Convolutions and applications for the offset linear canonical transform via Hermite weights

Cite as: AIP Conference Proceedings **2046**, 020014 (2018); <https://doi.org/10.1063/1.5081534>
Published Online: 04 December 2018

L. P. Castro, L. T. Minh, and N. M. Tuan

View Online

Export Citation

ARTICLES YOU MAY BE INTERESTED IN

[New convolutions for an oscillatory integral operator on the half-line](#)

AIP Conference Proceedings **2046**, 020015 (2018); <https://doi.org/10.1063/1.5081535>

[On integral operators and equations generated by cosine and sine Fourier transforms](#)

AIP Conference Proceedings **2046**, 020013 (2018); <https://doi.org/10.1063/1.5081533>

[Stabilities for a class of higher order integro-differential equations](#)

AIP Conference Proceedings **2046**, 020012 (2018); <https://doi.org/10.1063/1.5081532>

Lock-in Amplifiers
up to 600 MHz

Convolutions and Applications for the Offset Linear Canonical Transform Via Hermite Weights

L.P. Castro^{1,a),b)}, L.T. Minh^{2,c)} and N.M. Tuan^{3,d)}

¹Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, Aveiro, Portugal.

²Department of Mathematics, Ha Noi Architectural University, Km 10, Nguyen Trai Rd., Thanh Xuan Dist., Ha Noi, Vietnam.

³Department of Mathematics, College of Education, Viet Nam National University, G7 Build., 144 Xuan Thuy Rd., Cau Giay Dist., Hanoi, Vietnam.

^{a)}Corresponding author: castro@ua.pt

^{b)}URL: <http://sweet.ua.pt/castro/>

^{c)}tienminh.hau@gmail.com

^{d)}nguyentuan@vnu.edu.vn

Abstract. The main purpose of this paper is to present three new convolutions for the offset linear canonical transform, with the Hermite weights, and to illustrate their potential applications. In view of this, new factorization theorems are obtained and new Young's convolution inequalities will be introduced. Within the more applied side, the way to design filters (including multiplicative filters in the time domain) is also discussed in the last section.

INTRODUCTION

The offset linear canonical transform (OLCT) (see [7]) of a signal $f(t)$ with real parameters $A = (a, b, c, d, u_0, \omega_0)$, (satisfying $ad - bc = 1$) is defined as

$$F_A(u) := \mathbb{O}_A\{f(t)\}(u) := \begin{cases} \int_{\mathbb{R}} f(t) \mathcal{K}_A(u, t) dt, & b \neq 0 \\ \sqrt{d} e^{j\frac{cd}{2}(u-u_0)^2 + j\omega_0 u} f(d(u-u_0)), & b = 0, \end{cases} \quad (1)$$

where $\mathcal{K}_A(u, t) := K_A e^{j(\frac{d}{2b}u^2 - \frac{1}{b}tu + \frac{a}{2b}t^2 + \frac{(b\omega_0 - du_0)}{b}u + \frac{u_0}{b}t)}$, and $K_A = \frac{e^{-j\frac{du_0^2}{2b}}}{\sqrt{2\pi b j}}$. The inverse of the OLCT is given by

$$f(t) = \mathbb{O}_{A^{-1}}\{F_A(u)\}(t) = C \int_{\mathbb{R}} F_A(u) \mathcal{K}_{A^{-1}}(u, t) du, \quad (2)$$

where $A^{-1} = (d, -b, -c, a, b\omega_0 - du_0, cu_0 - a\omega_0)$, and $C = e^{j\frac{1}{2}(cd u_0^2 - 2adu_0\omega_0 + ab\omega_0^2)}$. In this paper, we will always consider $b \neq 0$ since the OLCT becomes a chirp multiplication operation otherwise. We recall that the Fourier transform and its inverse are defined by $\Psi_{FT}(f(t))(u) = \int_{\mathbb{R}} f(t) e^{-jut} dt$ and $f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \Psi_{FT}(f(t))(u) \cdot e^{jut} du$, respectively. If $f, h \in L^1(\mathbb{R})$, then the classic (Fourier) convolution in the time domain is expressed as

$$(f * h)(t) := \int_{\mathbb{R}} f(\tau) h(t - \tau) d\tau, \quad (3)$$

and the factorization property as follows

$$\Psi_{FT}\{(f * h)(t)\}(u) = \Psi_{FT}\{f(t)\}(u) \cdot \Psi_{FT}\{h(t)\}(u). \quad (4)$$

For any real number $\lambda \neq 0$, we have

$$(f * h)(\lambda t) = \lambda f(\lambda t) * h(\lambda t). \quad (5)$$

We also have the Young's inequality (see [2]). If $f \in L^p(\mathbb{R})$, $h \in L^q(\mathbb{R})$, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ (with $p, q, r \geq 1$). Then, the following inequality holds

$$\|f * h\|_r \leq C_1 \|f\|_p \cdot \|h\|_q, \quad \text{for some } C_1 > 0. \quad (6)$$

We notice that when $u_0 = \omega_0 = 0$, $\mathbb{O}_A$ is the well-known linear canonical transform (LCT) (see [4]). Remind that if $A = (a, b, c, d, 0, 0)$, $|a + d| < 2$, and $\phi_n(t)$, μ_n are the eigenfunctions and the eigenvalues of the OLCT (or the LCT) (see [6]), then we have

$$\mu_n \phi_n(u) = \mathbb{O}_A\{\phi_n(t)\}(u), \quad (7)$$

where

$$\phi_n(t) := \frac{1}{\sqrt{\beta} 2^n n! \sqrt{\pi}} e^{-\frac{(1+j\alpha)}{2\beta^2} t^2} H_n\left(\frac{t}{\beta}\right), \quad \mu_n := e^{-j(\frac{1}{2}+n)\theta} \quad (n \in \mathbb{N}),$$

and H_n is the n -th Hermite polynomial. The constants α, β, θ can be taken from

$$\alpha := \frac{\text{sgn}(b) \cdot (a - d)}{\sqrt{4 - (a + d)^2}}, \quad \beta := \frac{2|b|}{\sqrt{4 - (a + d)^2}}, \quad \theta := \cos^{-1}\left(\frac{a + d}{2}\right). \quad (8)$$

Throughout this paper, for convenience, we denote

$$\mathcal{E}_A(t) := e^{j(\frac{a}{2b}t^2 + \frac{u_0}{b}t)}, \quad \bar{f}(t) := \mathcal{E}_A(t)f(t), \quad \mathcal{E}_A^m(t) := e^{jmt}\mathcal{E}_A(t) \quad (m \in \mathbb{R}).$$

The identity (1) becomes

$$F_A(u) = \mathbb{O}_A\{f(t)\}(u) = K_A e^{j(\frac{d}{2b}u^2 + \frac{(b\omega_0 - du_0)}{b}u)} \int_{\mathbb{R}} \bar{f}(t) e^{-\frac{jut}{b}} dt. \quad (9)$$

This paper is divided into four sections and organized as follows. In Section 2, we introduce the relationship between the Hermite functions and the OLCT, which are displayed in Theorem 1 and Theorem 2. Three new convolutions for the OLCT with the Hermite weights and their product theorems are studied in the Section 3. Some special cases of these convolutions are also obtained. In the last section, we propose some applications of these convolutions as well as new Young's convolution inequalities and designing multiplicative filter in the time domain.

HERMITE FUNCTIONS AND THE OLCT

For $\lambda \neq 0$, it is easy to realize that $1 = ad - bc = (a\lambda)(\frac{d}{\lambda}) - (\frac{b}{\lambda})(c\lambda)$. Let $\lambda \neq 0$, and the parameters $A_\lambda := (a\lambda, \frac{b}{\lambda}, c\lambda, \frac{d}{\lambda}, 0, 0)$ satisfy

$$\left|a\lambda + \frac{d}{\lambda}\right| < 2. \quad (10)$$

Under the condition (10), let $\phi_n^\lambda(t)$, μ_n^λ be the eigenfunctions and the corresponding eigenvalues of the OLCT with parameters $A_\lambda = (a\lambda, \frac{b}{\lambda}, c\lambda, \frac{d}{\lambda}, 0, 0)$. We then have $\mu_n^\lambda \cdot \phi_n^\lambda(u) = \mathbb{O}_{A_\lambda}\{\phi_n^\lambda(t)\}(u)$. The eigenfunctions $\phi_n^\lambda(t)$ and the eigenvalues μ_n^λ corresponding parameters A_λ can be calculated as in (8).

Theorem 1 *Let the parameters $A_1 = (a, b, c, d, 0, 0)$, and one of the following conditions is satisfied:*

(i) $|a + d| < 2$; (ii) $|a + d| \geq 2$ and $1 - ad > 0$.

Then, there exists a constant $\lambda > 0$ such that the following relation holds

$$\phi_n^\lambda(u) = \frac{1}{\mu_n^\lambda \cdot \sqrt{\lambda}} \mathbb{O}_{A_1}\left\{\phi_n^\lambda\left(\frac{t}{\lambda}\right)\right\}(u). \quad (11)$$

Proof. If $|a + d| < 2$ then from relation (7) we choose $\lambda = 1$. Thus, (11) is fulfilled. If $|a + d| \geq 2$ and $1 - ad > 0$. By changing the variable $t = \lambda\tau$ ($\lambda > 0$), we get

$$\mathcal{K}_{A_1}(u, t) = K_{A_1} e^{j\left(\frac{d}{2b}u^2 - \frac{1}{b}tu + \frac{a}{2b}t^2\right)} = K_{A_1} e^{j\left(\frac{d}{2b}u^2 - \frac{1}{b}\tau u + \frac{a}{2b}\lambda^2\tau^2\right)} = K_{A_1} e^{j\left(\frac{\frac{d}{\lambda}}{2\left(\frac{b}{\lambda}\right)}u^2 - \frac{1}{\left(\frac{b}{\lambda}\right)}\tau u + \frac{\frac{a}{\lambda}}{2\left(\frac{b}{\lambda}\right)}\tau^2\right)} = \frac{1}{\sqrt{\lambda}} \mathcal{K}_{A_\lambda}(u, \tau).$$

It follows

$$\mathcal{K}_{A_\lambda}(u, \tau) = \sqrt{\lambda} \mathcal{K}_{A_1}(u, t). \quad (12)$$

Assume that the condition (10) is satisfied. We then have

$$(a\lambda^2 - 2\lambda + d)(a\lambda^2 + 2\lambda + d) < 0. \quad (13)$$

If $a = 0$, then from (13) we derive $|\lambda| > \frac{|d|}{2}$. Thus, there exists $\lambda > 0$ such as (10) is satisfied. If $a \neq 0$, since $\delta = 1 - ad > 0$ then we denote $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ be four solutions of the following equation $(a\lambda^2 - 2\lambda + d)(a\lambda^2 + 2\lambda + d) = 0$. Solving this equation, we receive

$$\lambda_i \in \left\{ \frac{-1 \pm \sqrt{\delta}}{a}, \frac{1 \pm \sqrt{\delta}}{a} \right\}, i \in \{1, 2, 3, 4\}.$$

From (13), we deduce $\lambda \in (\lambda_1, \lambda_2) \cup (\lambda_3, \lambda_4)$, and $\lambda_4 = \frac{1 + \sqrt{\delta}}{|a|} > 0$. Hence, there exists $\lambda > 0$ such that (10) is fulfilled. Therefore, the OLCT with parameters A_λ has the eigenfunctions $\phi_n^\lambda(t)$ and the eigenvalues μ_n^λ :

$$\mu_n^\lambda \cdot \phi_n^\lambda(u) = \int_{\mathbb{R}} \mathcal{K}_{A_\lambda}(u, \tau) \cdot \phi_n^\lambda(\tau) d\tau. \quad (14)$$

Substituting the relation (12) into (14) results in

$$\mu_n^\lambda \cdot \phi_n^\lambda(u) = \sqrt{\lambda} \int_{\mathbb{R}} \mathcal{K}_{A_1}(u, t) \cdot \phi_n^\lambda\left(\frac{t}{\lambda}\right) d\left(\frac{t}{\lambda}\right).$$

That implies $\phi_n^\lambda(u) = \frac{1}{\mu_n^\lambda \cdot \sqrt{\lambda}} \int_{\mathbb{R}} \mathcal{K}_{A_1}(u, t) \cdot \phi_n^\lambda\left(\frac{t}{\lambda}\right) dt$. Hence $\phi_n^\lambda(u) = \frac{1}{\mu_n^\lambda \cdot \sqrt{\lambda}} \mathbb{O}_{A_1} \{\phi_n^\lambda\left(\frac{t}{\lambda}\right)\}(u)$. The proof is completed.

Theorem 2 Let $A = (a, b, c, d, u_0, \omega_0)$, and one of the following conditions be fulfilled: (i) $|a + d| < 2$; (ii) $|a + d| \geq 2$, $a + d \neq 2$ and $1 - ad > 0$. Then, there exists a positive constant λ , such that the following relation holds

$$e^{jm_1 u} \cdot \phi_n^\lambda(u - m_2) = \frac{e^{j\left(m_3 - \frac{du_0}{2b}\right)}}{\mu_n^\lambda \cdot \sqrt{\lambda}} \mathbb{O}_A \left\{ e^{jm_1 t} \cdot \phi_n^\lambda\left(\frac{t - m_2}{\lambda}\right) \right\}(u), \quad (15)$$

provided

$$\begin{cases} m_1 = \frac{(a-1)(b\omega_0 - du_0) + u_0(1-d)}{b(a+d-2)} \\ m_2 = -\frac{b\omega_0 - du_0 + u_0}{a+d-2} \\ m_3 = \frac{(b\omega_0 - du_0 + u_0)^2}{2b(a+d-2)}. \end{cases} \quad (16)$$

Proof. We realize that

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{K}_A(u, t) \cdot e^{j(m_1 t + m_3)} e^{-jm_1 u} \cdot \phi_n^\lambda\left(\frac{t - m_2}{\lambda}\right) dt &= K_A \int_{\mathbb{R}} e^{j\left(\frac{d}{2b}u^2 - \frac{1}{b}tu + \frac{a}{2b}t^2 + \frac{(b\omega_0 - du_0)}{b}u + \frac{u_0}{b}t\right)} e^{j(m_1 t + m_3)} e^{-jm_1 u} \cdot \phi_n^\lambda\left(\frac{t - m_2}{\lambda}\right) dt \\ &= K_A \int_{\mathbb{R}} e^{j\left(\frac{d}{2b}u^2 - \frac{1}{b}tu + \frac{a}{2b}t^2 + \left(\frac{b\omega_0 - du_0}{b} - m_1\right)u + \left(\frac{u_0}{b} + m_1\right)t + m_3\right)} \cdot \phi_n^\lambda\left(\frac{t - m_2}{\lambda}\right) dt. \end{aligned} \quad (17)$$

Let

$$\frac{d}{2b}u^2 - \frac{1}{b}tu + \frac{a}{2b}t^2 + \left(\frac{b\omega_0 - du_0}{b} - m_1\right)u + \left(\frac{u_0}{b} + m_1\right)t + m_3 = \frac{d}{2b}(u - m_2)^2 - \frac{1}{b}(t - m_2)(u - m_2) + \frac{a}{2b}(t - m_2)^2, \quad (18)$$

then

$$\begin{cases} m_1 - \frac{d-1}{b}m_2 = \frac{b\omega_0 - du_0}{b} \\ m_1 + \frac{a-1}{b}m_2 = -\frac{u_0}{b} \\ m_3 = m_2\left(\frac{d}{2b} - \frac{1}{b} + \frac{a}{2b}\right). \end{cases}$$

Remind that $a + d \neq 2$. Then, the solution of this system equations is given as

$$\begin{cases} m_1 = \frac{(a-1)(b\omega_0 - du_0) + u_0(1-d)}{b(a+d-2)} \\ m_2 = -\frac{b\omega_0 - du_0 + u_0}{a+d-2} \\ m_3 = \frac{(b\omega_0 - du_0 + u_0)^2}{2b(a+d-2)}. \end{cases}$$

Substituting equation (18) into equation (17), we obtain

$$\int_{\mathbb{R}} \mathcal{K}_A(u, t) e^{j(m_1 t + m_3)} e^{-jm_1 u} \phi_n^\lambda\left(\frac{t - m_2}{\lambda}\right) dt = e^{\frac{jdu_0^2}{2b}} \int_{\mathbb{R}} \mathcal{K}_{A_1}(u - m_2, t - m_2) \phi_n^\lambda\left(\frac{t - m_2}{\lambda}\right) dt = e^{\frac{jdu_0^2}{2b}} \mathbb{O}_{A_1} \left\{ \phi_n^\lambda\left(\frac{t}{\lambda}\right) \right\} (u - m_2).$$

Thanks to equation (11), we derive $\int_{\mathbb{R}} \mathcal{K}_A(u, t) \cdot e^{j(m_1 t + m_3)} e^{-jm_1 u} \cdot \phi_n^\lambda\left(\frac{t - m_2}{\lambda}\right) dt = e^{\frac{jdu_0^2}{2b}} \cdot \mu_n^\lambda \cdot \sqrt{\lambda} \cdot \phi_n^\lambda(u - m_2)$, which implies that $e^{-jm_3} e^{\frac{jdu_0^2}{2b}} \mu_n^\lambda \cdot \sqrt{\lambda} \cdot \phi_n^\lambda(u - m_2) = \int_{\mathbb{R}} \mathcal{K}_A(u, t) \cdot e^{jm_1 t} e^{-jm_1 u} \cdot \phi_n^\lambda\left(\frac{t - m_2}{\lambda}\right) dt$. This means

$$e^{jm_1 u} \cdot \phi_n^\lambda(u - m_2) = \frac{e^{j(m_3 - \frac{du_0^2}{2b})}}{\mu_n^\lambda \cdot \sqrt{\lambda}} \mathbb{O}_A \left\{ e^{jm_1 t} \cdot \phi_n^\lambda\left(\frac{t - m_2}{\lambda}\right) \right\} (u).$$

The theorem is achieved.

Remark 1 If $a + d = 2$, and $b\omega_0 - (1 - a)u_0 = 0$, then the relation (15) holds for the following conditions $m_1 = \frac{(1-a)}{b}m_2 - \frac{u_0}{b}$, $m_2 \in \mathbb{R}$ and $m_3 = 0$.

Example 1 Consider the case $A = (-\frac{2}{3}, \frac{1}{3}, -9, 3, 1, 3)$, $A_1 = (-\frac{2}{3}, \frac{1}{3}, -9, 3, 0, 0)$. Since $|a + d| = \frac{7}{3} > 2$, then it is easily seen that

$$\lambda \in \left(\frac{-3 - 3\sqrt{3}}{2}, \frac{3 - 3\sqrt{3}}{2} \right) \cup \left(\frac{3\sqrt{3} - 3}{2}, \frac{3 + 3\sqrt{3}}{2} \right).$$

If we choose $\lambda = \frac{3}{2}$, then $A_\lambda = (-1, \frac{2}{9}, -\frac{27}{2}, 2, 0, 0)$. The Hermite functions and the values μ_n^λ can be expressed as

$$\phi_n^\lambda(t) = \frac{3}{2\sqrt{2^n n!} \sqrt{3\pi}} e^{-\frac{243(1-j\sqrt{3})t^2}{32}} H_n\left(\frac{9\sqrt{3}t}{4}\right), \quad \mu_n^\lambda = e^{-j(\frac{1}{2}+n)\frac{\pi}{3}} \quad (n \in \mathbb{N}).$$

Therefore, the relation (11) becomes $\phi_n^\lambda(t) = \sqrt{\frac{2}{3}} e^{j(\frac{1}{2}+n)\frac{\pi}{3}} \cdot \mathbb{O}_{A_1} \left\{ \phi_n^\lambda\left(\frac{2t}{3}\right) \right\} (u)$. From (16), we obtain $m_1 = 12, m_2 = 3, m_3 = \frac{9}{2}$. The relation (15) gives $e^{12jt} \phi_n^\lambda(t - 3) = \sqrt{\frac{2}{3}} e^{j(\frac{1}{2}+n)\frac{\pi}{3}} \cdot \mathbb{O}_A \left\{ e^{12jt} \phi_n^\lambda\left(\frac{2t-6}{3}\right) \right\} (u)$.

CONVOLUTIONS FOR THE OFFSET LINEAR CANONICAL TRANSFORM WITH HERMITE WEIGHTS

In this section, the space $L^p(\mathbb{R})$ will be endowed with the norm $\|\cdot\|_p$ defined by $\|f\|_p := \left(\int_{\mathbb{R}} |f(t)|^p dt \right)^{\frac{1}{p}}$, $p \geq 1$. Assume that the conditions of Theorem 1 and Theorem 2 are satisfied. The convolution for the OLCT of two signals $f(t)$ and $h(t)$ is defined by

$$(f \overset{1}{\otimes} h)(t) := \frac{K_A^2 e^{j(m_3 - \frac{du_0^2}{2b})} (\mathcal{E}_A(t))^{-1}}{\mu_n^\lambda \cdot \sqrt{\lambda}} \int_{\mathbb{R}^2} \bar{f}(\tau) \bar{h}(v) \cdot \mathcal{E}_A^{m_1}(t - \tau - v) \phi_n^\lambda\left(\frac{t - \tau - v - m_2}{\lambda}\right) d\tau dv, \quad (19)$$

provided that the integral in (19) is well-defined. Moreover, if $f, h \in L^1(\mathbb{R})$, then the function defined in (19) belongs to $L^1(\mathbb{R})$, and $\|f \overset{1}{\otimes} h\|_1 \leq C_2 \|f\|_1 \cdot \|h\|_1$, where C_2 is a positive constant. We have the following product theorem.

Theorem 3 Assume that $f, h \in L^1(\mathbb{R})$, F_A and H_A denote the OLCT of the signals $f(t)$ and $h(t)$ with parameters A , respectively. We have

$$\mathbb{O}_A \left\{ (f \overset{1}{\otimes} h)(t) \right\} (t) = e^{jm_1 u} e^{-j\left(\frac{d}{b}u^2 + 2\frac{(b\omega_0 - du_0)}{b}u\right)} \cdot \phi_n^\lambda(u - m_2) \cdot F_A(u) \cdot H_A(u).$$

Moreover, if $e^{jm_1 u} e^{-j\left(\frac{d}{b}u^2 + 2\frac{(b\omega_0 - du_0)}{b}u\right)} \cdot \phi_n^\lambda(u - m_2) \cdot F_A(u) \cdot H_A(u) \in \mathbb{O}_A(L^1(\mathbb{R}))$, then

$$(f \overset{1}{\otimes} h)(t) = \mathbb{O}_{A^{-1}} \left\{ e^{jm_1 u} e^{-j\left(\frac{d}{b}u^2 + 2\frac{(b\omega_0 - du_0)}{b}u\right)} \cdot \phi_n^\lambda(u - m_2) \cdot F_A(u) \cdot H_A(u) \right\} (t). \quad (20)$$

Proof. Using the identity (9) and Theorem 2, we realize that

$$\begin{aligned} & e^{jm_1 u} e^{-j\left(\frac{d}{b}u^2 + 2\frac{(b\omega_0 - du_0)}{b}u\right)} \cdot \phi_n^\lambda(u - m_2) \cdot F_A(u) \cdot H_A(u) \\ &= e^{jm_1 u} e^{-j\left(\frac{d}{b}u^2 + 2\frac{(b\omega_0 - du_0)}{b}u\right)} \cdot \phi_n^\lambda(u - m_2) K_A^2 e^{j\left(\frac{d}{b}u^2 + \frac{2(b\omega_0 - du_0)}{b}u\right)} \int_{\mathbb{R}^2} \bar{f}(\tau) \bar{h}(v) e^{-\frac{j\tau}{b}} e^{-\frac{jv}{b}} d\tau dv \\ &= \frac{e^{j\left(m_3 - \frac{du_0^2}{2b}\right)}}{\mu_n^\lambda \cdot \sqrt{\lambda}} K_A^3 e^{j\left(\frac{d}{2b}u^2 + \frac{(b\omega_0 - du_0)}{b}u\right)} \int_{\mathbb{R}} e^{-\frac{j\tau}{b}} \cdot \mathcal{E}_A^{m_1}(t) \cdot \phi_n^\lambda\left(\frac{t - m_2}{\lambda}\right) dt \int_{\mathbb{R}^2} \bar{f}(\tau) \bar{h}(v) e^{-\frac{j\tau}{b}} e^{-\frac{jv}{b}} d\tau dv \\ &= \frac{e^{j\left(m_3 - \frac{du_0^2}{2b}\right)}}{\mu_n^\lambda \cdot \sqrt{\lambda}} K_A^3 e^{j\left(\frac{d}{2b}u^2 + \frac{(b\omega_0 - du_0)}{b}u\right)} \int_{\mathbb{R}^3} e^{-j\frac{(t+\tau+v)u}{b}} \cdot \bar{f}(\tau) \bar{h}(v) \cdot \mathcal{E}_A^{m_1}(t) \cdot \phi_n^\lambda\left(\frac{t - m_2}{\lambda}\right) d\tau dv dt. \end{aligned}$$

By making $\tau = \tau$, $v = v$ and $s = t + \tau + v$, we obtain

$$\begin{aligned} & e^{jm_1 u} e^{-j\left(\frac{d}{b}u^2 + 2\frac{(b\omega_0 - du_0)}{b}u\right)} \cdot \phi_n^\lambda(u - m_2) \cdot F_A(u) \cdot H_A(u) \\ &= K_A e^{-j\left(\frac{d}{b}u^2 + 2\frac{(b\omega_0 - du_0)}{b}u\right)} \int_{\mathbb{R}} e^{-\frac{jus}{b}} \mathcal{E}_A(s) \left\{ \frac{K_A^2 e^{j\left(m_3 - \frac{du_0^2}{2b}\right)} (\mathcal{E}_A(s))^{-1}}{\mu_n^\lambda \cdot \sqrt{\lambda}} \int_{\mathbb{R}^2} \bar{f}(\tau) \bar{h}(v) \mathcal{E}_A^{m_1}(s - \tau - v) \cdot \phi_n^\lambda\left(\frac{s - \tau - v - m_2}{\lambda}\right) d\tau dv \right\} ds \\ &= \int_{\mathbb{R}} \mathcal{K}_A(u, s) \left\{ \frac{K_A^2 e^{j\left(m_3 - \frac{du_0^2}{2b}\right)} (\mathcal{E}_A(s))^{-1}}{\mu_n^\lambda \cdot \sqrt{\lambda}} \int_{\mathbb{R}^2} \bar{f}(\tau) \bar{h}(v) \cdot \mathcal{E}_A^{m_1}(s - \tau - v) \phi_n^\lambda\left(\frac{s - \tau - v - m_2}{\lambda}\right) d\tau dv \right\} ds \\ &= \int_{\mathbb{R}} \mathcal{K}_A(u, s) \cdot (f \overset{1}{\otimes} h)(s) ds = \mathbb{O}_A \left\{ (f \overset{1}{\otimes} h)(u) \right\}. \end{aligned}$$

The proof is concluded.

By using the same method as in Theorem 3, we derive the next result.

Theorem 4 Let $f, h \in L^1(\mathbb{R})$, F_A and H_A denote the OLCT of the signals $f(t)$ and $h(t)$ with parameters A , respectively. The transform

$$(f \overset{2}{\otimes} h)(t) := \frac{K_A^2 e^{j\left(m_3 - \frac{du_0^2}{2b}\right)} (\mathcal{E}_A(t))^{-1}}{\mu_n^\lambda \cdot \sqrt{\lambda}} \int_{\mathbb{R}^2} \bar{f}(\tau) \bar{h}(v) \cdot \mathcal{E}_A^{m_1}(t - \tau - v + 2(b\omega_0 - du_0)) \phi_n^\lambda\left(\frac{t - \tau - v - m_2 + 2(b\omega_0 - du_0)}{\lambda}\right) d\tau dv \quad (21)$$

defines a convolution belonging $L^1(\mathbb{R})$, and turns possible the following factorization identity

$$\mathbb{O}_A \left\{ (f \overset{2}{\otimes} h)(t) \right\} (t) = e^{jm_1 u} e^{-j\frac{d}{b}u^2} \cdot \phi_n^\lambda(u - m_2) \cdot F_A(u) \cdot H_A(u).$$

The convolution for the OLCT of two signals $f(t)$ and $h(t)$ associated with the Hermite functions $\phi_n^\lambda(\frac{u}{\sqrt{3}} - m_2)$ scaled by the chirp $e^{-\frac{j\mu_1 u}{\sqrt{3}}}$, is defined as

$$(f \overset{3}{\otimes} h)(t) := \frac{\sqrt{3} K_A^2 e^{j\left(m_3 - \frac{du_0^2}{2b}\right)} (\mathcal{E}_A(t))^{-1}}{\mu_n^\lambda \cdot \sqrt{\lambda}} \int_{\mathbb{R}^2} \bar{f}(\tau) \bar{h}(v) \cdot \mathcal{E}_A^{m_1}(\sqrt{3}t - \tau - v + \kappa) \phi_n^\lambda\left(\frac{\sqrt{3}t - \tau - v - m_2 + \kappa}{\lambda}\right) d\tau dv, \quad (22)$$

where $\kappa = (3 - \sqrt{3})(b\omega_0 - du_0)$ (as long as the integral in (22) is well-defined). Moreover, if $f, h \in L^1(\mathbb{R})$ then $(f \overset{3}{\otimes} h)(t) \in L^1(\mathbb{R})$ since $\|f \overset{3}{\otimes} h\|_1 \leq C_3 \|f\|_1 \cdot \|h\|_1$ for some $C_3 > 0$.

Theorem 5 Let $f, h \in L^1(\mathbb{R})$, F_A and H_A denote the OLCT of the signals $f(t)$ and $h(t)$, with parameter A , respectively. The following factorization identity holds

$$\mathbb{O}_A \left\{ (f \overset{3}{\otimes} h)(t) \right\} (u) = e^{\frac{jm_1 u}{\sqrt{3}}} \cdot \phi_n^\lambda \left(\frac{u}{\sqrt{3}} - m_2 \right) \cdot F_A \left(\frac{u}{\sqrt{3}} \right) \cdot H_A \left(\frac{u}{\sqrt{3}} \right).$$

Moreover, if $e^{\frac{jm_1 u}{\sqrt{3}}} \cdot \phi_n^\lambda \left(\frac{u}{\sqrt{3}} - m_2 \right) \cdot F_A \left(\frac{u}{\sqrt{3}} \right) \cdot H_A \left(\frac{u}{\sqrt{3}} \right) \in \mathbb{O}_A(L^1(\mathbb{R}))$, then

$$(f \overset{3}{\otimes} h)(t) = \mathbb{O}_{A^{-1}} \left\{ e^{\frac{jm_1 u}{\sqrt{3}}} \cdot \phi_n^\lambda \left(\frac{u}{\sqrt{3}} - m_2 \right) \cdot F_A \left(\frac{u}{\sqrt{3}} \right) \cdot H_A \left(\frac{u}{\sqrt{3}} \right) \right\} (t). \quad (23)$$

Proof. Based on (9) and (16), we have

$$\begin{aligned} e^{jm_1 u} \cdot \phi_n^\lambda(u - m_2) \cdot F_A(u) \cdot H_A(u) &= e^{jm_1 u} \cdot \phi_n^\lambda(u - m_2) \cdot K_A^2 e^{j\left(\frac{d}{b}u^2 + \frac{2(b\omega_0 - du_0)}{b}u\right)} \int_{\mathbb{R}^2} \bar{f}(\tau) \bar{h}(v) e^{-\frac{j\tau}{b}} e^{-\frac{jv}{b}} d\tau dv \\ &= \frac{e^{j\left(m_3 - \frac{du_0^2}{2b}\right)} K_A^3}{\mu_n^\lambda \cdot \sqrt{\lambda}} e^{3j\left(\frac{d}{2b}u^2 + \frac{(b\omega_0 - du_0)}{b}u\right)} \int_{\mathbb{R}} e^{-\frac{j\tau}{b}} \cdot \mathcal{E}_A^{m_1}(t) \cdot \phi_n^\lambda\left(\frac{t - m_2}{\lambda}\right) dt \int_{\mathbb{R}^2} \bar{f}(\tau) \bar{h}(v) e^{-\frac{j\tau}{b}} e^{-\frac{jv}{b}} d\tau dv \\ &= \frac{e^{j\left(m_3 - \frac{du_0^2}{2b}\right)} K_A^3}{\mu_n^\lambda \cdot \sqrt{\lambda}} e^{j\left(\frac{d}{2b}(u\sqrt{3})^2 + \frac{(b\omega_0 - du_0)}{b}(u\sqrt{3})\right)} \int_{\mathbb{R}^3} e^{-\frac{j\left(t + \tau + v - (3 - \sqrt{3})(b\omega_0 - du_0)\right)u}{b}} \bar{f}(\tau) \bar{h}(v) \mathcal{E}_A^{m_1}(t) \cdot \phi_n^\lambda\left(\frac{t - m_2}{\lambda}\right) d\tau dv dt. \end{aligned}$$

Performing the change of variables $\tau = \tau$, $v = v$ and $s = t + \tau + v - \kappa$, we achieve

$$\begin{aligned} e^{jm_1 u} \phi_n^\lambda(u - m_2) \cdot F_A(u) \cdot H_A(u) &= K_A e^{j\left(\frac{d}{2b}(u\sqrt{3})^2 + \frac{(b\omega_0 - du_0)}{b}(u\sqrt{3})\right)} \int_{\mathbb{R}} e^{-\frac{j\tau}{b}} \left(\frac{s}{\sqrt{3}}\right) \cdot \mathcal{E}_A\left(\frac{s}{\sqrt{3}}\right) \left\{ \frac{K_A^2 e^{j\left(m_3 - \frac{du_0^2}{2b}\right)} \left(\mathcal{E}_A\left(\frac{s}{\sqrt{3}}\right)\right)^{-1}}{\mu_n^\lambda \cdot \sqrt{\lambda}} \times \right. \\ &\quad \left. \int_{\mathbb{R}^2} \bar{f}(\tau) \bar{h}(v) \cdot \mathcal{E}_A^{m_1}(s - \tau - v + \kappa) \cdot \phi_n^\lambda\left(\frac{s - \tau - v - m_2 + \kappa}{\lambda}\right) d\tau dv \right\} ds \\ &= \int_{\mathbb{R}} \mathcal{K}_A\left(u\sqrt{3}, \frac{s}{\sqrt{3}}\right) \left\{ \frac{K_A^2 e^{j\left(m_3 - \frac{du_0^2}{2b}\right)} \left(\mathcal{E}_A\left(\frac{s}{\sqrt{3}}\right)\right)^{-1}}{\mu_n^\lambda \cdot \sqrt{\lambda}} \int_{\mathbb{R}^2} \bar{f}(\tau) \bar{h}(v) \cdot \mathcal{E}_A^{m_1}(s - \tau - v + \kappa) \phi_n^\lambda\left(\frac{s - \tau - v - m_2 + \kappa}{\lambda}\right) d\tau dv \right\} ds \\ &= \int_{\mathbb{R}} \mathcal{K}_A\left(u\sqrt{3}, \frac{s}{\sqrt{3}}\right) \cdot (f \overset{3}{\otimes} h)\left(\frac{s}{\sqrt{3}}\right) \cdot d\left(\frac{s}{\sqrt{3}}\right) = \mathbb{O}_A \left\{ (f \overset{3}{\otimes} h) \right\} (u\sqrt{3}), \end{aligned}$$

which proves the theorem.

Corollary 1 Let $f, h \in L^1(\mathbb{R})$, $k \in \{1, 2\}$, F_{A_1} and H_{A_1} denote the LCT of the signals $f(t)$ and $h(t)$ with parameters A_1 , respectively. The convolution of two signals $f(t)$, $h(t)$ for the LCT is defined as follows

$$(f \overset{k}{\otimes} h)(t) = \frac{K_{A_1}^2 (\mathcal{E}_{A_1}(t))^{-1}}{\mu_n^\lambda \cdot \sqrt{\lambda}} \int_{\mathbb{R}^2} \bar{f}(\tau) \bar{h}(v) \cdot \phi_n^\lambda\left(\frac{t - \tau - v}{\lambda}\right) d\tau dv. \quad (24)$$

and we have

$$\mathbb{O}_{A_1} \left\{ (f \overset{k}{\otimes} h)(t) \right\} (u) = \phi_n^\lambda(u) \cdot e^{-j\frac{d}{2b}u^2} \cdot F_{A_1}(u) \cdot H_{A_1}(u). \quad (25)$$

Corollary 2 Let $f, h \in L^1(\mathbb{R})$, F_{A_1} and H_{A_1} are the LCT of the signals $f(t)$ and $h(t)$ with parameters A_1 . The convolution of two signals $f(t)$, $h(t)$ for the LCT with the Hermite weights $\phi_n^\lambda(\frac{u}{\sqrt{3}})$ is defined by

$$(f \overset{3}{\otimes} h)(t) = \frac{\sqrt{3} K_{A_1}^2 (\mathcal{E}_{A_1}(t))^{-1}}{\mu_n^\lambda \cdot \sqrt{\lambda}} \int_{\mathbb{R}^2} \bar{f}(\tau) \bar{h}(v) \cdot \phi_n^\lambda\left(\frac{\sqrt{3}t - \tau - v}{\lambda}\right) d\tau dv. \quad (26)$$

and the following relation holds

$$\mathbb{O}_{A_1} \left\{ (f \overset{3}{\otimes} h)(t) \right\} (u) = \phi_n^\lambda\left(\frac{u}{\sqrt{3}}\right) \cdot F_{A_1}\left(\frac{u}{\sqrt{3}}\right) \cdot H_{A_1}\left(\frac{u}{\sqrt{3}}\right). \quad (27)$$

APPLICATIONS

Young's Convolution Inequalities

Note that $1 = |e^{jt}| = |\mathcal{E}_A(t)| = |\mathcal{E}_A^m(t)| = |\mu_n^\lambda|$, and $|f(t)| = |\bar{f}(t)|$. We derive the following theorem (see [1, 3]).

Theorem 6 Suppose that $p, q, r, s \geq 1$, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, $k \in \{1, 2, 3\}$. Then,

- (i) $\|f \otimes^k h\|_s \leq C_4 \|\phi_n\|_s \cdot \|f\|_1 \cdot \|h\|_1$, for any $f, h \in L^1(\mathbb{R})$.
- (ii) $\|f \otimes^k h\|_r \leq C_5 \|\phi_n\|_1 \cdot \|f\|_p \cdot \|h\|_q$, for any $f \in L^p(\mathbb{R}), h \in L^q(\mathbb{R})$, where C_1, C_2 are some positive constants.

Proof. We will present the proof for the case $k = 3$. The cases $k \in \{1, 2\}$ will be omitted because the proofs are analogous. Remind that $\phi_n(t)$ are rapidly decreasing functions. By applying the Minkowski's integral inequality and changing variable we obtain

$$\begin{aligned} \left[\int_{\mathbb{R}} \left| (f \otimes^3 h)(t) \right|^s dt \right]^{1/s} &= \left| \frac{\sqrt{3}K_A^2}{\sqrt{\lambda}} \right| \left[\int_{\mathbb{R}} \left| \int_{\mathbb{R}^2} \bar{f}(\tau) \bar{h}(v) \cdot \phi_n^\lambda \left(\frac{\sqrt{3}t - \tau - v - m_2 + \kappa}{\lambda} \right) d\tau dv \right|^s dt \right]^{1/s} \\ &\leq \left| \frac{\sqrt{3}K_A^2}{\sqrt{\lambda}} \right| \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} \left| \phi_n^\lambda \left(\frac{\sqrt{3}t - \tau - v - m_2 + \kappa}{\lambda} \right) \right|^s \cdot |\bar{f}(\tau)|^s \cdot |\bar{h}(v)|^s dt \right)^{1/s} d\tau dv \\ &= \left| \frac{\sqrt{3}K_A^2}{\sqrt{\lambda}} \right| \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} \left| \phi_n^\lambda \left(\frac{\sqrt{3}t - \tau - v - m_2 + \kappa}{\lambda} \right) \right|^s dt \right)^{1/s} \cdot |\bar{f}(\tau)| \cdot |\bar{h}(v)| d\tau dv \\ &\leq C_4 \|\phi_n\|_s \int_{\mathbb{R}^2} |f(\tau)| \cdot |h(v)| d\tau dv = C_4 \|\phi_n\|_s \cdot \|f\|_1 \cdot \|h\|_1, \end{aligned}$$

where C_4 is a positive constant. Thus, we obtain (i).

Now, we turn to the proof of (ii). Due to the formula (5) the convolution (22) can be also expressed as

$$(f \otimes^3 h)(t) = 3K_A (\mathcal{E}_A(t))^{-1} \cdot (\bar{f}(\sqrt{3}t) * \bar{h}(\sqrt{3}t) * \mathcal{G}_\kappa(\sqrt{3}t)), \quad (28)$$

where $\mathcal{G}_\kappa(t) := \frac{\sqrt{3}K_A e^{j(m_3 - \frac{du_0^2}{2b})}}{\mu_n^\lambda \sqrt{\lambda}} \cdot \mathcal{E}_A^{m_1}(t + \kappa) \cdot \phi_n^\lambda \left(\frac{t - m_2 + \kappa}{\lambda} \right)$. Remind that $f \in L^p(\mathbb{R}), h \in L^q(\mathbb{R})$. By performing a change of variable, we realize that $\bar{f}(\sqrt{3}t) \in L^p(\mathbb{R}), \bar{h}(\sqrt{3}t) \in L^q(\mathbb{R})$. Applying the Young's inequality (6) for the case $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, we have $\bar{f}(\sqrt{3}t) * \bar{h}(\sqrt{3}t) \in L^r(\mathbb{R})$. Since the Hermite functions $\phi_n(t)$ are rapidly decreasing functions then, applying the Young's inequality (6) for the case $\frac{1}{r} + \frac{1}{1} = \frac{1}{r} + 1$, we get $(\bar{f}(\sqrt{3}t) * \bar{h}(\sqrt{3}t)) * \mathcal{G}_\kappa(\sqrt{3}t) \in L^r(\mathbb{R})$. Moreover, we also achieve

$$\|f \otimes^3 h\|_r \leq C_5 \|\phi_n\|_1 \cdot \|f\|_p \cdot \|h\|_q,$$

where C_5 is a positive constant. The proof is completed.

The Multiplicative Filter in the OLCT Domain

In this subsection, we will discuss an application of the new convolution to the design of multiplicative filters in the OLCT domain (see [7]). We only consider the convolution (19) when $n = 0$. The Hermite function $\phi_0^\lambda(t)$ and the value μ_0 are given by

$$\phi_0^\lambda(t) = \frac{1}{\sqrt{\beta} \sqrt{\pi}} e^{-\frac{(1+j\alpha)}{2\beta^2} t^2}, \quad \mu_0 = e^{-\frac{j\theta}{2}},$$

where α, β, θ can be taken from (8). We shall denote by $r_{in}(t)$ and $r_{out}(t)$ the input signal and output signal, respectively. From (20), the output signal can be expressed as

$$r_{out}(t) = \mathbb{O}_{A^{-1}} \left\{ \mathbb{O}_A \{r_{in}(t)\} (u) \cdot e^{jm_1 u} e^{-j(\frac{d}{b} u^2 + 2\frac{(bu_0 - du_0)}{b} u)} \cdot \phi_0^\lambda(u - m_2) \cdot H_A(u) \right\} (t). \quad (29)$$

FIGURE 1. Method of achieving multiplicative filter in time domain.

Let us now denote

$$\tilde{H}_A(u) = e^{jm_1 u} e^{-j(\frac{d}{b}u^2 + 2\frac{(b\omega_0 - d\omega_0)}{b}u)} \cdot \phi_0^\lambda(u - m_2) \cdot H_A(u). \quad (30)$$

Then, it follows $H_A(u) = e^{-jm_1 u} e^{j(\frac{d}{b}u^2 + 2\frac{(b\omega_0 - d\omega_0)}{b}u)} \cdot (\phi_0^\lambda(u - m_2))^{-1} \cdot \tilde{H}_A(u)$.

Based on different transforms $\tilde{H}(u)$, there are many ways to design a multiplicative filter. For instance, we can choose the function $h(t)$ such that $\tilde{H}_A(u)$ is constant over $[-\Omega, \Omega]$, and zero or with rapid decay outside that region. Let T be a constant and

$$\tilde{H}_A(u) = \begin{cases} T, & u \in [-\Omega, \Omega] \\ 0, & u \notin [-\Omega, \Omega]. \end{cases} \quad (31)$$

Thus, we obtain

$$r_{out}(t) = T \cdot \mathbb{O}_{A^{-1}}\{\mathbb{O}_A\{r_{in}(t)\}(u)\}(t).$$

From (3), the output signal $r_{out}(t)$ can be rewritten as

$$r_{out}(t) = K_A(E_A(t))^{-1}(\bar{r}_{in} * \ell)(t), \quad (32)$$

where the convolution function $\ell(t)$ is given by

$$\ell(t) = \frac{K_A e^{j(m_3 - \frac{d\omega_0^2}{2b})}}{\mu_0^\lambda \cdot \sqrt{\lambda}} \int_{\mathbb{R}} \bar{h}(v) \cdot E_A^{m_1}(t - v) \cdot \phi_0^\lambda\left(\frac{t - v - m_2}{\lambda}\right) dv. \quad (33)$$

This shows that we can achieve the multiplicative filter through the classic Fourier convolution of $r_{int}(t)$ and $\ell(t)$ in the time domain. A realization of the method is displayed in Figure 1 (see also [7]).

Using the expression (2), we obtain

$$\begin{aligned} h(t) &= CK_{A^{-1}} \int_{\mathbb{R}} H_A(u) e^{-j(\frac{d}{2b}u^2 - \frac{1}{b}tu + \frac{a}{2b}t^2 + \frac{(b\omega_0 - d\omega_0)}{b}u + \frac{u_0}{b}t)} du \\ &= CK_{A^{-1}} \int_{\mathbb{R}} e^{-jm_1 u} e^{j(\frac{d}{b}u^2 + 2\frac{(b\omega_0 - d\omega_0)}{b}u)} \cdot (\phi_0^\lambda(u - m_2))^{-1} \cdot \tilde{H}_A(u) e^{-j(\frac{d}{2b}u^2 - \frac{1}{b}tu + \frac{a}{2b}t^2 + \frac{(b\omega_0 - d\omega_0)}{b}u + \frac{u_0}{b}t)} du \\ &= CK_{A^{-1}} (E_A(t))^{-1} \int_{\mathbb{R}} e^{-jm_1 u} e^{j(\frac{d}{2b}u^2 + \frac{(b\omega_0 - d\omega_0)}{b}u)} \cdot (\phi_0^\lambda(u - m_2))^{-1} \cdot \tilde{H}_A(u) e^{\frac{jut}{b}} du. \end{aligned}$$

Then,

$$\bar{h}(t) = CK_{A^{-1}} \int_{\mathbb{R}} e^{-jm_1 u} e^{j(\frac{d}{2b}u^2 + \frac{(b\omega_0 - d\omega_0)}{b}u)} \cdot (\phi_0^\lambda(u - m_2))^{-1} \cdot \tilde{H}_A(u) e^{\frac{jut}{b}} du. \quad (34)$$

FIGURE 2. Wigner distributions of desired and observed signals.

Substituting (34) into (33), gives rise to

$$\begin{aligned}
 \ell(t) &= \frac{K_A e^{j(m_3 - \frac{du_0^2}{2b})}}{\mu_0^\lambda \cdot \sqrt{\lambda}} \int_{\mathbb{R}} \bar{h}(v) \cdot \mathcal{E}_A^{m_1}(t-v) \cdot \phi_0^\lambda\left(\frac{t-v-m_2}{\lambda}\right) dv \\
 &= \frac{K_A e^{j(m_3 - \frac{du_0^2}{2b})}}{\mu_0^\lambda \cdot \sqrt{\lambda}} \int_{\mathbb{R}} \left(CK_{A^{-1}} \int_{\mathbb{R}} e^{-jm_1 u} e^{j(\frac{d}{2b}u^2 + \frac{(b\omega_0 - du_0)}{b}u)} \cdot (\phi_0^\lambda(u-m_2))^{-1} \tilde{H}_A(u) e^{\frac{juv}{b}} du \right) \mathcal{E}_A^{m_1}(t-v) \cdot \phi_0^\lambda\left(\frac{t-v-m_2}{\lambda}\right) dv \\
 &= \frac{CK_{A^{-1}} K_A e^{j(m_3 - \frac{du_0^2}{2b})}}{\mu_0^\lambda \cdot \sqrt{\lambda}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-jm_1 u} e^{j(\frac{d}{2b}u^2 + \frac{(b\omega_0 - du_0)}{b}u)} \cdot (\phi_0^\lambda(u-m_2))^{-1} \cdot \tilde{H}_A(u) e^{\frac{juv}{b}} e^{jm_1(t-v)} \mathcal{E}_A(t-v) \cdot \phi_0^\lambda\left(\frac{t-v-m_2}{\lambda}\right) dv du \\
 &= \frac{CK_{A^{-1}} e^{j(m_3 - \frac{du_0^2}{2b})}}{\mu_0^\lambda \cdot \sqrt{\lambda}} \int_{\mathbb{R}} \left(e^{-jm_1 u} (\phi_0^\lambda(u-m_2))^{-1} \cdot \tilde{H}_A(u) \right) \left(K_A e^{j(\frac{d}{2b}u^2 + \frac{(b\omega_0 - du_0)}{b}u)} \int_{\mathbb{R}} e^{\frac{juv}{b}} e^{jm_1(t-v)} \mathcal{E}_A(t-v) \cdot \phi_0^\lambda\left(\frac{t-v-m_2}{\lambda}\right) dv \right) du.
 \end{aligned}$$

By taking $s = t - v$, it shows that

$$\begin{aligned}
 K_A e^{j(\frac{d}{2b}u^2 + \frac{(b\omega_0 - du_0)}{b}u)} \int_{\mathbb{R}} e^{\frac{jus}{b}} e^{jm_1(t-s)} \mathcal{E}_A(t-s) \phi_0^\lambda\left(\frac{t-s-m_2}{\lambda}\right) ds &= e^{\frac{jut}{b}} K_A e^{j(\frac{d}{2b}u^2 + \frac{(b\omega_0 - du_0)}{b}u)} \int_{\mathbb{R}} e^{-\frac{jus}{b}} \mathcal{E}_A(s) \left(e^{jm_1 s} \phi_0^\lambda\left(\frac{s-m_2}{\lambda}\right) \right) ds \\
 &= e^{\frac{jut}{b}} \odot_A \left\{ e^{jm_1 s} \phi_0^\lambda\left(\frac{s-m_2}{\lambda}\right) \right\} (u). \tag{35}
 \end{aligned}$$

Manipulating (15), we have

$$K_A e^{j(\frac{d}{2b}u^2 + \frac{(b\omega_0 - du_0)}{b}u)} \int_{\mathbb{R}} e^{\frac{jus}{b}} e^{jm_1(t-s)} \mathcal{E}_A(t-s) \cdot \phi_0^\lambda\left(\frac{t-s-m_2}{\lambda}\right) ds = \mu_0^\lambda \cdot \sqrt{\lambda} e^{-j(m_3 - \frac{du_0^2}{2b})} e^{\frac{jut}{b}} e^{jm_1 u} \cdot \phi_0^\lambda(u-m_2). \tag{36}$$

Therefore, $\ell(t) = CK_{A^{-1}} \int_{\mathbb{R}} \tilde{H}_A(u) e^{\frac{jut}{b}} du$. Based on the relation (31), we derive

$$\ell(t) = CK_{A^{-1}} \int_{-\Omega}^{\Omega} T e^{\frac{jut}{b}} du = 2bCK_{A^{-1}} T \cdot \frac{\sin(\frac{t\Omega}{b})}{t}.$$

Hence $\ell(t) = \frac{T}{\pi K_A} \cdot \frac{\sin(\frac{t\Omega}{b})}{t}$.

In the following example, we shall use the proposed multiplicative filter to restore an observed signal $r_{in}(t) = X(t) + N(t)$, where $X(t), N(t)$ denote the desired signal and the additive noise, respectively.

FIGURE 3. Result of multiplicative filter achieved by using convolution (19).

Example 2 We use $r_{in}(t) = e^{-t^2} \cdot \sin(1.5t) + e^{i(t+10)^2}$, $X(t) = e^{-t^2} \cdot \sin(1.5t)$, and $N(t) = e^{i(t+10)^2}$. For convenience, let $u_0 = \omega_0 = 0$. The Wigner distributions of $X(t)$ and $r_{in}(t)$ are shown in Figure 2. Thus, we can choose (see [5]) $a = -\frac{2}{3}$, $b = \frac{1}{3}$, $\Omega = 2$. The transfer function reads

$$\tilde{H}_A(u) = \begin{cases} 1, & u \in [-2, 2] \\ 0, & u \notin [-2, 2], \end{cases}$$

and $\ell(t) = \sqrt{\frac{2i}{3\pi}} \cdot \frac{\sin 6t}{t}$. The output signal can be expressed as $r_{out}(t) = \sqrt{\frac{3}{2\pi i}} e^{it^2} \cdot (\tilde{r}_{in} * \ell)(t)$. The consequent result of the multiplicative filter is displayed in Figure 3.

ACKNOWLEDGMENTS

This work was supported in part by FCT–Portuguese Foundation for Science and Technology through the Center for Research and Development in Mathematics and Applications (CIDMA) of Universidade de Aveiro, within project UID/MAT/04106/2013, and by the Viet Nam National Foundation for Science and Technology Development (NAFOSTED).

REFERENCES

- [1] P. K. Anh, L. P. Castro, P. T. Thao and N. M. Tuan, Inequalities and consequences of new convolutions for the fractional Fourier transform with Hermite weights, *American Institute of Physics, AIP Proceedings* **1798**(1), (2017).
- [2] W. Beckner, Inequalities in Fourier analysis, *Annals of Math* **102**(1), 159–182 (1975).
- [3] L. P. Castro, L. T. Minh and N. M. Tuan, New convolutions for quadratic-phase Fourier integral operators and their applications, *Mediterranean Journal of Mathematics* **15**:13, 17pp. (2018).
- [4] A. Koc, H. M. Ozaktas, C. Candan and M. A. Kutay, Digital computation of linear canonical transforms, *IEEE Trans. Signal Process* **56**(6) 2383–2394 (2008).
- [5] S. C. Pei and J. J. Ding, Relations between fractional operations and time-frequency distributions, and their applications, *IEEE Trans. Signal Processing* **49**(8), 1638–1655 (2001).
- [6] S. C. Pei and J. J. Ding, Eigenfunctions of linear canonical transform, *IEEE Trans. Signal Process* **50**(1) 11–26 (2002).
- [7] Q. Xiang and K. Qin, Convolution, correlation, and sampling theorems for the offset linear canonical transform, *Signal, Image and Video Processing* **8**(3), 433–442 (2014).