reminded us of that problem and the 2004 Euro – the football championship hosted in Portugal – inspired us for the adaptation presented in the book:

How to use the four 4's to write the number of each of the 23 players of the Portuguese squad?

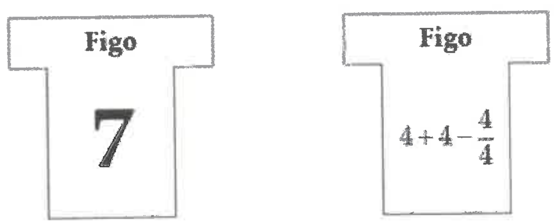


Figure 6: The traditional and the "new jersey" of Figo in the 2004 Euro.

It is an easily exciting riddle, because each number uncovered motivates the reader (and frequently brings new ideas/strategies) to find the others. We have already shown this problem in several presentations, to adults and children, in classrooms and in public spaces, and it is usually very well received. The football players' names bring memories to the elders and some curiosity to the youngsters. On the one hand, the calculations use mainly

basic operations and are accessible to almost everyone; on the other hand, some complex calculations are a challenge to the persistent ones. Another interesting point is the fact that there can be several correct answers to the same number. And, at the end of the process, the most distracted reader is practicing mental calculation and strategies, while having some fun.

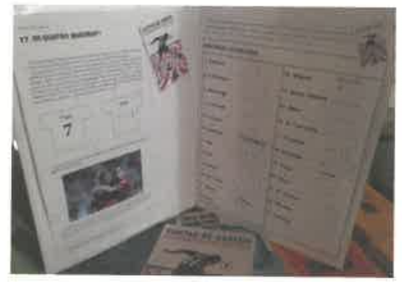


Figure 7: The four 4's riddle in a public presentation.

After a while, we noticed that the fictional situation we created with the Portuguese players was already, somehow, a reality in the Romanian football team:

Some final considerations:

- There are many interesting problems/puzzles that are true classics that can be used nowadays because the math content is still attractive;
- Many of those problems could be adapted to new updated contexts with a little imagination and work, such as, for example, sports, technology, social networks, transportation and everyday life;
- It is important to choose themes that are truly meaningful and interesting to the target audience (usually children and adolescents);
- In this book the theme of football was chosen; others could prefer other themes; the important thing is choosing "whatever works" (with more tools available, there is a greater chance of success in attracting people to math).

The book was presented to the public in a special event at the City of Football (the FPF headquarters), on the 14th of March – Pi day. The SPM and the UNESCO (Portuguese section) joined this event, highlighting the importance of Mathematics and Education. A group of high-school students was invited to participate in the event. Two futsal players, who had recently won the European championship, shared their life journey with the students emphasising the importance of academic background (both are

Romania's soccer team puts math problems instead of player numbers on jerseys

"The Romanian men's soccer team recently took on a pair of noteworthy challenges. While it prepared for a friendly Sunday with powerhouse Spain, the squad also tried to help kids back home get excited about math. [...] Instead of putting the usual player numbers on the backs of its training jerseys, Romania substituted math problems, the solutions to which revealed those numbers. [...] Romania has one of the highest dropout rates, nearly 20 percent, in the European Union.

"Football and mathematics are not mutually exclusive," said the Romanian Football Federation president, Razvan Burleanu (via The Guardian). "We must look at sports and education as not only complementary but fundamental elements integrated in the training and perfection of children. We want to have healthy generation and smart students who achieve performance and tools through tailored passions." [...] The training jerseys are part of a program that aims to use soccer to make math more appealing to Romanian youngsters."

Figure 8: Whashington Post, March 27, 2016.

graduates in sports). The presence of the Minister of Education brought media interest to the presentation.

We believe that the outcome of this project is quite positive for all partners. Our project materialized and gained visibility in the media because of its connection with football. FPF and SPM started a partnership that we hope will be very beneficial. Further, we hope that the readers embrace these challenges with the enthusiasm of Football and the perseverance of Mathematics.

Acknowledgments

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Figure 9: Speech of the Portuguese Minister of Education at the event (FPF, March 14, 2018): https://youtu.be/WE0r-zhMUFY

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