CARTESIAN CLOSED EXACT COMPLETIONS IN TOPOLOGY

MARIA MANUEL CLEMENTINO, DIRK HOFMANN, AND WILLIAN RIBEIRO

Abstract. Using generalized enriched categories, in this paper we show that Rosický's proof of cartesian closedness of the exact completion of the category of topological spaces can be extended to a wide range of topological categories over **Set**, like metric spaces, approach spaces, ultrametric spaces, probabilistic metric spaces, and bitopological spaces. In order to do so we prove a sufficient criterion for exponentiability of $(\mathbb{T},V)$-categories and show that, under suitable conditions, every injective $(\mathbb{T},V)$-category is exponentiable in $(\mathbb{T},V)$-**Cat**.

1. Introduction

As Lawvere has shown in his celebrated paper [Law73], when V is a closed category the category V-**Cat** of V-enriched categories and V-functors is also monoidal closed. This result extends neither to the cartesian structure nor to the more general setting of $(\mathbb{T},V)$-categories. Indeed, cartesian closedness of V does not guarantee cartesian closedness of V-**Cat**: take for instance the category of (Lawvere's) metric spaces P_+-**Cat**, where P_+ is the complete real half-line, ordered with the $\geq$ relation, and equipped with the monoidal structure given by addition $+$; P_+ is cartesian closed but P_+-**Cat** is not (see [CH06] for details); and, even when the monoidal structure of V is the cartesian one, the category $(\mathbb{T},V)$-**Cat** of $(\mathbb{T},V)$-categories and $(\mathbb{T},V)$-functors (see [CT03]) does not need to be cartesian closed, as it is the case of the category **Top** of topological spaces and continuous maps, that is $(\mathbb{U},2)$-**Cat** for $\mathbb{U}$ the ultrafilter monad.

Rosický showed in [Ros99] that **Top** is weakly cartesian closed, and, consequently, that its exact completion is cartesian closed. Weak cartesian closedness of **Top** follows from the existence of enough injectives in its full subcategory $\mathbf{Top}_0$ of $T0$-spaces and the fact that they are exponentiable, and this feature, together with several good properties of **Top**, gives cartesian closedness of its exact completion. More precisely, Rosický has shown in [Ros99] the following theorem.

Theorem 1.1. *Let* $\mathbf{C}$ *be a complete, infinitely extensive and well-powered category with (reg epi, mono)-factorizations such that* $f\times 1$ *is an epimorphism whenever* f *is a regular epimorphism. Then the exact completion of* $\mathbf{C}$ *is cartesian closed provided that* $\mathbf{C}$ *is weakly cartesian closed.*

In this paper we use the setting of $(\mathbb{T},V)$-categories, for a quantale V and a **Set**-monad $\mathbb{T}$ laxly extended to V-**Rel** to conclude, in a unified way, that several topological categories over **Set** share with **Top** the cartesian closedness of the exact completion. This was recently used by Adámek and Rosický in the study of free completions of categories [AR18]. In fact, the category $(\mathbb{T},V)$-**Cat** is topological over **Set** [CH03, CT03], hence complete and with (reg epi, mono)-factorizations such that $f\times 1$ is an epimorphism whenever f is, and it is infinitely extensive [MST06]. To assure weak

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cartesian closedness of $(\mathbb{T},V)$-**Cat** we consider two distinct scenarios, either restricting to the case that V is a frame – so that its monoidal structure is the cartesian one – or considering the case that the lax extension is determined by a $\mathbb{T}$-algebraic structure on V, as introduced in [Hof07] under the name of topological theory. In the latter case the proof generalizes Rosický's proof for $\mathbf{Top}_0$, after observing that, using the Yoneda embedding of [CH09, Hof11], every separated $(\mathbb{T},V)$-category can be embedded in an injective one, and, moreover, these are exponentiable in $(\mathbb{T},V)$-**Cat**. For general $(\mathbb{T},V)$-categories one proceeds again as in [Ros99], using the fact that the reflection of $(\mathbb{T},V)$-**Cat** into its full subcategory of separated $(\mathbb{T},V)$-categories preserves finite products. As observed by Rosický, the exact completion of **Top** relates to the cartesian closed category of equilogical spaces [BBS04]. Analogously, our approach leads to the study of generalized equilogical spaces, as developed in [Rib18].

The paper is organized as follows. In Section 2 we introduce $(\mathbb{T},V)$-categories and list their properties used throughout the paper. In Section 3 we revisit the exponentiability problem in $(\mathbb{T},V)$-**Cat**, establishing a sufficient criterion for exponentiability which generalizes the results obtained in [Hof07, HS15]. In Section 4 we study the properties of injective $(\mathbb{T},V)$-categories which will be used in the forthcoming section to conclude that, under suitable assumptions, injective $(\mathbb{T},V)$-categories are exponentiable (Theorem 5.8). This result will allow us to conclude, in Theorem 6.3, that $(\mathbb{T},V)$-**Cat** is weakly cartesian closed, and, finally, thanks to Theorem 1.1, that the exact completion of $(\mathbb{T},V)$-**Cat** is cartesian closed. We conclude our paper with a section on examples, which include, among others, metric spaces, approach spaces, probabilistic metric spaces, and bitopological spaces.

2. The category of $(\mathbb{T},V)$-categories

Throughout *V is a commutative and unital quantale*, i.e V is a complete lattice with a symmetric and associative tensor product $\otimes$, with unit k and right adjoint hom, so that $u\otimes v\leq w$ if, and only if, $v\leq \hom(u,w)$, for all $u,v,w\in V$. Further assume that *V is a Heyting algebra*, so that $u\wedge -$ also has a right adjoint, for every $u\in V$. We denote by V-**Rel** the 2-category of *V-relations* (or V-matrices), having as objects sets, as 1-cells V-relations $r:X\,{\to\!\!\!\!\!\!\mapsto}\,Y$, i.e. maps $r:X\times Y\to V$, and 2-cells $\varphi:r\to r'$ given by componentwise order $r(x,y)\leq r'(x,y)$. Composition of 1-cells is given by relational composition. V-**Rel** has an involution, given by transposition: the transpose of $r:X\,{\to\!\!\!\!\!\!\mapsto}\,Y$ is $r^\circ:Y\,{\to\!\!\!\!\!\!\mapsto}\,X$ with $r^\circ(y,x)=r(x,y)$.

We fix a *non-trivial monad* $\mathbb{T}=(T,m,e)$ *on* **Set** *satisfying (BC)*, i.e. T preserves weak pullbacks and the naturality squares of the natural transformation m are weak pullbacks (see [CHJ14]). In general we do not assume that T preserves products. Later we will make use of the comparison map $\mathrm{can}_{X,Y}:T(X\times Y)\to TX\times TY$ defined by $\mathrm{can}_{X,Y}(\mathfrak{w})=(T\pi_X(\mathfrak{w}),T\pi_Y(\mathfrak{w}))$ for all $\mathfrak{w}\in T(X\times Y)$, where π_X and π_Y are the product projections. Moreover, we assume that $\mathbb{T}$ *has an extension to* V-**Rel**, which we also denote by $\mathbb{T}$, in the following sense:

- there is a lax functor $T:V\text{-}\mathbf{Rel}\to V\text{-}\mathbf{Rel}$ which extends $T:\mathbf{Set}\to\mathbf{Set}$;
- $T(r^\circ)=(Tr)^\circ$ for all V-relations r;

– the natural transformations $e : 1_{V\text{-}\mathbf{Rel}} \to T$ and $m : T^2 \to T$ become op-lax; that is, for every $r : X \nrightarrow Y$,

$$e_Y \cdot r \leq Tr \cdot e_X, \qquad\qquad m_Y \cdot TTr \leq Tr \cdot m_X.$$

$$\begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ {\scriptstyle r}\downarrow & \leq & \downarrow{\scriptstyle Tr} \\ Y & \xrightarrow[e_Y]{} & TY \end{array} \qquad\qquad \begin{array}{ccc} TTX & \xrightarrow{m_X} & TX \\ {\scriptstyle TTr}\downarrow & \leq & \downarrow{\scriptstyle Tr} \\ TTY & \xrightarrow[m_Y]{} & TY \end{array}$$

We note that our conditions are stronger than those used in [HST14].

A $(\mathbb{T}, V)$-*category* is a pair (X, a) where X is a set and $a : TX \nrightarrow X$ is a V-relation such that

$$\begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ & {\scriptstyle 1_X}\searrow \;\leq & \downarrow{\scriptstyle a} \\ & & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} T^2X & \xrightarrow{m_X} & TX \\ {\scriptstyle Ta}\downarrow & \leq & \downarrow{\scriptstyle a} \\ TX & \xrightarrow[a]{} & X \end{array}$$

that is, the map $a : TX \times X \to V$ satisfies the conditions:

(R) for each $x \in X$, $k \leq a(e_X(x), x)$;

(T) for each $\mathfrak{X} \in T^2X, \mathfrak{x} \in TX, x \in X$, $Ta(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \leq a(m_X(\mathfrak{X}), x)$.

Given $(\mathbb{T}, V)$-categories (X, a), (Y, b), a $(\mathbb{T}, V)$-*functor* $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ such that

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle a}\downarrow & \leq & \downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array}$$

that is, for each $\mathfrak{x} \in TX$ and $x \in X$, $a(\mathfrak{x}, x) \leq b(Tf(\mathfrak{x}), f(x))$; f is said to be *fully faithful* when this inequality is an equality.

$(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-functors form the category $(\mathbb{T}, V)$-**Cat**. If $(X, a : TX \nrightarrow X)$ satisfies (R) (and not necessarily (T)), we call it a $(\mathbb{T}, V)$-*graph*. The category $(\mathbb{T}, V)$-**Gph**, of $(\mathbb{T}, V)$-graphs and $(\mathbb{T}, V)$-functors, contains $(\mathbb{T}, V)$-**Cat** as a full reflective subcategory.

We present the examples in detail in the last section. We mention here, however, that the leading examples are obtained when one considers the quantale $2 = (\{0, 1\}, \leq, \&, 1)$ and Lawvere's real half-line $P_+ = ([0, \infty], \geq, +, 0)$, the identity monad $\mathbb{I}$ and the ultrafilter monad $\mathbb{U}$ on **Set**. Thus we obtain the following examples:

– $(\mathbb{I}, V)$-**Cat** is the category of V-categories and V-functors; in particular, $(\mathbb{I}, 2)$-**Cat** is the category **Ord** of (pre)ordered sets and monotone maps, while $(\mathbb{I}, P_+)$-**Cat** is the category **Met** of Lawvere's metric spaces and non-expansive maps (see [Law73]).

– $(\mathbb{U}, 2)$-**Cat** is the category **Top** of topological spaces and continuous maps.

– $(\mathbb{U}, P_+)$-**Cat** is the category **App** of Lowen's approach spaces and non-expansive maps (see [Low97]).

We recall (see [AHS90, Definition 21.1]) that a functor $G : \mathbf{A} \to \mathbf{B}$ is said to be *topological* if every source $(f_i : B \to GA_i)_{i \in I}$ in **B** has a unique G-initial lift $(\overline{f_i} : A \to A_i)_{i \in I}$. The following was proved in [CH03] (see also [CT03]).

Theorem 2.1. *The forgetful functors* $(\mathbb{T}, V)$-**Cat** $\to$ **Set** *and* $(\mathbb{T}, V)$-**Gph** $\to$ **Set** *are topological.*

This shows, in particular, that (see [AHS90, Chapter 21] for details):

– $(\mathbb{T}, V)$-**Cat** is complete and cocomplete.

– Monomorphisms in $(\mathbb{T},V)$-**Cat** are the morphisms whose underlying map is injective; therefore, since the $(\mathbb{T},V)$-structures on any set form a set, $(\mathbb{T},V)$-**Cat** is well-powered.
– Every topological category over **Set** has two factorization systems, (reg epi, mono) and (epi, reg mono); in $(\mathbb{T},V)$-**Cat** the former one is in general not stable (that is, regular epimorphisms need not be stable under pullback – **Top** is such an example), but the latter one is. Indeed, epimorphisms in $(\mathbb{T},V)$-**Cat** are the $(\mathbb{T},V)$-functors which are surjective as maps, the forgetful functor $(\mathbb{T},V)$-**Cat** $\to$ **Set** preserves pullbacks, and surjective maps are stable under pullback in **Set**. Therefore, as $f\times 1_Z$ is the pullback of $f:X\to Y$ along $\pi_Y:Y\times Z\to Y$, we conclude that $f\times 1_Z$ is an epimorphism provided f is.

$(\mathbb{T},V)$-**Cat** has a natural structure of 2-category: for $(\mathbb{T},V)$-functors $f,g:(X,a)\to(Y,b)$, $f\le g$ if $g\cdot a\le b\cdot Tf$. This condition can be equivalently written as $k\le b(e_Y(f(x)),g(x))$ for every $x\in X$ (see [CT03] for details). We write $f\simeq g$ if $f\le g$ and $g\le f$.

Extensivity of $(\mathbb{T},V)$-**Cat** was studied in [MST06]:

Theorem 2.2. $(\mathbb{T},V)$-**Cat** *is infinitely extensive.*

In general $(\mathbb{T},V)$-**Cat** is not cartesian closed, while $(\mathbb{T},V)$-**Gph** is. In fact, the following was proved in [CHT03]:

Theorem 2.3. $(\mathbb{T},V)$-**Gph** *is a quasi-topos.*

We also note that the tensor product of V induces a canonical structure c on $X\times Y$ defined by

$$c(\mathfrak{w},(x,y))=a(T\pi_X(\mathfrak{w}),x)\otimes b(T\pi_Y(\mathfrak{w}),y),$$

where $\mathfrak{w}\in T(X\times Y)$, $x\in X$, $y\in Y$. We put

$$(X,a)\otimes(Y,b)=(X\times Y,c),$$

and this construction is in an obvious way part of a functor $\otimes:(\mathbb{T},V)$-**Gph** $\times(\mathbb{T},V)$-**Gph** $\to(\mathbb{T},V)$-**Gph**. However, the tensor product of two $(\mathbb{T},V)$-categories is in general not a $(\mathbb{T},V)$-category (see [Hof07, Lemma 6.1]).

Weak cartesian closedness of $(\mathbb{T},V)$-**Cat** needs a thorough study of injective $(\mathbb{T},V)$-categories and some extra conditions. This is the subject of the following sections.

3. Exponentiable $(\mathbb{T},V)$-categories

Recall that an object C of a category $\mathbf{C}$ with finite products is *exponentiable* whenever the functor $C\times-:\mathbf{C}\to\mathbf{C}$ has a right adjoint. The category $\mathbf{C}$ is *cartesian closed* if every object C of $\mathbf{C}$ is exponentiable. Equivalently, if for each pair of objects A,B of $\mathbf{C}$ there exists an object $\langle A,B\rangle$ and a morphism $\mathrm{ev}:\langle A,B\rangle\times A\to B$ such that, for each morphism $f:C\times A\to B$ there exists a unique morphism $\overline{f}:C\to\langle A,B\rangle$ with $\mathrm{ev}\cdot(\overline{f}\times 1_A)=f$. Dropping uniqueness of $\overline{f}$ gives the notion of *weakly cartesian closed category.*

In this section we present a sufficient condition for a $(\mathbb{T},V)$-category X to be exponentiable in $(\mathbb{T},V)$-**Cat**, which generalises [Hof06, Theorem 4.3] and [Hof07, Theorem 6.5]. To start, we recall that $(\mathbb{T},V)$-**Cat** can be fully embedded into the cartesian closed category $(\mathbb{T},V)$-**Gph**. Here, for $(\mathbb{T},V)$-graphs (X,a) and (Y,b), the exponential $\langle(X,a),(Y,b)\rangle$ has as underlying set

$$Z:=\{h:(X,a)\times(1,e_1^\circ)\to(Y,b)\mid h\text{ is a }(\mathbb{T},V)\text{-functor}\},$$

which becomes a $(\mathbb{T},V)$-graph when equipped with the largest structure b^a making the evaluation map

$$\mathrm{ev}:Z\times X\to Y,\ (h,x)\mapsto h(x)$$

a $(\mathbb{T}, V)$-functor: for $\mathfrak{p} \in TZ$ and $h \in Z$, put

$$b^a(\mathfrak{p}, h) = \bigvee \{ v \in V \mid \forall \mathfrak{q} \in (T\pi_Z)^{-1}(\mathfrak{p}), x \in X \;.\; a(T\pi_X(\mathfrak{q}), x) \wedge v \leq b(T\mathrm{ev}(\mathfrak{q}), h(x)) \},$$

where π_X and π_Z are the product projections. Note that the supremum above is even a maximum since $- \wedge -$ distributes over suprema.

Given V-relations $r : X \nrightarrow X'$ and $s : Y \nrightarrow Y'$, we define in V-**Rel** $r \owedge s : X \times Y \nrightarrow X' \times Y'$ by $(r \owedge s)((x, y), (x', y')) = r(x, x') \wedge s(y, y')$. That is, $r \owedge s = (\pi_{X'}^\circ \cdot r \cdot \pi_X) \wedge (\pi_{Y'}^\circ \cdot s \cdot \pi_Y)$ in the ordered set V-**Rel**$(X \times Y, X' \times Y')$.

Theorem 3.1. *Assume that the diagram*

$$\begin{array}{ccc} T(X \times Y) & \xrightarrow{\mathrm{can}_{X,Y}} & TX \times TY \\ {\scriptstyle T(r \owedge s)} \big\downarrow & & \big\downarrow {\scriptstyle (Tr) \owedge (Ts)} \\ T(X' \times Y') & \xrightarrow[\mathrm{can}_{X',Y'}]{} & TX' \times TY' \end{array} \tag{3.i}$$

commutes, for all V-relations $r : X \nrightarrow X'$ and $s : Y \nrightarrow Y'$. Then a $(\mathbb{T}, V)$-category (X, a) is exponentiable provided that

$$\bigvee_{\mathfrak{x} \in TX} (Ta(\mathfrak{X}, \mathfrak{x}) \wedge u) \otimes (a(\mathfrak{x}, x) \wedge v) \geq a(m_X(\mathfrak{X}), x) \wedge (u \otimes v), \tag{3.ii}$$

for all $\mathfrak{X} \in TTX$, $x \in X$ and $u, v \in V$.

Proof. We show that the $(\mathbb{T}, V)$-graph structure b^a on Z is transitive, for each $(\mathbb{T}, V)$-category (Y, b). To this end, let $\mathfrak{P} \in TTZ$, $\mathfrak{p} \in TZ$, $h \in Z$, $x \in X$ and $\mathfrak{w} \in T(Z \times X)$ with $T\pi_Z(\mathfrak{w}) = m_Z(\mathfrak{P})$. We have to show that

$$(T(b^a)(\mathfrak{P}, \mathfrak{p}) \otimes b^a(\mathfrak{p}, h)) \wedge a(T\pi_X(\mathfrak{w}), x) \leq b(T\,\mathrm{ev}(\mathfrak{w}), h(x)).$$

Since m has (BC), there is some $\mathfrak{Q} \in TT(Z \times X)$ with $TT\pi_Z(\mathfrak{Q}) = \mathfrak{P}$ and $m_{Z \times X}(\mathfrak{Q}) = \mathfrak{w}$. Hence, $m_X(TT\pi_X(\mathfrak{Q})) = T\pi_X(\mathfrak{w})$, and we calculate:

$$\begin{aligned}
&(T(b^a)(\mathfrak{P}, \mathfrak{p}) \otimes b^a(\mathfrak{p}, h)) \wedge a(T\pi_X(\mathfrak{w}), x) \\
&\quad \leq \bigvee_{\mathfrak{x} \in TX} ((T(b^a)(TT\pi_Z(\mathfrak{Q}), \mathfrak{p}) \wedge Ta(TT\pi_X(\mathfrak{Q}), \mathfrak{x})) \otimes (b^a(\mathfrak{p}, h) \wedge a(\mathfrak{x}, x)) && \text{(by (3.ii))} \\
&\quad \leq \bigvee_{\mathfrak{x} \in TX} \; \bigvee_{\mathfrak{q} \in \mathrm{can}^{-1}(\mathfrak{p}, \mathfrak{x})} T(b^a \owedge a)(T\,\mathrm{can}_{Z,X}(\mathfrak{Q}), \mathfrak{q}) \otimes (b^a \owedge a)(\mathrm{can}_{Z,X}(\mathfrak{q}), (h, x)) && \text{(using (3.i))} \\
&\quad = \bigvee_{\mathfrak{q} \in (T\pi_Z)^{-1}(\mathfrak{p})} T(b^a \owedge a)(T\,\mathrm{can}_{Z,X}(\mathfrak{Q}), \mathfrak{q}) \otimes (b^a \owedge a)(\mathrm{can}_{Z,X}(\mathfrak{q}), (h, x)) \\
&\quad = \bigvee_{\mathfrak{q} \in (T\pi_Z)^{-1}(\mathfrak{p})} T(b^a \times a)(\mathfrak{Q}, \mathfrak{q}) \otimes (b^a \times a)(\mathfrak{q}, (h, x)) \\
&\quad \leq \bigvee_{\mathfrak{q} \in (T\pi_Z)^{-1}(\mathfrak{p})} Tb(TT\,\mathrm{ev}(\mathfrak{Q}), T\,\mathrm{ev}(\mathfrak{q})) \otimes b(T\,\mathrm{ev}(\mathfrak{q}), h(x)) \\
&\quad \leq b(m_Y \cdot TT\,\mathrm{ev}(\mathfrak{Q}), h(x)) = b(T\,\mathrm{ev}(\mathfrak{w}), h(x)).
\end{aligned}$$

□

Remark 3.2. We note that the inequality $\mathrm{can}_{X',Y'} \cdot T(r \owedge s) \leq ((Tr) \owedge (Ts)) \cdot \mathrm{can}_{X,Y}$ is automatically true. Firstly, this inequality is equivalent to $T(r \owedge s) \leq \mathrm{can}^\circ_{X',Y'} \cdot ((Tr) \owedge (Ts)) \cdot \mathrm{can}_{X,Y}$; secondly,

$$\begin{aligned} T(r \owedge s) &= T((\pi^\circ_{X'} \cdot r \cdot \pi_X) \wedge (\pi^\circ_{Y'} \cdot s \cdot \pi_Y)) \\ &\leq T(\pi^\circ_{X'} \cdot r \cdot \pi_X) \wedge T(\pi^\circ_{Y'} \cdot s \cdot \pi_Y) \\ &\leq \mathrm{can}^\circ_{X',Y'} \cdot ((Tr) \owedge (Ts)) \cdot \mathrm{can}_{X,Y}\,. \end{aligned}$$

It is worthwhile to notice that, when V is a frame, that is $\otimes = \wedge$, the condition above is equivalent to

$$\bigvee_{\mathfrak{x} \in TX} Ta(\mathfrak{X}, \mathfrak{x}) \wedge a(\mathfrak{x}, x) \geq a(m_X(\mathfrak{X}), x),$$

for all $\mathfrak{X} \in TTX$ and $x \in X$. Therefore:

Corollary 3.3. *When V is a frame and* (3.i) *commutes for all V-relations $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow X'$ and $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Y'$, a $(\mathbb{T}, V)$-category (X, a) is exponentiable provided that*

$$a \cdot m_X = a \cdot Ta.$$

4. Injective and representable $(\mathbb{T}, V)$-categories

In this section we recall an important class of $(\mathbb{T}, V)$-categories, the so-called *representable* ones. More information on this type of $(\mathbb{T}, V)$-categories can be found in [CCH15, HST14]. We also recall from [CH09, Hof07, Hof11] that every injective $(\mathbb{T}, V)$-category is representable.

Based on the lax extension of the **Set**-monad $\mathbb{T} = (T, m, e)$ to V-**Rel**, $\mathbb{T}$ admits a natural extension to a monad on V-**Cat**, in the sequel also denoted by $\mathbb{T} = (T, m, e)$ (see [Tho09]). Here the functor $T : V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$ sends a V-category (X, a_0) to (TX, Ta_0), and $e_X : X \to TX$ and $m_X : TTX \to TX$ become V-functors for each V-category X. The Eilenberg–Moore algebras for this monad can be described as triples (X, a_0, α) where (X, a_0) is a V-category and (X, α) is an algebra for the **Set**-monad $\mathbb{T}$ such that $\alpha : T(X, a_0) \to (X, a_0)$ is a V-functor. For $\mathbb{T}$-algebras (X, a_0, α) and (Y, b_0, β), a map $f : X \to Y$ is a homomorphism $f : (X, a_0, \alpha) \to (Y, b_0, \beta)$ precisely if f preserves both structures, that is, whenever $f : (X, a_0) \to (Y, b_0)$ is a V-functor and $f : (X, \alpha) \to (Y, \beta)$ is a $\mathbb{T}$-homomorphism.

There are canonical adjoint functors

$$(V\text{-}\mathbf{Cat})^{\mathbb{T}} \underset{M}{\overset{K}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} (\mathbb{T}, V)\text{-}\mathbf{Cat}.$$

The functor K associates to each $X = (X, a_0, \alpha)$ in $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ the $(\mathbb{T}, V)$-category $KX = (X, a)$, where $a = a_0 \cdot \alpha$, and keeps morphisms unchanged. Its left adjoint $M : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to (V\text{-}\mathbf{Cat})^{\mathbb{T}}$ sends a $(\mathbb{T}, V)$-category (X, a) to $(TX, Ta \cdot m^\circ_X, m_X)$ and a $(\mathbb{T}, V)$-functor f to Tf. Via the adjunction $M \dashv K$ one obtains a lifting of the **Set**-monad $\mathbb{T} = (T, m, e)$ to a monad on $(\mathbb{T}, V)$-**Cat**, also denoted by $\mathbb{T} = (T, m, e)$.

In this setting we can define 'duals' in $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ and carry them into $(\mathbb{T}, V)$-**Cat**. Indeed, since $T : V\text{-}\mathbf{Rel} \to V\text{-}\mathbf{Rel}$ commutes with the involution $(-)^\circ$: for every $\mathbb{T}$-algebra $X = (X, a_0, \alpha)$ also (X, a°_0, α) is a $\mathbb{T}$-algebra. Moreover, if (X, a) is a $(\mathbb{T}, V)$-category, we define X^{op} by mapping (X, a) into $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ via M, dualizing the image in $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$, and then carrying it back to $(\mathbb{T}, V)$-**Cat**; that is,

$$X^{\mathrm{op}} = K((M(X, a))^{\mathrm{op}}) = (TX, m_X \cdot (Ta)^\circ \cdot m_X).$$

Since the monad $\mathbb{T} = (T, m, e)$ on $(\mathbb{T}, V)$-**Cat** is lax idempotent (i.e, of Kock-Zöberlein type), an algebra structure $\alpha : TX \to X$ on a $(\mathbb{T}, V)$-category X is left adjoint to the unit $e_X : X \to TX$.

We call a $(\mathbb{T},V)$-category X *representable* whenever $e_X : X \to TX$ has a left adjoint in $(\mathbb{T},V)$-**Cat**; equivalently, whenever there is some $(\mathbb{T},V)$-functor $\alpha : TX \to X$ with $\alpha \cdot e_X \simeq 1_X$, since then

$$e_X \cdot \alpha = T\alpha \cdot e_{TX} \geq T\alpha \cdot Te_X \simeq 1_{TX}.$$

However, a left adjoint $\alpha : TX \to X$ to e_X is in general only a pseudo-algebra structure on X, that is,

$$\alpha \cdot e_X \simeq 1_X \qquad \text{and} \qquad \alpha \cdot T\alpha \simeq \alpha \cdot m_X.$$

For every representable $(\mathbb{T},V)$-category (X,a), the structure $a : TX \rightarrow\!\!\!\!\!\!\mapsto X$ can be decomposed as $a = a_0 \cdot \alpha$, where $a_0 = a \cdot e_X$ denotes the underlying V-category structure.

A $(\mathbb{T},V)$-category X is *injective* whenever, for each fully faithful $h : A \to B$ in $(\mathbb{T},V)$-**Cat** and each $(\mathbb{T},V)$-functor $f : A \to X$, there is a $(\mathbb{T},V)$-functor $g : B \to X$ with $g \cdot h \simeq f$.

Proposition 4.1. *Every injective $(\mathbb{T},V)$-category is representable.*

Proof. Let X be an injective $(\mathbb{T},V)$-category. The $(\mathbb{T},V)$-functor $e_X : (X,a) \to (TX, Ta \cdot m_X^\circ \cdot m_X)$ is an embedding. Indeed, e_X is injective because the monad T is non-trivial, and it is fully faithful:

$$e_X^\circ \cdot Ta \cdot m_X^\circ \cdot m_X \cdot Te_X \leq a \cdot Ta \cdot m_X^\circ \leq a \cdot m_X \cdot m_X^\circ \leq a.$$

Hence, there is a $(\mathbb{T},V)$-functor $\alpha : TX \to X$ with $\alpha \cdot e_X \simeq 1_X$, and so X is representable. □

5. Injective $(\mathbb{T},V)$-categories are exponentiable

In Section 6 we will show that, under some conditions, $(\mathbb{T},V)$-**Cat** is weakly cartesian closed. Notably, we will use that every $(\mathbb{T},V)$-category can be embedded into an injective one; which, by the main result of this section, implies that every $(\mathbb{T},V)$-category can be embedded into an exponentiable one. We hasten to remark that this is easily seen to be fulfilled for $\mathbb{T}$ being the identity monad, witnessed by the *Yoneda embedding* (see [Law73])

$$y_X : X \to PX := V^{X^{\mathrm{op}}}.$$

Here PX is the free cocompletion of X; being cocomplete, PX is injective.

To treat the general case, we *will consider from now on only* extensions of the monad $\mathbb{T}$ to V-**Rel** given by a $\mathbb{T}$-algebra structure $\xi : TV \to V$ on V, so that we are dealing with a *strict topological theory* in the sense of [Hof07]. In this case, the extension of $T : \mathbf{Set} \to \mathbf{Set}$ to V-**Rel** is defined by

$$Tr : TX \times TY \to V$$
$$(\mathfrak{x},\mathfrak{y}) \mapsto \bigvee \left\{ \xi \cdot Tr(\mathfrak{w}) \;\middle|\; \mathfrak{w} \in T(X \times Y), T\pi_X(\mathfrak{w}) = \mathfrak{x}, T\pi_Y(\mathfrak{w}) = \mathfrak{y} \right\}$$

for each V-relation $r : X \times Y \to V$.

In order to obtain a Yoneda embedding, we consider the $\mathbb{T}$-algebra $(V, \hom, \xi)$ which is mapped by K into the important $(\mathbb{T},V)$-category $(V, \hom_\xi)$, where $\hom_\xi = \hom \cdot \xi$ (see Section 4). The proof of the following result can be found in [CH09] and [Hof11].

Theorem 5.1. *If the extension of $\mathbb{T}$ to V-**Rel** is induced by a strict topological theory, then, for every $(\mathbb{T},V)$-category (X,a), the V-relation $a : TX \rightarrow\!\!\!\!\!\!\mapsto X$ defines a $(\mathbb{T},V)$-functor*

$$a : X^{\mathrm{op}} \otimes X \to (V, \hom_\xi).$$

Moreover, the $\otimes$-exponential mate $y_X = \ulcorner a \urcorner : X \to V^{X^{\mathrm{op}}}$ of a is fully faithful, and the $(\mathbb{T},V)$-category $PX = V^{X^{\mathrm{op}}}$ is injective.

*In fact, this construction defines a functor $P : (\mathbb{T},V)$-**Cat** $\to (\mathbb{T},V)$-**Cat** and $y = (y_X)_X$ is a natural transformation $y : 1_{(\mathbb{T},V)\text{-}\mathbf{Cat}} \to P$.*

Since y_X is fully faithful, when X is injective there exists a $(\mathbb{T},V)$-functor $\mathrm{Sup}_X : PX \to X$ such that $\mathrm{Sup}_X \cdot y_X \simeq 1_X$. As shown in [Hof11, Theorem 2.7], $\mathrm{Sup}_X \dashv y_X$. Moreover, for each $(\mathbb{T},V)$-category (X,a), y_X is one-to-one if, and only if, (X,a) is *separated*, i.e. for every $f,g : (Y,b) \to (X,a)$, $f \simeq g$ only if $f = g$ (see [HT10], for example). It follows immediately that, for an injective $(\mathbb{T},V)$-functor $f : X \to Y$ where Y is separated, also X is.

Lemma 5.2. *The $\otimes$-exponential Y^X is separated, for every separated $(\mathbb{T},V)$-category Y and every representable $(\mathbb{T},V)$-category X; in particular, PX is separated, for every $(\mathbb{T},V)$-category X.*

Proof. See [HT10, Corollary 4.12 (2)]. □

Corollary 5.3. *Every separated $(\mathbb{T},V)$-category embeds into an injective $(\mathbb{T},V)$-category.*

In Section 2 we introduced the tensor product $X \otimes Y$ of $(\mathbb{T},V)$-graphs X and Y. We remark that, in the setting of a strict topological theory, $X \otimes Y$ is a $(\mathbb{T},V)$-category provided that X and Y are so (see [Hof07]).

The result promised in the title of this section was shown in [Hof14, Proposition 2.7] for the special case of $\otimes = \wedge$:

Proposition 5.4. *If the quantale V is a frame and* (3.i) *commutes for all V-relations $r : X \relbar\joinrel\mapsto X'$ and $s : Y \relbar\joinrel\mapsto Y'$, then every representable $(\mathbb{T},V)$-category is exponentiable. In particular, in this case every injective $(\mathbb{T},V)$-category is exponentiable.*

To treat the general case, we will make use of the following conditions:

Assumptions 5.5.
(1) The diagram (3.i) commutes, for all V-relations $r : X \relbar\joinrel\mapsto X'$ and $s : Y \relbar\joinrel\mapsto Y'$.
(2) For all $u,v,w \in V$,
$$w \wedge (u \otimes v) = \bigvee\{u' \otimes v' \mid u' \leq u, v' \leq v, u' \otimes v' \leq w\};$$
or, equivalently, every injective V-category is exponentiable: see [HR13, Theorem 5.3].
(3) For every V-relation $a : X \relbar\joinrel\mapsto Y$ and $u \in V$,
$$T(a \otimes u) = Ta \otimes u,$$
where $a \otimes u$ is the V-relation defined by $(a \otimes u)(x,y) = a(x,y) \otimes u$.
(4) The maps $V \otimes V \xrightarrow{\otimes} V$ and $X \xrightarrow{(-,u)} X \otimes V$ are $(\mathbb{T},V)$-functors, for all $u \in V$.

These morphisms induce an interesting action of V on every injective $(\mathbb{T},V)$-category (X,a) as follows. The $(\mathbb{T},V)$-functor
$$X^{\mathrm{op}} \otimes X \otimes V \xrightarrow{a \otimes 1} V \otimes V \xrightarrow{\otimes} V$$
induces a $(\mathbb{T},V)$-functor $\tilde{a} : X \otimes V \to PX$. We denote the composite
$$X \otimes V \xrightarrow{\tilde{a}} PX \xrightarrow{\mathrm{Sup}_X} X$$
by $\oplus$, and
$$X \xrightarrow{(-,u)} X \otimes V \xrightarrow{\tilde{a}} PX \xrightarrow{\mathrm{Sup}_X} X,$$
assigning to each $x \in X$ an element $x \oplus u$ in X, by $- \oplus u$.

Analogously we will write $\mathfrak{x} \oplus u$ for $T(- \oplus u)(\mathfrak{x})$, for every $\mathfrak{x} \in TX$ and $u \in V$. Note that $(\mathbb{T},V)$-functoriality of $- \oplus u$ can be written as
$$a(\mathfrak{x},y) \leq a(\mathfrak{x} \oplus u, y \oplus u),$$
for every $\mathfrak{x} \in TX$ and $y \in X$.

Lemma 5.6. *Assuming 5.5 (4), for an injective $(\mathbb{T},V)$-category (X,a), with $a=a_0\cdot\alpha$ as usual, the following holds, for every $x,y\in X$, $\mathfrak{x}\in TX$ and $u\in V$:*

(1) $a_0(x\oplus u,y)=\hom(u,a_0(x,y))$;
(2) $a_0(x,y\oplus u)\geq a_0(x,y)\otimes u$;
(3) $a(\mathfrak{x}\oplus u,y)\geq \hom(u,a(\mathfrak{x},y))$;
(4) $a(\mathfrak{x},y\oplus u)\geq a(\mathfrak{x},y)\otimes u$.

Moreover, if, in addition, 5.5 (3) holds, then, for every $\mathfrak{X}\in T^2X$, $\mathfrak{y}\in TX$, $u\in V$,

(5) $Ta(\mathfrak{X},\mathfrak{y}\oplus u)\geq Ta(\mathfrak{X},\mathfrak{y})\otimes u$.

Proof. (1) For every $x,y\in X$ and $u\in V$,

$$\begin{aligned}
a_0(x\oplus u,y)&=a_0(\mathrm{Sup}_X(\tilde{a}(x,u)),y) && \text{(by definition of }\oplus)\\
&=[\tilde{a}(x,u),y_X(y)] && \text{(because }\mathrm{Sup}_X\dashv y_X)\\
&=\bigwedge_{\mathfrak{x}\in TX}\hom(\tilde{a}(x,u)(\mathfrak{x}),y_X(y)(\mathfrak{x})) && \text{(by definition of }[\ ,\ ])\\
&=\bigwedge_{\mathfrak{x}\in TX}\hom(a(\mathfrak{x},x)\otimes u,a(\mathfrak{x},y)) && \text{(by definition of }\tilde{a}\text{ and }y_X(y))\\
&=\hom(u,a_0(x,y)),
\end{aligned}$$

because, using the fact that $a=a_0\cdot\alpha$ and

$$a_0(\alpha(\mathfrak{x}),x)\otimes u\otimes\hom(u,a_0(x,y))\leq a_0(\alpha(\mathfrak{x}),x)\otimes a_0(x,y)\leq a_0(\alpha(\mathfrak{x}),y),$$

for $\mathfrak{x}\in TX$, we can conclude that

$$\hom(u,a_0(x,y))\leq\bigwedge_{\mathfrak{x}\in TX}\hom(a_0(\alpha(\mathfrak{x}),x)\otimes u,a_0(\alpha(\mathfrak{x}),y)).$$

Taking $\mathfrak{x}=e_X(x)$, we see that this inequality is in fact an equality as claimed.

(2) Since, by hypothesis, $-\oplus u$ is a $(\mathbb{T},V)$-functor, and so, in particular, a V-functor $(X,a_0)\to(X,a_0)$,

$$a_0(x,y)\leq a_0(x\oplus u,y\oplus u)=\hom(u,a_0(x,y\oplus u)),$$

and then

$$a_0(x,y)\otimes u\leq\hom(u,a_0(x,y\oplus u))\otimes u\leq a_0(x,y\oplus u).$$

(3) One has

$$\begin{aligned}
k&\leq a_0(\alpha(\mathfrak{x}),\alpha(\mathfrak{x}))=a(\mathfrak{x},\alpha(\mathfrak{x}))\\
&\leq a(\mathfrak{x}\oplus u,\alpha(\mathfrak{x})\oplus u)\\
&= a_0(\alpha(\mathfrak{x}\oplus u),\alpha(\mathfrak{x})\oplus u).
\end{aligned}$$

Using (1) we conclude that

$$\begin{aligned}
\hom(u,a(\mathfrak{x},y))&=a_0(\alpha(\mathfrak{x})\oplus u,y)\\
&\leq a_0(\alpha(\mathfrak{x}\oplus u),\alpha(\mathfrak{x})\oplus u)\otimes a_0(\alpha(\mathfrak{x})\oplus u,y)\\
&\leq a_0(\alpha(\mathfrak{x}\oplus u),y)=a(\mathfrak{x}\oplus u,y).
\end{aligned}$$

(4) follows directly from (2), while (5) follows from (4). □

Lemma 5.7. *Let $\varphi: V\to W$ be a surjective quantale homomorphism; that is, φ preserves the tensor, the neutral element, and suprema. Then, if V satisfies condition 5.5 (2), so does W.*

Theorem 5.8. *Under Assumptions 5.5, every injective $(\mathbb{T},V)$-category is exponentiable in $(\mathbb{T},V)$-**Cat**.*

Proof. Let $\mathfrak{X} \in T^2X$, $x \in X$ and $u, v \in V$. In order to conclude that

$$\bigvee_{\mathfrak{x}\in TX} (Ta(\mathfrak{X},\mathfrak{x}) \wedge u) \otimes (a(\mathfrak{x},x) \wedge v) \geq a(m_X(\mathfrak{X}),x) \wedge (u \otimes v),$$

we make use of Hypothesis 5.5 (2). Let $u', v' \in V$ with $u' \leq u$, $v' \leq v$ and $u' \otimes v' \leq a(m_X(\mathfrak{X}),x)$. First we note that

$$\begin{aligned} Ta(\mathfrak{X}, T\alpha(\mathfrak{X}) \oplus u') \wedge u &\geq (Ta(\mathfrak{X}, T\alpha(\mathfrak{X})) \otimes u') \wedge u && \text{(by 5.6 (5))}\\ &= (Ta_0(T\alpha(\mathfrak{X}), T\alpha(\mathfrak{X})) \otimes u') \wedge u\\ &\geq (k \otimes u') \wedge u = u', \end{aligned}$$

and

$$\begin{aligned} a(T\alpha(\mathfrak{X}) \oplus u', x) &\geq \hom(u', a(T\alpha(\mathfrak{X}), x)) && \text{(by 5.6 (3))}\\ &= \hom(u', a_0(\alpha(T\alpha(\mathfrak{X})), x))\\ &= \hom(u', a_0(\alpha(m_X(\mathfrak{X})), x))\\ &= \hom(u', a(m_X(\mathfrak{X}), x)). \end{aligned}$$

Now, from $u' \otimes v' \leq a(m_X(\mathfrak{X}),x)$ and $v' \leq v$ we get

$$v' \leq \hom(u', a(m_X(\mathfrak{X}), x)) \wedge v \leq a(T\alpha(\mathfrak{X}) \oplus u', x) \wedge v,$$

hence

$$u' \otimes v' \leq (Ta(\mathfrak{X}, T\alpha(\mathfrak{X}) \oplus u') \wedge u) \otimes (a(T\alpha(\mathfrak{X}) \oplus u', x) \wedge v).$$

Therefore $a(m_X(\mathfrak{X}),x) \wedge (u \otimes v) \leq \bigvee_{\mathfrak{x}\in TX} (Ta(\mathfrak{X},\mathfrak{x}) \wedge u) \otimes (a(\mathfrak{x},x) \wedge v)$. □

Remark 5.9. Under Assumptions 5.5, it follows from Lemma 5.2 that the exponential $\langle (X,a),(Y,b)\rangle$ is separated, for all separated injective $(\mathbb{T},V)$-categories (X,a) and (Y,b). In fact, with $a = a_0 \cdot \alpha$, the epimorphism $(X,\alpha) \to (X,a)$ in $(\mathbb{T},V)$-**Cat** is mapped to the monomorphism

$$\langle (X,a),(Y,b)\rangle \longrightarrow \langle (X,\alpha),(Y,b)\rangle = (Y,b)^{(X,\alpha)},$$

which proves that $\langle (X,a),(Y,b)\rangle$ is separated.

6. $(\mathbb{T},V)$-**Cat** is weakly cartesian closed

Building on the results of the previous section, in this section we show that, under some conditions, $(\mathbb{T},V)$-**Cat** is weakly cartesian closed. We start by proving this property for the full subcategory $(\mathbb{T},V)$-$\mathbf{Cat}_{\mathrm{sep}}$ of $(\mathbb{T},V)$-**Cat** of separated $(\mathbb{T},V)$-categories.

Theorem 6.1. *Under Assumptions 5.5,* $(\mathbb{T},V)$-$\mathbf{Cat}_{\mathrm{sep}}$ *is weakly cartesian closed.*

Proof. For X, Y separated $(\mathbb{T},V)$-categories, consider the Yoneda embeddings $y_X : X \to PX$ and $y_Y : Y \to PY$, and the exponential $\langle PX, PY\rangle$. The elements of its underlying set can be identified with $(\mathbb{T},V)$-functors $E \times PX \to PY$ (where $E = (1, e_1^\circ)$ is the generator of $(\mathbb{T},V)$-**Cat**), and the universal morphism $\mathrm{ev} : \langle PX, PY\rangle \times PX \to PY$ with the evaluation map: $\mathrm{ev}(\varphi,\mathfrak{x}) = \varphi(\mathfrak{x})$ (where, for simplicity, we identify the set $E \times PX$ with PX). We can therefore consider

$$\ll X, Y \gg = \{\varphi : E \times PX \to PY \mid \varphi(y_X(X)) \subseteq y_Y(Y)\},$$

with the initial structure with respect to the inclusion $\iota : \ll X, Y \gg \to \langle PX, PY\rangle$. Moreover, the morphism

$$\ll X, Y \gg \times X \xrightarrow{\ \iota \times y_X\ } \langle PX, PY\rangle \times PX \xrightarrow{\ \mathrm{ev}\ } PY$$

factors through y_Y via a morphism

$$\ll X, Y \gg \times X \xrightarrow{\widetilde{\mathrm{ev}}} Y.$$

Next we show that this is a weak exponential in $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{sep}}$.

Given any separated $(\mathbb{T}, V)$-category Z, and a $(\mathbb{T}, V)$-functor $f : Z \times X \to Y$, by injectivity of PY there exists a $(\mathbb{T}, V)$-functor $f' : Z \times PX \to PY$ making the square below commute. Then, by universality of the evaluation map ev, there exists a unique $(\mathbb{T}, V)$-functor $\overline{f} : Z \to \langle PX, PY\rangle$ making the bottom triangle commute.

$$\begin{array}{ccc}
Z \times X & \xrightarrow{f} & Y \\
{\scriptstyle 1_Z \times y_X}\downarrow & & \downarrow{\scriptstyle y_Y} \\
Z \times PX & \xrightarrow{f'} & PY \\
{\scriptstyle \overline{f} \times 1_{PX}}\downarrow & \nearrow{\scriptstyle \mathrm{ev}} & \\
\langle PX, PY\rangle \times PX & &
\end{array}$$

The map $\overline{f} : Z \to \langle PX, PY\rangle$, assigning to each $z \in Z$ a map $\overline{f}(z) : PX \to PY$, is such that $\overline{f}(z)(y_X(x)) = \mathrm{ev}(\overline{f}(z), y_X(x)) = y_Y(f(z, x))$; that is, $\overline{f}(z)(y_X(X)) \subseteq y_Y(Y)$, and this means that $\overline{f}(z) \in \ll X, Y \gg$. Hence we can consider the corestriction $\tilde{f}$ of $\overline{f}$ to $\ll X, Y \gg$, which is again a $(\mathbb{T}, V)$-functor since $\ll X, Y \gg$ has the initial structure with respect to $\langle PX, PY\rangle$, so that the following diagram commutes.

$$\begin{array}{ccc}
\ll X, Y \gg \times X & \xrightarrow{\widetilde{\mathrm{ev}}} & Y \\
{\scriptstyle \tilde{f} \times 1_X}\uparrow & \nearrow{\scriptstyle f} & \\
Z \times X & &
\end{array}$$

□

In order to show that $(\mathbb{T}, V)$-$\mathbf{Cat}$ is weakly cartesian closed, we follow the proof of [Ros99]. Hence, first we show that:

Proposition 6.2. *The reflector* $R : (\mathbb{T}, V)$-$\mathbf{Cat} \to (\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{sep}}$ *preserves finite products.*

Proof. We recall that, for any $(\mathbb{T}, V)$-category (X, a), $R(X, a) = (\tilde{X}, \tilde{a})$, with $\tilde{X} = X/\sim$, where $x \sim y$ if $k \leq a(e_X(x), y) \wedge a(e_X(y), x)$, and $\tilde{a} = \eta_X \cdot a \cdot (T\eta_X)^\circ$, with $\eta_X : X \to \tilde{X}$ the projection. This structure makes η_X both an initial and a final morphism (see [HST14] for details).

Let $f : R(X \times Y) \to RX \times RY$ be the unique morphism such that $f \cdot \eta_{X\times Y} = \eta_X \times \eta_Y$.

$$\begin{array}{ccc}
(X \times Y, c) & \xrightarrow{\eta_{X\times Y}} & (R(X \times Y), \tilde{c}) \\
& \searrow{\scriptstyle \eta_X \times \eta_Y} & \downarrow{\scriptstyle f} \\
& & (RX \times RY, d)
\end{array}$$

From $c(e_{X\times Y}(x, y), (x', y')) = a(e_X(x), x') \wedge b(e_Y(y), y')$ it is immediate that $(x, y) \sim (x', y')$ in $X \times Y$ if, and only if, $x \sim x'$ in X and $y \sim y'$ in Y. Therefore, f is a bijection. Assuming the Axiom of Choice, so that T preserves surjections, we have, for every $\mathfrak{z} \in T(R(X \times Y))$, $(x, y) \in X \times Y$,

$$\begin{array}{rcll}
\tilde{c}(\mathfrak{z}, [(x, y)]) & = & c(\mathfrak{w}, (x, y)) & \text{(for any } \mathfrak{w} \in (T\eta_{X\times Y})^{-1}(\mathfrak{z})) \\
& = & d(T(\eta_X \times \eta_Y)(\mathfrak{w}), ([x], [y])) & \text{(because } \eta_X \times \eta_Y \text{ is initial)} \\
& = & d(Tf(\mathfrak{z}), ([x], [y]); &
\end{array}$$

that is, f is initial and therefore an isomorphism. □

Theorem 6.3. *Under Assumptions 5.5,* $(\mathbb{T},V)$-**Cat** *is weakly cartesian closed.*

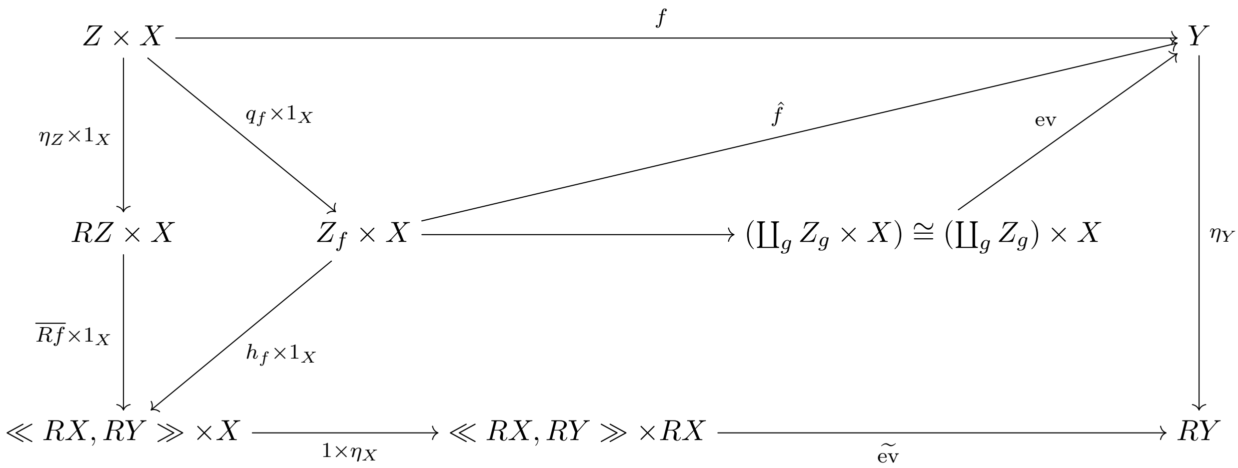
Proof. Given $(\mathbb{T},V)$-categories (X,a), (Y,b), to build the weak exponential $\ll X,Y \gg$ we will show the *cosolution set condition* for the functor $-\times(X,a)$.

For each $(\mathbb{T},V)$-functor $f:(Z,c)\times(X,a)\to(Y,b)$ we consider its reflection $Rf:RZ\times RX\cong R(Z\times X)\to RY$ and we factorise it through the weak evaluation in $(\mathbb{T},V)$-$\mathbf{Cat}_{\rm sep}$, $Rf=\widetilde{\rm ev}\cdot(\overline{Rf}\times 1_{RX})$, so that in the diagram below the outer rectangle commutes.

Then we define $Z_f=Z/\sim$ by

$$z\sim z' \text{ if both } f(z,x)=f(z',x),\ \text{for every } x\in X,\ \text{and } \overline{Rf}(\eta_Z(z))=\overline{Rf}(\eta_Z(z')),$$

and equip it with the final structure for the projection $q_f:Z\to Z_f$. Then $h_f:Z_f\to\ll RX,RY\gg$, with $h_f([z])=\overline{Rf}(\eta_Z(z))$, is a $(\mathbb{T},V)$-functor since its composition with q_f is $\overline{Rf}\cdot\eta_Z$ and q_f is final. Then we factorise f via the surjection $q_f\times 1_X:Z\times X\to Z_f\times X$ as in the diagram below. Moreover, the map $\hat{f}:Z_f\times X\to Y$, with $\hat{f}([z],x)=f(z,x)$, is a $(\mathbb{T},V)$-functor because $\eta_Y\cdot\hat{f}=\widetilde{\rm ev}\cdot(h_f\times\eta_X)$ is and η_Y is initial.



Since the cardinality of Z_f is bounded by the cardinality of the set $|\ll RX,RY\gg|\times|Y|^{|X|}$, as witnessed by the injective map

$$\begin{array}{rcl} Z_f & \to & |\ll RX,RY\gg|\times|Y|^{|X|},\\ [z] & \mapsto & (\overline{Rf}(\eta_Z(z)),f(z,-)) \end{array}$$

there is only a set of possible $(\mathbb{T},V)$-categories Z_f. Hence we can form its coproduct, as in the diagram above, and consider the induced $(\mathbb{T},V)$-functor ${\rm ev}:(\coprod_g Z_g)\times X\cong\coprod_g(Z_g\times X)\to Y$ (note that the isomorphism follows from extensivity of $(\mathbb{T},V)$-**Cat**). □

7. Examples

In this section we use Theorem 6.3 to present examples of weakly cartesian closed categories. Hence, in conjunction with the following theorem established in [Ros99], we obtain examples of categories with cartesian closed exact completion since all other conditions of that theorem are trivially satisfied in these examples.

Theorem 7.1. *Let* **C** *be a complete, infinitely extensive and well-powered category in which every morphism factorizes as a regular epi followed by a mono, and where* $f\times 1$ *is an epimorphism for every regular epimorphism* $f:A\to B$ *in* **C**. *Then, if* **C** *is weakly cartesian closed, the exact completion* $\mathbf{C}_{\rm ex}$ *of* **C** *is cartesian closed.*

We note that, in order to conclude that $(\mathbb{T}, V)$-**Cat** is weakly cartesian closed, we have to check whether V and $\mathbb{T}$ satisfy Assumptions 5.5.

First we analyse examples where $\mathbb{T}$ is the identity monad. In this particular setting we only have to check that 5.5 (2) holds. The category V-**Cat** is always monoidal closed, as shown in [Law73]. Therefore, when V is a frame considered as a quantale, then V-**Cat** is cartesian closed. This is the case of 2, and so one concludes that **Ord** *is cartesian closed*. Moreover, for V the lattice $([0,\infty], \geq)$ with $\otimes = \wedge$, V-**Cat** is *the category of ultrametric spaces*, which *is* therefore also *cartesian closed*.

When $V = P_+$, V-**Cat** is the category **Met** of Lawvere's metric spaces [Law73], which is not cartesian closed (see [CH06] for details). However, the quantale P_+ satisfies 5.5 (2), and so from Theorem 6.3 it follows that **Met** *is weakly cartesian closed*.

Metric and ultrametric spaces can be also viewed as categories enriched in a quantale based on the complete lattice $[0,1]$ with the usual "less or equal" relation $\leq$, which is isomorphic to $[0,\infty]$ via the map $[0,1] \to [0,\infty]$, $u \mapsto -\ln(u)$ where $-\ln(0) = \infty$. More in detail, we consider the following quantale operations on $[0,1]$ with neutral element 1.

(1) For $\otimes = *$ being the ordinary multiplication, via the isomorphism $[0,1] \simeq [0,\infty]$, this quantale is isomorphic to the quantale P_+, hence $[0,1]$-**Cat** $\simeq$ **Met**.
(2) For the tensor $\otimes = \wedge$ being infimum, the isomorphism $[0,1] \simeq [0,\infty]$ establishes an equivalence between $[0,1]$-**Cat** and the category of ultrametric spaces and non-expansive maps.
(3) Another interesting multiplication on $[0,1]$ is the *Łukasiewicz tensor* $\otimes = \odot$ given by $u \odot v = \max(0, u+v-1)$. Via the lattice isomorphism $[0,1] \to [0,1]$, $u \mapsto 1-u$, this quantale is isomorphic to the quantale $[0,1]$ with "greater or equal" relation $\geq$ and tensor $u \otimes v = \min(1, u+v)$ truncated addition. Therefore $[0,1]$-**Cat** is equivalent to the *category of bounded-by-*1 *metric spaces and non-expansive maps*. Moreover, with respect to the "greater or equal" relation and truncated addition on $[0,1]$, the map

$$[0,\infty] \to [0,1],\ u \mapsto \min(1,u)$$

is a surjective quantale morphism; therefore, by Lemma 5.7, also $[0,1]$ with the Łukasiewicz tensor satisfies 5.5 (2).
(4) More generally, every continuous quantale structure $\otimes$ on the lattice $[0,1]$ (with Euclidean topology and the usual "less or equal" relation) with neutral element 1 satisfies 5.5 (2). This can be shown using the fact, proven in [Fau55] and [MS57], that every such tensor $\otimes : [0,1] \times [0,1] \to [0,1]$ is a combination of the three operations on $[0,1]$ described above. More precise:
   (a) For $u, v \in [0,1]$ and $e \in [0,1]$ idempotent with $u \leq e \leq v$: $u \otimes v = \min(u,v) = u$.
   (b) For every non-idempotent $u \in [0,1]$, there exist idempotents e and f with $e < u < f$ and such that the interval $[e,f]$ (with the restriction of the tensor on $[0,1]$ and with neutral element f) is isomorphic to $[0,1]$ either with multiplication or Łukasiewicz tensor.

   Now let $w, u, v \in [0,1]$. We may assume $u \leq v$. If $u \otimes v \leq w$, then clearly

$$w \wedge (u \otimes v) = u \otimes v = \bigvee \{u' \otimes v' \mid u' \leq u, v' \leq v, u' \otimes v' \leq w\}.$$

   We consider now $w < u \otimes v \leq u \leq v$. If w is idempotent, then

$$w = w \otimes v, \quad w \leq u, \quad v \leq v;$$

   otherwise there are idempotents e and f with $e < w < f$ and $[e,f]$ is isomorphic to $[0,1]$ either with multiplication or Łukasiewicz tensor.
   Case 1: $v \leq f$. Then 5.5 (2) holds since $w, u \otimes v, u, v \in [e,f]$.

Case 2: $f < v$. Then $w = w \wedge v = w \otimes v$, $w \leq u$ and $v \leq v$.

We conclude that $[0,1]$-**Cat** *is weakly cartesian closed*, for every continuous quantale structure $\otimes$ on the lattice $[0,1]$ with neutral element 1.

Now let $V = \Delta$ be the *quantale of distribution functions* (see [HR13, CH17] for details). As observed in [HR13], it verifies 5.5 (2), and so we can conclude from Theorem 6.3 that *the category* Δ-**Cat** *of probabilistic metric spaces and non-expansive maps is weakly cartesian closed.*

When $\mathbb{T}$ is not the identity monad, some further work is need to guarantee Assumptions 5.5.

Theorem 7.2. *(1) The tensor product on the quantale V defines a $(\mathbb{T},V)$-functor $\otimes : V \otimes V \to V$.*

(2) Let $u \in V$ satisfying $u\cdot! \geq \xi \cdot Tu$.

$$\begin{array}{ccc} T1 & \xrightarrow{Tu} & TV \\ {\scriptstyle !}\downarrow & \geq & \downarrow{\scriptstyle \xi} \\ 1 & \xrightarrow[u]{} & V \end{array}$$

Then $(-,u) : X \to X \times V$ is a $(\mathbb{T},V)$-functor, for every $(\mathbb{T},V)$-category X.

(3) Let $u \in V$ satisfying $u\cdot! = \xi \cdot Tu$. Then $T(r \otimes u) = (Tr) \otimes u$, for every V-relation $r : X \nrightarrow Y$.

Proof. The first assertion is [Hof11, Proposition 1.4(1)]. To see (2), assume that $u \in V$ with $u\cdot! \geq \xi \cdot Tu$. Let (X,a) be a $(\mathbb{T},V)$-category, $\mathfrak{x} \in TX$ and $x \in X$. Considering the map $X \xrightarrow{!} 1 \xrightarrow{u} V$, we have to show that

$$a(\mathfrak{x}, x) \leq a(\mathfrak{x}, x) \otimes \hom(T(u\cdot !)(\mathfrak{x}), u),$$

which follows immediately from $u\cdot! \geq \xi \cdot Tu$. Finally, to prove (3), let $r : X \nrightarrow Y$ be a V-relation and $u \in V$ with $u\cdot! = \xi \cdot Tu$. Note that the V-relation $r \otimes u : X \nrightarrow Y$ is given by

$$X \times Y \xrightarrow{\ r\ } V \xrightarrow{\langle 1_V, u\cdot!\rangle} V \times V \xrightarrow{\ \otimes\ } V.$$

Hence, applying the **Set**-functor T to the functions $r : X \times Y \to V$ and $r \otimes u : X \times Y \to V$, we obtain

$$\begin{aligned} \xi \cdot T(r \otimes u) &= \xi \cdot T(\otimes) \cdot T\langle 1_V, u\cdot!\rangle \cdot Tr \\ &= \otimes \cdot (\xi \times \xi) \cdot \mathrm{can}_{X,Y} \cdot T\langle 1_V, u\cdot!\rangle \cdot Tr \\ &= \otimes \cdot \langle \xi, u\cdot! \cdot \xi\rangle \cdot Tr \\ &= \otimes \cdot \langle 1_V, u\cdot!\rangle \cdot \xi \cdot Tr. \end{aligned}$$

Therefore, returning to V-relations, we conclude that $T(r \otimes u) = (Tr) \otimes u$. □

Remark 7.3. If $T1 = 1$, then $u\cdot! = \xi \cdot Tu$ for every $u \in V$.

In order to guarantee Assumptions 5.5 (1), we need an extra condition on ξ.

Proposition 7.4. *Assume that*

$$\begin{array}{ccc} T(V \times V) & \xrightarrow{T(\wedge)} & TV \\ {\scriptstyle \langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle}\downarrow & \leq & \downarrow{\scriptstyle \xi} \\ V \times V & \xrightarrow[\wedge]{} & V. \end{array}$$

Then, for all V-relations $r : X \nrightarrow X'$ and $s : Y \nrightarrow Y'$,

$$\begin{array}{ccc} T(X\times Y) & \xrightarrow{\mathrm{can}_{X,Y}} & TX\times TY \\ {\scriptstyle T(r\owedge s)}\big\downarrow\!\!\!\!\!\!\;+ & \geq & +\!\!\!\!\!\!\;\big\downarrow{\scriptstyle Tr\owedge Ts} \\ T(X'\times Y') & \xrightarrow[\mathrm{can}_{X',Y'}]{} & TX'\times TY'. \end{array}$$

Proof. First we note that, from the preservation of weak pullbacks by T, it follows that the commutative diagram

$$\begin{array}{ccc} T(A\times B) & \xrightarrow{T(f\times g)} & T(X\times Y) \\ {\scriptstyle \mathrm{can}_{A,B}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{can}_{X,Y}} \\ TA\times TB & \xrightarrow[Tf\times Tg]{} & TX\times TY \end{array}$$

is also a weak pullback.

Let $\mathfrak{w} \in T(X \times Y)$, $\mathfrak{x}' \in TX'$ and $\mathfrak{y}' \in TY'$. Put $(\mathfrak{x}, \mathfrak{y}) = \mathrm{can}_{X,Y}(\mathfrak{w})$. By the definition of the extension of T and since V is a Heyting algebra,

$$Tr(\mathfrak{x},\mathfrak{x}')\wedge Ts(\mathfrak{y},\mathfrak{y}') = \bigvee\left\{ \xi\cdot Tr(\mathfrak{w}_1)\wedge \xi\cdot Ts(\mathfrak{w}_2) \;\middle|\; \begin{array}{l} \mathfrak{w}_1\in T(X\times X') : \mathfrak{w}_1\mapsto \mathfrak{x}, \mathfrak{w}_1\mapsto\mathfrak{x}' \\ \mathfrak{w}_2\in T(Y\times Y') : \mathfrak{w}_2\mapsto \mathfrak{y}, \mathfrak{w}_2\mapsto\mathfrak{y}' \end{array}\right\}.$$

Note that in

$$\begin{array}{ccccccc} & & T(X\times Y\times X'\times Y') & & & & \\ & & {\scriptstyle\cong}\big\downarrow & & & & \\ T(X\times Y) & \xleftarrow{T(\pi_X\times\pi_Y)} & T(X\times X'\times Y\times Y') & \xrightarrow{T(r\times s)} & T(V\times V) & \xrightarrow{T(\wedge)} & TV \\ {\scriptstyle\mathrm{can}}\big\downarrow & & {\scriptstyle\mathrm{can}}\big\downarrow & & {\scriptstyle\mathrm{can}}\big\downarrow & & \big\downarrow \\ TX\times TY & \xleftarrow[T\pi_X\times T\pi_Y]{} & T(X\times X')\times T(Y\times Y') & \xrightarrow[Tr\times Ts]{} & TV\times TV & & \big\downarrow{\scriptstyle\xi} \\ & & & & {\scriptstyle\xi\times\xi}\big\downarrow & \leq & \big\downarrow \\ & & & & V\times V & \xrightarrow[\wedge]{} & V \end{array}$$

the left hand side is a weak pullback, the middle diagram commutes, and in the right hand side we have "lower path" $\leq$ "upper path" as indicated. Therefore, for such $\mathfrak{w}_1 \in T(X \times X')$ and $\mathfrak{w}_2 \in T(Y \times Y')$, there exists some $\mathfrak{v} \in T(X \times X' \times Y \times Y')$ which projects to $\mathfrak{w} \in T(X \times Y)$ and to $(\mathfrak{w}_1, \mathfrak{w}_2) \in T(X \times X') \times T(Y \times Y')$. Hence, taking also into account the definition of the V-relation $T(r \owedge s)$,

$$\begin{aligned} Tr(\mathfrak{x},\mathfrak{x}')\wedge Ts(\mathfrak{y},\mathfrak{y}') &\leq \bigvee\left\{ \xi\cdot T(\wedge)\cdot T(r\times s)(\mathfrak{v}) \;\middle|\; \mathfrak{v}\in T(X\times Y\times X'\times Y'); \begin{array}{c} \mathfrak{v}\mapsto\mathfrak{w} \\ \mathfrak{v}\mapsto\mathfrak{x}', \mathfrak{v}\mapsto\mathfrak{y}' \end{array}\right\} \\ &\leq \bigvee\{T(r\owedge s)(\mathfrak{w},\mathfrak{w}') \mid \mathfrak{w}'\in T(X'\times Y'),\ \mathrm{can}_{X',Y'}(\mathfrak{w}') = (\mathfrak{x}',\mathfrak{y}')\}. \end{aligned}$$

□

Remark 7.5. We note that the inequality

$$\begin{array}{ccc} T(V\times V) & \xrightarrow{T(\wedge)} & TV \\ {\scriptstyle \langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle}\big\downarrow & \geq & \big\downarrow{\scriptstyle \xi} \\ V\times V & \xrightarrow[\wedge]{} & V \end{array}$$

is always true.

Corollary 7.6. *If the quantale V satisfies Assumption 5.5 (2) and the diagrams*

$$\begin{array}{ccc} T(V\times V) & \xrightarrow{T(\wedge)} & TV \\ {\scriptstyle \langle \xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle}\downarrow & & \downarrow{\scriptstyle \xi} \\ V\times V & \xrightarrow[\wedge]{} & V \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} T1 & \xrightarrow{Tu} & TV \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \xi} \\ 1 & \xrightarrow[u]{} & V \end{array}$$

commute, for all $u \in V$, then all Assumptions 5.5 are satisfied.

Let $\mathbb{T}$ be the ultrafilter monad $\mathbb{U} = (U, m, e)$. Then, when V is any of the quantales listed above but Δ, all the needed conditions are satisfied. Therefore, in particular we can conclude that:

Examples 7.7. (1) *The category* **Top** $= (\mathbb{U}, 2)$-**Cat** *of topological spaces and continuous maps is weakly cartesian closed* (as shown by Rosický in [Ros99]).

(2) *The category* **App** $= (\mathbb{U}, P_+)$-**Cat** *of approach spaces and non-expansive maps is weakly cartesian closed.*

(3) In fact, for each continuous quantale structure on the lattice $([0,1], \leq) \simeq ([0,\infty], \geq)$, $(\mathbb{U}, [0,1])$-**Cat** is weakly cartesian closed. In particular, *the category of non-Archimedean approach spaces and non-expansive maps studied in* [CVO17] *is weakly cartesian closed.*

(4) If V is a completely distributive complete lattice with $\otimes = \wedge$, then, with

$$\xi : UV \to V,\ \mathfrak{x} \mapsto \bigwedge_{A\in\mathfrak{x}} \bigvee A,$$

all the conditions of Theorem 6.3 are satisfied (see [Hof07, Theorem 3.3]) and therefore $(\mathbb{U}, V)$-**Cat** is weakly cartesian closed. In particular, with $V = P2$ being the powerset of a 2-element set, we obtain that *the category* **BiTop** *of bitopological spaces and bicontinuous maps is weakly cartesian closed* (see [HST14]).

Remark 7.8. For $V = \Delta$ the quantale of distribution functions, we do not know whether there is an appropriate compact Hausdorff topology $\xi : UV \to V$ satisfying the conditions of this section.

Now let $\mathbb{T}$ be the free monoid monad $\mathbb{W} = (W, m, e)$. For each quantale V, we consider

$$\xi : WV \to V,\ (v_1, \dots, v_n) \mapsto v_1 \otimes \dots \otimes v_n,\ () \mapsto k$$

which induces the extension $W : V$-**Rel** $\to V$-**Rel** sending $r : X \nrightarrow Y$ to the V-relation $Wr : WX \nrightarrow WY$ given by

$$Wr((x_1, \dots, x_n), (y_1, \dots, y_m)) = \begin{cases} r(x_1, y_1) \otimes \dots \otimes r(x_n, y_n) & \text{if } n = m, \\ \bot & \text{if } n \neq m. \end{cases}$$

The category $(\mathbb{W}, 2)$-**Cat** is equivalent to the category **MultiOrd** of *multi-ordered sets* and their morphisms (see [HST14]), more generally, $(\mathbb{W}, V)$-categories can be interpreted as multi-V-categories and their morphisms. The representable multi-ordered sets are precisely the ordered monoids, which is a special case of [Her00, Her01] describing monoidal categories as representable multi-categories (see also [CCH15]). We recall that the separated injective multi-ordered sets are precisely the quantales (see [LBKR12] and also [Sea10]), and we conclude:

Proposition 7.9. *Every quantale is exponentiable in* **MultiOrd**.

Theorem 7.10. *If the quantale V is a frame, then* $(\mathbb{W}, V)$-**Cat** *is weakly cartesian closed. In particular,* **MultiOrd** *is weakly cartesian closed.*

Finally, for a monoid $(H,\cdot,h)$, we consider the monad $\mathbb{H}=(-\times H,m,e)$, with $m_X:X\times H\times H\to X\times H$ given by $m_X(x,a,b)=(x,a\cdot b)$ and $e_X:X\to X\times H$ given by $e_X(x)=(x,h)$. Here we consider

$$\xi:V\times H\to V,\ (v,a)\mapsto v,$$

which leads to the extension $-\times H:V\text{-}\mathbf{Rel}\to V\text{-}\mathbf{Rel}$ sending the V-relation $r:X\nrightarrow Y$ to the V-relation $r\times H:X\times H\nrightarrow Y\times H$ with

$$r\times H((x,a),(y,b))=\begin{cases}r(x,y) & \text{if } a=b,\\ \bot & \text{if } a\neq b.\end{cases}$$

In particular, $(\mathbb{H},2)$-categories can be interpreted as *H-labelled ordered sets* and *equivariant maps*.

For every quantale V and every $v:1\to V$, the diagrams

$$\begin{array}{ccc} V\times V\times H & \xrightarrow{\wedge\times 1_H} & V\times H\\ {\scriptstyle \pi_{1,2}}\big\downarrow & & \big\downarrow{\scriptstyle \xi=\pi_1}\\ V\times V & \xrightarrow[\wedge]{} & V\end{array}\qquad\text{and}\qquad\begin{array}{ccc} 1\times H & \xrightarrow{v\times 1_H} & V\times H\\ {\scriptstyle !}\big\downarrow & & \big\downarrow{\scriptstyle \xi}\\ 1 & \xrightarrow[v]{} & V\end{array}$$

commute, therefore we obtain:

Theorem 7.11. *For every quantale V satisfying Assumption 5.5* (2)*, the category* $(\mathbb{H},V)$-**Cat** *is weakly cartesian closed.*

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CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
Email address: `mmc@mat.uc.pt`

CIDMA, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
Email address: `dirk@ua.pt`

CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
Email address: `willian.ribeiro.vs@gmail.com`