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Stabilities for a Class of Higher Order Integro-Differential Equations

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Abstract. This work is devoted to analyse different kinds of stabilities for higher order integro-differential equations within appropriate metric spaces. We will consider the σ -semi-Hyers-Ulam stability which is a new kind of stability somehow between the Hyers-Ulam and the Hyers-Ulam-Rassias stabilities. Sufficient conditions are obtained in view to guarantee Hyers-Ulam, σ -semi-Hyers-Ulam and Hyers-Ulam-Rassias stabilities for such a class of integro-differential equations. We will be considering finite and infinite intervals as integration domains. Among the used techniques, we have fixed point arguments and generalizations of the Bielecki metric.

INTRODUCTION

The study of properties associated with stability for functional, differential, integral and integro-differential has been widely diffused and has been a subject of great interest in the last seven decades earning particular interest due to their great number of applications in elasticity, semiconductors, heat conduction, fluid flow, scattering theory, chemical reactions and population dynamic, and others (see [1, 3, 6, 8, 9, 10, 11, 12, 16, 17, 18, 20, 21, 22, 23, 24]). In part, this is because of their potential applicability in mathematical models where we cannot expect to easily obtain the exact solution of the problem but we may expect to obtain an approximate solution which should be stable in a certain specific sense. Hyers-Ulam and Hyers-Ulam-Rassias are among the most studied types of stabilities. These two stabilities were originated from a famous question raised by S. M. Ulam [28] at the University of Wisconsin in 1940, about to know when a solution of an equation differing “slightly” from a given one must be somehow near to the solution of the given equation. A first partial answer to this question was given by D. H. Hyers, for Banach spaces, in the case of the additive Cauchy equation $f(x+y) = f(x) + f(y)$. It is for this reason that the obtained result is nowadays called the Hyers-Ulam stability. After that preliminary answer, other approaches emerged, and new orientations were introduced by Th. M. Rassias, see [25], introducing therefore the so-called Hyers-Ulam-Rassias stability. Different generalizations were obtained by other researchers, by considering the possibility of using different involved norms and others types of equations, in particular by Aoki [2], Gajda [19], and Rassias [26]. The interested reader can obtain a detailed description of these advances in [4].

In this paper, we study the Hyers-Ulam stability, the σ -semi-Hyers-Ulam stability and the Hyers-Ulam-Rassias stability for the following class of higher order integro-differential equation,

$$y^{(n)}(x) = f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right), \quad y^{(j)}(a) = 0, \quad j = 0, \dots, n-1 \quad (1)$$

with $y \in C^n([a, b])$, for $x \in [a, b]$ where, for starting, a and b are fixed real numbers, $f : [a, b] \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ and $k : [a, b] \times [a, b] \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ are continuous functions, and $\alpha : [a, b] \rightarrow [a, b]$ is a continuous delay function which fulfills $\alpha(\tau) \leq \tau$ for all $\tau \in [a, b]$.

The formal definition of the above mentioned stabilities are now introduced for the higher order integro-differential equation (1). If for each continuously differentiable function y satisfying $\left| y^{(n)}(x) - f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right) \right| \leq \sigma(x)$, $x \in [a, b]$, where σ is a non-negative function, there is a solution y_0 of the higher order integro-differential equation and a constant $C > 0$ independent of y and y_0 such that $|y(x) - y_0(x)| \leq C \sigma(x)$, for all $x \in [a, b]$, then we say that the higher order integro-differential equation (1) has the *Hyers-Ulam-Rassias stability*.

If for each continuously differentiable function y satisfying $\left| y^{(n)}(x) - f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right) \right| \leq \theta$, $x \in [a, b]$, where $\theta \geq 0$, there is a solution y_0 of the higher order integro-differential equation and a constant $C > 0$ independent of y and y_0 such that $|y(x) - y_0(x)| \leq C \theta$, for all $x \in [a, b]$, then we say that the higher order integro-differential equation has the *Hyers-Ulam stability*.

We also use a stability in-between the two just mentioned stabilities of Hyers-Ulam-Rassias and Hyers-Ulam, introduced in [14], in the following way:

Definition 1 *Let σ a non-decreasing function defined on $[a, b]$. If for each continuously differentiable function y satisfying $\left| y^{(n)}(x) - f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right) \right| \leq \theta$, $x \in [a, b]$, where $\theta \geq 0$, there is a solution y_0 of the higher order integro-differential equation (1) and a constant $C > 0$ independent of y and y_0 such that $|y(x) - y_0(x)| \leq C \sigma(x)$, $x \in [a, b]$, then we say that the higher order integro-differential equation (1) has the σ -semi-Hyers-Ulam stability.*

Some of the usual techniques to study the stability of functional, integral and integro-differential equations are a combination of fixed point results with a generalized metric in appropriate settings (cf. [5, 8, 9, 10, 11, 12, 13, 15]). In view of this, and just for the sake of completeness, let us recall the definition of a generalized metric and the corresponding Banach Fixed Point Theorem.

Definition 2 *Let X a nonempty set. A function $d : X \times X \rightarrow [0, +\infty]$ is called a generalized metric on X if and only if d satisfies: (i) $d(x, y) = 0$ if and only if $x = y$; (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$; (iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.*

Theorem 3 *Let (X, d) be a generalized complete metric space and $T : X \rightarrow X$ a strictly contractive operator with a Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(T^{k+1}x, T^kx) < \infty$ for some $x \in X$, then the following three propositions hold true: (i) the sequence $(T^n x)_{n \in \mathbb{N}}$ converges to a fixed point x^* of T ; (ii) x^* is the unique fixed point of T in $X^* = \{y \in X : d(T^k x, y) < \infty\}$; (iii) if $y \in X^*$, then*

$$d(y, x^*) \leq \frac{1}{1-L} d(Ty, y). \quad (2)$$

HYERS-ULAM-RASSIAS STABILITY IN THE FINITE INTERVAL CASE

In this section we will present sufficient conditions for the Hyers-Ulam-Rassias stability of the higher order integro-differential equation (1), where $x \in [a, b]$, for some fixed real numbers a and b .

We will consider the space of continuously differentiable functions $C^n([a, b])$ on $[a, b]$ endowed with a generalization of the Bielecki metric

$$d(u, v) = \sup_{x \in [a, b]} \frac{|u(x) - v(x)|}{\sigma(x)}, \quad (3)$$

where σ is a non-decreasing continuous function $\sigma : [a, b] \rightarrow (0, \infty)$. If instead of σ in (3) we have $e^{p(\cdot - a)}$ with $p > 0$ then we have the well-known Bielecki metric.

We recall that $(C^n([a, b]), d)$ is a complete metric space (cf., [7], [27]).

Theorem 4 *Let $\alpha : [a, b] \rightarrow [a, b]$ be a continuous delay function with $\alpha(t) \leq t$ for all $t \in [a, b]$ and $\sigma : [a, b] \rightarrow (0, \infty)$ a non-decreasing continuous function. In addition, suppose that there is $\beta \in \mathbb{R}$ such that*

$$\int_a^x \sigma(\tau) d\tau \leq \beta \sigma(x), \quad (4)$$

for all $x \in [a, b]$. Moreover, suppose that $f : [a, b] \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ is a continuous function satisfying the Lipschitz condition $|f(x, u(x), g(x)) - f(x, v(x), h(x))| \leq M(|u(x) - v(x)| + |g(x) - h(x)|)$ with $M > 0$ and the kernel $k : [a, b] \times [a, b] \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ is a continuous function satisfying the Lipschitz condition $|k(x, t, u(t), u(\alpha(t))) - k(x, t, v(t), v(\alpha(t)))| \leq L|u(\alpha(t)) - v(\alpha(t))|$ with $L > 0$.

If $y \in C^n([a, b])$ is such that

$$\left| y^{(n)}(x) - f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right) \right| \leq \sigma(x), \quad x \in [a, b], \quad (5)$$

and $M(\beta^n + L\beta^{n+1}) < 1$, then there is a unique function $y_0 \in C^n([a, b])$ such that $y_0^{(n)}(x) = f\left(x, y_0(x), \int_a^x k(x, \tau, y_0(\tau), y_0(\alpha(\tau)))d\tau\right)$ and

$$|y(x) - y_0(x)| \leq \frac{\beta^n}{1 - M(\beta^n + L\beta^{n+1})} \sigma(x) \quad (6)$$

for all $x \in [a, b]$. This means that under the above conditions, the higher order integro-differential equation (1) has the Hyers-Ulam-Rassias stability.

Proof. By integration we have that $y^{(n)}(x) = f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right)$ is equivalent to $y(x) = \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} f\left(s_1, y(s_1), \int_a^{s_1} k(s_1, \tau, y(\tau), y(\alpha(\tau)))d\tau\right) ds_1 \dots ds_{n-2} ds_{n-1} ds_n$ where we are using the conditions $y^{(j)}(a) = 0, j = 0, \dots, n-1$. So, we will consider the operator $T : C^n([a, b]) \rightarrow C^n([a, b])$, defined by

$$(Tu)(x) = \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} f\left(s_1, u(s_1), \int_a^{s_1} k(s_1, \tau, u(\tau), u(\alpha(\tau)))d\tau\right) ds_1 \dots ds_{n-2} ds_{n-1} ds_n, \quad (7)$$

for all $x \in [a, b]$ and $u \in C^n([a, b])$.

Note that for any continuous function u , Tu is also continuous. Indeed, $|(Tu)(x) - (Tu)(x_0)| = \left| \int_x^{x_0} \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} f\left(s_1, u(s_1), \int_a^{s_1} k(s_1, \tau, u(\tau), u(\alpha(\tau)))d\tau\right) ds_1 \dots ds_{n-2} ds_{n-1} ds_n \right| \rightarrow 0$, when $x \rightarrow x_0$. Note that by the hypothesis (4) we have

$$\begin{aligned} \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} \int_a^{s_1} \sigma(\tau) d\tau ds_1 \dots ds_{n-2} ds_{n-1} ds_n &\leq \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} \beta \sigma(s_1) ds_1 \dots ds_{n-2} ds_{n-1} ds_n \\ &\leq \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_3} \beta^2 \sigma(s_2) ds_2 \dots ds_{n-2} ds_{n-1} ds_n \leq \dots \leq \int_a^x \beta^n \sigma(s_n) ds_n \leq \beta^{n+1} \sigma(x). \end{aligned}$$

Under the present conditions, we will deduce that the operator T is strictly contractive with respect to the metric (3). Indeed, for all $u, v \in C^n([a, b])$, we have,

$$\begin{aligned} d(Tu, Tv) &= \sup_{x \in [a, b]} \frac{|(Tu)(x) - (Tv)(x)|}{\sigma(x)} \\ &\leq \sup_{x \in [a, b]} \frac{1}{\sigma(x)} \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} \left| f\left(s_1, u(s_1), \int_a^{s_1} k(s_1, \tau, u(\tau), u(\alpha(\tau)))d\tau\right) \right. \\ &\quad \left. - f\left(s_1, v(s_1), \int_a^{s_1} k(s_1, \tau, v(\tau), v(\alpha(\tau)))d\tau\right) \right| ds_1 \dots ds_{n-2} ds_{n-1} ds_n \\ &\leq M \sup_{x \in [a, b]} \frac{1}{\sigma(x)} \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} |u(s_1) - v(s_1)| ds_1 \dots ds_{n-2} ds_{n-1} ds_n \\ &\quad + M \sup_{x \in [a, b]} \frac{1}{\sigma(x)} \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} \int_a^{s_1} |k(s_1, \tau, u(\tau), u(\alpha(\tau))) \\ &\quad - k(s_1, \tau, v(\tau), v(\alpha(\tau)))| d\tau ds_1 \dots ds_{n-2} ds_{n-1} ds_n \\ &\leq M \sup_{x \in [a, b]} \frac{1}{\sigma(x)} \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} |u(s_1) - v(s_1)| ds_1 \dots ds_{n-2} ds_{n-1} ds_n \end{aligned}$$

$$\begin{aligned}
& +ML \sup_{x \in [a,b]} \frac{1}{\sigma(x)} \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} \int_a^{s_1} |u(\alpha(\tau)) - v(\alpha(\tau))| d\tau ds_1 \dots ds_{n-2} ds_{n-1} ds_n \\
= & M \sup_{x \in [a,b]} \frac{1}{\sigma(x)} \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} \int_a^{s_1} \sigma(s_1) \frac{|u(s_1) - v(s_1)|}{\sigma(s_1)} ds_1 \dots ds_{n-2} ds_{n-1} ds_n \\
& +ML \sup_{x \in [a,b]} \frac{1}{\sigma(x)} \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} \int_a^{s_1} \sigma(\tau) \frac{|u(\alpha(\tau)) - v(\alpha(\tau))|}{\sigma(\tau)} d\tau ds_1 \dots ds_{n-2} ds_{n-1} ds_n \\
\leq & M \sup_{s_1 \in [a,b]} \frac{|u(s_1) - v(s_1)|}{\sigma(s_1)} \sup_{x \in [a,b]} \frac{1}{\sigma(x)} \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} \sigma(s_1) ds_1 \dots ds_{n-2} ds_{n-1} ds_n \\
& +ML \sup_{\tau \in [a,b]} \frac{|u(\tau) - v(\tau)|}{\sigma(\tau)} \sup_{x \in [a,b]} \frac{1}{\sigma(x)} \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} \int_a^{s_1} \sigma(\tau) d\tau ds_1 \dots ds_{n-2} ds_{n-1} ds_n \\
\leq & Md(u, v)\beta^n + MLd(u, v) \sup_{x \in [a,b]} \frac{\beta^{n+1}\sigma(x)}{\sigma(x)} \\
= & M(\beta^n + L\beta^{n+1})d(u, v).
\end{aligned}$$

Due to the fact that $M(\beta^n + L\beta^{n+1}) < 1$ it follows that T is strictly contractive. Thus, we can apply the above mentioned Banach Fixed Point Theorem, which ensures that we have the Hyers-Ulam-Rassias stability for the higher order integro-differential equation.

Additionally, (6) follows from (2) and (5). Indeed, from (5), we have

$$-\sigma(x) \leq y^{(n)}(x) - f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right) \leq \sigma(x), \quad x \in [a, b], \quad (8)$$

and so, using integration, we obtain

$$\begin{aligned}
& \left| y(x) - \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} f\left(s_1, y(s_1), \int_a^{s_1} k(s_1, \tau, y(\tau), y(\alpha(\tau)))d\tau\right) ds_1 \dots ds_{n-2} ds_{n-1} ds_n \right| \\
& \leq \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} \sigma(s_1) ds_1 \dots ds_{n-2} ds_{n-1} ds_n \leq \beta^n \sigma(x).
\end{aligned} \quad (9)$$

Therefore, having in mind (7) and (9), we conclude that

$$|y(x) - (Ty)(x)| \leq \beta^n \sigma(x), \quad x \in [a, b]. \quad (10)$$

Now we can apply again the Banach Fixed Point Theorem and from (2) we have that

$$d(y, y_0) \leq \frac{1}{1 - M(\beta^n + L\beta^{n+1})} d(Ty, y). \quad (11)$$

From the definition of the metric d and by (10) follows that

$$\sup_{x \in [a,b]} \frac{|y(x) - y_0(x)|}{\sigma(x)} \leq \frac{\beta^n}{1 - M(\beta^n + L\beta^{n+1})} \quad (12)$$

and consequently (6) holds.

σ -SEMI-HYERS-ULAM AND HYERS-ULAM STABILITIES IN THE FINITE INTERVAL CASE

In this section we will present sufficient conditions for the σ -semi-Hyers-Ulam stability and for the Hyers-Ulam stability of the higher order integro-differential equation (1).

Theorem 5 Let $\alpha : [a, b] \rightarrow [a, b]$ be a continuous delay function with $\alpha(t) \leq t$ for all $t \in [a, b]$ and $\sigma : [a, b] \rightarrow (0, \infty)$ a non-decreasing continuous function. In addition, suppose that there is $\beta \in \mathbb{R}$ such that

$$\int_a^x \sigma(\tau) d\tau \leq \beta \sigma(x), \quad (13)$$

for all $x \in [a, b]$. Moreover, suppose that $f : [a, b] \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ is a continuous function satisfying the Lipschitz condition $|f(x, u(x), g(x)) - f(x, v(x), h(x))| \leq M(|u(x) - v(x)| + |g(x) - h(x)|)$ with $M > 0$ and the kernel $k : [a, b] \times [a, b] \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ is a continuous function satisfying the Lipschitz condition $|k(x, t, u(t), u(\alpha(t))) - k(x, t, v(t), v(\alpha(t)))| \leq L|u(\alpha(t)) - v(\alpha(t))|$ with $L > 0$.

If $y \in C^n([a, b])$ is such that

$$\left| y^{(n)}(x) - f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau))) d\tau\right) \right| \leq \theta, \quad x \in [a, b], \quad (14)$$

where $\theta \geq 0$ and $M(\beta^n + L\beta^{n+1}) < 1$, then there is a unique function $y_0 \in C^n([a, b])$ such that $y_0^{(n)}(x) = f\left(x, y_0(x), \int_a^x k(x, \tau, y_0(\tau), y_0(\alpha(\tau))) d\tau\right)$ and

$$|y(x) - y_0(x)| \leq \frac{(b-a)^n \theta}{(1 - M(\beta^n + L\beta^{n+1})) \sigma(a)} \sigma(x) \quad (15)$$

for all $x \in [a, b]$. This means that under the above conditions, the higher order integro-differential equation (1) has the σ -semi-Hyers-Ulam stability.

Proof. We will consider the operator $T : C^n([a, b]) \rightarrow C^n([a, b])$, defined by

$$(Tu)(x) = \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} f\left(s_1, u(s_1), \int_a^{s_1} k(s_1, \tau, u(\tau), u(\alpha(\tau))) d\tau\right) ds_1 \dots ds_{n-2} ds_{n-1} ds_n, \quad (16)$$

for all $x \in [a, b]$ and $u \in C^n([a, b])$.

By the same procedure as above, we have that T is strictly contractive with respect to the metric (3) due to the fact that $M(\beta^n + L\beta^{n+1}) < 1$. Thus, we can apply the Banach Fixed Point Theorem, which ensures that we have the σ -semi-Hyers-Ulam stability for the higher order integro-differential equation (1).

On the other hand, from (14) and using integration we obtain

$$\begin{aligned} & \left| y(x) - \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} f\left(s_1, y(s_1), \int_a^{s_1} k(s_1, \tau, y(\tau), y(\alpha(\tau))) d\tau\right) ds_1 \dots ds_{n-2} ds_{n-1} ds_n \right| \\ & \leq \int_a^x \int_a^{s_n} \int_a^{s_{n-1}} \dots \int_a^{s_2} \theta ds_1 \dots ds_{n-2} ds_{n-1} ds_n \leq \theta(b-a)^n. \end{aligned} \quad (17)$$

Therefore, having in mind (16) and (17), we conclude that

$$|y(x) - (Ty)(x)| \leq \theta(b-a)^n, \quad x \in [a, b]. \quad (18)$$

From (2), the definition of the metric d and by (18) follows that

$$\sup_{x \in [a, b]} \frac{|y(x) - y_0(x)|}{\sigma(x)} \leq \frac{1}{1 - M(\beta^n + L\beta^{n+1})} \sup_{x \in [a, b]} \frac{\theta(b-a)^n}{\sigma(x)}. \quad (19)$$

and consequently by the definition of σ we have that (15) holds.

Corollary 6 Let $\alpha : [a, b] \rightarrow [a, b]$ be a continuous delay function with $\alpha(t) \leq t$ for all $t \in [a, b]$ and $\sigma : [a, b] \rightarrow (0, \infty)$ a non-decreasing continuous function. In addition, suppose that there is $\beta \in \mathbb{R}$ such that

$$\int_a^x \sigma(\tau) d\tau \leq \beta \sigma(x), \quad (20)$$

for all $x \in [a, b]$. Moreover, suppose that $f : [a, b] \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ is a continuous function satisfying the Lipschitz condition $|f(x, u(x), g(x)) - f(x, v(x), h(x))| \leq M(|u(x) - v(x)| + |g(x) - h(x)|)$ with $M > 0$ and the kernel $k : [a, b] \times [a, b] \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ is a continuous function satisfying the Lipschitz condition

$$|k(x, t, u(t), u(\alpha(t))) - k(x, t, v(t), v(\alpha(t)))| \leq L|u(\alpha(t)) - v(\alpha(t))| \quad (21)$$

with $L > 0$.

If $y \in C^n([a, b])$ is such that

$$\left| y^{(n)}(x) - f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right) \right| \leq \theta, \quad x \in [a, b], \quad (22)$$

where $\theta \geq 0$ and $M(\beta^n + L\beta^{n+1}) < 1$, then there is a unique function $y_0 \in C^n([a, b])$ such that $y_0^{(n)}(x) = f\left(x, y_0(x), \int_a^x k(x, \tau, y_0(\tau), y_0(\alpha(\tau)))d\tau\right)$ and

$$|y(x) - y_0(x)| \leq \frac{(b-a)^n \sigma(b)}{(1 - M(\beta^n + L\beta^{n+1}))\sigma(a)} \theta \quad (23)$$

for all $x \in [a, b]$. This means that under the above conditions, the higher order integro-differential equation (1) has the Hyers-Ulam stability.

HYERS-ULAM-RASSIAS STABILITY IN THE INFINITE INTERVAL CASE

In this section, we analyse the Hyers-Ulam-Rassias stability of the higher order integro-differential equation (1) but, instead of considering a finite interval $[a, b]$ (with $a, b \in \mathbb{R}$), we will consider the infinite interval $[a, \infty)$, for some fixed $a \in \mathbb{R}$.

Thus, we will now be dealing with the higher order integro-differential equations,

$$y^{(n)}(x) = f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right), \quad y^{(j)}(a) = 0, \quad j = 0, \dots, n-1, \quad (24)$$

with $y \in C^n([a, \infty))$, $x \in [a, \infty)$ where a is a fixed real number, $f : [a, \infty) \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ and $k : [a, \infty) \times [a, \infty) \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ are continuous functions, and $\alpha : [a, \infty) \rightarrow [a, \infty)$ is a continuous delay function which therefore fulfills $\alpha(\tau) \leq \tau$ for all $\tau \in [a, \infty)$.

Our strategy will be based on a recurrence procedure due to the already obtained result for the corresponding finite interval case.

Let us consider a fixed non-decreasing continuous function $\sigma : [a, \infty) \rightarrow (\varepsilon, \omega)$, for some $\varepsilon, \omega > 0$ and the space $C_b^n([a, \infty))$ of bounded differentiable functions endowed with the metric

$$d_b(u, v) = \sup_{x \in [a, \infty)} \frac{|u(x) - v(x)|}{\sigma(x)}. \quad (25)$$

Theorem 7 Let $\alpha : [a, \infty) \rightarrow [a, \infty)$ be a continuous delay function with $\alpha(t) \leq t$ for all $t \in [a, \infty)$ and $\sigma : [a, \infty) \rightarrow (\varepsilon, \omega)$, for some $\varepsilon, \omega > 0$, a non-decreasing continuous function. In addition, suppose that there is $\beta \in \mathbb{R}$ such that $\int_a^x \sigma(\tau)d\tau \leq \beta\sigma(x)$, for all $x \in [a, \infty)$. Moreover, suppose that $f : [a, \infty) \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ is a continuous function satisfying the Lipschitz condition $|f(x, u(x), g(x)) - f(x, v(x), h(x))| \leq M(|u(x) - v(x)| + |g(x) - h(x)|)$ with $M > 0$ and the kernel $k : [a, \infty) \times [a, \infty) \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ is a continuous function so that $\int_a^x k(x, \tau, z(\tau), z(\alpha(\tau)))d\tau$ is a bounded continuous function for any bounded continuous function z . In addition, suppose that k satisfies the Lipschitz condition $|k(x, t, u(t), u(\alpha(t))) - k(x, t, v(t), v(\alpha(t)))| \leq L|u(\alpha(t)) - v(\alpha(t))|$ with $L > 0$.

If $y \in C_b^n([a, \infty))$ is such that

$$\left| y^{(n)}(x) - f\left(x, y(x), \int_a^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right) \right| \leq \sigma(x), \quad x \in [a, \infty), \quad (26)$$

and $M(\beta^n + L\beta^{n+1}) < 1$, then there is a unique function $y_0 \in C_b^n([a, \infty))$ such that

$$y_0^{(n)}(x) = f\left(x, y_0(x), \int_a^x k(x, \tau, y_0(\tau), y_0(\alpha(\tau)))d\tau\right) \quad (27)$$

and

$$|y(x) - y_0(x)| \leq \frac{\beta^n}{1 - M(\beta^n + L\beta^{n+1})}\sigma(x) \quad (28)$$

for all $x \in [a, \infty)$. This means that under the above conditions, the higher order integro-differential equation (24) has the Hyers-Ulam-Rassias stability.

Proof. For any $m \in \mathbb{N}$, we will define $I_m = [a, a + m]$. By Theorem 4, there exists a unique bounded differentiable function $y_{0,m} : I_m \rightarrow \mathbb{C}$ such that

$$y_{0,m}^{(n)}(x) = f\left(x, y_{0,m}(x), \int_a^x k(x, \tau, y_{0,m}(\tau), y_{0,m}(\alpha(\tau)))d\tau\right) \quad (29)$$

and

$$|y(x) - y_{0,m}(x)| \leq \frac{\beta^n}{1 - M(\beta^n + L\beta^{n+1})}\sigma(x) \quad (30)$$

for all $x \in I_m$. The uniqueness of $y_{0,m}$ implies that if $x \in I_m$ then

$$y_{0,m}(x) = y_{0,m+1}(x) = y_{0,m+2}(x) = \dots \quad (31)$$

For any $x \in [a, \infty)$, let us define $m(x) \in \mathbb{N}$ as $m(x) = \min\{m \in \mathbb{N} : x \in I_m\}$. We also define a function $y_0 : [a, \infty) \rightarrow \mathbb{C}$ by

$$y_0(x) = y_{0,m(x)}(x). \quad (32)$$

For any $x_1 \in [a, \infty)$, let $m_1 = m(x_1)$. Then $x_1 \in \text{Int } I_{m_1+1}$ and there exists an $\epsilon > 0$ such that $y_0(x) = y_{0,m_1+1}(x)$ for all $x \in (x_1 - \epsilon, x_1 + \epsilon)$. By Theorem 4, y_{0,m_1+1} is continuous at x_1 , and so it is y_0 .

Now, we will prove that y_0 satisfies

$$y_0^{(n)}(x) = f\left(x, y_0(x), \int_a^x k(x, \tau, y_0(\tau), y_0(\alpha(\tau)))d\tau\right) \quad (33)$$

and

$$|y(x) - y_0(x)| \leq \frac{\beta^n}{1 - M(\beta^n + L\beta^{n+1})}\sigma(x). \quad (34)$$

For an arbitrary $x \in [a, \infty)$ we choose $m(x)$ such that $x \in I_{m(x)}$. By (29) and (32), we have

$$\begin{aligned} y_0(x) = y_{0,m(x)}(x) &= f\left(x, y_{0,m(x)}(x), \int_a^x k(x, \tau, y_{0,m(x)}(\tau), y_{0,m(x)}(\alpha(\tau)))d\tau\right) \\ &= f\left(x, y_0(x), \int_a^x k(x, \tau, y_0(\tau), y_0(\alpha(\tau)))d\tau\right). \end{aligned} \quad (35)$$

Note that $m(\tau) \leq m(x)$, for any $\tau \in I_{m(x)}$, and it follows from (31) that $y_0(\tau) = y_{0,m(\tau)}(\tau) = y_{0,m(x)}(\tau)$, so, the last equality in (35) holds.

To prove (28), by (32) and (30), we have that for all $x \in [a, \infty)$, $|y(x) - y_0(x)| = |y(x) - y_{0,m(x)}(x)| \leq \frac{\beta^n}{1 - M(\beta^n + L\beta^{n+1})}\sigma(x)$. Finally, we will prove the uniqueness of y_0 . Let us consider another bounded differentiable function y_1 which satisfies (27) and (28), for all $x \in [a, \infty)$. By the uniqueness of the solution on $I_{m(x)}$ for any $m(x) \in \mathbb{N}$ we have that $y_0|_{I_{m(x)}} = y_{0,m(x)}$ and $y_1|_{I_{m(x)}}$ satisfies (27) and (28) for all $x \in I_{m(x)}$, so $y_0(x) = y_0|_{I_{m(x)}}(x) = y_1|_{I_{m(x)}}(x) = y_1(x)$.

Remark 8 With the necessary adaptations, Theorem 7 also holds true for infinite intervals of the type $(-\infty, b]$, with $b \in \mathbb{R}$, as well as for $(-\infty, \infty)$.

Illustrative Examples

In this section we will present some examples simply to illustrate that the conditions of the above results are possible to attain.

For 2-differentiable functions $y : [0, 1] \rightarrow \mathbb{R}$, let us start by considering the integro-differential equation

$$y''(x) = -2y(x) + 2 + \int_0^x ((\tau - x)y(\alpha(\tau))) d\tau, \quad x \in [0, 1], \quad y^{(j)}(0) = 0, \quad j = 0, 1 \quad (36)$$

as well as the continuous function $\sigma : [0, 1] \rightarrow (0, \infty)$ defined by $\sigma(x) = 1.1e^{10x}$ and the continuous function $\alpha : [0, 1] \rightarrow [0, 1]$ given by $\alpha(x) = x$.

We have all the conditions of Theorem 5 being satisfied. In fact, such $\alpha : [0, 1] \rightarrow [0, 1]$ defined by $\alpha(x) = x$ is a continuous function, and obviously $\alpha(x) \leq x$. Moreover, for $\beta = 0.1(1 - e^{-10})$ we have that $\sigma : [0, 1] \rightarrow (0, \infty)$ defined by $\sigma(x) = 1.1e^{10x}$ is a continuous function fulfilling

$$\int_0^x 1.1e^{10\tau} d\tau \leq \beta 1.1e^{10x} = 0.1(1 - e^{-10})\sigma(x), \quad x \in [0, 1]; \quad (37)$$

$f : [0, 1] \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ defined by $f(x, y(x), g(x)) = -2y(x) + 2 + g(x)$ is a continuous function which fulfills $|f(x, u(x), g(x)) - f(x, v(x), h(x))| \leq 2(|u(x) - v(x)| + |g(x) - h(x)|)$, $x \in [0, 1]$ (and so the previous constant M is here taking the value 2); the kernel $k : [0, 1] \times [0, 1] \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ defined by $k(x, \tau, y(\tau), y(\alpha(\tau))) = (\tau - x)y(\alpha(\tau))$ is a continuous function which fulfills the condition $|k(x, \tau, u(\tau), u(\alpha(\tau))) - k(x, \tau, v(\tau), v(\alpha(\tau)))| \leq |u(\alpha(\tau)) - v(\alpha(\tau))|$, $\tau \in [0, x]$, $x \in [0, 1]$ (where we may identify 1 as the constant previously denoted by L). Thus, $M(\beta^2 + L\beta^3) \approx 0.021998 < 1$.

If we choose $y(x) = 100/99 x \sin x$, it follows $\left| y''(x) - f(x, y(x), \int_0^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau) \right| = \frac{2}{99} =: \theta$, $x \in [0, 1]$.

Therefore, from Theorem 5, we have the σ -semi-Hyers-Ulam stability of the integro-differential equation (36), and from Corollary 6 we have the consequent Hyers-Ulam stability. In particular, having in mind the exact solution $y_0(x) = x \sin x$ of (36), it follows that $|y(x) - y_0(x)| = \left| \frac{1}{99} x \sin x \right| \leq \frac{\theta}{(1 - 2(\beta^2 + \beta^3))\sigma(0)} \sigma(x)$, $x \in [0, 1]$, and $|y(x) - y_0(x)| = \left| \frac{1}{99} x \sin x \right| \leq \frac{\sigma(1)}{(1 - 2(\beta^2 + \beta^3))\sigma(0)} \theta$, $x \in [0, 1]$.

Still within this first example associated with the integro-differential equation (36), and using the same M and L , we will now show that all the conditions of Theorem 4 being satisfied. Indeed, let us now consider for σ the continuous function $\sigma : [0, 1] \rightarrow [0, \infty)$ defined by $\sigma(x) = 0.0002 + 0.007x + 0.03x^2$. If this is the case, instead of (37), we have now

$$\int_0^x 0.0002 + 0.007\tau + 0.03\tau^2 d\tau \leq 0.36828 (0.0002 + 0.007x + 0.03x^2) = \beta \sigma(x), \quad x \in [0, 1], \quad (38)$$

and so still having $M(\beta^2 + L\beta^3) \approx 0.37116 < 1$.

Considering now $y(x) = 0.99 x \sin(1.01x)$, it follows

$$\left| y''(x) - f\left(x, y(x), \int_0^x k(x, \tau, y(\tau), y(\alpha(\tau)))d\tau\right) \right| = \left| \frac{401979699}{5151505000} \cos(1.01x) - \frac{3999699}{10201000000} x \sin(1.01x) - \frac{80602}{1030301} \right| \leq \sigma(x), \quad (39)$$

for all $x \in [0, 1]$, and so we have the Hyers-Ulam-Rassias stability of the integro-differential equation (36). Having in mind the exact solution $y_0(x) = x \sin x$ of (36), it follows that

$$|y(x) - y_0(x)| = |0.99 x \sin(1.01x) - x \sin x| \leq \frac{\beta^2}{1 - 2(\beta^2 + \beta^3)} \sigma(x), \quad x \in [0, 1]. \quad (40)$$

Let us now turn to other example in which the Hyers-Ulam-Rassias stability is illustrated. For a 2-differentiable function $y : [0.1, 1] \rightarrow \mathbb{R}$, we shall consider the integro-differential equation

$$y''(x) = 2 - \frac{x}{12} y(x) \sin x + \frac{1}{12000} \sin x + \int_{0.1}^x (\sin x y(\alpha(\tau))) d\tau, \quad x \in [0.1, 1], \quad y^{(j)}(0) = 0, \quad j = 0, 1 \quad (41)$$

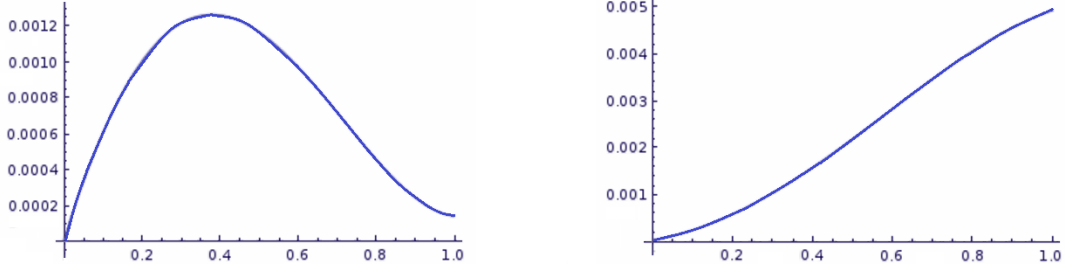


FIGURE 1. On the left, we have $z_1(x) = \sigma(x) - \left| y''(x) - f(x, y(x), \int_0^x k(x, \tau, y(\tau), y(\alpha(\tau))) d\tau) \right|$ to illustrate the inequality (39); on the right, we have $z_2(x) = \frac{\beta^2}{1-2(\beta^2+\beta^3)}\sigma(x) - |y(x) - y_0(x)|$ to illustrate the inequality (40).

as well as the continuous function $\sigma : [0.1, 1] \rightarrow (0, \infty)$ defined by $\sigma(x) = 0.02e^{0.9x}$ and the continuous delay function $\alpha : [0.1, 1] \rightarrow [0.05, 0.5]$ given by $\alpha(x) = x/2$.

We realize that all the conditions of Theorem 4 are here satisfied. In fact, $\alpha(x) = x/2$ is obviously continuous and satisfies $\alpha(x) \leq x$; for e.g. $\beta = 0.617$, $\sigma : [0.1, 1] \rightarrow [0, \infty)$ defined by $\sigma(x) = 0.02e^{0.9x}$ is a continuous function which fulfills $\int_{0.1}^x 0.02e^{0.9\tau} d\tau \leq 0.617 \times 0.02e^{0.9x} = \beta\sigma(x)$, $x \in [0.1, 1]$; $f : [0.1, 1] \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ defined by $f(x, y(x), g(x)) = 2 - \frac{x}{12}y(x)\sin x + \frac{1}{12000}\sin x + g(x)$ is a continuous function which fulfills $|f(x, u(x), g(x)) - f(x, v(x), h(x))| \leq |u(x) - v(x)| + |g(x) - h(x)|$, $x \in [0.1, 1]$ (where we may identify $M = 1$); the kernel $k : [0.1, 1] \times [0.1, 1] \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ defined by $k(x, \tau, y(\tau), y(\alpha(\tau))) = \sin x y(\alpha(\tau))$ is a continuous function which fulfills the condition $|k(x, \tau, u(\tau), u(\alpha(\tau))) - k(x, \tau, v(\tau), v(\alpha(\tau)))| \leq \sin 1 |u(\alpha(\tau)) - v(\alpha(\tau))|$, $\tau \in [0.1, x]$, $x \in [0.1, 1]$, where we make the identification $L = \sin 1$. Thus, $M(\beta^2 + L\beta^3) \approx 0.578338 < 1$. Moreover, if we choose $y(x) = 0.99x^2$, it follows,

$$\left| y''(x) - f\left(x, y(x), \int_0^x k(x, \tau, y(\tau), y(\alpha(\tau))) d\tau\right) \right| = |-0.02 - 8.3 \times 10^{-7} \sin x| \leq \sigma(x), \quad x \in [0.1, 1]. \quad (42)$$

Therefore, the integro-differential equation (41) exhibits the Hyers-Ulam-Rassias stability. In particular, by using the exact solution $y_0(x) = x^2$, we realize that

$$|y(x) - y_0(x)| = 0.01x^2 \leq \frac{\beta^2}{1 - (\beta^2 + \sin 1 \beta^3)} \sigma(x). \quad (43)$$

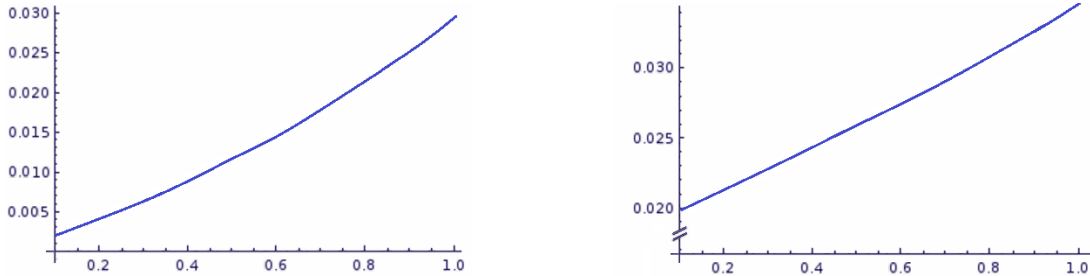


FIGURE 2. On the left, we have $z_3(x) = \sigma(x) - \left| y''(x) - f\left(x, y(x), \int_{0.1}^x k(x, \tau, y(\tau), y(\alpha(\tau))) d\tau\right) \right|$ to illustrate the inequality (42); on the right, we have $z_4(x) = \frac{\beta^2}{1-(\beta^2+\sin 1\beta^3)}\sigma(x) - |y(x) - y_0(x)|$ to illustrate the inequality (43).

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