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Asymptotically flat spinning scalar, Dirac and Proca stars

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ABSTRACT

Einstein's gravity minimally coupled to free, massive, classical fundamental fields admits particle-like solutions. These are asymptotically flat, everywhere non-singular configurations that realise Wheeler's concept of a *geon*: a localised lump of self-gravitating energy whose existence is anchored on the non-linearities of general relativity, trivialising in the flat spacetime limit. In [1] the key properties for the existence of these solutions (also referred to as *stars* or *self-gravitating solitons*) were discussed – which include a harmonic time dependence in the matter field –, and a comparative analysis of the stars arising in the Einstein-Klein-Gordon, Einstein-Dirac and Einstein-Proca models was performed, for the particular case of static, spherically symmetric spacetimes. In the present work we generalise this analysis for *spinning* solutions. In particular, the spinning Einstein-Dirac stars are reported here for the first time. Our analysis shows that the high degree of universality observed in the spherical case remains when angular momentum is allowed. Thus, as classical field theory solutions, these self-gravitating solitons are rather insensitive to the fundamental fermionic or bosonic nature of the corresponding field, displaying similar features. We describe some physical properties and, in particular, we observe that the angular momentum of the spinning stars satisfies the quantisation condition $J = mN$, for all models, where N is the particle number and m is an integer for the bosonic fields and a half-integer for the Dirac field. The way in which this quantisation condition arises, however, is more subtle for the non-zero spin fields.

1. Introduction

In vacuum Einstein's general relativity, the only physically reasonable stationary solution describing a localised lump of energy is provided by the Kerr black hole [2,3]. A simple application of a Komar integral [4] and the positive energy theorem [5,6] shows that there are no everywhere regular localised lumps of energy in vacuum, as realised (in a different way) long ago by Lichnerowicz [7].

With some caveats (see, *e.g.* the discussion in [8]), the situation is similar if Einstein's gravity is minimally coupled to a massless, free, fundamental field. This includes, in particular, electrovacuum. But a rather distinct situation becomes possible if the fundamental field is massive and with enough degrees of freedom. Considering a massive complex Klein-Gordon, or Dirac or Proca field, minimally coupled to Einstein's gravity, everywhere regular localised solutions are possible – see [9–12], for the original references.[1] We shall refer to these self-gravitating solitonic solutions as, respectively, scalar, Dirac or Proca stars, which provide explicit realisations of Wheeler's *geons* [13]. Naturally, they were originally computed under the assumption of a spherically symmetric, static spacetime. Yet, rotation is ubiquitous, for all objects, in all scales. Thus, despite the higher technical complexity, it is of interest to study rotating scalar, Dirac or Proca stars. For the bosonic fields, the corresponding spinning stars were first computed in [12,14,15,17,18], whereas for the Dirac case they will be described herein for the first time. This is one of the main purposes of this work.

It turns out that a spinning Dirac star is somewhat more natural than the static spinless one. Indeed, since a single fermion possesses an intrinsic angular momentum, the matter content re-

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[1] The inclusion of matter self-interactions opens the possibility of particle-like objects with finite energy also in flat spacetime – see [16] for a review – albeit only bosonic such solutions have been so far considered. In this work we shall restrict ourselves to free matter fields.

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quired to obtain a spinless solution consists of (at least) two fermionic fields which allows for an angular momentum cancellation. To study spinning Dirac stars, on the other hand we need a single Dirac field. With respect to their bosonic counterparts, which can be regarded as *'macroscopic quantum states'* prevented from gravitationally collapsing by Heisenberg's uncertainty principle, the interpretation of the Dirac stars is more delicate and has been considered in [1]. As classical field theory solutions, however, Dirac stars are in many ways similar to the bosonic ones, an observation already established in [1] for the static case and confirmed here for the spinning solutions. For instance, rotating Dirac stars have an intrinsic toroidal topology in their energy distribution, which parallels that of the rotating scalar stars [14]; for all cases, moreover, the star's angular momentum J is quantised as $J = mQ$, where m is an integer and Q the Noether charge, that also becomes an integer $Q = N$ upon quantisation. To make this comparison more meaningful, following [1], we analyse the three types of stars under a unified framework. Thus, the mathematical description of each of the three models is made in parallel to emphasise the similarities. The physical interpretation is only distinct when quantisation is taken into account, which distinguishes fermions and bosons. Then, in particular, whereas the bosonic configurations form a continuous sequence or family of solutions for a given field mass, fermionic solutions do not, due to Pauli's exclusion principle [1]. To be clear, the Dirac stars we are considering, upon quantisation, correspond to the gravitational field of a single fermion, rather than a quantum fermionic star – see [19] for work on the latter.

This paper is organised as follows. In Section 2 we describe the basic equations of each of the three different models. In Section 3 we introduce the spacetime and matter fields ansatz. In Section 4 we discuss the global quantities and the angular momentum-Noether charge relation which is universal for the three models but appears in a more contrived way in the cases with non-zero spin. In Section 5 we construct the spinning stars by solving numerically the field equations subject to specified boundary conditions. We also clarify the physical interpretation of the sequences of fermionic solutions. Concluding remarks and some open questions are presented in Section 6.

2. The model

Let us first describe the three models. The discussion and conventions follow closely those in [1] where a few more details are provided. Einstein's gravity in $3+1$ dimensional spacetime is minimally coupled with a spin-s field, where s takes one of the values $s = 0, \frac{1}{2}, 1$. The action is (with $c = 1 = \hbar$)

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} + \mathcal{L}_{(s)}\right] , \tag{2.1}$$

where the three possible matter Lagrangians are:

$$\mathcal{L}_{(0)} = -g^{\alpha\beta}\bar{\Phi}_{,\,\alpha}\Phi_{,\,\beta} - \mu^2\bar{\Phi}\Phi \,,$$

$$\mathcal{L}_{(1)} = -\frac{1}{4}\mathcal{F}_{\alpha\beta}\bar{\mathcal{F}}^{\alpha\beta} - \frac{\mu^2}{2}\mathcal{A}_\alpha\bar{\mathcal{A}}^\alpha \,, \tag{2.2}$$

$$\mathcal{L}_{(1/2)} = -i\left[\frac{1}{2}\left(\{\hat{\not{D}}\overline{\Psi}\}\Psi - \overline{\Psi}\hat{\not{D}}\Psi\right) + \mu\overline{\Psi}\Psi\right] . \tag{2.3}$$

Here, Φ is a complex scalar field; Ψ is a Dirac 4-spinor, with four complex components; $\hat{\not{D}} \equiv \gamma^\mu\hat{D}_\mu$, where γ^μ are the curved spacetime gamma matrices, $\hat{D}_\mu = \partial_\mu - \Gamma_\mu$ is the spinor covariant derivative and Γ_μ are the spinor connection matrices [20]; $\mathcal{A}$ is a complex 4-potential, with the field strength $\mathcal{F} = d\mathcal{A}$. In all cases, $\mu > 0$ corresponds to the mass of the field(s). For the scalar and Proca fields, the overbar denotes complex conjugation; $\overline{\Psi}$ denotes the Dirac conjugate [20].

Variation of (2.1) with respect to the metric leads to the Einstein field equations

$$E_{\alpha\beta} = G_{\alpha\beta} - 8\pi G\, T^{(s)}_{\alpha\beta} = 0\,, \tag{2.4}$$

where $G_{\alpha\beta}$ denotes, as usual, the Einstein tensor and $T^{(s)}_{\alpha\beta}$ is the energy-momentum tensor:

$$T^{(0)}_{\alpha\beta} = \bar{\Phi}_{,\alpha}\Phi_{,\beta} + \bar{\Phi}_{,\beta}\Phi_{,\alpha} - g_{\alpha\beta}\left[\frac{1}{2}g^{\gamma\delta}(\bar{\Phi}_{,\gamma}\Phi_{,\delta} + \bar{\Phi}_{,\delta}\Phi_{,\gamma}) + \mu^2\bar{\Phi}\Phi\right] , \tag{2.5}$$

$$T^{(1/2)}_{\alpha\beta} = -\frac{i}{2}\left[\overline{\Psi}\gamma_{(\alpha}\hat{D}_{\beta)}\Psi - \left\{\hat{D}_{(\alpha}\overline{\Psi}\right\}\gamma_{\beta)}\Psi\right] , \tag{2.6}$$

$$T^{(1)}_{\alpha\beta} = \frac{1}{2}(\mathcal{F}_{\alpha\sigma}\bar{\mathcal{F}}_{\beta\gamma} + \bar{\mathcal{F}}_{\alpha\sigma}\mathcal{F}_{\beta\gamma})g^{\sigma\gamma} - \frac{1}{4}g_{\alpha\beta}\mathcal{F}_{\sigma\tau}\bar{\mathcal{F}}^{\sigma\tau} + \frac{\mu^2}{2}\left[\mathcal{A}_\alpha\bar{\mathcal{A}}_\beta + \bar{\mathcal{A}}_\alpha\mathcal{A}_\beta - g_{\alpha\beta}\mathcal{A}_\sigma\bar{\mathcal{A}}^\sigma\right] . \tag{2.7}$$

The corresponding matter field equations are:

$$\nabla^2\Phi - \mu^2\Phi = 0\,, \qquad \hat{\not{D}}\Psi - \mu\Psi = 0\,, \qquad \nabla_\alpha\mathcal{F}^{\alpha\beta} - \mu^2\mathcal{A}^\beta = 0\,. \tag{2.8}$$

In the Proca case, the field eqs. (2.8) imply the Lorentz condition, $\nabla_\alpha\mathcal{A}^\alpha = 0$.

The matter field action, in all cases, possesses a global $U(1)$ invariance, under the transformation $\{\Phi, \Psi, \mathcal{A}\} \to e^{ia}\{\Phi, \Psi, \mathcal{A}\}$, where a is a constant. By Noether's theorem this implies the existence of a conserved 4-current:

$$j^\alpha_{(0)} = -i(\bar{\Phi}\partial^\alpha\Phi - \Phi\partial^\alpha\bar{\Phi})\,, \qquad j^\alpha_{(1/2)} = \bar{\Psi}\gamma^\alpha\Psi\,,$$

$$j^\alpha_{(1)} = \frac{i}{2}\left[\bar{\mathcal{F}}^{\alpha\beta}\mathcal{A}_\beta - \mathcal{F}^{\alpha\beta}\bar{\mathcal{A}}_\beta\right] . \tag{2.9}$$

Indeed, the field equations imply $j^\alpha_{(s);\alpha} = 0$. Then, integrating the timelike component of this 4-current on a spacelike hypersurface Σ yields a conserved *Noether charge*:

$$Q_{(s)} = \int_\Sigma n_\mu j^\mu_{(s)}\,, \tag{2.10}$$

where n is the unit normal to the Cauchy surface. The Noether charge become an integer after quantisation, $Q = N$, where N is the particle number.

3. The ansatz

We seek spacetimes with two commuting Killing vector fields, ξ and η, with $\xi = \partial_t$, and $\eta = \partial_\varphi$, in a coordinate system adapted to the isometries, where t and φ are the time and azimuthal coordinates, respectively. General relativity solutions with these symmetries are usually studied within the following metric ansatz: $ds^2 = -e^{-2U(\rho,z)}(dt + \Omega(\rho,z)d\varphi)^2 + e^{2U(\rho,z)}\big(e^{2k(\rho,z)}(d\rho^2 + dz^2) + S^2(\rho,z)d\varphi^2\big)$, where (ρ, z) correspond, asymptotically, to standard cylindrical coordinates. In the electrovacuum case, it is always possible to set $S \equiv \rho$, such that only three independent metric functions appear in the equations, and (ρ, z) become the canonical Weyl coordinates [21]. For the matter sources in this work, however, a generic metric ansatz with four independent functions is needed. Also, it turns out to be more convenient for numerics to use ‘spheroidal-type’ coordinates (r,θ) defined as $\rho = r\sin\theta$, $z = r\cos\theta$, instead of (ρ, z), with the usual range $0 \leqslant r < \infty$, $0 \leqslant \theta \leqslant \pi$. After a suitable redefinition of the metric functions, this leads to the following metric ansatz:

$$ds^2 = -e^{2F_0}dt^2 + e^{2F_1}\left(dr^2 + r^2 d\theta^2\right) + e^{2F_2} r^2 \sin^2\theta \left(d\varphi - \frac{W}{r}dt\right)^2 , \tag{3.11}$$

which has been employed in the study of $s = 0$ [22] and $s = 1$ [12, 17] spinning stars. The four metric functions $(F_i; W)$, $i = 0, 1, 2$, are functions of the variables r and θ only, chosen such that the trivial angular and radial dependence of the line element is already factorised. The symmetry axis of the spacetime is given by $\eta^2 = 0$ and corresponds to $\theta = 0, \pi$. The Minkowski spacetime background is approached for $r \to \infty$, where the asymptotic values are $F_i = 0$, $W = 0$.

For the Dirac stars case ($s = 1/2$), we shall employ the following orthonormal tetrad for the metric (3.11)

$$\mathbf{e}^0_\mu dx^\mu = e^{F_0} dt , \qquad \mathbf{e}^1_\mu dx^\mu = e^{F_1} dr ,$$
$$\mathbf{e}^2_\mu dx^\mu = e^{F_1} r d\theta , \qquad \mathbf{e}^3_\mu dx^\mu = e^{F_2} r \sin\theta \left(d\varphi - \frac{W}{r}dt\right) , \tag{3.12}$$

such that $ds^2 = \eta_{ab}(\mathbf{e}^a_\mu dx^\mu)(\mathbf{e}^b_\nu dx^\nu)$, where $\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1)$.

Let us now consider the ansatz for the three mater fields. In the scalar case, the matter field ansatz which is compatible with an axially symmetric geometry is written in terms of a single real function $\phi(r,\theta)$, and reads:

$$\Phi = e^{i(m\varphi - wt)}\phi(r,\theta) . \tag{3.13}$$

In the Proca case, the ansatz introduces four real potentials [12]:

$$\mathcal{A} = e^{i(m\varphi - wt)}\left(iV(r,\theta)dt + \frac{H_1(r,\theta)}{r}dr + H_2(r,\theta)d\theta + iH_3(r,\theta)\sin\theta d\varphi\right) . \tag{3.14}$$

In the case of a Dirac field, the ansatz also contains four real functions[2]

$$\Psi = e^{i(m\varphi - wt)}\begin{pmatrix} \psi_1(r,\theta) \\ \psi_2(r,\theta) \\ -i\psi_1^*(r,\theta) \\ -i\psi_2^*(r,\theta)\end{pmatrix} ,$$

$$\text{with } \psi_1(r,\theta) = P(r,\theta) + iQ(r,\theta) , \quad \psi_2(r,\theta) = X(r,\theta) + iY(r,\theta) . \tag{3.15}$$

For $s = 0, 1$, the parameter m in an integer, while for the Dirac field m is a half-integer; w is the field’s frequency in all three cases, which we shall take to be positive.

4. Global charges and the J-Q relation

Given the above ansatz, let us consider the explicit form for two relevant physical quantities. The first one is the temporal component of the current density:

$$j^t_{(0)} = 2e^{-2F_0}\left(w - \frac{mW}{r}\right)\phi^2 , \tag{4.16}$$

$$j^t_{(1/2)} = 2e^{-F_0}(P^2 + Q^2 + X^2 + Y^2) , \tag{4.17}$$

$$j^t_{(1)} = \frac{e^{-2(F_0+F_2)}}{r^2}H_3\left(wH_3 + \frac{mV}{\sin\theta}\right) + \frac{e^{-2(F_0+F_1)}}{r^3}\Big\{r(H_1^2 + H_2^2)\left(w - \frac{mW}{r}\right) + \cos\theta H_2 H_3 W + rH_1(rV_{,r} + \sin\theta W H_{3,r}) + H_2(rV_{,\theta} + \sin\theta W H_{3,\theta})\Big\} . \tag{4.18}$$

The second one is the angular momentum density:

$$T_\varphi^{t(0)} = 2e^{-2F_0} m\left(w - \frac{mW}{r}\right)\phi^2 , \tag{4.19}$$

$$T_\varphi^{t(1/2)} = e^{-F_0}m(P^2 + Q^2 + X^2 + Y^2) + e^{-F_0-F_1+F_2}\sin\theta\Big\{(PX + QY)[1 + r(F_{2,r} - F_{0,r})] - \frac{1}{2}(P^2 + Q^2 - X^2 - Y^2)(\cot\theta + F_{2,\theta} - F_{0,\theta}) + 2e^{-F_0+F_1}r\left(w - \frac{mW}{r}\right)(QX - PY)\Big\} , \tag{4.20}$$

$$T_\varphi^{t(1)} = -\frac{\mu^2}{r}e^{-2F_0}H_3\sin\theta(rV + H_3\sin\theta W) + \frac{e^{-2(F_0+F_1)}}{r}\Big\{\frac{H_1}{r}\sin\theta(-rw + 2mW)H_{3,r} + (mH_1 - r\sin\theta H_{3,r})V_{,r} - W[\sin^2\theta H_{3,r}^2 + \frac{1}{r^2}(\cos\theta H_3 + \sin\theta H_{3,\theta})^2] + \frac{mH_2 V_{,\theta}}{r} + \frac{1}{r^2}(\cos\theta H_3 + \sin\theta H_{3,\theta})[H_2(rw - 2mW) + rV_{,\theta}] + \frac{m}{r}(H_1^2 + H_2^2)\left(w - \frac{mW}{r}\right)\Big\} . \tag{4.21}$$

The ADM mass M and the angular momentum J of the solutions are read off from the asymptotic expansion:

$$g_{tt} = -1 + \frac{2M}{r} + \ldots , \qquad g_{\varphi t} = -\frac{2J}{r}\sin^2\theta + \ldots . \tag{4.22}$$

The total angular momentum can also be computed as the integral of the corresponding density[3]

$$J \equiv J_{(s)} = 2\pi\int_0^\infty dr\int_0^\pi d\theta \sin\theta r^2 e^{F_0+2F_1+F_2}T_\varphi^{t(s)} . \tag{4.23}$$

The explicit form of the Noether charge, as computed from (2.10), is

$$Q \equiv Q_{(s)} = 2\pi\int_0^\infty dr\int_0^\pi d\theta \sin\theta r^2 e^{F_0+2F_1+F_2} j^t_{(s)} . \tag{4.24}$$

For a scalar field one can easily see that J and Q are proportional,

$$J = mQ , \tag{4.25}$$

[2] Ansatz (3.15) is compatible with the (circular) metric form (3.11). Also, the ansatz considered in [1,11] in the study of spherically symmetric stars is recovered for $m = \pm 1/2$, with a factorised angular dependence.

[3] The ADM mass can also be computed as volume integral; however, this is less relevant in the context of this work.

since the corresponding densities (4.16), (4.19), are identical up to a factor of m. It turns out that this relation also holds for the Dirac and Proca case, but the result is less obvious, since the angular momentum *density* and Noether charge *density* are *not* proportional. Nonetheless, the proportionality still holds at the level of the integrated quantities. Indeed, in both cases the angular momentum density and Noether charge density (multiplied by the azimuthal index m) differ by a total divergence,[4]

$$T_\varphi^t = mj^t + \nabla_\alpha P^\alpha \ , \tag{4.26}$$

with

$$P^\alpha = \mathcal{A}_\varphi \bar{\mathcal{F}}^{\alpha t} + \bar{\mathcal{A}}_\varphi \mathcal{F}^{\alpha t} \ , \tag{4.27}$$

for the Proca field [17], and

$$P^\alpha = -\frac{i}{4}\overline{\Psi}\gamma_\varphi \gamma^\alpha \gamma^t \Psi \ , \tag{4.28}$$

for the Dirac field. The total divergence is non-zero locally; however, its volume integral vanishes for the solutions subject to the boundary conditions described in the next Section. As a result, (4.25) still holds for a Proca and Dirac fields. Observe, nonetheless, the implicit differences in this relation. The bosonic solutions with $m = 0$ are static; but for the Dirac stars m is a half-integer and thus cannot be zero – they are necessarily rotating (recall static Proca stars require at least two Dirac fields).

The solutions satisfy also a first law of thermodynamics of the type:

$$dM = w dQ \ , \tag{4.29}$$

which provides a test of numerical accuracy.

5. The solutions

In solving the equations of motion we exploit some symmetries thereof. Let us briefly comment on these, following [1]. Firstly, the factor of $4\pi G$ in the Einstein field equations can be set to unity by a redefinition of the matter functions

$$\{\Phi, \mathcal{A}, \Psi\} \to \frac{1}{\sqrt{4\pi G}}\{\Phi, \mathcal{A}, \Psi\} \ . \tag{5.30}$$

Secondly, the field equations remain invariant under the transformation

$$(*): \quad r \to \lambda r \ , \quad W \to \lambda W \ , \quad F_i \to F_i \ , \ \{w, \mu\} \to \frac{1}{\lambda}\{w, \mu\} \ ,$$
$$\left\{ \begin{array}{l} \Phi \to \Phi \ , \\ \mathcal{A} \to \frac{1}{\sqrt{\lambda}}\mathcal{A} \ , \\ \Psi \to \Psi \ . \end{array} \right\} , \tag{5.31}$$

where λ is a positive constant. In all three cases the ratio w/μ is left invariant by the $(*)$ symmetry. This $(*)$-invariance is used to work in units set by the field mass,

$$\bar{\mu} = 1 \ , \ \text{ i.e. } \lambda = \frac{1}{\mu} \ . \tag{5.32}$$

Then, to recover the physical quantities from those obtained in the numerical solution, a set of relations are used, identical to the ones described in [1].

[4] In deriving (4.26) one uses also the matter field equations.

5.1. The boundary conditions and numerical method

Given the matter ansatz (3.13)-(3.15), all components of the energy momentum tensor are zero, except for T_{rr}, $T_{r\theta}$, $T_{\varphi\varphi}$, T_{tt} and $T_{\varphi t}$, which possess a (r, θ)-dependence only. Then, the Einstein field equations with the energy momentum-tensors (2.5)-(2.7), plus the matter field equations (2.8), together with the ansatz (3.13)-(3.15), lead to a system of five (eight) coupled partial differential equations for the scalar (Dirac and Proca) cases. There are four equations for the metric functions F_i, W; these are found by taking suitable combinations of the Einstein equations: $E_r^r + E_\theta^\theta = 0$, $E_\varphi^\varphi = 0$, $E_t^t = 0$ and $E_\varphi^t = 0$; additionally, there is one (four) equations for the matter functions. Apart from these, there are two more Einstein equations $E_\theta^r = 0$, $E_r^r - E_\theta^\theta = 0$, which are not solved in practice. Following an argument originally proposed in [23], one can, however, show that the identities $\nabla_\nu E^{\nu r} = 0$ and $\nabla_\nu E^{\nu\theta} = 0$, imply the Cauchy-Riemann relations $\partial_{\bar{r}}\mathcal{P}_2 + \partial_\theta \mathcal{P}_1 = 0$, $\partial_{\bar{r}}\mathcal{P}_1 - \partial_\theta \mathcal{P}_2 = 0$, with $\mathcal{P}_1 = \sqrt{-g}E_\theta^r$, $\mathcal{P}_2 = \sqrt{-g}r(E_r^r - E_\theta^\theta)/2$ and $d\bar{r} = dr/r$. Therefore the weighted constraints E_θ^r and $E_r^r - E_\theta^\theta$ still satisfy Laplace equations in $(\bar{r}, \theta)$ variables. Then they are fulfilled, when one of them is satisfied on the boundary and the other at a single point [23]. From the boundary conditions below, it turns out that this is the case for all three models, *i.e.* the numerical scheme is self-consistent.

The boundary conditions are found by considering an approximate construction of the solutions on the boundary of the domain of integration together with the assumption of regularity and asymptotic flatness.[5] The metric functions satisfy

$$\partial_r F_i\big|_{r=0} = W\big|_{r=0} = 0 \ , \qquad F_i\big|_{r=\infty} = W\big|_{r=\infty} = 0 \ ,$$
$$\partial_\theta F_i\big|_{\theta=0,\pi} = \partial_\theta W\big|_{\theta=0,\pi} = 0 \ . \tag{5.33}$$

The scalar field amplitude vanishes on the boundary of the domain of integration (see *e.g.* [15])

$$\phi\big|_{r=0} = \phi\big|_{r=\infty} = \phi\big|_{\theta=0,\pi} = 0 \ . \tag{5.34}$$

The boundary conditions in the Proca case are [12,17],

$$H_i|_{r=0} = V|_{r=0} = 0 \ , \qquad H_i|_{r=\infty} = V|_{r=\infty} = 0 \ ,$$
$$H_1|_{\theta=0,\pi} = \partial_\theta H_2\big|_{\theta=0,\pi} = \partial_\theta H_3\big|_{\theta=0,\pi} = V|_{\theta=0,\pi} = 0 \ , \tag{5.35}$$

where the last set of conditions applies to the lowest $m = 1$ states. For a Dirac field, one imposes

$$P\big|_{r=0} = Q\big|_{r=0} = X\big|_{r=0} = Y\big|_{r=0} = 0 \ ,$$
$$P\big|_{r=\infty} = Q\big|_{r=\infty} = X\big|_{r=\infty} = Y\big|_{r=\infty} = 0 \ , \tag{5.36}$$

and, for $m = 1/2$,

$$\partial_\theta P\big|_{\theta=0} = \partial_\theta Q\big|_{\theta=0} = X\big|_{\theta=0} = Y\big|_{\theta=0} = 0 \ ,$$
$$P\big|_{\theta=\pi} = Q\big|_{\theta=\pi} = \partial_\theta X\big|_{\theta=\pi} = \partial_\theta Y\big|_{\theta=\pi} = 0 \ . \tag{5.37}$$

In all three cases, the solutions are found by using a fourth order finite difference scheme. The system of five/eight equations is discretised on a grid with $N_r \times N_\theta$ points (where typically $N_r \sim 200$, $N_\theta \sim 50$). We also introduce a new radial coordinate $x = r/(r + c)$, which maps the semi-infinite region $[0, \infty)$ onto the unit interval $[0, 1]$ (with c some constant of order one). The bosonic stars were constructed by using the professional package FIDISOL/CADSOL [24] which uses a Newton-Raphson method. The

[5] In particular, the matter field equations in the far field reveal that the solutions satisfy the condition $w < \mu$.

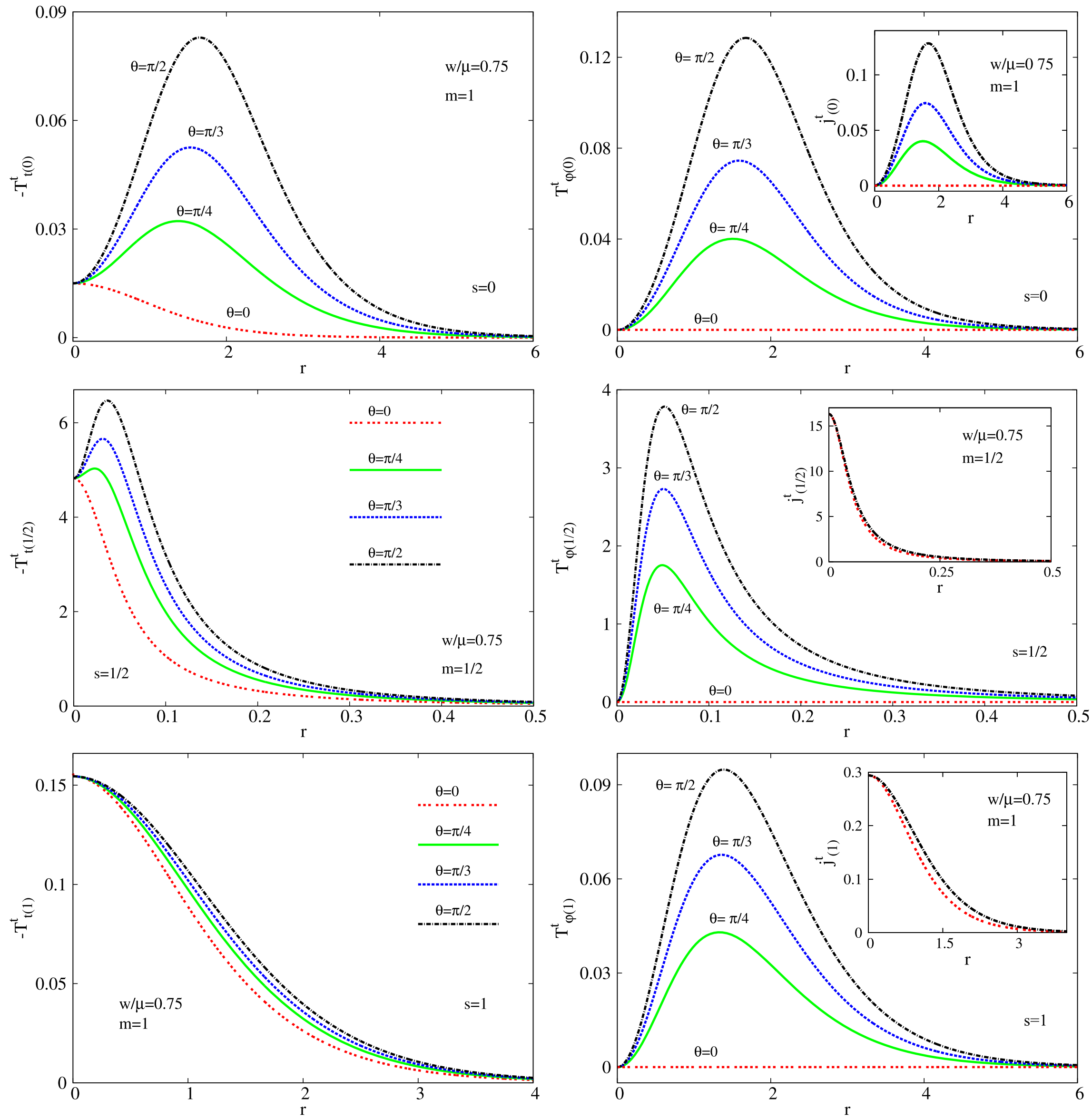

Fig. 1. The components T_t^t (left panels) and T_φ^t (right panels) of the energy-momentum tensor, and the 4-current component j^t (right panels inset) are shown for a fundamental branch solution of the scalar (top panels), Dirac (middle panels) and Proca (bottom panels) model, all with the same frequency, $w/\mu = 0.75$.

Einstein-Dirac system is solved with the Intel MKL PARDISO sparse direct solver [25], and using the CESDSOL[6] library. In all cases, the typical errors are of order of 10^{-4}.

The data shown in this work correspond to fundamental states, all matter functions being nodeless.[7] For the solutions herein, the geometry and the matter/current distributions are invariant under a reflexion in the equatorial plane ($\theta = \pi/2$), thus possessing a $\mathbb{Z}_2$ symmetry. Also, we shall consider solutions with the lowest number m (except for the Dirac stars in Fig. 3, right panel).

5.2. *Numerical results: basic properties and domain of existence*

In Fig. 1 we display the components T_t^t and T_φ^t of the energy-momentum tensor related to the mass-energy and angular momentum density, together with the temporal component j^t of the current for a typical (fundamental branch) solution of each model, all with $w/\mu = 0.75$ and the lowest allowed value of $m > 0$. One can see that, unlike for the scalar case, for Dirac and Proca stars, T_φ^t and j^t are not proportional, with the maximum of T_φ^t located on the equatorial plane, while j^t posses an almost spherical shape (the last feature, however, changes for higher m). A qualitative difference is that both scalar and Dirac stars possess an intrinsic toroidal shape in what concerns their energy distribution; for the Proca case, however, this distribution is almost spherical. Another qualitative difference is that for the scalar stars $j^t = 0$ on the symmetry axis.

As seen in Fig. 2, in all three cases, when considering a mass M/angular momentum J/Noether charge Q, *vs.* frequency w, diagram, the domain of existence of the solutions corresponds to a smooth curve. This curve starts from $M = 0$ ($J = 0$) for $w = \mu$, in which limit the fields becomes very diluted and the solution trivialises. At some intermediate frequency, a maximal mass (angular momentum) is attained. The parameters of these particular solutions are given in the 2nd-4th columns of Table 1. As can be seen

[6] Complex Equations – Simple Domain partial differential equations SOLver is a C++ package being developed by one of us (I.P.).

[7] For a given w, a discrete set of solutions may exist, indexed by the number of nodes, n, of (some of) the matter function(s). Such excited solutions were reported for $s = 0$ (see *e.g.* [30]) and $s = 1$ fields (see [12,17]).

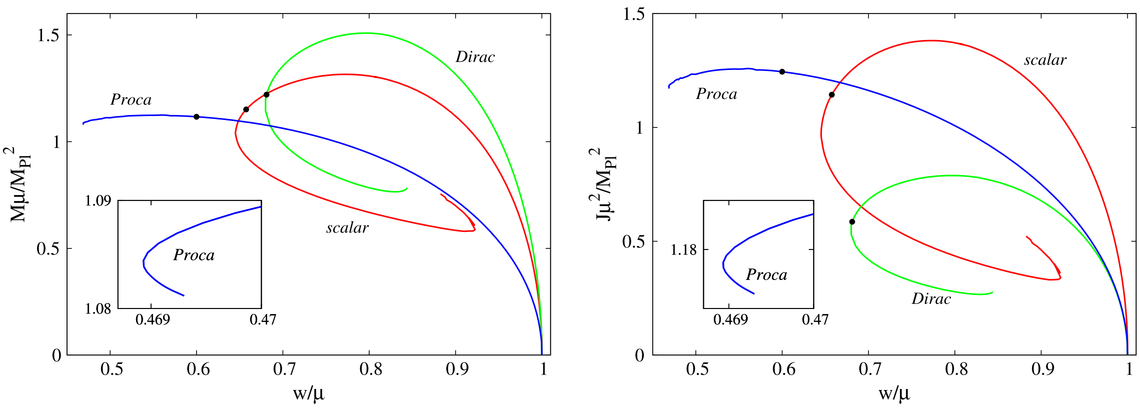

Fig. 2. The ADM mass M (left panel) and the angular momentum J (right panel) *vs.* field frequency w for the scalar (red line), vector (blue line) and spinor (green line) models. In each case the dot marks the particular solutions where an ergoregion first occurs, when moving from the maximal frequency $w/\mu = 1$ towards the centre of the spiral. The inset provides a zoom on the backbending of the curves, for the Proca case.

Table 1
1st column: the three different models. 2nd, 3rd and 4th columns: mass, angular momentum and frequency of the solution with maximal mass and angular momentum; 5th, 6th and 7th columns: frequency, mass and angular momentum of the minimal frequency solution – first backbending in the diagrams of Fig. 1; 8th-9th columns: mass/Noether charge and frequency of the solution with equal ADM mass and Noether charge (the data for Proca stars is missing in this case). All quantities are presented in units of μ, G.

	$M^{\max}$	$J^{\max}$	$w(M^{\max}, J^{\max})$	$w^{\min}$	$M(w^{\min})$	$J(w^{\min})$	$M = Q$	w^{crossing}
scalar	1.315	1.381	0.775	0.645	1.041	0.975	1.166	0.661
Dirac	1.509	0.789	0.795	0.680	1.198	0.569	1.303	0.692
Proca	1.125	1.259	0.562	0.469	1.086	1.180	–	–

there, the behaviour is not monotonic with spin. In each case there is also a minimal frequency, below which no solutions are found. The minimal frequencies and the corresponding M, J are shown in the 5th-7th columns of Table 1. After reaching the minimal frequency, the spiral backbends into a second branch. For the scalar and Dirac fields we were able to obtain further backbendings and branches. For a Proca field, however, we have not been able to construct these secondary branches. For any value of s, we conjecture that, similarly to the spherically symmetric case, the $M(w)$ (and $Q(w)$) curves describe spirals which approach, at their centre, a critical singular solution.

As expected, in all three cases, rotating solutions in the strong gravity region possess an ergo-region of toroidal shape [42]. The position of the critical solutions for which the ergo-region emerges is shown with a dot in Fig. 2. All remaining solutions, starting from that particular configuration up to the putative solution at the centre of the spiral, have an $S^1 \times S^1$ ergo-surface.

Although a detailed stability analysis of this solutions is technically challenging and beyond the scope of this paper, some simple observations can be done based on energetic arguments. The Noether charge measures the particle number. If this quantity multiplied by the field mass μ is smaller than the ADM mass M, then the solution has excess, rather than binding, energy and it should be unstable against fission. In all three cases, close to the maximal frequency, $w = \mu$ the solutions are stable under this criterion: there is binding energy, a necessary, albeit not sufficient, condition for stability. For scalar and spinor fields, we have found that at some point, the Noether charge and ADM mass curves cross and M becomes larger than Q corresponding to solutions with excess energy and hence unstable. The corresponding parameters of these particular solutions are given in the 8th-9th columns of Table 1. A similar picture should exist for Proca stars as well, but so far we have not been able to construct the corresponding solutions.

We emphasise that solutions with binding energy may, nonetheless, be perturbatively unstable. This has been clarified so far only for spherically symmetric configurations – see Refs. [31,32] for $s = 0$, Refs. [12,33] for $s = 1$ and Ref. [11] for $s = 1/2$.

5.3. *Bosonic* vs. *fermionic nature*

What if one tries to go beyond the classical field theory analysis and impose the quantum nature of fermions, which demands $Q = 1$ for Dirac stars? This condition can also be imposed for scalar and Proca stars, although in those cases it is not a mandatory requirement. Then, as discussed in [1], the spiral in Fig. 2 is not a sequence of solutions with constant μ and varying Q – recall that here $J = mQ$ –; rather, it is a sequence with constant Q and varying μ. Thus, since μ is a parameter in the action, it represents a sequence of solutions of different models. Consequently, there cannot be a difference of orders of magnitude between M, the physical mass of the star, and μ, the mass of the field. They should be of the same order of magnitude, unlike the macroscopic quantum states that may occur in the bosonic case. This is illustrated in Fig. 3 (left panel), where we plot the same data as in Fig. 1 but imposing the single particle condition.

Considering the stars as one particle microscopic classical configurations, the mass of the field μ becomes bounded, and, for fundamental states, never exceeds, $\sim M_{Pl}$. Thus, for these single particle configurations, the particle's size (measured by its Compton wavelength) cannot be smaller than $\sim$ Planck length. This upper μ bound can be pushed further up by considering configurations with higher values of m, making these configurations increasingly trans-Planckian. The corresponding masses for the Dirac model with $m = 1/2, 3/2$ and $5/2$ are shown in Fig. 3 (right panel).

6. Further remarks

The main purpose of this work was to provide a comparative analysis of three different types of spinning solitonic solutions of General Relativity coupled with matter fields of spin 0, 1 and 1/2, respectively. In particular, the Einstein-Dirac spinning configurations are reported here for the first time. In all cases there is a

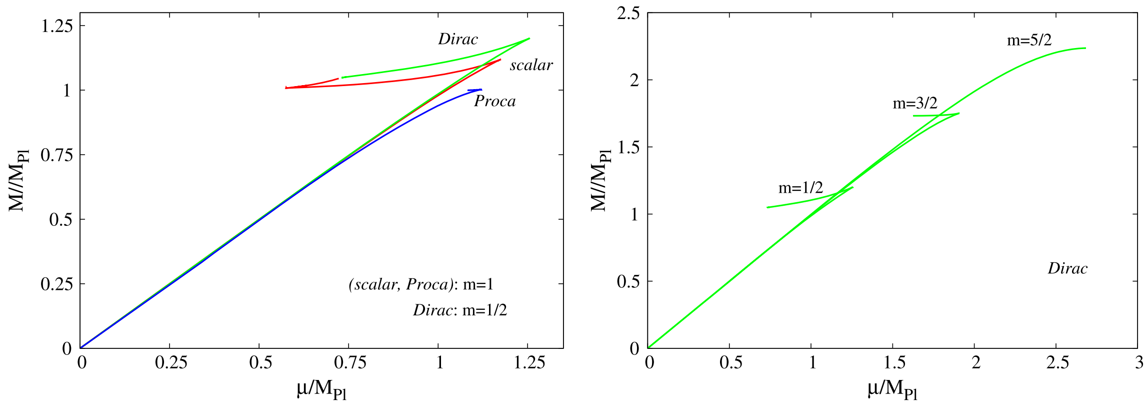

Fig. 3. Consequences of the single particle condition $Q = 1$. (Left panel) ADM mass *vs.* scalar field mass, in Planck units, for the three families of stars. (Right panel) Same for the first three states of the Dirac field ($m = 1/2, 3/2$ and $m = 5/2$).

harmonic time dependence in the fields (with a frequency w), together with a confining mechanism, as provided by a mass μ of the elementary quanta of the field.

Our results confirm that, when considered as classical field theory solutions, the stars share the same universal pattern, insensitive to the fermionic/bosonic nature of the fields. That is, when ignoring Pauli's exclusion principle, the (field frequency-mass/Noether charge)-diagram of the solutions looks similar for both bosonic and fermionic stars.[8] This generalises the results in [1] for spherically symmetric configurations. Introducing spin, another universal feature is the relation (4.25), *i.e.* the angular momentum and the particle number are always proportional (although the situation is more subtle for Proca and Dirac fields). We conjecture that similar configurations may exist for *any* spin, given a consistent matter model minimally coupled to GR, likely with similar properties. In particular, this should hold for $s = 3/2$: Rarita-Schwinger stars should exist, which, for a single field, should also satisfy relation (4.25).

On the other hand, if one imposes that the configuration describes a single particle, which is a consequence of the quantum nature of fermions, one finds that for each field mass there is a discrete set of states, up to a maximal field mass.

As noticed in [1] for the spherically symmetric case, the observed similarities between bosonic and fermionic solitons remain in the absence of gravity as long as appropriate self-interactions of the matter fields are allowed. For the matter fields in this work, spinning flat space solitons are known for $s = 0$ only [36,37], but should exist for $s = 1/2, 1$ as well. Moreover, one can show that the relation (4.25) is still satisfied. A preliminary numerical analysis indicates the existence of spinning flat space Dirac solitons, which generalise the solutions in [26] for a *single* spinor with a quartic self-interaction.

An important difference between bosonic and fermionic solutions is the following. Scalar or Proca stars can be in equilibrium with a black hole horizon at their centre, if both are rotating synchronously, leading to black holes with scalar or Proca hair [17,27]. This does not seem to be the case for a Dirac star. Conventional wisdom may attempt to relate this putative impossibility to the absence of superradiance for a fermionic field on the Kerr background [34]. However, spinning black holes with scalar hair exist even in the absence of the superradiant instability, the hair being intrinsically non-linear [28,29]. Therefore one cannot rule out, based on this association, that Dirac stars could allow for black hole generalisations. A more convincing obstacle is provided by the following argument. When assuming the existence of a power series expansion of the Einstein-matter field equations in the vicinity the event horizon,[9] the case of a Dirac field appears to be special. On the one hand, for a bosonic field ($s = 0, 1$), the synchronization condition $w = m\Omega_H$ (with Ω_H the event horizon velocity) occurs naturally, allowing for non-zero values of the matter fields at the horizon together with finite values for relevant quantities (*e.g.* j^t). As a result, a consistent local, non-trivial solution exists, in term of the values taken at the horizon. On the other hand, this is not the case for a Dirac field, where the condition $w = m\Omega_H$ (which still occurs naturally) is not enough to assure regularity at the horizon. It turns out that the spinor components are forced to vanish there order by order, yielding only the trivial solution. Despite this suggestive argument, a rigorous proof of the impossibility of endowing a Kerr black hole with synchronous Dirac hair is still lacking.

Beyond the matter contents discussed in this work, it is worth mentioning the case of $SU(2)$ Yang-Mills fields. While this nonlinear model possesses no flat spacetime solitons [38], the coupling to gravity allows for particle-like solutions [39]. Spinning generalisations of these solutions, however, do not exist [40], a rather unique situation amongst field theory models. Nonetheless, spinning Einstein-Yang-Mills configurations are found when adding a rotating horizon at the centre of a static soliton [41].

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[8] As discussed in [35], this holds also for the higher dimensional spherical stars.

[9] Here it is convenient to consider again spheroidal coordinates together with a non-extremal horizon.

References

[1] C.A.R. Herdeiro, A.M. Pombo, E. Radu, Phys. Lett. B 773 (2017) 654, arXiv:1708.05674 [gr-qc].
[2] R.P. Kerr, Phys. Rev. Lett. 11 (1963) 237.
[3] P.T. Chrusciel, J. Lopes Costa, M. Heusler, Living Rev. Relativ. 15 (2012) 7, arXiv:1205.6112 [gr-qc].
[4] A. Komar, Phys. Rev. 113 (1959) 934.
[5] R. Schon, S.T. Yau, Commun. Math. Phys. 65 (1979) 45.
[6] E. Witten, Commun. Math. Phys. 80 (1981) 381.
[7] A. Lichnerowicz, Théories Relativiste de la Gravitation et de l'Électromagnétisme, Masson, Paris, 1955.
[8] C.A.R. Herdeiro, J.M.S. Oliveira, Class. Quantum Gravity 36 (10) (2019) 105015, arXiv:1902.07721 [gr-qc].
[9] D.J. Kaup, Phys. Rev. 172 (1968) 1331.
[10] R. Ruffini, S. Bonazzola, Phys. Rev. 187 (1969) 1767.
[11] F. Finster, J. Smoller, S.T. Yau, Phys. Rev. D 59 (1999) 104020, arXiv:gr-qc/9801079.
[12] R. Brito, V. Cardoso, C.A.R. Herdeiro, E. Radu, Phys. Lett. B 752 (2016) 291, arXiv:1508.05395 [gr-qc].
[13] J.A. Wheeler, Phys. Rev. 97 (1955) 511.
[14] F.E. Schunck, E.W. Mielke, Phys. Lett. A 249 (1998) 389.
[15] S. Yoshida, Y. Eriguchi, Phys. Rev. D 56 (1997) 762.
[16] E. Radu, M.S. Volkov, Phys. Rep. 468 (2008) 101, arXiv:0804.1357 [hep-th].
[17] C. Herdeiro, E. Radu, H. Runarsson, Class. Quantum Gravity 33 (15) (2016) 154001, arXiv:1603.02687 [gr-qc].
[18] C.A.R. Herdeiro, E. Radu, Phys. Rev. Lett. 119 (26) (2017) 261101, arXiv:1706.06597 [gr-qc].
[19] T.D. Lee, Y. Pang, Phys. Rev. D 35 (1987) 3678.
[20] S.R. Dolan, D. Dempsey, Class. Quantum Gravity 32 (18) (2015) 184001, arXiv:1504.03190 [gr-qc].
[21] D. Kramer, H. Stephani, E. Herlt, M. MacCallum, Exact Solutions of Einstein's Field Equations, Cambridge University Press, Cambridge, 1980.
[22] C. Herdeiro, E. Radu, Class. Quantum Gravity 32 (14) (2015) 144001, arXiv:1501.04319 [gr-qc].
[23] T. Wiseman, Class. Quantum Gravity 20 (2003) 1137, arXiv:hep-th/0209051.
[24] W. Schönauer, R. Weiß, J. Comput. Appl. Math. 27 (279) (1989) 279;
M. Schauder, R. Weiß, W. Schönauer, Universität Karlsruhe, Interner Bericht Nr. 46/92, 1992.
[25] N.I.M. Gould, J.A. Scott, Y. Hu, ACM Trans. Math. Softw. 33 (2007) 10;
O. Schenk, K. Gärtner, Future Gener. Comput. Syst. 20 (3) (2004) 475.
[26] M. Soler, Phys. Rev. D 1 (1970) 2766.
[27] C.A.R. Herdeiro, E. Radu, Phys. Rev. Lett. 112 (2014) 221101, arXiv:1403.2757 [gr-qc].
[28] Y. Brihaye, C. Herdeiro, E. Radu, Phys. Lett. B 739 (2014) 1, arXiv:1408.5581 [gr-qc].
[29] C. Herdeiro, J. Kunz, E. Radu, B. Subagyo, Phys. Lett. B 748 (2015) 30, arXiv:1505.02407 [gr-qc].
[30] J. Kunz, I. Perapechka, Y. Shnir, arXiv:1904.07630 [gr-qc].
[31] M. Gleiser, R. Watkins, Nucl. Phys. B 319 (1989) 733.
[32] T.D. Lee, Y. Pang, Nucl. Phys. B 315 (1989) 477.
[33] N. Sanchis-Gual, C. Herdeiro, E. Radu, J.C. Degollado, J.A. Font, Phys. Rev. D 95 (10) (2017) 104028, arXiv:1702.04532 [gr-qc].
[34] R. Brito, V. Cardoso, P. Pani, Lect. Notes Phys. 906 (2015) 1, arXiv:1501.06570 [gr-qc].
[35] J.L. Blazquez-Salcedo, C. Knoll, E. Radu, Phys. Lett. B 793 (2019) 161, arXiv:1902.05851 [gr-qc].
[36] M.S. Volkov, E. Wohnert, Phys. Rev. D 66 (2002) 085003, arXiv:hep-th/0205157.
[37] B. Kleihaus, J. Kunz, M. List, Phys. Rev. D 72 (2005) 064002, arXiv:gr-qc/0505143.
[38] S.R. Coleman, Commun. Math. Phys. 55 (1977) 113.
[39] R. Bartnik, J. Mckinnon, Phys. Rev. Lett. 61 (1988) 141.
[40] J.J. Van der Bij, E. Radu, Int. J. Mod. Phys. A 17 (2002) 1477, arXiv:gr-qc/0111046.
[41] B. Kleihaus, J. Kunz, Phys. Rev. Lett. 86 (2001) 3704, arXiv:gr-qc/0012081.
[42] C. Herdeiro, E. Radu, Phys. Rev. D 89 (12) (2014) 124018, arXiv:1406.1225 [gr-qc].