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ON THE COMPETITIVE HARVESTING OF MARINE RESOURCES*

ALBERTO BRESSAN† AND VASILE STAICU‡

Abstract. The paper is concerned with the optimal harvesting of a marine resource, described by an elliptic equation with Neumann boundary conditions and a nonlinear source term. We first consider a single agent, whose harvesting effort at various locations is described by a positive Radon measure. Necessary conditions for optimality are derived, complementing the existence result proved in [A. Bressan, G. Coclite, and W. Shen, *SIAM J. Control Optim.*, 51 (2013), pp. 1186–1202]. The second part of the paper deals with a competitive scenario, where several groups of fishermen, from different coastal towns and hence with different cost functions, harvest the same marine resource. We prove the existence of a Nash equilibrium, which is characterized in terms of a suitable variational inequality.

Key words. optimal harvesting of a marine resource, necessary conditions for optimality, existence of Nash equilibrium, variational inequality, obstacle problem

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1. Introduction.

Mathematical models of fisheries have received much attention in recent literature [11, 15, 22, 23, 25]. Following [7], we consider here a harvesting problem in a multidimensional domain Ω. The density of fish (or some other marine resource) is recovered as the solution u to the semilinear elliptic equation with measure-valued coefficients

$$\Delta u + f(x,u) - u\,\mu \;=\; 0, \qquad\qquad x \in \Omega, \tag{1.1}$$

and Neumann boundary conditions

$$\mathbf{n}(x)\cdot\nabla u(x) \;=\; 0, \qquad\qquad x \in \partial\Omega. \tag{1.2}$$

By $\mathbf{n}(x)$ we denote the unit outer normal vector at the boundary point $x \in \partial\Omega$. The first term in (1.1) accounts for diffusion, while $f = f(x,u)$ yields the growth rate. The last term describes the decrease in the population due to harvesting. Here μ is a measure, describing the harvesting effort at various locations.

We remark that, by the nature of the problem, there is no reason to assume a priori that μ should be absolutely continuous w.r.t. Lebesgue measure. For example, μ may concentrate a positive mass along the boundary $\partial\Omega$. From a mathematical point of view, this makes the problem more challenging, since one has to rely on the theory of elliptic equations with measure-valued coefficients.

Throughout, we assume the following:

(A1) *$\Omega \subset \mathbb{R}^d$ is a bounded, connected open set with $\mathcal{C}^2$ boundary.*

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†Department of Mathematics, Penn State University, University Park, PA 16802 (bressan@math.psu.edu).

‡CIDMA and Department of Mathematics, University of Aveiro, Campus Universitário de Santiago, 3810-193 Aveiro, Portugal (vasile@ua.pt).

(A2) *$f = f(x,u)$ is a $\mathcal{C}^2$ function such that, for some constants M, K, one has*

$$f(x,0) \geq 0, \quad f_u(x,0) > 0, \quad f_{uu}(x,u) < 0, \quad f(x,M) \leq 0, \quad |f(x,u)| \leq K \tag{1.3}$$

for all $x \in \overline{\Omega}$ and $u \in [0,M]$.

(A3) *μ is a positive Radon measure, absolutely continuous w.r.t. capacity, supported on the compact set $\overline{\Omega}$.*

Elliptic problems with measure data have been studied in several papers [5, 6, 10, 12] and are now fairly well understood. Following [5, 6] we denote by $\mathcal{M}_0$ the family of measures which vanish on Borel sets with zero capacity, so that

$$\mathrm{cap}_2(V) = 0 \qquad \Longrightarrow \qquad \mu(V) = 0. \tag{1.4}$$

For the definition and basic properties of capacity, we refer to [2, 17].

DEFINITION 1.1. *Let μ be a measure in $\mathcal{M}_0$. A function $u \in \mathbf{L}^\infty(\Omega) \cap H^1(\Omega)$ is a solution to the elliptic problem* (1.1)–(1.2) *if*

$$-\int_\Omega \nabla u \cdot \nabla \phi \, dx + \int_\Omega f(x,u)\phi \, dx - \int_{\overline{\Omega}} u\phi \, d\mu = 0 \tag{1.5}$$

for every test function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$.

We recall that, for a function $u \in H^1(\Omega)$, the pointwise values of u on the closure $\overline{\Omega}$ can be determined by the formula

$$u(x) = \lim_{r \to 0+} \frac{1}{\mathrm{Vol}(B_r(x) \cap \Omega)} \int_{B_r(x)\cap\Omega} u(y)\, dy, \tag{1.6}$$

where $B_r(x)$ denotes the open ball centered at x with radius r. Since Ω is a domain with $\mathcal{C}^2$ boundary, this limit exists at all points $x \in \overline{\Omega}$ except possibly on a set of zero capacity [17, 18]. Assuming that (1.4) holds, the last integral in (1.5) is thus meaningful.

We also observe that, by (1.3), if u is a solution to (1.1), then the function $x \mapsto u(x) + K|x|^2$ is subharmonic. In turn (see [2]), this implies that the map u is upper semicontinuous.

Example 1.2. Consider the open domain $\Omega =]0,1[$, take $f(x,u) = 1$, and let the measure μ be a single Dirac mass at the origin. Then the function $u(x) = 1 + x - \frac{x^2}{2}$ satisfies the requirements of Definition 1.1. Indeed, for every $\phi \in \mathcal{C}^\infty$ one has

$$-\int_\Omega \nabla u \cdot \nabla \phi \, dx + \int_\Omega f(x,u)\phi \, dx - \int_{\overline{\Omega}} u\phi \, d\mu = -\int_0^1 (1-x)\cdot\phi' \, dx + \int_0^1 \phi \, dx - \phi(0) = 0.$$

Notice that, since μ concentrates a positive mass at $0 \in \partial\Omega$, at the origin the boundary condition (1.2) is not satisfied in a classical sense. Indeed,

$$\lim_{x \to 0+} u'(x) = 1 \neq 0.$$

However, the solution satisfies the balance equation

$$\int_\Omega f \, dx - \int_{\overline{\Omega}} u \, d\mu = \int_0^1 1 \, dx - u(0) = 0,$$

showing that there is no flux across the boundary.

To formulate an optimization problem, we consider, with an upper bound on the harvesting capacity,

$$\mu(\overline{\Omega}) \ \leq \ \kappa_0\,, \tag{1.7}$$

together with a cost function γ which satisfies the following:

(A4) $\gamma : \overline{\Omega} \mapsto \mathbb{R}_+$ *is a strictly positive* $\mathcal{C}^2$ *function. Namely,*

$$\gamma(x) \ \geq \ c_0 \ > \ 0 \qquad \text{for all } x \in \overline{\Omega}. \tag{1.8}$$

We think of $\gamma(x)$ as the cost for a harvesting at the point $x \in \overline{\Omega}$. In a realistic example, $\gamma(x)$ could depend on the distance to reach x from the harbor where the fishing company is located. As in [7], we consider the following:

(OHP) Optimal Harvesting Problem.

$$\text{maximize:} \qquad \mathcal{J}(u,\mu) \ \doteq \ \int_{\overline{\Omega}} \big(u(x) - \gamma(x)\big)\, d\mu \tag{1.9}$$

over all couples (u,μ)*, where* $\mu \in \mathcal{M}_0$ *is a positive measure on* $\overline{\Omega}$ *with total mass bounded by* (1.7)*, and* u *is a corresponding solution of* (1.1).

By the analysis in [7], one has the following.

THEOREM 1.3. *Under the assumptions* (A1), (A2), *and* (A4), *the optimal harvesting problem* **(OHP)** *has at least one optimal solution.*

Indeed, the proof given in [7] covers more general cases where f need not be concave w.r.t. u, and the cost function γ is only assumed to be lower semicontinuous.

The purpose of the present paper is two-fold. In section 2 we derive necessary conditions satisfied by an optimal solution (u^*, μ^*) to the harvesting problem. The remaining three sections deal with a competitive scenario, where a large number of small fishing companies harvest the same marine resource.

We start by considering the case where all players share the same cost function $\gamma = \gamma(x)$. In this setting, an equilibrium solution is constructed by solving a family of variational inequalities, and establishing a suitable continuity property in terms of the parameter. In section 4 we study the case of finitely many groups of players with different costs $\gamma_1, \ldots, \gamma_N$. This model can describe several groups of fishermen based at different coastal towns $y_1, \ldots, y_N$. In this case, the cost $\gamma_i(x)$ will depend on the distance of the point x from y_i. The existence of an equilibrium solution is here proved by a topological argument as in [8], writing the problem in the form of a variational inequality.

Finally, in section 5 we consider a continuum of small players, each with a different cost function $\gamma(x,\theta)$, $\theta \in [0,\kappa]$. After introducing a definition of equilibrium solution, we prove the existence of such a solution within a class of generalized strategies.

We remark that, in all the above cases, the uniqueness of the equilibrium solution remains an interesting open problem for future investigation.

2. Necessary conditions for optimality. Consider the Optimal Harvesting Problem **(OHP)**, assuming (A1), (A2), and (A4).

Let (u^*, μ) be an optimal pair. To derive necessary conditions, we first outline the main argument. Consider a family of perturbed measures of the form

$$\mu_\varepsilon \ = \ \mu + \varepsilon\nu. \tag{2.1}$$

Let

$$u_\varepsilon \;=\; u^* + \varepsilon v + o(\varepsilon)$$

be the corresponding solution of

$$\Delta u_\varepsilon + f(x, u_\varepsilon) - u_\varepsilon\, \mu_\varepsilon \;=\; 0\,, \tag{2.2}$$

with Neumann boundary conditions (1.2), always in the sense of Definition 1.1. A formal linearization procedure shows that the first order perturbation v satisfies the linear, nonhomogeneous equation

$$\begin{cases} \Delta v + f_u(x, u^*)v - v\mu \;=\; u^*\nu, & x \in \Omega\,, \\ \mathbf{n}\cdot\nabla v \;=\; 0, & x \in \partial\Omega\,. \end{cases} \tag{2.3}$$

Differentiating the harvest functional w.r.t. ε, we obtain

$$\frac{d}{d\varepsilon}\mathcal{J}(u_\varepsilon, \mu_\varepsilon)\Big|_{\varepsilon=0} \;=\; \int_{\overline{\Omega}} (u^* - \gamma)\, d\nu + \int_{\overline{\Omega}} v\, d\mu. \tag{2.4}$$

Observe that (2.3) can be written as

$$Lv \;=\; u^*\nu,$$

where L is a linear, self-adjoint operator. Hence, denoting by ψ the solution to the linear problem

$$L\psi \;=\; \mu,$$

one has

$$\langle v\,,\ \mu\rangle \;=\; \langle L^{-1}u^*\nu\,,\ \mu\rangle \;=\; \langle u^*\nu\,,\ L^{-1}\mu\rangle \;=\; \langle u^*\nu\,,\ \psi\rangle\,.$$

We can thus write

$$\frac{d}{d\varepsilon}\mathcal{J}(u_\varepsilon, \mu_\varepsilon)\Big|_{\varepsilon=0} \;=\; \int_{\overline{\Omega}} (u^* - \gamma)\, d\nu + \int_{\overline{\Omega}} \psi\, u^*\, d\nu, \tag{2.5}$$

where $\psi \in H^1(\Omega)$ provides the solution to the linear, nonhomogeneous problem

$$\begin{cases} \Delta\psi + f_u(x, u^*)\psi - \psi\mu \;=\; \mu, & x \in \Omega\,, \\ \nabla\psi\cdot\mathbf{n} \;=\; 0, & x \in \partial\Omega\,. \end{cases} \tag{2.6}$$

In view of the above analysis, the necessary conditions for optimality can now be stated as follows.

THEOREM 2.1. *Let the assumptions* (A1), (A2), *and* (A4) *hold, and let* $(u^*, \mu) \in H^1(\Omega) \times \mathcal{M}_0$ *be an optimal pair. Call* ψ *the solution to the adjoint linear problem* (2.6). *Then there exists a constant* $p \geq 0$ *such that*

$$\begin{cases} (1 + \psi(x))u^*(x) - \gamma(x) \;\leq\; p & \text{a.e. w.r.t. the Lebesgue measure } \mathcal{L}^d \text{ on } \Omega\,, \\ (1 + \psi(x))u^*(x) - \gamma(x) \;=\; p & \text{a.e. w.r.t. the measure } \mu \text{ on } \overline{\Omega}\,. \end{cases} \tag{2.7}$$

Proof. If μ is the zero measure, the conclusion of the theorem is straightforward. In the remainder of the proof we thus assume that μ is a bounded, nonzero Radon measure on $\overline{\Omega}$, absolutely continuous w.r.t. capacity.

1. We start by showing that u^* is uniformly positive on $\overline{\Omega}$. Indeed, by (1.8) one has $\gamma(x) \geq c_0 > 0$. Call $\chi_{\{u^* \leq \gamma\}}$ the characteristic function of the set

$$\big\{x \in \overline{\Omega}\,;\quad u^*(x) \leq \gamma(x)\big\}. \tag{2.8}$$

By the formula (1.6), since $\gamma \in \mathcal{C}^2$, the set (2.8) is measurable w.r.t. Lebesgue measure, and also w.r.t. the measure $\mu \in \mathcal{M}_0$.

Consider the reduced measure

$$\tilde{\mu} \;=\; \chi_{\{u^* \geq \gamma\}} \cdot \mu.$$

Since $\tilde{\mu} \leq \mu$, the system (1.1)–(1.2) has a solution $\tilde{u} \geq u^*$. If $\tilde{\mu} \neq \mu$, recalling (1.9) we have

$$\mathcal{J}(u^*, \mu) \;<\; \mathcal{J}(\tilde{u}, \tilde{\mu}),$$

contradicting the optimality of (u^*, μ). We thus conclude

$$u^*(x) \;\geq\; \gamma(x) \;\geq\; c_0 \tag{2.9}$$

for every point x in the support of μ.

Next, by the assumptions (1.3) we can choose another positive constant δ_0, with $0 < \delta_0 < c_0$ such that

$$f(x,u) \;>\; 0 \qquad \text{for all } x \in \overline{\Omega}, \quad 0 < u \leq \delta_0\,. \tag{2.10}$$

Consider the set

$$\Omega_0 \;\doteq\; \{x \in \Omega\,;\quad u^*(x) < \delta_0\}, \tag{2.11}$$

which is open because u^* is upper semicontinuous. On this set, u^* is a solution to

$$\Delta u^* \;=\; -\,f(x, u^*(x)) \;\leq\; 0, \tag{2.12}$$

with boundary conditions

$$\mathbf{n}(x) \cdot \nabla u^*(x) \;=\; 0, \qquad x \in \partial\Omega_0 \cap \partial\Omega. \tag{2.13}$$

We now observe that the constant function $v(x) = \delta_0$ is a subsolution to (2.12), satisfying the Neumann conditions (2.13) together with

$$v(x) \;\leq\; u^*(x), \qquad x \in \partial\Omega_0 \cap \Omega,$$

on the remaining portion of the boundary $\partial\Omega_0$. A comparison argument now yields

$$u^*(x) \;\geq\; v(x) \;=\; \delta_0 \qquad \text{for all } x \in \Omega_0\,.$$

By the above definition of Ω_0, this implies $u^*(x) \geq \delta_0$ for all $x \in \Omega$.

2. Following [10], it will be convenient to work in the Hilbert space $X_\mu \doteq H^1(\Omega) \cap \mathbf{L}^2_\mu(\overline{\Omega})$, consisting of all functions $u \in H^1(\Omega)$ such that $u \in \mathbf{L}^2_\mu(\overline{\Omega})$. This definition makes sense because, as remarked in step **1**, the pointwise values of any function $u \in H^1(\Omega)$ are uniquely determined up to a set of zero capacity.

On X_μ we shall use the norm generated by the inner product

$$\langle u, v \rangle \;\doteq\; \int_\Omega \nabla u \cdot \nabla v\, dx + \int_{\overline{\Omega}} uv\, d\mu\,. \tag{2.14}$$

We claim that this is equivalent to the norm $\|u\|_{H^1(\Omega)} + \|u\|_{\mathbf{L}^2_\mu(\overline{\Omega})}$. Indeed, in the opposite case there would exist a sequence $u_n \in X_\mu$ such that

(2.15)
$$\lim_{n\to\infty} \left(\int_\Omega |\nabla u_n(x)|^2 \, dx + \int_{\overline{\Omega}} |u_n(x)|^2 \, d\mu \right) \;=\; 0, \qquad \|u_n\|_{\mathbf{L}^2(\Omega)} \;=\; 1 \quad \text{for all } n \geq 1.$$

Call a_n the average value of u_n on Ω. Since Ω is bounded and connected, by Poincaré's inequality [16] there exists a constant C such that

$$\|u_n - a_n\|_{\mathbf{L}^2(\Omega)} \;\leq\; C\, \|\nabla u_n\|_{\mathbf{L}^2(\Omega)} \;\to\; 0. \tag{2.16}$$

By (2.15) and (2.16) one has the convergence $\|u_n - a_n\|_{H^1(\Omega)} \to 0$. Therefore,

$$\int_\Omega |a_n|^2 \, d\mu \;\leq\; \int_\Omega \Big(|u_n(x)|^2 + |u_n(x) - a_n|^2 \Big) \, d\mu \;\to\; 0.$$

Since μ is a positive nonzero measure, this implies $a_n \to 0$ and hence $\|u_n\|_{\mathbf{L}^2(\Omega)} \to 0$, contradicting the last condition in (2.15).

3. The fact that X_μ is complete can be proved as in [10]. The dual space of X_μ will be denoted by X'_μ. For every $\eta \in X'_\mu$, by Riesz' theorem, there exists $w \in X_\mu$ such that

$$\big\langle w, v \big\rangle \;\doteq\; \int_\Omega \nabla w \cdot \nabla v \, dx + \int_{\overline{\Omega}} w\, v \, d\mu \;=\; \eta(v) \qquad \text{for all } v \in X_\mu\,.$$

We denote by $\eta \mapsto \Lambda(\eta)$ this bounded linear operator from X'_μ into X_μ.

For the sake of convenience, we also introduce the function

$$g(x,u) \;\doteq\; \frac{f(x,u)}{u}\,. \tag{2.17}$$

By the properties of f at (1.3) and (1.8), for $u \in [c_0, +\infty[$, the function $u \mapsto g(x,u)$ is $\mathcal{C}^2$ positive, and strictly decreasing.

4. Next, consider any bounded Borel measurable functions $\varphi_1, \varphi_2 : \Omega \mapsto \mathbb{R}$, with $\varphi_1 \geq 0$. Set $\nu \doteq \varphi_1 \mathcal{L}^d + \varphi_2\, \mu$, and define the 1-parameter family of measures

$$\mu_\varepsilon \;=\; \mu + \varepsilon\nu \;=\; \mu + \varepsilon\varphi_1 \mathcal{L}^d + \varepsilon\varphi_2 \mu\,. \tag{2.18}$$

We observe that the two maps

$$u \;\mapsto\; g(x,u)u, \qquad\qquad (u,\varepsilon) \;\mapsto\; \varepsilon(\varphi_1 \mathcal{L}^d + \varepsilon\varphi_2\mu)\, u$$

are both continuously differentiable (for u close to u^*) as maps from X_μ into X_μ. By a slight abuse of notation, we regard $g(x,u)u$ and $\varepsilon(\varphi_1\mathcal{L}^d + \varepsilon\varphi_2\mu)\, u$ also as elements of X'_μ, given by the maps

$$v \;\mapsto\; \int_\Omega g(x,u)u(x)v(x)\, dx, \qquad\qquad v \;\mapsto\; \int_\Omega \varepsilon u(x)v(x)\varphi_1 \, dx + \int_{\overline{\Omega}} \varepsilon\, u\, v\, \varphi_2 \, d\mu\,.$$

5. Call u_ε the solution to (2.2), taking μ_ε as in (2.18). Our next goal is to prove that the map $\varepsilon \mapsto u_\varepsilon$ is differentiable for ε close to zero. This will be achieved by expressing u_ε as the fixed point of a differentiable transformation, and using the implicit

function theorem. Indeed, with the previous notation, according to Definition 1.1 one has

$$u_\varepsilon \;=\; \Lambda\Big(g(x,u_\varepsilon)u_\varepsilon\Big) - \varepsilon\Lambda\Big((\varphi_1\mathcal{L}^d + \varphi_2\mu)u_\varepsilon\Big). \tag{2.19}$$

Since $g \in \mathcal{C}^2$, the map

$$(u,\varepsilon) \;\mapsto\; u - \Lambda\Big(g(x,u)u\Big) + \varepsilon\Lambda\Big((\varphi_1\mathcal{L}^d + \varphi_2\mu)u\Big)$$

is differentiable from $X_\mu \times \mathbb{R}$ into X_μ. By the implicit function theorem [13, 14], to prove that at $\varepsilon = 0$ the derivative (2.4) is well defined, it thus suffices to show that at $u = u^*$, the linear operator

$$I - \Psi, \qquad\qquad \Psi \;\doteq\; D_u\Lambda\Big(g(x,u^*)\,u^*\Big),$$

has a bounded inverse. More precisely, Ψ is the linear map

$$v \;\mapsto\; \Psi(v) \;\doteq\; \Lambda\Big(g_u(x,u^*)u^*v + g(x,u^*)v\Big) \tag{2.20}$$

from X_μ into itself.

6. Since u^* is a uniformly positive, bounded solution to (1.1) and $g_u < 0$, one has

$$\Delta u^* - u^*\mu + [g_u(x,u^*)u^*]u^* + g(x,u^*)u^* \;=\; g_u(x,u^*)(u^*)^2 \;\leq\; -2\delta u^* \tag{2.21}$$

for some $\delta > 0$ and all $x \in \Omega$. By continuity, there exists $\varepsilon > 0$ such that

$$\Delta u^* - u^*\mu + \lambda^{-1}\big[g_u(x,u^*)u^* + g(x,u^*)\big]u^* \;\leq\; -\delta u^* \tag{2.22}$$

for all $|\lambda - 1| \leq \varepsilon$ and $x \in \Omega$.

We observe that Λ is self-adjoint, hence the same is true of Ψ. To prove that $I - \Psi$ has a bounded inverse, we will show that any eigenvalue λ of Ψ must satisfy the inequality

$$|\lambda - 1| \;>\; \varepsilon\,. \tag{2.23}$$

Assume that $\Psi(v) = \lambda v$ for some $v \in X_\mu$ and $\lambda > 0$. According to Definition 1.1, this means

$$\begin{cases} \Delta v - v\mu + \lambda^{-1}\big[g_u(x,u^*)u^* + g(x,u^*)\big]v \;=\; 0, & x \in \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad \mathbf{n}\cdot\nabla v \;=\; 0, & x \in \partial\Omega. \end{cases} \tag{2.24}$$

Since g, g_u are bounded and $\mu \geq 0$, it follows that v is bounded.

Consider the linear semigroup generated by the parabolic equation

$$\begin{cases} w_t \;=\; \Delta w - w\mu + \lambda^{-1}\big[g_u(x,u^*)u^* + g(x,u^*)\big]w, & x \in \Omega\,, \\ \qquad \mathbf{n}\cdot\nabla w \;=\; 0, & x \in \partial\Omega\,. \end{cases} \tag{2.25}$$

If (2.23) fails, to reach a contradiction we argue as follows. On the one hand, the function v in (2.24) is a time-invariant solution of (2.25). On the other hand, by (2.22) the function

$$U(t,x) \;=\; e^{-\delta t}u^*(x)$$

is an upper solution of (2.25), while $-U$ is a lower solution. Choosing C such that

$$-C\,u^*(x) \;\leq\; v(x) \;\leq\; Cu^*(x), \tag{2.26}$$

a comparison argument yields

$$-Ce^{-\delta t}u^*(x) \;\leq\; v(x) \;\leq\; Ce^{-\delta t}u^*(x). \tag{2.27}$$

Hence $v = 0$, achieving a contradiction.

The inequalities (2.27) follow from corresponding inequalities valid for the resolvent operators. For $\epsilon > 0$ small, consider the linear operator $w \mapsto w_\epsilon \doteq E_\epsilon(w)$, defined by

$$w_\epsilon \;=\; w + \epsilon\Big(\Delta w_\epsilon - w_\epsilon\mu + \lambda^{-1}\big[g_u(x,u^*)u^* + g(x,u^*)\big]w_\epsilon\Big), \tag{2.28}$$

with Neumann boundary conditions on $\partial\Omega$. Notice that E_ϵ yields the backward Euler approximation of step ϵ for the linear parabolic equation (2.25). We also observe that E_ϵ is positive in the sense that

$$w \;\geq\; 0 \qquad \Longrightarrow \qquad E_\epsilon w \;\geq\; 0. \tag{2.29}$$

We claim that

$$v_\epsilon \;\doteq\; E_\epsilon v \;=\; v\,, \qquad\qquad u_\epsilon \;\doteq\; E_\epsilon u^* \;\leq\; \frac{u^*}{1+\epsilon\delta} \tag{2.30}$$

for all $\epsilon \geq 0$ small enough. Indeed, the first identity follows trivially from (2.24). On the other hand, setting $\tilde u = (1+\epsilon\delta)^{-1}u^*$, by (2.22) we obtain

$$\begin{aligned} &\tilde u - \epsilon\Big(\Delta\tilde u - \tilde u\mu + \lambda^{-1}\big[g_u(x,u^*)u^* + g(x,u^*)\big]\tilde u\Big) \\ &\qquad\geq\; (1+\epsilon\delta)^{-1}u^* + \epsilon\delta(1+\epsilon\delta)^{-1}u^* \;=\; u^* \end{aligned} \tag{2.31}$$

for all $\epsilon > 0$ small enough. We can write $u_\epsilon = \tilde u + u^\sharp$, where

$$u^\sharp \;=\; E_\epsilon\Big(u^* - \tilde u + \epsilon\big(\Delta\tilde u - \tilde u\mu + \lambda^{-1}\big[g_u(x,u^*)u^* + g(x,u^*)\big]\tilde u\big)\Big).$$

By linearity, (2.29) and (2.31) imply $u^\sharp \leq 0$. Hence $u_\epsilon \leq \tilde u$.

Since $u \geq v$ and the backward Euler operator E_ϵ preserves ordering, we have

$$v \;=\; E_\epsilon v \;\leq\; E_\epsilon u \;\leq\; \tilde u \;=\; \frac{u^*}{1+\epsilon\delta}\,. \tag{2.32}$$

Calling $u(t,\cdot)$ the solution of (2.25) with initial data $u(0,\cdot) = u^*$, the representation formula

$$u(t,\cdot) \;=\; \lim_{n\to\infty}(E_{t/n})^n(u^*)$$

together with (2.26) and (2.32) yields

$$v(t,\cdot) \;\leq\; C\lim_{n\to\infty}\left(\frac{1}{1+tn^{-1}\delta}\right)^n u^* \;=\; Ce^{-\delta t}u^*.$$

This proves that all of the eigenvalues of the self-adjoint linear operator Ψ are bounded away from 1. Hence $I - \Psi$ has a bounded inverse, as claimed. We thus

conclude that the derivative in (2.5) is well defined for ε in a neighborhood of the origin.

7. Next, let φ be any Borel measurable function on $\overline{\Omega}$ such that

$$\|\varphi\|_{\mathbf{L}^\infty(\mu)} \,<\, 1, \qquad\qquad \int_{\overline{\Omega}} \varphi \, d\mu \,=\, 0. \tag{2.33}$$

For $|\varepsilon| \leq 1$, consider the measures

$$\mu_\varepsilon \,\doteq\, (1+\varepsilon\varphi)\mu. \tag{2.34}$$

Note that each μ_ε is a nonnegative Radon measure which satisfies (1.7).

Let $u_\varepsilon \in H^1(\Omega)$ be the solution to (2.2) with Neumann boundary conditions (1.2), always in the sense of Definition 1.1. The optimality condition implies

$$0 \,=\, \frac{d}{d\varepsilon}\mathcal{J}(u_\varepsilon,\mu_\varepsilon)\Big|_{\varepsilon=0} \,=\, \int_{\overline{\Omega}} (v + u^*\varphi - \gamma\varphi)\, d\mu \,=\, \int_{\overline{\Omega}} \big[(\psi+1)u^* - \gamma\big]\varphi \, d\mu\,. \tag{2.35}$$

Here the function ψ is the solution to the adjoint problem (2.6). Indeed, this formula corresponds to (2.5), with $\nu = \varphi\,\mu$.

If the function $(\psi+1)u^* - \gamma$ does not coincide with a constant a.e. w.r.t. the measure μ, we can find a function $\varphi \in \mathbf{L}^\infty_\mu(\overline{\Omega})$ satisfying (2.33) such that the right-hand side of (2.35) is nonzero, attaining a contradiction.

For some constant p we thus have

$$(\psi(x)+1)u^*(x) - \gamma(x) \,=\, p, \tag{2.36}$$

μ-a.e. on the domain $\overline{\Omega}$. This proves the second statement in (2.7).

8. Finally, if the first statement in (2.7) fails, then there exists $\delta > 0$ and a Borel set $V \subset \Omega$ with Lebesgue measure $\mathrm{meas}(V) > 0$ such that

$$(1+\psi(x))u^*(x) - \gamma(x) \,\geq\, p + \delta \qquad \text{for all } x \in V. \tag{2.37}$$

Choose a set $V' \subset \overline{\Omega}$, disjoint from V, such that $\mu(V') > 0$. Define the functions φ_1, φ_2 by setting

$$\varphi_1(x) \,=\, \begin{cases} [\mathrm{meas}(V)]^{-1} & \text{if } \ x \in V, \\ 0 & \text{if } \ x \notin V, \end{cases} \qquad \varphi_2(x) \,=\, \begin{cases} [\mu(V')]^{-1} & \text{if } \ x \in V', \\ 0 & \text{if } \ x \notin V'. \end{cases}$$

For all $\varepsilon > 0$ small enough, the measure

$$\mu_\varepsilon \,\doteq\, \mu + \varepsilon\varphi_1 \mathcal{L}^d - \varepsilon\varphi_2\mu$$

is admissible. Indeed, $\mu_\varepsilon(\overline{\Omega}) = \mu(\overline{\Omega})$. Calling u_ε the corresponding solution to (1.1) and using (2.35)–(2.37), we now compute

$$\begin{aligned} \frac{d}{d\varepsilon}\mathcal{J}(u_\varepsilon,\mu_\varepsilon)\Big|_{\varepsilon=0} \,&=\, \int_\Omega \big[(\psi+1)u^* - \gamma\big]\varphi_1\, dx - \int_{\overline{\Omega}} \big[(\psi+1)u^* - \gamma\big]\varphi_2\, d\mu \\ &\geq\, \int_V (p+\delta)\,\varphi_1\, dx - \int_{V'} p\,\varphi_2\, d\mu \,=\, (p+\delta) - p \,=\, \delta \,>\, 0\,. \end{aligned} \tag{2.38}$$

This contradicts the optimality of the pair (u^*,μ), proving the first inequality in (2.7). □

3. A noncooperative game. Next, instead of a single fishing company, we consider a large number of small, competing companies. We denote by $\kappa > 0$ the joint harvesting capability of these companies, while μ will describe the combined harvesting effort. In this setting, the concept of a Nash equilibrium solution can be introduced, where each player has the choice of fishing at some point x or remaining idle.

DEFINITION 3.1 (Nash equilibrium with several small players having the same cost). *Consider a harvesting game governed by* (1.1)–(1.2). *Let $\kappa > 0$ be the total harvesting capability, and let $\gamma(x)$ be the cost of fishing at x.*

We say that (u, μ) is an equilibrium solution *to the harvesting game provided that:*

- *μ is a positive measure on $\overline{\Omega}$, with $0 \leq \mu(\overline{\Omega}) \leq \kappa$.*
- *u is a strictly positive solution to the elliptic equation* (1.1) *with Neumann boundary conditions* (1.2).
- *There exists a constant $p \geq 0$ such that*

$$\begin{cases} u(x) - \gamma(x) \;=\; p & \text{for all } \; x \in \text{Supp}(\mu), \\ u(x) - \gamma(x) \;\leq\; p & \text{for all } \; x \in \overline{\Omega}. \end{cases} \tag{3.1}$$

Moreover, one has the implication

$$p \;>\; 0 \qquad \Longrightarrow \qquad \mu(\overline{\Omega}) \;=\; \kappa\,. \tag{3.2}$$

Remark 3.2. Here $u(x) - \gamma(x)$ is the payoff for a unit effort of fishing at the point x. Since we are assuming that (u, μ) represents a Nash equilibrium, this payoff must be constant over the support of μ, and smaller at points outside $\text{Supp}(\mu)$ where nobody is fishing.

If the total mass of μ equals κ, this means that all fishermen are actively working. On the other hand, if $\mu(\overline{\Omega}) < \kappa$, some fishermen remain idle. This is possible precisely when the payoff satisfies $u(x) - \gamma(x) \leq 0$ for all $x \in \overline{\Omega}$. In this case, there is no advantage in working rather than remaining idle.

THEOREM 3.3. *In the setting of Definition* 3.1*, let $\kappa > 0$ be given, and assume that* (A1), (A2)*, and* (A4) *hold. Then the harvesting game admits an equilibrium solution (u, μ).*

Proof. An equilibrium solution will be constructed in several steps.

1. Fix $p \geq 0$. We first show the existence of a unique uniformly positive solution to the obstacle problem (see Figure 1):

$$u(x) \;\leq\; \gamma(x) + p\,, \tag{3.3}$$

$$\begin{cases} \Delta u + f(x,u) \;\geq\; 0 & \text{if} \quad x \in \Omega\,, \\ \mathbf{n}(x) \cdot \nabla u(x) \;\leq\; 0 & \text{if} \quad x \in \partial\Omega\,, \end{cases} \tag{3.4}$$

$$u(x) \;<\; \gamma(x) + p \qquad \Longrightarrow \qquad \begin{cases} \Delta u + f(x,u) \;=\; 0 & \text{if} \quad x \in \Omega\,, \\ \mathbf{n}(x) \cdot \nabla u(x) \;=\; 0 & \text{if} \quad x \in \partial\Omega\,. \end{cases} \tag{3.5}$$

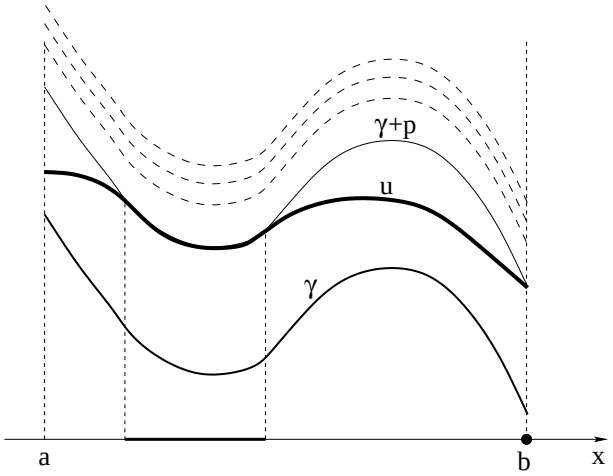

FIG. 1. *An example of a solution to the obstacle problem* (3.3)–(3.4). *Note that* $u_x(a) = 0$, *but at the point* b *one has* $u(b) = \gamma(b) + p$ *and* $\gamma_x(b) < u_x(b) < 0$. *If the measure* μ *concentrates a positive mass on the boundary* $\partial\Omega$, *solutions to the variational inequality* (3.6) *need not satisfy the boundary condition* (1.2) *in the classical sense. Theorem* 3.3 *is proved by solving the obstacle problem* (3.6) *with various choices of* $p \geq 0$, *and finding a value of* p *for which* (3.1)–(3.2) *hold.*

Following [24], the problem (3.3)–(3.5) can be reformulated as a differential inequality. Consider the closed convex domain

$$\mathcal{K}_p \doteq \left\{ v \in H^1(\Omega), \quad v(x) \leq \gamma(x) + p \quad \text{for a.e. } x \in \Omega \right\}.$$

We then seek $u \in \mathcal{K}_p$ such that

$$\int_\Omega \Big[-\nabla u \cdot \nabla(u - v) + f(x,u)(u - v) \Big]\, dx \;\geq\; 0 \tag{3.6}$$

for all $v \in \mathcal{K}_p$. Notice that, in the smooth case,

$$\begin{aligned} &\int_\Omega \Big[-\nabla u \cdot \nabla(u - v) + f(x,u)(u - v) \Big]\, dx \\ &\qquad = \int_\Omega \Big(\Delta u + f(x,u) \Big)(u - v)\, dx - \int_{\partial\Omega} (\mathbf{n} \cdot \nabla u)(u - v)\, d\sigma. \end{aligned}$$

On regions where $u(x) < \gamma(x) + p$, one can choose $v \in \mathcal{K}_p$ so that $u - v$ takes both positive and negative signs, thus obtaining (3.5). On the other hand, we can always choose $v \in \mathcal{K}_p$ so that $u - v$ is an arbitrary negative function. In this way one obtains (3.4).

2. By the assumptions (A1)–(A2), we can find $0 < \delta_0 \leq \min_{x \in \overline{\Omega}} \gamma(x)$ such that $f(x,u) > 0$ for all $x \in \overline{\Omega}$ and $0 < u \leq \delta_0$. The functions

$$u^{upper}(x) \;=\; \min\{M,\, \gamma(x) + p\}, \qquad u^{lower}(x) \;=\; \delta_0$$

are thus an upper and a lower solution, respectively. By a comparison argument, the variational inequality admits a solution $u : \overline{\Omega} \mapsto [\delta_0, M]$. By the regularity theory for solutions to variational inequalities with smooth obstacles (see [9, 19, 24] or Theorem 6.1 in [26]), it follows that $u \in W^{2,\infty}_{loc}(\Omega)$. In particular, u is continuously

differentiable on the open set Ω. Furthermore, u is globally Lipschitz on the closure $\overline{\Omega}$.

To prove uniqueness, let $u_1, u_2 : \overline{\Omega} \mapsto [\delta_0, M]$ be two solutions. To fix the ideas, assume $u_1(y) < u_2(y)$ for some $y \in \overline{\Omega}$. By continuity and compactness, there exists $\bar{x} \in \overline{\Omega}$, where the ratio u_1/u_2 attains its minimum, say

$$0 \ < \ \lambda \ \doteq \ \min_{x\in\overline{\Omega}} \frac{u_1(x)}{u_2(x)} \ = \ \frac{u_1(\bar{x})}{u_2(\bar{x})} \ < \ 1.$$

Therefore,

$$0 \ = \ u_1(\bar{x}) - \lambda\, u_2(\bar{x}) \ \leq \ u_1(x) - \lambda\, u_2(x) \qquad \text{for all} \quad x \in \overline{\Omega}. \tag{3.7}$$

Two cases will be considered.

Case 1. $\bar{x} \in \Omega$ is an interior point. The strict concavity of f implies

$$f(\bar{x}, u_1(\bar{x})) - \frac{u_1(\bar{x})}{u_2(\bar{x})} f(\bar{x}, u_2(\bar{x})) \ > \ 0. \tag{3.8}$$

By continuity, there exist $r, \delta_0 > 0$ small enough so that

$$-\, f(\bar{x}, u_1(x)) + \lambda f(\bar{x}, u_2(x)) \ \leq \ -\,\delta_0 \qquad \text{for all } x \in B(\bar{x}, r). \tag{3.9}$$

Consider the function $u_1 - \lambda u_2 \in W^{2,\infty}$. For a.e. $x \in B(\bar{x}, r)$ we have

$$\Delta u_1(x) - \lambda \Delta u_2(x) \ \leq \ -\, f(x, u_1(x)) - \lambda f(x, u_2(x)) \ \leq \ -\,\delta_0 \ < \ 0. \tag{3.10}$$

On the other hand, for points on the boundary of the ball $x \in \partial B(\bar{x}, r)$ one has

$$u_1(x) - \lambda u_2(x) \ \geq \ 0. \tag{3.11}$$

Together, (3.10) and (3.11) imply the strict inequality

$$u_1(x) - \lambda u_2(x) \ > \ 0 \qquad \text{for} \quad |x - \bar{x}| \ < \ r.$$

This yields a contradiction with the first identity in (3.7).

Case 2. $\bar{x} \in \partial\Omega$ is a boundary point. Recalling (3.9), and observing that u_1 does not touch the obstacle in a neighborhood of $\bar{x}$, we can find $r, \delta_0 > 0$ such that

$$\mathbf{n}(x) \cdot \nabla (u_1(x) - \lambda u_2(x)) \ = \ -\lambda \mathbf{n}(x) \cdot u_2(x) \ \geq \ 0 \qquad \text{for all } x \in \partial\Omega \cap B(\bar{x}, r), \tag{3.12}$$

$$\Delta u_1(x) - \lambda \Delta u_2(x) \ \leq \ -\,\delta_0 \ < \ 0 \qquad \text{for a.e. } x \in \Omega \cap B(\bar{x}, r). \tag{3.13}$$

By (3.7) we have

$$u_1(x) - \lambda u_2(x) \ \geq \ 0 \qquad \text{for all } x \in \partial(\Omega \cap B(\bar{x}, r)).$$

Hence (3.12)–(3.13) imply

$$u_1(x) - \lambda u_1(x) \ > \ 0 \qquad \text{for all } x \in \overline{\Omega}, \quad |x - \bar{x}| < r.$$

This yields again a contradiction with the first identity in (3.7), thus proving uniqueness.

3. For each $p \geq 0$, denote by $u^{(p)}$ the unique, strictly positive solution to this obstacle problem. Continuity in $\mathcal{C}^0$ can be proved by a comparison argument. Indeed, let $0 \leq \bar{p} < p$. Then $u^{(\bar{p})}$ is a lower solution to the obstacle problem satisfied by $u^{(p)}$. This immediately implies $u^{(\bar{p})}(x) \leq u^{(p)}(x)$. On the other hand, consider a point $\bar{x} \in \overline{\Omega}$, where γ attains its (strictly positive) minimum. Define

$$\lambda \doteq \frac{\gamma(\bar{x}) + \bar{p}}{\gamma(\bar{x}) + p}, \qquad v(x) \doteq \lambda\, u^{(p)}(x).$$

We claim that $v \leq u^{(\bar{p})}$. Indeed, by construction we have

$$v(x) = \lambda\, u^{(\bar{p})}(x) \leq \frac{\gamma(\bar{x}) + \bar{p}}{\gamma(\bar{x}) + p} \cdot (\gamma(x) + p) \leq \gamma(x) + \bar{p}$$

for all $x \in \overline{\Omega}$. Moreover,

$$\mathbf{n}(x) \cdot \nabla v(x) = \lambda\, \mathbf{n}(x) \cdot \nabla v(x) \leq 0 \qquad \text{for all } x \in \partial\Omega.$$

Next, at any point $x \in \Omega$, by the concavity of f w.r.t. u we have

$$\lambda\,(\Delta v + f(x, v(x))) \geq \Delta u^{(p)}(x) + \lambda\, f(x, \lambda u^{(p)}(x)) \geq \Delta u^{(p)}(x) + f(x, u^{(p)}(x)) \geq 0.$$

This proves that v is a subsolution, hence $v \leq u^{(\bar{p})}$. The above arguments show that the map $p \mapsto u^{(p)}$ is continuous with values in $\mathcal{C}^0(\overline{\Omega})$.

4. Given the solution $u^{(p)}$, define the corresponding measure $\mu^{(p)}$ by setting

(3.14)
$$\begin{aligned}\mu^{(p)}(A) &= \int_{A\cap\Omega} \frac{1}{u^{(p)}}\Big(\Delta u^{(p)} + f(x, u^{(p)})\Big)\, dx - \int_{A\cap\partial\Omega} \frac{1}{u^{(p)}}\mathbf{n}(x)\cdot\nabla u^{(p)}\, d\sigma \\ &= \int_{A\cap\Omega\cap\{u(x)=\gamma(x)\}} \frac{1}{u^{(p)}}\Big(\Delta\gamma + f(x, u^{(p)})\Big)\, dx - \int_{A\cap\partial\Omega} \frac{1}{u^{(p)}}\mathbf{n}(x)\cdot\nabla u^{(p)}\, d\sigma\end{aligned}$$

for every Borel set A. Notice that the above equality holds because $u \in W^{2,\infty}_{loc}(\Omega)$. We observe that $\Delta u^{(p)} \in \mathbf{L}^\infty(\Omega)$. Indeed, for a.e. $x \in \Omega$, one has

$$\Delta u^{(p)}(x) = \begin{cases} \Delta\gamma(x) & \text{if } u^{(p)}(x) = \gamma(x) + p, \\ -f(x, u^{(p)}(x)) & \text{if } u^{(p)}(x) < \gamma(x) + p. \end{cases} \tag{3.15}$$

We claim that the function

$$p \mapsto \Phi(p) \doteq \mu^{(p)}(\overline{\Omega}) \tag{3.16}$$

is continuous. Indeed, we can write

$$\Phi(p) = \Phi_1(p) + \Phi_2(p) \doteq \int_\Omega \frac{\Delta u^{(p)}(x) + f(x, u^{(p)}(x))}{\gamma(x) + p}\, dx - \int_{\partial\Omega} \frac{\mathbf{n}\cdot\nabla u^{(p)}}{u^{(p)}(x)}\, d\sigma\,. \tag{3.17}$$

By Theorem B in [3], the map $p \mapsto \mathbf{n}\cdot\nabla u^{(p)}$ is continuous with values in $\mathbf{L}^1(\partial\Omega)$. Together with the uniform continuity of the map $p \mapsto u^{(p)}$ with values in $\mathcal{C}^0(\Omega)$, proved in the previous step, this yields the continuity of the map $p \mapsto \Phi_2(p)$.

To prove the continuity of Φ_1, let $\bar{p} \geq 0$ and $\varepsilon > 0$ be given. Choose $\delta > 0$ so that, calling

$$\Omega_\delta \doteq \{x \in \Omega\,;\ d(x, \partial\Omega) < \delta\},$$

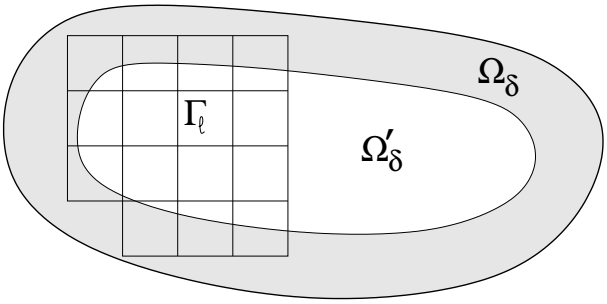

FIG. 2. *The decomposition of the domain $\Omega = \Omega_\delta \cup \Omega'_\delta$, where the inner domain is covered by finitely many hypercubes Γ_ℓ.*

one has

$$\int_{\Omega_\delta} \left| \frac{\Delta\gamma(x) + f(x, \gamma(x)+p)}{\gamma(x)+p} \right| dx \;<\; \varepsilon \tag{3.18}$$

for all p such that $|p - \bar p| < \delta$.

Next, cover the remaining set $\Omega'_\delta \doteq \Omega \setminus \Omega_\delta$ with hypercubes of side ρ, choosing $\rho > 0$ small enough so that the oscillation of $f, u^{(p)}$ on each such cube Γ_ℓ is $\leq \varepsilon$ (see Figure 2). Namely,

$$\sup_{x,y\in\Gamma_\ell} \big|f(x,\gamma(x)+p) - f(y,\gamma(y)+p)\big| \;<\; \varepsilon, \qquad\qquad \sup_{x,y\in\Gamma_\ell} \big|u^{(p)}(x) - u^{(p)}(y)\big| \;<\; \varepsilon, \tag{3.19}$$

for every Γ_ℓ and every p such that $|p - \bar p| < \delta$. We now compute

$$\begin{aligned} A \;&\doteq\; \left| \int_{\Omega'_\delta} \frac{\Delta u^{(p)}(x) + f(x, u^{(p)}(x))}{u^{(p)}(x)} - \frac{\Delta u^{(\bar p)}(x) + f(x, u^{(\bar p)}(x))}{u^{(\bar p)}(x)}\, dx \right| \\ &\leq\; \int_{\Omega'_\delta} \left| \frac{f(x, u^{(p)}(x))}{u^{(p)}(x)} - \frac{f(x, u^{(\bar p)}(x))}{u^{(\bar p)}(x)} \right| dx + \sum_\ell \left| \int_{\Gamma_\ell} \frac{\Delta u^{(p)}(x)}{u^{(p)}(x)} - \frac{\Delta u^{(\bar p)}(x)}{u^{(\bar p)}(x)}\, dx \right| \\ &\doteq\; A_1 + A_2\,. \end{aligned} \tag{3.20}$$

As $p \to \bar p$, the uniform convergence $u^{(p)} \to u^{(\bar p)}$ implies that $A_1 \to 0$. To estimate A_2, call $u_\ell^{(p)}$ and $u_\ell^{(\bar p)}$ the average values of $u^{(p)}$ and $u^{(\bar p)}$ on the cube Γ_ℓ, respectively. For each ℓ we now have

$$\begin{aligned} &\left| \int_{\Gamma_\ell} \frac{\Delta u^{(p)}(x)}{u^{(p)}(x)} - \frac{\Delta u^{(\bar p)}(x)}{u^{(\bar p)}(x)}\, dx \right| \\ &\quad\leq\; \left| \int_{\Gamma_\ell} \frac{\Delta u^{(p)}(x) - \Delta u^{(\bar p)}(x)}{u_\ell^{(\bar p)}}\, dx \right| + \int_{\Gamma_\ell} \left|\Delta u^{(p)}(x)\right| dx \cdot \sup_{x\in\Gamma_\ell} \left| \frac{1}{u^{(p)}(x)} - \frac{1}{u_\ell^{(\bar p)}} \right| \\ &\quad\leq\; \Big(\min_{x\in\overline\Omega} u^{(\bar p)}\Big)^{-1} \cdot \left| \int_{\partial\Gamma_\ell} \frac{\nabla u^{(p)}(x) - \nabla u^{(\bar p)}(x)}{u_\ell^{(\bar p)}}\, dx \right| \\ &\quad+ \textstyle\int_{\Gamma_\ell} \left|\Delta u^{(p)}(x)\right| dx \cdot \sup_{x\in\Gamma_\ell} \left| \frac{1}{u^{(p)}(x)} - \frac{1}{u_\ell^{(\bar p)}} \right|. \end{aligned}$$

We now observe that the quantities

$$\Big(\min_{x\in\overline\Omega} u^{(\bar p)}\Big)^{-1}, \qquad\qquad \int_\Omega \left|\Delta u^{(p)}(x)\right| dx$$

remain uniformly bounded. Indeed, the first bound is a consequence of the strict inequality $f_u(x,0) > 0$ in (1.3) and of (1.8). By (3.5), the second bound follows from

$$\int_\Omega \left|\Delta u^{(p)}(x)\right| dx \;\leq\; \int_\Omega \max\left\{\left|\Delta\gamma(x)\right|, \left|f(x,u^{(p)}(x))\right|\right\} dx.$$

Since A_2 is a sum over finitely many cubes, we conclude that

$$A_2 \;=\; \mathcal{O}(1)\cdot\sum_\ell \left|\int_{\partial\Gamma_\ell} \frac{\nabla u^{(p)}(x) - \nabla u^{(\bar p)}(x)}{u_\ell^{(\bar p)}}\, dx\right| + \mathcal{O}(1)\cdot \sup_{x\in\Gamma_\ell}\left|\frac{1}{u^{(p)}(x)} - \frac{1}{u_\ell^{(\bar p)}}\right|.$$

Given $\varepsilon > 0$, we first choose the diameter of each cube small enough, so that

$$\sup_\ell \sup_{x\in\Gamma_\ell}\left|\frac{1}{u^{(p)}(x)} - \frac{1}{u_\ell^{(\bar p)}}\right| \;<\; \varepsilon.$$

By a result in [3], the map $p \mapsto \mathbf{n}\cdot\nabla u^{(p)}$ is continuous with values in $\mathbf{L}^1(\partial\Omega)$. Letting $p \to \bar p$, each term of the summation on the right-hand side thus converges to zero. We conclude that, as $p \to \bar p$, both terms A_1, A_2 approach zero, proving the continuity of the map $p \mapsto \Phi(p)$.

5. Using the fact that Φ is continuous, we can now conclude as follows.

For $p > 0$ large enough, we have $\gamma(x) + p > M$ for all $x \in \overline{\Omega}$. Hence the measure $\mu^{(p)}$ vanishes and $\Phi(p) = 0$. Two cases can now arise.

Case 1. $\Phi(0) \geq \kappa$. In this case, by continuity there exists some value $p \geq 0$ such that $\Phi(p) = \kappa$. In this case, the measure $\mu^{(p)}$ defined at (3.14) satisfies all requirements.

Case 2. $\Phi(0) < \kappa$. In this case, we simply take $p = 0$. The measure $\mu^{(p)}$ defined at (3.14) again satisfies our requirements. □

Remark 3.4. The uniqueness of the equilibrium solution is left as an open question. According to the above analysis, uniqueness would follow from the strict monotonicity of the map $p \mapsto \Phi(p)$ in (3.16). However, there is no obvious reason for Φ to be strictly decreasing. In this context, it is worth observing that the coincidence sets

$$\Sigma^{(p)} \;\doteq\; \{x \in \overline{\Omega}\,;\;\; u^{(p)}(x) \,=\, \gamma(x) + p\}$$

may not become smaller as p increases.

Remark 3.5. For any given $\kappa > 0$, the total payoff achieved by competing players cannot be greater than the maximum payoff achieved by one single player with total harvesting capacity κ. From the above analysis it follows that, whenever

$$\kappa \;>\; \max_{p\in[0,M]} \mu^{(p)}(\overline{\Omega}),$$

the fish density must decrease to a level where the profit for every competing agent is exactly zero. Namely,

$$u(x) \;\leq\; \gamma(x), \qquad\qquad x \in \overline{\Omega}.$$

This is a form of the so-called "tragedy of commons": because of competition, the natural resources are depleted to a level where no profit is possible [21].

If harvesting is cheap, i.e., if the harvesting cost $\gamma(x) > 0$ is small, then the total amount of fish caught will be small as well. Indeed, the concavity assumption in (1.3) implies

$$\int_{\overline{\Omega}} u\, d\mu \;=\; \int_\Omega f(x,u(x))\, dx \;\leq\; \int_\Omega f_u(x,0)\,\gamma(x)\, dx\,.$$

4. Players with different costs. In the previous sections we assumed that all fishermen had the same cost $\gamma(x)$ for fishing at the location x. This may not be a realistic assumption. For example, think of N groups of fishermen, located at coastal towns $y_1, y_2, \ldots, y_N \in \overline{\Omega}$. To members of the ith group, the cost $\gamma_i(x)$ for fishing at a point x will depend on the distance from x to the harbor, say

$$\gamma_i(x) \;=\; \sqrt{1+|x-y_i|^2}, \qquad\qquad i=1,\ldots,N.$$

We shall extend Definition 3.1 to cover this more general situation. In the following, for $i=1,\ldots,N$, we call κ_i the joint harvesting capability of all fishermen in the i-th group, while μ_i will be the combined harvesting effort of all members of the i-th group.

DEFINITION 4.1 (Nash equilibrium with different groups of small players). *Consider a harvesting game governed by* (1.1)–(1.2)*, with N groups of players. Let $\kappa_i > 0$ be the harvesting capability of the ith group, and let $\gamma_i(x)$ be the corresponding cost for fishing at x.*

We say that $(u, \mu_1, \ldots, \mu_N)$ is an equilibrium solution *to the harvesting game provided that:*

- *Each μ_i is a positive measure on $\overline{\Omega}$, with $\mu_i(\overline{\Omega}) \leq \kappa_i$.*
- *u is a strictly positive solution to the elliptic equation* (1.1) *with Neumann boundary conditions* (1.2)*, where $\mu = \mu_1 + \cdots + \mu_N$.*
- *There exist constants $p_i \geq 0$ such that*

$$u(x) - \gamma_i(x) \;=\; p_i \qquad \text{on } \ \mathrm{Supp}(\mu_i), \tag{4.1}$$

$$u(x) - \gamma_i(x) \;\leq\; p_i \qquad \text{for all } \ x \in \Omega. \tag{4.2}$$

Moreover, one notes the implication

$$p_i \;>\; 0 \qquad \Longrightarrow \qquad \mu_i(\overline{\Omega}) \;=\; \kappa_i\,. \tag{4.3}$$

THEOREM 4.2. *Let the assumptions* (A1)–(A2) *hold, and assume that all cost functions γ_i, $i=1,\ldots,N$ satisfy the same assumptions as in* (A4)*. Then the harvesting game with with different groups of small players has an equilibrium solution.*

Proof. **1.** For every N-tuple $p=(p_1,\ldots,p_N) \in [0,M]^N$, call $u^{(p)}$ the solution to the semilinear problem (3.3)–(3.5), where the constraint $u(x) \leq \gamma(x)+p$ is replaced by

$$u(x) \;\leq\; \gamma^{(p)}(x) \;\doteq\; \min_i \,\big(\gamma_i(x)+p_i\big). \tag{4.4}$$

By arguments very similar to those in section 3, one can prove that the this problem has a unique positive solution, which depends continuously on p. In more detail, the existence of a solution is proved by solving the variational inequality (3.6) on the domain

$$\mathcal{K}_{(p_1,\ldots,p_N)} \;\doteq\; \Big\{ v\in H^1(\Omega)\,, \quad v(x) \leq \min_i (\gamma_i(x)+p_i) \quad \text{for a.e. } x\in\Omega \Big\}\,. \tag{4.5}$$

We now observe that the solution u has the same regularity as in the previous case. Namely, u lies in $W^{2,\infty}_{loc}(\Omega)$ and is Lipschitz on the closure $\overline{\Omega}$. Indeed, in order to

apply the regularity theorems in [24, 26] one only needs to check that the positive part of the measure $\Delta\gamma^{(p)} + f$ is absolutely continuous w.r.t. Lebesgue measure, with density in $\mathbf{L}^\infty$. This is clearly true because the source term f is bounded above and below, and moreover, for any choice of $(p_1, \ldots, p_N)$, one has

$$\Big\|(\Delta\gamma^{(p)})^+\|_{\mathbf{L}^\infty(\Omega)} \;\leq\; \max_i \sup_{x\in\Omega} \,(\Delta\gamma_i(x))^+ \;\leq\; C\,.$$

By the regularity property $\Delta u \in \mathbf{L}^\infty(\Omega)$ and the strict convexity of f w.r.t. u, the same arguments as in step **2** of the proof of Theorem 3.3 yield the uniqueness of the solution $u^{(p)}$ to the obstacle problem.

2. Let $\mu^{(p)}$ be the positive measure defined as in (3.14). In the present setting, $\mu^{(p)}$ describes the combined harvesting effort of all fishermen together. Repeating the arguments in steps **3** and **4** of the proof of Theorem 3.3, we obtain

(i) the continuity of the map $p = (p_1, \ldots, p_N) \mapsto u^{(p)}$, as a map from $\mathbb{R}^N$ into $\mathcal{C}^0(\overline{\Omega})$ and

(ii) the continuity of the map $p \mapsto \mu^{(p)}(\overline{\Omega})$.

In particular, by (i) it follows that the multifunctions

$$p \;\mapsto\; \Sigma_i^{(p)} \;\doteq\; \Big\{x \in \overline{\Omega}\,;\;\; u^{(p)}(x) = \gamma_i(x) + p_i\Big\}, \qquad\qquad i = 1, \ldots, N, \tag{4.6}$$

have a closed graph.

3. For every $p = (p_1, \ldots, p_N)$, in connection with the positive measure $\mu^{(p)}$, define the set of coefficients

$$\Lambda^{(p)} \doteq \Big\{(\lambda_1, \ldots, \lambda_N)\,;\text{there exist positive measures } \mu_1, \ldots, \mu_N \text{ such that } \mu_i(\overline{\Omega}) = \lambda_i\,,$$
$$\mu^{(p)} = \mu_1 + \cdots + \mu_N\,,\; \mathrm{Supp}(\mu_i) \subseteq \{x\,;\; u^{(p)}(x) = \gamma_i(x) + p_i\},\;\; i = 1, \ldots, N\Big\}. \tag{4.7}$$

We claim that the map $p \mapsto \Lambda^{(p)}$ defined at (4.7) is an upper semicontinuous multifunction with nonempty, closed, convex values.

To prove that each set $\Lambda^{(p)}$ is nonempty, consider the coincidence sets $\Sigma_1^{(p)}, \ldots, \Sigma_N^{(p)}$ introduced at (4.6). Define

$$\lambda_1 \doteq\; \mu^{(p)}\left(\Sigma_1^{(p)}\right), \qquad \lambda_i \;\doteq\; \mu^{(p)}\Big(\Sigma_i^{(p)} \setminus \Sigma_{i-1}^{(p)}\Big), \qquad i = 2, \ldots, N.$$

Then $(\lambda_1, \ldots, \lambda_N) \in \Lambda^{(p)}$.

To prove that $\Lambda^{(p)}$ is convex, assume $(\lambda_1, \ldots, \lambda_N)$, $(\tilde\lambda_1, \ldots, \tilde\lambda_N) \in \Lambda^{(p)}$. Let

$$\mu_1 + \cdots + \mu_N \;=\; \mu^{(p)} \;=\; \tilde\mu_1 + \cdots + \tilde\mu_N$$

be the corresponding measures. For any $\sigma \in [0, 1]$ and $i \in \{1, \ldots, N\}$, the measure

$$\mu_i^\sigma \;=\; \sigma\mu_i + (1 - \sigma)\tilde\mu_i$$

is supported on the set where $u(x) = c_i(x) + p_i$. Moreover,

$$\mu_1^\sigma + \cdots + \mu_N^\sigma \;=\; \mu^{(p)}.$$

This shows that each set $\Lambda^{(p)}$ is convex.

4. To prove upper semicontinuity, we need to check that the graph of $p \mapsto \Lambda^{(p)}$ is closed. Toward this goal, we recall that the multifunctions $p \mapsto \Sigma_i^{(p)}$ introduced at (4.6) have a closed graph.

According to (3.14), the measure $\mu^{(p)}$ is the sum of two parts. The restriction of $\mu^{(p)}$ to the open set Ω is absolutely continuous with density

(4.8)
$$\phi(x) \doteq \frac{\Delta u^{(p)}(x) + f(x, u^{(p)}(x))}{u^{(p)}(x)} = \frac{\Delta \gamma^{(p)}(x) + f(x, u^{(p)}(x))}{u^{(p)}(x)} \cdot \chi_{\{u^{(p)}(x)=\gamma^{(p)}(x)\}}$$

w.r.t. Lebesgue measure, while the restriction of $\mu^{(p)}$ to the boundary $\partial\Omega$ has density

$$\psi(x) \doteq -\frac{\mathbf{n}(x) \cdot \nabla u^{(p)}(x)}{u^{(p)}(x)} \tag{4.9}$$

w.r.t. $(d-1)$-dimensional surface measure. In (4.8) we denote by χ_A the characteristic function of a set A, while $\gamma^{(p)}$ is the obstacle function introduced at (4.4).

We now consider a convergent sequence

$$(p_{1,n}, \ldots, p_{N,n}) \to (p_1, \ldots, p_N).$$

Let $(\lambda_{1,n}, \ldots \lambda_{N,n}) \in \Lambda^{(p_n)}$ for all $n \geq 1$, and assume the convergence $(\lambda_{1,n}, \ldots \lambda_{N,n}) \to (\lambda_1, \ldots \lambda_N)$. We need to show that $(\lambda_1, \ldots \lambda_N) \in \Lambda^{(p)}$.

By the uniform convergence

$$\gamma^{(p_n)}(x) \doteq \min_i \big(\gamma_i(x) + p_{i,n}\big) \to \min_i \big(\gamma_i(x) + p_i\big) \doteq \gamma^{(p)}(x), \qquad u^{(p_n)} \to u^{(p)},$$

the convergence of the corresponding densities in (4.8)–(4.9) follows, namely

$$\phi_n \to \phi, \qquad \psi_n \to \psi,$$

in $\mathbf{L}^1(\Omega)$ and in $\mathbf{L}^1(\partial\Omega)$, respectively. Here ϕ, ψ are the densities of the limit measure $\mu^{(p)}$.

For each $n \geq 1$, by assumption there exist positive measures $\mu_{1,n}, \ldots, \mu_{N,n}$ such that

(4.10)
$$\mu_{i,n}(\overline{\Omega}) = \lambda_{i,n}\,, \quad \mathrm{Supp}(\mu_{i,n}) \subseteq \{x\,;\, u^{(p_n)}(x) = \gamma_i(x) + p_i\}, \quad \mu^{(p_n)} = \mu_{1,n} + \cdots + \mu_{N,n}\,.$$

By possibly taking a subsequence, one has the weak convergence $(\mu_{1,n}, \ldots, \mu_{N,n}) \rightharpoonup (\mu_1, \ldots, \mu_N)$. The conditions (4.10), together with the upper semicontinuity of the sets Σ_i in (4.6), imply

$$\mu_i(\overline{\Omega}) = \lambda_i, \qquad \mathrm{Supp}(\mu_i) \subseteq \{x\,;\, u^{(p)}(x) = \gamma_i(x) + p_i\}, \qquad \mu^{(p)} = \mu_1 + \cdots + \mu_N,$$

proving that $(\lambda_1, \ldots, \lambda_N) \in \Lambda^{(p)}$, as claimed.

5. On the N-dimensional cube $\Gamma \doteq [0, M]^N$ we now define the multivalued vector field

$$\mathbf{v}(p) = \Big\{(\lambda_1, \ldots, \lambda_N) - (\kappa_1, \ldots, \kappa_N)\,;\quad (\lambda_1, \ldots, \lambda_N) \in \Lambda^{(p)}\Big\}.$$

By the previous step, this is an upper semicontinuous map with compact, convex values. By the general theory of variational inequalities [20, 24], there exists a point $p^* \in \Gamma$ and a vector

$$\mathbf{v}^* = (v_1^*, \ldots, v_N^*) = (\lambda_1^* - \kappa_1, \ldots, \lambda_N^* - \kappa_N) \in \mathbf{v}(p^*)$$

such that

$$\langle \mathbf{v}^* ,\ p - p^* \rangle \ \le\ 0 \qquad \text{for all } p \in \Gamma. \tag{4.11}$$

We claim that this provides the required solution. Indeed, consider any $i \in \{1, \dots, N\}$. If $\lambda_i^* = \kappa_i$ we are done. Otherwise we consider two cases.

Case 1. $\lambda_i^* < \kappa_i$. We then use (4.11), choosing the vector $p = (p_1^*, \dots, p_{i-1}^*, 0, p_{i+1}^*, \dots, p_N^*)$. This yields

$$\langle \mathbf{v}^* ,\ p - p^* \rangle \ =\ -\, v_i^*\, p_i^* \ =\ -\,(\lambda_i^* - \kappa_i) p_i^* \ \le\ 0.$$

This implies $p_i^* = 0$, because otherwise a contradiction is obtained.

Case 2. $\lambda_i^* > \kappa_i$. We then use (4.11), choosing the vector $p = (p_1^*, \dots, p_{i-1}^*, M, p_{i+1}^*, \dots, p_N^*)$. This yields

$$\langle \mathbf{v}^* ,\ p - p^* \rangle \ =\ -\, v_i^*\,(M - p_i^*) \ =\ (\lambda_i^* - \kappa_i)(M - p_i^*) \ \le\ 0.$$

This implies $p_i^* = M$, because otherwise a contradiction is obtained.

On the other hand, if $p_i^* = M$, then the measure μ_i has total mass

$$\mu_i(\overline{\Omega}) \ =\ \lambda_i^* \ >\ \kappa_i \ >\ 0 \tag{4.12}$$

and is supported in the set

$$\{x \in \overline{\Omega}\,;\quad u(x) \ge \gamma_i(x) + M\}.$$

Since $u(x) \le M$ for all x, the above set is empty, reaching a contradiction with (4.12).

The previous arguments show that, for every $i = 1, \dots, N$, either $p_i^* = 0$ or else $\mu_i(\overline{\Omega}) = \kappa_i$, as required by (4.3). This completes the proof. □

5. A continuum of different players. We consider here a continuum of fishermen, labeled by a variable $\theta \in [0, \kappa]$. We let $\gamma(x, \theta)$ be the cost to fisherman θ for harvesting at x. Throughout, we assume the following:

(A5) *The map $\theta \mapsto \gamma(\cdot, \theta)$ is measurable and uniformly bounded, from $[0, \kappa]$ into $\mathcal{C}^2(\overline{\Omega})$. Moreover,*

$$\gamma(x, \theta) \ \ge\ c_0 \ > 0 \tag{5.1}$$

for all $x \in \overline{\Omega}$ and $\theta \in [0, \kappa]$.

By a Lusin type theorem, this implies that, for every $\varepsilon > 0$, there exists a closed set $K_\varepsilon \subset [0, \kappa]$, with $\text{meas}(K_\varepsilon) > \kappa - \varepsilon$, such that the restriction $\gamma : K_\varepsilon \mapsto \mathcal{C}^2(\overline{\Omega})$ is continuous.

DEFINITION 5.1 (Nash equilibrium with infinitely many, different small players). *We say that (u, μ^θ) is a* generalized equilibrium solution *to the harvesting game* (1.1)–(1.2) *for a continuum of players with costs $\gamma = \gamma(x, \theta)$ provided that:*

(i) *u is a solution to the elliptic equation* (1.1) *with Neumann boundary conditions* (1.2)*, where μ is the measure defined by*

$$\mu(V) \ \doteq\ \int_0^\kappa \mu^\theta(V)\, d\theta. \tag{5.2}$$

(ii) *For a.e.* $\theta \in [0,\kappa]$, μ^θ *is a positive measure on* $\overline{\Omega}$ *such that*

$$\mu^\theta(\overline{\Omega}) \ \in \ [0,1], \tag{5.3}$$

(5.4)

$$Supp(\mu^\theta) \ \subseteq \ \Big\{x \in \overline{\Omega}\,; \ \ u(x) - \gamma(x,\theta) \ = \ \max_{y\in\overline{\Omega}} \ \big\{u(y) - \gamma(y,\theta)\big\} \ \geq 0\Big\},$$

$$\max_{y\in\overline{\Omega}} \ \big\{u(y) - \gamma(y,\theta)\big\} \ > \ 0 \qquad \Longrightarrow \qquad \mu^\theta(\overline{\Omega}) \ = \ 1. \tag{5.5}$$

Remark 5.2. As in the previous sections, here κ corresponds to the maximum harvesting capacity. For all fishermen, the payoff from harvesting at a point x is proportional to the density of fish at x, namely $u(x)$. On the other hand, the cost to player θ for harvesting at x is $\gamma(x,\theta)$.

According to (5.3)–(5.5), any fisherman $\theta \in [0,\kappa]$ can choose to work during a fraction $\mu^\theta(\overline{\Omega})$ of his time, and remain idle for the remaining portion $1 - \mu^\theta(\overline{\Omega})$. If $\mu^\theta(\overline{\Omega}) > 0$, he will harvest at points $x \in \overline{\Omega}$, distributed according to a measure μ^θ, supported within the set where the profit $u(x) - \gamma(x,\theta)$ attains a nonnegative maximum.

By (5.5), if there are points $x \in \overline{\Omega}$ where this profit is strictly positive, then fisherman θ will work full time, i.e., $\mu^\theta(\overline{\Omega}) = 1$.

THEOREM 5.3. *Let the assumptions* (A1)–(A2) *hold, together with* (A5). *Then the harvesting game with infinitely many small players has a generalized equilibrium solution.*

Proof. We shall use Theorem 4.2 together with an approximation argument.

1. Let $\varepsilon > 0$ be given. By the assumption (A5) there exists a closed set $K_\varepsilon \subset [0,\kappa]$ with $\text{meas}(K_\varepsilon) > \kappa - \varepsilon$ such that the map

$$\theta \ \mapsto \ \gamma(\cdot,\theta)$$

is continuously restricted to K_ε. Hence it can be approximated with a function which is piecewise constant w.r.t. θ.

More precisely, there exist cost functions $\gamma_1,\ldots,\gamma_N$ and numbers

$$0 \ \doteq \ \theta_0 \ < \ \theta_1 \ < \ \ldots, \ \theta_N \ = \ \kappa$$

such that

$$\big|\gamma(x,\theta) - \gamma_i(x)\big| < \varepsilon \qquad \text{for all} \ \ \theta \in]\theta_{i-1},\, \theta_i] \cap K_\varepsilon\,.$$

By Theorem 4.2, the harvesting game with N groups of fishermen having costs γ_i and sizes

$$\kappa_i \ \doteq \ \text{meas}\Big([\theta_{i-1},\theta_i] \cap K_\varepsilon\Big) \tag{5.6}$$

has at least one Nash equilibrium solution. Let $(\mu_1,\ldots,\mu_N)$ be the measures on $\overline{\Omega}$ corresponding to this solution. Each of these measures can be represented as the push-forward of Lebesgue measure on $\Theta_i \subseteq]\theta_{i-1},\theta_i] \cap K_\varepsilon$ through some measurable map $\xi_i(\cdot)$.

Call $\Theta \doteq \cup_i \Theta_i$, and let $\xi : \Theta \mapsto \overline{\Omega}$ be the map whose restriction to Θ_i coincides with ξ_i. We then define the family of measures $\{\mu^\theta\,; \ \ \theta \in [0,\kappa]\}$ by setting the following:

- If $\theta \in \Theta$, then μ^θ is a unit mass at the point $\xi(\theta)$.
- If $\theta \notin \Theta$, then μ^θ is the zero measure.

2. We now take a decreasing sequence $\varepsilon_n \downarrow 0$. For each n, following the above procedure we construct an approximate equilibrium solution. Let $u_n(\cdot)$ be the corresponding sequence of fish densities. The uniform bound $\|\gamma(\cdot,\theta)\|_{\mathcal{C}^2} \leq C$ for all $\theta \in [0,\kappa]$ implies that all u_n have uniformly bounded $\mathcal{C}^1$ norm. In particular, they are uniformly Lipschitz continuous on $\overline{\Omega}$. By possibly taking a subsequence we can assume the uniform convergence $u_n \to u$. Moreover, without loss of generality we can assume that the domains $K_{\varepsilon_n} \subset [0,\kappa]$ are monotone increasing:

$$K_{\varepsilon_n} \;\subseteq\; K_{\varepsilon_{n+1}}\,.$$

By construction, a.e. $\theta \in [0,\kappa]$ is a Lebesgue point of some K_{ε_n}.

For each $n \geq 1$, consider the measure μ_n on the product space $[0,\kappa]\times\overline{\Omega}$ defined by

$$\mu_n(A) \;\doteq\; \int_0^\kappa \mu_n^\theta\Big(\{x\in\overline{\Omega}\,;\quad (x,\theta)\in A\}\Big)\,d\theta\,.$$

By extracting a further subsequence, we can assume the weak convergence $\mu_n \rightharpoonup \mu_\infty$. By a disintegration theorem (see section 2.5 in [1]), this limit measure μ_∞ on $[0,\kappa]\times\overline{\Omega}$ can be represented in the form

$$\mu_\infty(A) \;\doteq\; \int_0^\kappa \mu^\theta\Big(\{x\in\overline{\Omega}\,;\quad (x,\theta)\in A\}\Big)\,d\theta.$$

In the remaining part of the proof we show that, inserting the above measures μ^θ into (5.2)–(5.5), all requirements of Definition 5.1 are satisfied.

3. We observe that, since

$$\mu_n\big(V\times\overline{\Omega}\big) \;\leq\; \text{meas}(V)$$

for every Borel subset $V \subseteq [0,\kappa]$, this implies that the same holds for the limit measure μ_∞. Hence

$$\mu^\theta(\overline{\Omega}) \;\leq\; 1$$

for all $\theta \in [0,\kappa]$. This proves (5.3).

4. Next, we claim that u provides the solution to (1.1)–(1.2), in connection with the measure μ defined at (5.2). This is an immediate consequence of the weak convergence $\mu_n \rightharpoonup \mu$ and the uniform convergence $u_n \to u$.

To prove (5.4) and (5.5), consider a value $\theta^* \in [0,\kappa]$ which is a density point of some set K_{ε_m}.

Then the map $\theta \mapsto \gamma(\cdot,\theta)$ is quasi-continuous at θ^*, namely

$$\lim_{r\to 0}\frac{1}{2r}\int_{\theta^*-r}^{\theta^*+r}\big\|\gamma(\cdot,\theta)-\gamma(\cdot,\theta^*)\big\|_{\mathcal{C}^2(\Omega)} \;=\; 0.$$

By continuity, for any given $\varepsilon > 0$ and all $n \geq 1$ large enough we have

$$\text{Supp}(\mu_n^\theta) \;\subseteq\; \Big\{x\in\overline{\Omega}\,;\quad u_n(x)-\gamma(x,\theta)-\varepsilon \;\leq\; \max_{y\in\overline{\Omega}}\,\big[u_n(y)-\gamma(y,\theta)\big]\Big\}$$

provided $\theta \in K_{\varepsilon_n}$.

By the continuity of γ on K_{ε_n}, and the uniform convergence $u_n \to u$, this yields

$$\mathrm{Supp}(\mu^\theta) \;\subseteq\; \Big\{x \in \overline{\Omega}\,;\;\; u_n(x) - \gamma(x,\theta) - \varepsilon \;\leq\; \max_{y\in\overline{\Omega}} \Big[u(y) - \gamma(y,\theta)\Big]\Big\}$$

for all θ which are density points of some set K_{ε_m}. Hence this is true for a.e. $\theta \in [0,\kappa]$. Since $\varepsilon > 0$ can be chosen arbitrarily small, the conclusion holds for a.e. θ. This proves (5.4).

6. Finally, let θ^* be a density point of some set K_{ε_m}, and assume that

$$\max_{y\in\overline{\Omega}} \big[u(y) - \gamma(y,\theta^*)\big] \;>\; 0.$$

Then

$$\max_{y\in\overline{\Omega}} \big(u_n(y) - \gamma(y,\theta^*)\big) \;>\; 0$$

for all n sufficiently large. In turn, this implies

$$\mu^\theta_n(\overline{\Omega}) \;=\; 1$$

for all n large and all $\theta \in K_{\varepsilon_m}$ sufficiently close to θ^*. Taking a weak limit, this yields $\mu^{\theta^*}(\overline{\Omega}) = 1$, proving (5.5). □

Remark 5.4. By a theorem of Skorokhod (see, for example, [4, pp. 185–187]), the measure μ constructed in Theorem 3.3 can be represented as the push forward of the Lebesgue measure on an interval $[0,\kappa']$, with $\kappa' \leq \kappa$. In other words, there exists a measurable map $\phi : [0,\kappa'] \mapsto \overline{\Omega}$ such that

$$\mu(V) \;=\; \mathrm{meas}\Big(\{\theta \in [0,\kappa']\,;\;\; \phi(\theta) \in V\}\Big)$$

for every Borel set $V \subset \mathbb{R}^d$. We can think of θ as a Lagrangian variable labeling the various fishermen. In the equilibrium solution, each fisherman $\theta \in [0,\kappa']$ works full time at the single location $\phi(x)$, while if $\kappa' < \kappa$, all fishermen $\theta \in]\kappa',\kappa]$ remain idle.

In contrast with the results proved in the previous sections, Theorem 5.3 only establishes the existence of an equilibrium solution within generalized strategies. To each player $\theta \in [0,\kappa]$ we associate a positive measure μ^η with total mass ≤ 1. In general, it is not possible to choose μ^η a Dirac mass (meaning the harvesting effort is concentrated at a single point) or the zero measure (meaning that the fisherman remains idle). As shown by the following example, this is not a shortcoming of the analysis, but reflects the nature of the problem.

Example 5.5. On the unit circle $\Omega = \{x \in \mathbb{R}^2\,;\; |x| < 1\}$, consider a concave function $f = f(u)$ such that

$$f(0) \;=\; f(3) \;=\; 0, \qquad \begin{cases} f(u) \;=\; 1/2 & \text{if} \quad u \in [1,2], \\ f''(u) \;\leq\; 0 & \text{if} \quad u \in [0,3]. \end{cases}$$

As cost functions, working in polar coordinates (r,α), assume

$$\gamma(r,\alpha;\theta) \;\doteq\; 2 - r^2\cos^2(\alpha-\theta), \qquad\qquad \theta \in [0,\pi].$$

Then the function

$$u(r,\alpha) \;=\; 1+\frac{1-r^2}{8}$$

satisfies

$$\left\{\begin{array}{lll} \Delta u + f(u) &=& 0, \qquad\qquad x\in\Omega,\\ \mathbf{n}\cdot\nabla u &=& -1/4, \qquad\qquad x\in\partial\Omega. \end{array}\right. \tag{5.7}$$

The equilibrium solution is provided by the family of measures $\{\mu^\theta\,;\;\theta\in[0,\pi]\}$, where μ^θ is the atomic measure containing two masses, each of size $1/4$, namely the two points

$$P^\theta \;=\; (\cos\theta,\sin\theta), \qquad Q^\theta \;=\; (-\cos\theta,-\sin\theta).$$

Notice that none of these measures μ^θ is a probability measure. We observe that, for each $\theta\in[0,\pi]$, the maximum of the profit $u(\cdot)-\gamma(\cdot,\theta)$ to player θ is

$$\max_{r\leq 1,\,\alpha\in[0,\,2\pi]}\left\{1+\frac{1-r^2}{8}-\Big(2-r^2\cos^2(\alpha-\theta)\Big)\right\} \;=\; 0,$$

and it is attained precisely at the two points $(r,\alpha)=(1,\theta)$ or $(1,\theta+\pi)$.

The corresponding measure μ in (5.2) is uniformly distributed along the unit circumference, with density $1/4$ w.r.t. arc length. This is indeed consistent with (5.7).

We remark that there is no way to obtain the measure μ as the push-forward of the Lebesgue measure on a subset $\Theta\subseteq[0,\pi]$ by some (single-valued) measurable map $\theta\mapsto\xi(\theta)$ such that

$$\xi(\theta) \;\in\; \{P^\theta,\,Q^\theta\}$$

for every $\theta\in\Theta$. Indeed, as θ varies in $]0,\pi]$, the sets $\{P^\theta,\,Q^\theta\}$ are all mutually disjoint. Hence, for any map $\xi:[0,\pi]\mapsto\partial\Omega$, the corresponding measure μ^ξ is given by

$$\mu^\xi(V) \;=\; m_1(V\cap A_\xi),$$

where m_1 is the 1-dimensional measure on $\partial\Omega$, and $A_\xi=\{\xi(\theta)\,;\;\;\theta\in\Theta\}$ denotes the range of $\xi(\cdot)$. Clearly, $\mu^\xi\neq\mu$ for any choice of the map $\xi(\cdot)$.

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