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Balancing a static black ring with a phantom scalar field

Burkhard Kleihaus [a], Jutta Kunz [a], Eugen Radu [b,*]

[a] Institut für Physik, Universität Oldenburg, Postfach 2503 D-26111 Oldenburg, Germany
[b] Departamento de Física da Universidade de Aveiro and CIDMA, Campus de Santiago, 3810-183 Aveiro, Portugal

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ABSTRACT

All known five dimensional, asymptotically flat, static black rings possess conical singularities. However, there is no fundamental obstruction forbidding the existence of balanced configurations, and we show that the Einstein–Klein-Gordon equations admit (numerical) solutions describing static asymptotically flat black rings, which are regular on and outside the event horizon. The scalar field is 'phantom', which creates the self-repulsion necessary to balance the black rings. Similar solutions are likely to exist in other spacetime dimensions, the basic properties of a line element describing a four dimensional, asymptotically flat black ring geometry being discussed.

1. Introduction and motivation

In 2001 Emparan and Reall have found a remarkable new static, vacuum black hole (BH) solution of Einstein equations in $4+1$ dimensions [1]. Different from the Schwarzschild-Tangherlini BH [2], this solution has an event horizon with $S^2 \times S^1$ topology and describes an asymptotically flat black ring (BR). However, the solution in [1] is not fully satisfactory, since it contains a conical singularity in the form of a disc (*i.e.* a negative tension source) that sits inside the ring, supporting it against collapse. This feature can be understood based on the heuristic construction of a BR starting with a black string (*i.e.* a four dimensional Schwarzschild BH extending into the fifth dimension) which is bent to form a circle. Then, without the tension, this loop would contract, decreasing the radius of S^1, due to its gravitational self-attraction.[1]

Nonvacuum generalizations of the static BR solution are known, see *e.g.* [4], [5], [6]; however, they still possess conical singularities. Moreover, as shown in [7], the same result holds also for (static) BRs in Einstein-Gauss-Bonnet theory, in which case a region of negative '*effective energy density*' (sourced by the Gauss-Bonnet term in the action) occurs. Although the absolute value of the conical excess decreases as the Gauss-Bonnet coupling constant α increases, the solutions stop to exist for some α_{max}, before approaching a balanced configuration.

So far, the only known mechanism to obtain an asymptotically flat configuration which is free of conical singularities is to set the ring into rotation [8], in which case the centrifugal force manages to balance the massive ring's self-attraction.

Static, balanced BRs may exist, however, in a non-asymptotically flat background. For example, as discussed in [9], by submerging a charged static BR into an electric/magnetic background field, the conical singularities can be eliminated and the static black ring stabilized. However, this construction has the drawback that, due to the backreaction of the background electromagnetic field, the BR approaches at infinity a Melvin-type background. Although an explicit construction is still missing, static BRs without conical singularity should also exist in a de Sitter spacetime, the cosmological expansion acting against the tension and assuring balance for a critical ring size [10]. Also, an exact solution describing a static, balanced BR with Kaluza-Klein magnetic monopole asymptotics has been reported in [11].

However, there is no fundamental obstruction forbidding the existence of static, balanced BRs also in a Minkowski spacetime background. In fact, such line elements can easily be obtained by considering (rather mild) modifications of the Emparan-Reall solution in [1]. For example, let us consider the following metric

$$ds^2 = \frac{R^2}{(x-y)^2}\Big(\frac{dx^2}{1-x^2} + \frac{1+\lambda x}{(y^2-1)(1+\lambda y)}dy^2 + U(x)d\varphi^2 + (1+\lambda x)(y^2-1)d\psi^2\Big) - \frac{1+\lambda y}{1+\lambda x}dt^2\,, \qquad (1)$$

where x, y are ring coordinates, with the usual range $-\infty \le y \le -1$, $-1 \le x \le 1$, φ and ψ are angular directions and t is the time

* Corresponding author.
E-mail address: eugen.radu@ua.pt (E. Radu).

[1] An analogous construction exists for a special class of *non-gravitating* solitons in four spacetime dimensions – the vortons, which are made from loops of vortices, being sustained against collapse by the centrifugal force [3], similar to balanced BRs.

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coordinate. Also, λ is a free parameter of the solution, with $0 < \lambda < 1$, while $R > 0$ is the radius of the ring. The above line element possesses an event horizon of $S^2 \times S^1$ topology, located at $y = -1/\lambda < -1$, the asymptotic infinity corresponding to $x \to y \to -1$. The absence of conical singularities implies that the ψ-coordinate possesses a periodicity $\Delta\psi = 2\pi/\sqrt{1-\lambda}$. The situation is more complicated for the φ-coordinate, depending on the choice for the function $U(x)$. For

$$U(x) = (1 - x^2)(1 + \lambda x) \tag{2}$$

one recognizes the static, vacuum Emparan-Reall solution [1], in which case one cannot eliminate the conical singularities at both $x = -1$ and $x = 1$. However, no conical singularities are found for particular expressions of the function $U(x)$, the simplest choice being

$$U(x) = 1 - x^2. \tag{3}$$

Then the metric is regular at $x = \pm 1$ (the periodicity of φ being 2π), and, when evaluating various invariant quantities, no singularities are found on and outside the horizon, while the line element still possesses the proper asymptotic decay. Moreover, the mass and the Hawking temperature are the same for both (2) and (3), while the event horizon area changes accordingly.

However, the vacuum Einstein equations are *not* solved for the choice (3), the components E_x^x, $E_y^y = E_\psi^\psi$ and E_t^t of the Einstein tensor being nonzero, while the expression of the Ricci scalar is

$$\mathcal{R} = \frac{3\lambda}{R^2} \frac{y(1+x^2) - x(1+y^2)}{1+\lambda x}\,. \tag{4}$$

The Einstein equations are 'satisfied' by assuming a matter source with $T_\mu^\nu = E_\mu^\nu/(8\pi G)$, with $\rho = -T_t^t$ corresponding to the energy density as measured by a fundamental timelike observer. Then a direct computation shows that $\rho < 0$ for some region on and outside the horizon (heuristically, this provides the repulsive force required for the ring balance).

Although no field theory source can be associated with the corresponding stress-energy tensor, the result above suggests that static balanced BRs may exist indeed in some models with a matter source violating the weak energy condition. The main purpose of this letter is to report on the existence of such configurations in Einstein gravity minimally coupled with a *phantom* real scalar field. Such a field has a reverse sign in front of the kinetic energy part of the Lagrangian density, which leads to the generic occurrence of negative energy densities and gravitational repulsion. In the four dimensional case, this form of exotic matter has been considered in cosmology and also in wormhole physics, see *e.g.* [12], [13]. Moreover, (spherical) BH solutions with 'phantom' scalar field hair do also exist [14], circumventing the no-hair theorems in the Einstein-scalar field model [15] due to the violation of the energy conditions. Although a phantom scalar possesses some undesirable features, it may perhaps be regarded as corresponding to an effective field theory description resulting from a fundamental theory which is well defined [16] (see also [17]).

For the purposes of this work a phantom scalar field is of interest as the simplest source of gravitational repulsion. Then, our results show that, for a critical size of the ring, this provides the necessary force to keep the BR from collapsing, the resulting configuration being regular, on and outside the horizon. Since no exact solutions are likely to exist in this model, static balanced BRs are found by solving numerically the Einstein–Klein-Gordon equations, subject to a suitable set of boundary conditions.

This paper is organized as follows. In the next Section we describe the Einstein-scalar field model. For a better understanding of the problem, both spherical BHs and BRs are considered. Then, in Section 3 we construct the solutions and show the existence of static, balanced BRs. Concluding remarks and some open questions are presented in Section 4. In particular, an explicit expression is shown there for a four dimensional asymptotically flat BR geometry.

2. The model

2.1. Action, equations and boundary conditions

We consider the action of a self-interacting real scalar field ϕ coupled to Einstein gravity in five spacetime dimensions,

$$S = \int d^5x\sqrt{-g}\left[\frac{1}{16\pi G}\mathcal{R} - \frac{\epsilon}{2}g^{\mu\nu}\phi_{,\,\mu}\phi_{,\,\nu} - V(\phi)\right], \tag{5}$$

where R is the curvature scalar, G is Newton's constant, $V(\phi)$ denotes the scalar field potential, while $\epsilon = 1$ for a normal field and $\epsilon = -1$ for a phantom field. Using the principle of variation, one finds the coupled Einstein–Klein-Gordon equations

$$\begin{aligned} &E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} - 8\pi G\, T_{\mu\nu} = 0, \\ &\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right) = \epsilon\frac{\partial V}{\partial\phi}, \end{aligned} \tag{6}$$

where $T_{\mu\nu}$ is the stress-energy tensor of the scalar field

$$T_{\mu\nu} = \epsilon\phi_{,\,\mu}\phi_{,\,\nu} - g_{\mu\nu}\left[\frac{\epsilon}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + V(\phi)\right]. \tag{7}$$

The solutions in this work are static and axisymmetric, with a symmetry group $\mathbf{R} \times U(1) \times U(1)$ (where $\mathbf{R}$ denotes the time translation) and can be studied by using a metric Ansatz introduced in [19], with[2]

$$\begin{aligned} ds^2 = {} & f_1(r,\theta)(dr^2 + r^2 d\theta^2) + f_2(r,\theta)d\psi^2 + f_3(r,\theta)d\varphi^2 \\ & - f_0(r,\theta)dt^2, \end{aligned} \tag{8}$$

where the range of θ is $0 \le \theta \le \pi/2$ and with $0 \le (\psi, \varphi) \le 2\pi$. Also, r and t correspond to the radial and time coordinates, respectively. The range of r is $0 < r_H \le r < \infty$ (with r_H the event horizon radius); thus the (r,θ) coordinates have a rectangular boundary well suited for numerics. The scalar field is also a function of (r,θ), only.

An appropriate combination of the Einstein equations, $E_t^t = 0$, $E_r^r + E_\theta^\theta = 0$, $E_\psi^\psi = 0$, and $E_\varphi^\varphi = 0$, yields the following set of equations for the functions f_1, f_2, f_3 and f_0 (where we define $(\nabla U)\cdot(\nabla W) = \partial_r U \partial_r W + \frac{1}{r^2}\partial_\theta U \partial_\theta W$, and $\nabla^2 U = \partial_r^2 U + \frac{1}{r^2}\partial_\theta^2 U + \frac{1}{r}\partial_r U$):

$$\begin{aligned} &\nabla^2 f_0 - \frac{1}{2f_0}(\nabla f_0)^2 + \frac{1}{2f_2}(\nabla f_0)\cdot(\nabla f_2) + \frac{1}{2f_3}(\nabla f_0)\cdot(\nabla f_3) \\ &\quad + \frac{32\pi G}{3} f_0 f_1 V(\phi) = 0, \\ &\nabla^2 f_1 - \frac{1}{f_1}(\nabla f_1)^2 - \frac{f_1}{2f_0 f_2}(\nabla f_0)\cdot(\nabla f_2) \\ &\quad - \frac{f_1}{2f_0 f_3}(\nabla f_0)\cdot(\nabla f_3) - \frac{f_1}{2f_2 f_3}(\nabla f_2)\cdot(\nabla f_3) \end{aligned}$$

[2] Although one can write an Ansatz based on the ring coordinates (x, y), (which results in a much simpler form of the vacuum solution), its use in numerics is problematic, at least for the scheme employed in this work, the asymptotic infinity being approached at a single point.

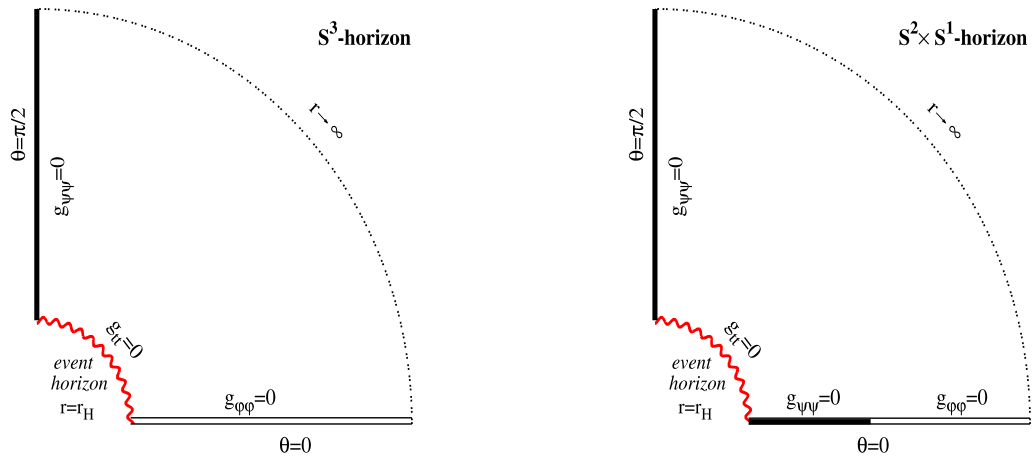

Fig. 1. The domain of integration for the coordinate system (8) is shown for a spherically symmetric black hole (left) and a static black ring (right).

$$
\begin{aligned}
&+8\pi G f_1\left(\epsilon(\nabla\phi)^2-\frac{2f_1}{3}V(\phi)\right)=0,\\
&\nabla^2 f_2-\frac{1}{2f_2}(\nabla f_2)^2+\frac{1}{2f_0}(\nabla f_0)\cdot(\nabla f_2)+\frac{1}{2f_3}(\nabla f_2)\cdot(\nabla f_3)\\
&\quad+\frac{32\pi G}{3}f_1 f_2 V(\phi)=0,\\
&\nabla^2 f_3-\frac{1}{2f_3}(\nabla f_3)^2+\frac{1}{2f_0}(\nabla f_0)\cdot(\nabla f_3)+\frac{1}{2f_2}(\nabla f_2)\cdot(\nabla f_3)\\
&\quad+\frac{32\pi G}{3}f_1 f_3 V(\phi)=0\,, \qquad (9)
\end{aligned}
$$

while the Klein-Gordon equation is

$$
\begin{aligned}
&\nabla^2\phi+\frac{1}{2f_0}(\nabla f_0)\cdot(\nabla\phi)+\frac{1}{2f_2}(\nabla f_2)\cdot(\nabla\phi)+\frac{1}{2f_3}(\nabla f_3)\cdot(\nabla\phi)\\
&\quad-\epsilon\frac{\partial V(\phi)}{\partial\phi}=0. \qquad (10)
\end{aligned}
$$

The remaining Einstein equations $E_\theta^r=0,\ E_r^r-E_\theta^\theta=0$ yield two constraints. However, following [18], one can show that they are satisfied as well, subject to the boundary conditions given below.

Both BHs with a spherical horizon topology and BRs can be described within the Ansatz (8). In the vacuum case ($\phi=V(\phi)=0$), the simplest solution is the (spherical) Schwarzschild-Tangherlini BH [2] written in isotropic coordinates, with

$$
f_0=\frac{(1-\frac{r_H^2}{r^2})^2}{(1+\frac{r_H^2}{r^2})^2},\quad f_1=\frac{f_2}{r^2\cos^2\theta}=\frac{f_3}{r^2\sin^2\theta}=\left(1+\frac{r_H^2}{r^2}\right)^2. \qquad (11)
$$

The corresponding expressions for the (static) Emparan-Reall solution are more complicated, with

$$
\begin{aligned}
f_0&=\frac{(1-\frac{r_H^2}{r^2})^2}{(1+\frac{r_H^2}{r^2})^2},\\
f_1&=\frac{(1+\frac{r_H^2}{r^2})^2}{(1+\frac{r_H^2}{R^2})^2 P}\left((1+\frac{r_H^4}{r^4})(1+\frac{r_H^4}{R^4})-\frac{4r_H^4}{r^2R^2}\cos 2\theta+\frac{2r_H^2}{R^2}P\right),\\
f_2&=\frac{1}{4f_3}r^4\left(1+\frac{r_H^2}{r^2}\right)^4\sin^2 2\theta,\\
f_3&=\frac{r^2}{2}\left(P+\frac{R^2}{r^2}(1+\frac{r_H^4}{R^4}-\frac{r_H^2}{R^2}(\frac{r^2}{r_H^2}+\frac{r_H^2}{r^2})\cos 2\theta)\right), \qquad (12)
\end{aligned}
$$

where

$$
\begin{aligned}
P=\frac{r^2}{2}\Bigg[&\left(1+\left(\frac{R}{r}\right)^4-2\cos 2\theta\left(\frac{R}{r}\right)^2\right)\left(1+\left(\frac{r_H^2}{rR}\right)^4\right.\\
&\left.-2\cos 2\theta\left(\frac{r_H^2}{rR}\right)^2\right)\Bigg]^{1/2},
\end{aligned}
$$

with $R>r_H$ a new parameter, the radius of the ring. Also, one can verify that the spherical solution (11) is approached as $R\to r_H$. Further properties of the static BR for the above parametrization, including the correspondence with the Weyl coordinates, can be found in Refs. [7], [19].

The solutions with $\phi\neq 0$ are found numerically, by solving the equations (9) subject to a set of boundary conditions which results from the requirement that the solutions describe asymptotically flat black objects with a regular horizon.[3] We assume that as $r\to\infty$, the Minkowski spacetime background (with $ds^2=dr^2+r^2(d\theta^2+\cos^2\theta d\psi^2+\sin^2\theta d\varphi^2)-dt^2$) is recovered, while the scalar field vanishes. This implies

$$
\begin{aligned}
&f_0|_{r=\infty}=1,\qquad f_1|_{r=\infty}=1,\qquad \lim_{r\to\infty}\frac{f_2}{r^2}=\cos^2\theta,\\
&\lim_{r\to\infty}\frac{f_3}{r^2}=\sin^2\theta,\ \ \phi|_{r=\infty}=0. \qquad (13)
\end{aligned}
$$

Also, we impose the existence of a nonextremal event horizon, which is located at a constant value of the radial coordinate, $r=r_H>0$. There we require

$$
\begin{aligned}
&f_0|_{r=r_H}=0,\qquad \partial_r f_1|_{r=r_H}=\partial_r f_2|_{r=r_H}=\partial_r f_3|_{r=r_H}=0,\\
&\partial_r\phi|_{r=r_H}=0. \qquad (14)
\end{aligned}
$$

The boundary conditions at $\theta=\pi/2$ are

$$
\begin{aligned}
&\partial_\theta f_0|_{\theta=\pi/2}=\partial_\theta f_1|_{\theta=\pi/2}=f_2|_{\theta=\pi/2}=\partial_\theta f_3|_{\theta=\pi/2}=0,\\
&\partial_\theta\phi|_{\theta=\pi/2}=0. \qquad (15)
\end{aligned}
$$

The absence of conical singularities requires also $r^2 f_1=f_2$ on that boundary.

[3] The imposed boundary conditions (13)-(17) are also compatible with an approximate form of the solutions on the boundaries of the domain of integration. This domain is shown in Fig. 1, together with the boundary conditions which determine the horizon topology.

The boundary conditions at $\theta = 0$ are more complicated. First, for a spherical BH one imposes

$$\partial_\theta f_0|_{\theta=0} = \partial_\theta f_1|_{\theta=0} = \partial_\theta f_2|_{\theta=0} = f_3|_{\theta=0} = 0, \qquad \partial_\theta \phi|_{\theta=0} = 0. \tag{16}$$

For a BR, a new input parameter, $R > r_H$, occurs, as for the vacuum solution. There, for $r_H < r < R$, we impose

$$\partial_\theta f_0|_{\theta=0} = \partial_\theta f_1|_{\theta=0} = f_2|_{\theta=0} = \partial_\theta f_3|_{\theta=0} = 0, \qquad \partial_\theta \phi|_{\theta=0} = 0. \tag{17}$$

2.2. Physical quantities

For any event horizon topology, the metric of a spatial cross-section of the horizon is

$$d\sigma^2 = f_1(r_H,\theta) r_H^2 d\theta^2 + f_2(r_H,\theta) d\psi^2 + f_3(r_H,\theta) d\varphi^2. \tag{18}$$

As we shall see, a spherical BH has $f_1(r_H,\theta) = f_{10}$, $f_2(r_H,\theta) = f_{10} r_H^2 \cos^2\theta$, $f_3(r_H,\theta) = f_{10} r_H^2 \sin^2\theta$, such that (18) parametrizes a round S^3. For a BR, the orbits of ψ shrink to zero at $\theta = 0$ and $\theta = \pi/2$, while the length of S^1-circle does not vanish anywhere, such that the topology of the horizon is $S^2 \times S^1$ (in fact, $f_2(r_H,\theta) \sim \sin^2 2\theta$ while $f_1(r_H,\theta)$ and $f_3(r_H,\theta)$ are strictly positive and finite functions). Also, we mention that although the constants (R, r_H) have no invariant meaning, they provide a rough measure for the radii of the S^1 and S^2 parts in the horizon metric (18).

For both BRs and spherical BHs, the event horizon area and the Hawking temperature[4] are given by

$$A_H = 4\pi^2 r_H \int_0^{\pi/2} d\theta \sqrt{f_1 f_2 f_3}\Big|_{r=r_H},$$
$$T_H = \frac{1}{2\pi} \lim_{r\to r_H} \sqrt{\frac{f_0}{(r-r_H)^2 f_1}}. \tag{19}$$

At infinity, the Minkowski background is approached. The ADM mass M of the solutions can be read from the asymptotic expression for the metric function f_0,

$$-g_{tt} = f_0 \sim 1 - \frac{8GM}{3\pi}\frac{1}{r^2} + \dots . \tag{20}$$

As usual, M can be expressed as the sum of the horizon mass and the mass stored in the matter field(s) outside the horizon, which results in the Smarr-type relation

$$M = \frac{3}{2} T_H \frac{1}{4G} A_H + M_{(\phi)}, \tag{21}$$

with

$$M_{(\phi)} = \frac{3}{2}\int_\Sigma d^4x \sqrt{-g}\left(\frac{1}{3}T_\nu^\nu - T_t^t\right)$$
$$= -4\pi^2 \int_{r_H}^\infty dr \int_0^{\pi/2} d\theta\, r f_1 \sqrt{f_0 f_2 f_3}\, V(\phi), \tag{22}$$

(where one integrates over a spacelike surface Σ bounded by the (spatial section of the) horizon and infinity). Also, we define the reduced dimensionless quantities, obtained by dividing out an appropriate power of M

$$a_H = \frac{3}{32}\sqrt{\frac{3}{2\pi}}\frac{A_H}{(GM)^{3/2}}, \qquad t_H = 4\sqrt{\frac{2\pi}{3}} T_H \sqrt{GM}, \tag{23}$$

such that $a_H = t_H = 1$ for the Schwarzschild-Tangherlini solution and $a_H = 1/t_H = 2Rr_H/(r_H^2 + R^2)$ for the Emparan-Reall static BR.

2.3. The potential, scaling properties and numerics

For a quantitative study of the solutions, we need to specify the expression of the potential $V(\phi)$. For any horizon topology, $V(\phi)$ should satisfy the following relation

$$\int_\Sigma d^4x \sqrt{-g}\left(\phi \nabla^2 \phi - \epsilon \phi \frac{\partial V(\phi)}{\partial \phi}\right) = 0, \tag{24}$$

which is found by multiplying the Klein-Gordon equation by ϕ and integrating it, the contribution of the boundary terms vanishing for static, regular solutions (with a scalar field that falls off sufficiently fast at infinity). This implies that $\phi \partial V/\partial \phi$ necessarily changes the sign outside the horizon and rules out a massless (or non-selfinteracting) field.

The results reported in this work correspond to the simplest polynomial potential which is compatible with (24); we also impose the discrete symmetry of the model $\phi \to -\phi$. Thus, for both normal and phantom fields, V is taken as the sum of a quadratic and a quartic term,

$$V(\phi) = \epsilon\left(\frac{1}{2}\mu^2\phi^2 - \frac{1}{4}\lambda\phi^4\right). \tag{25}$$

The first term (with $\mu^2 > 0$) provides a mass for the scalar field (and leads to an exponential decay of the scalar field), while λ is a positive parameter, as required by (24).

With the above choice of the potential, the system possesses two scaling symmetries (with c some positive constant)

$$(i) \quad r \to rc, \quad \mu \to \mu/c, \quad \lambda \to \lambda/c^2, \quad \text{and}$$
$$(ii) \quad \phi \to \phi c, \quad \lambda \to \lambda/c^2, \quad G \to G/c^2, \tag{26}$$

which are used to set to one the values of the constants μ and λ. This reveals the existence of the dimensionless parameter

$$\alpha^2 = \frac{4\pi G \mu^2}{\lambda} \tag{27}$$

characterizing a given model.[5]

The BRs are found by using an approach originally introduced in [27] and further employed *e.g.* in [6], [24], [28]. In this scheme, the required boundary behaviour of the metric functions is enforced by taking $f_i = f_i^{(0)} F_i$, with $f_i^{(0)}$ some suitable background functions, which, for the case in this work, are those of the vacuum BR as given by (12). The advantage of this approach is that the coordinate singularities are essentially subtracted, while imposing at the same time the $S^2 \times S^1$ event horizon topology. Then the numerics is done in terms of the new functions F_i, subject to a set of boundary conditions which follows directly from (13)-(17) together with (12). The equations for F_i are solved by employing a finite difference solver [20], which uses a Newton-Raphson method. This software provides an error estimate for each unknown function, which is the maximum of the discretization error divided by the

[4] The constraint equation $E_r^\theta = 0$ guarantees that the Hawking temperature T_H is a constant.

[5] Thus the Einstein equations solved numerically are $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 2\alpha^2\, T_{\mu\nu}$.

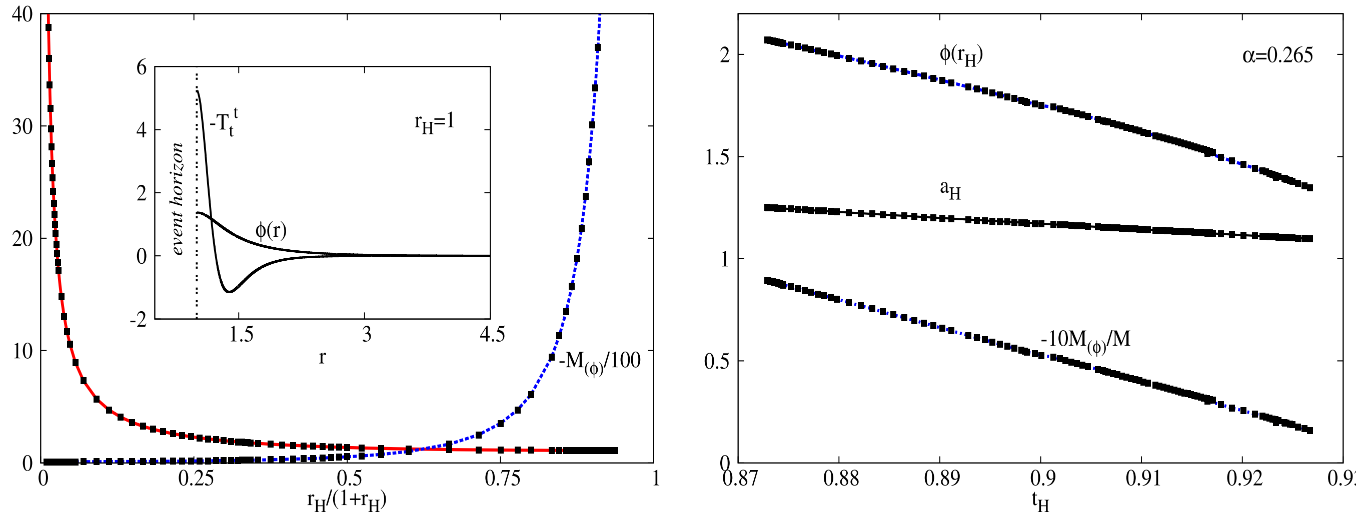

Fig. 2. *Left.* The value of the scalar field at the horizon $\phi(r_H)$ and the mass $M_{(\phi)}$ are shown for solutions in a fixed Schwarzschild-Tangherlini background with a given event horizon radius r_H. The inset shows the profile of a typical (non-gravitating) solution. *Right.* Some parameters of spherically symmetric black holes with phantom scalar hair are shown as a function of (reduced) temperature for a fixed value of the coupling constant α. Note, that in all figures in this work exhibiting results for families of solutions, the large dots represent the data points.

maximum of the function.[6] Further checks of numerics are provided by the Smarr relation (21) and by the constraint Einstein equations $E_\theta^r = 0$, $E_r^r - E_\theta^\theta = 0$. Based on that, the numerical error for the solutions reported in this work is estimated to be typically $< 10^{-3}$. However, similar to other cases [24], [28], the errors increase dramatically when studying BRs close to the critical point $R \to r_H$, whose accurate construction appears to require a different approach.

In the spherically symmetric case, the equations are solved by using a standard Runge-Kutta solver and implementing a shooting method.

Let us mention that the formalism described above holds for both values of ϵ. Also, we have considered solutions of the equations (9), (10) with $\epsilon = \pm 1$. However, we have failed to find balanced BR solutions with a normal scalar field (despite the occurrence of negative energy densities also in that case). Therefore, for the remainder of this work we shall consider the case of a phantom field only, $\epsilon = -1$.

3. The solutions

3.1. Spherically symmetric black holes

Let us start with a discussion of the spherically symmetric gravitating solutions. These configurations are easier to construct, while their study helps in understanding some of the BRs properties.

In this case, the scalar field is a function of r only, while the metric ansatz simplifies, with a factorized angular dependence

$$f_2 = f_1 r^2 \cos^2\theta, \quad f_3 = f_1 r^2 \sin^2\theta, \tag{28}$$

while f_0, f_1 depend on r only. The horizon of the black holes is located at $r = r_H > 0$, where the solutions have a power-series expansion (for completeness, here we restore the proper factors of μ, λ):

$$\phi(r) = \phi_0 + \frac{1}{4} f_{10}\phi_0(\mu^2 - \lambda\phi_0^2)(r - r_H)^2 + \dots,$$

$$f_0(r) = f_{02}(r - r_H)^2 - \frac{f_{02}}{r_H}(r - r_H)^2 + \dots,$$
$$f_1(r) = f_{10} - \frac{2f_{12}}{r_H}(r - r_H) + f_{10}(4 - \frac{1}{2}\alpha^2 f_{10} r_H^2\phi_0^2(\mu^2 - \frac{1}{2}\lambda\phi_0^2))(r - r_H)^2 + \dots, \tag{29}$$

in terms of three parameters $f_{10} = f_1(r_H)$, $f_{02} = f_0''(r_H)/2$ and $\phi_0 = \phi(r_H)$. One can write an approximate form of the solutions also for $r \to \infty$, with

$$f_0(r) = 1 + \frac{\bar f_{02}}{r^2} + \frac{\bar f_{02}^2}{2r^4} + \dots, \qquad f_1(r) = 1 - \frac{\bar f_{02}}{2r^2} + \frac{\bar f_{02}^2}{16r^4} + \dots,$$
$$\phi(r) = \bar\phi_1 \frac{e^{-\mu r}}{r^{3/2}} + \dots, \tag{30}$$

with $\bar f_{02}, \bar\phi$ two constants fixed by numerics.[7]

In the study of these solutions, it is useful to consider first the solutions of the Klein-Gordon equation (10) in a fixed BH background as given by the Schwarzschild-Tangherlini metric (11), *i.e.* the probe limit, $\alpha = 0$. The corresponding equation reads

$$\phi'' + \left(\frac{3 + \frac{r_H^4}{r^4}}{1 - \frac{r_H^4}{r^4}}\right)\frac{\phi'}{r} - (1 + \frac{r_H^2}{r^2})^2(\mu^2 - \lambda\phi^2)\phi = 0. \tag{31}$$

As seen in Fig. 2 (left panel), the solutions exist for very large (possible arbitrarily large) values of $r_H > 0$. However, the Minkowski spacetime limit $r_H \to 0$ is not well defined, with a divergent scalar field.[8] Also, the mass of these configurations, $M_{(\phi)} = -\int_\Sigma d^4x\sqrt{-g}T_t^t$, is always negative.

Including the backreaction leads to a fundamental branch of solutions describing BHs with scalar hair. As expected, the solutions with a given horizon size exist for a finite range of α. Moreover,

[6] The errors also depend on the order of consistency of the method, *i.e.* on the order of the discretisation of derivatives. For the solutions in this paper, this order was six.

[7] Note that only nodeless scalar field configurations are reported here (including the BR case). However, excited solutions do also exist.

[8] This results from the virial identity $T + 2\mu^2 V_1 - \lambda V_2 = 0$, with the strictly positive quantities $T = \int_0^\infty dr r^3\phi'^2$, $V_1 = \int_0^\infty dr r^3\phi^2$, $V_2 = \int_0^\infty dr r^3\phi^4$. Since the Bekenstein-type relation (24) implies $T + \mu^2 V_1 - \lambda V_2 = 0$, one finds $V_1 = 0$, and thus $\phi = 0$.

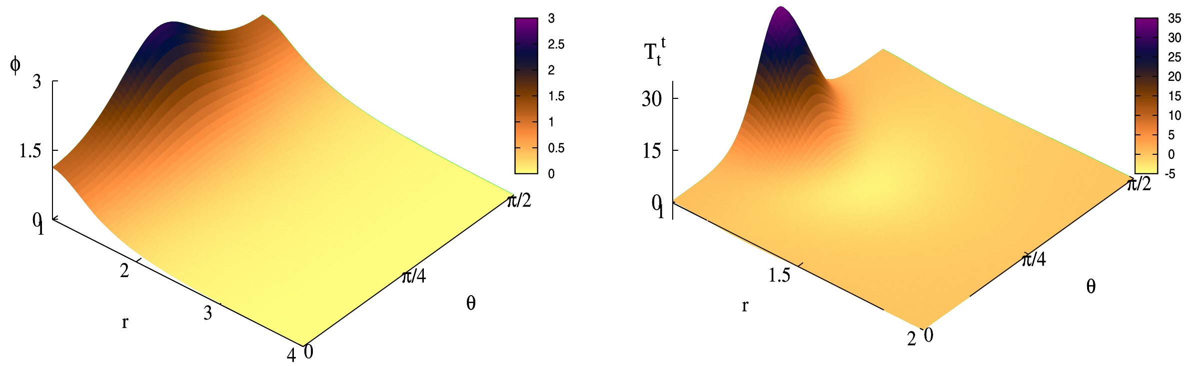

Fig. 3. The profile of a typical (non-gravitating) solution in a fixed black ring background with $r_H = 1$, $R = 2$.

for given α, more than one solution with the same value of r_H (or even the same horizon size) may exist. This can be understood by noticing that the limit $\alpha \to 0$ can be approached as $G \to 0$ (*i.e.* no backreaction, a fixed BH background) or as $\mu \to 0$ (which corresponds to a model with a massless scalar field). Moreover, these branches are not always connected. Also, we mention that BHs with $M < 0$ are also found, in which case the mass stored in the scalar field $M_{(\phi)}$ dominates in (21) over the horizon mass (typically found on the branch connected with the $G \to 0$ limit).

In Fig. 2 (right panel) we show some properties of the solutions with a given α, as a function of the scaled temperature t_H. As $t_H \to t_H^{(min)}$, the numerics becomes increasingly difficult, a singular solution being approached, with a divergent Kretschmann scalar as $r \to r_H$. No singularities are found as $t_H \to t_H^{(max)}$, in which limit the solutions seem to continue into a branch of wormhole configurations. A systematic discussion of the spherically symmetric solutions with $\epsilon = \pm 1$ will be presented elsewhere.

3.2. The black rings

Starting again with the probe limit, we have solved the equation for ϕ in a vacuum BR background as given by (12). For a given horizon radius r_H, the solutions were found up to a maximal value of the radius R, where the errors become large. The profile of a typical solution is shown in Fig. 3. One can see both the scalar field and the energy density possess a non-trivial angular dependence, with a maximum located at the horizon for some intermediate value of θ.

The backreacting generalizations of these solutions are found again by increasing from zero the parameter α. As in the spherical case, this results in a complicated branch structure, and more than one solution may exist for the same input parameters $(\alpha; r_H, R)$. The BRs are regular on and outside the horizon and show no sign of a singular behaviour. However, as expected, the generic configurations possess a conical singularity. As one can see from the boundary conditions (13), in this work we have chosen[9] to locate the conical singularity at $\theta = 0$, $r_H < r < R$, where we find a conical singularity, as measured by the parameter

$$\delta = 2\pi\left(1 - \lim_{\theta\to 0}\frac{f_2}{\theta^2 r^2 f_1}\right) \neq 0\,. \tag{32}$$

(Note that a vacuum BR has $\delta = -4\pi r_H^2/(R^2 - r_H^2) < 0$, with δ diverging in the Schwarzschild-Tangherlini limit.) This can be interpreted as a disk preventing the collapse of the configurations. Although the presence of a conical singularity is an undesirable feature, it has been argued in [21], [22], that such asymptotically flat black objects still admit a thermodynamical description (see also [23]). Moreover, when working with the appropriate set of thermodynamical variables, the Bekenstein-Hawking law still holds, while the parameter δ enters the first law of thermodynamics, corresponding to a pressure term P, with the conjugate extensive variable $\mathcal{A}$,

$$P = -\frac{\delta}{8\pi} \quad \text{and} \quad \mathcal{A} = Area\; T_H\,, \tag{33}$$

where *Area* is the space-time area of the conical singularity's world-volume, as computed from the line-element $d\sigma^2 = -f_0 dt^2 + f_1 dr^2 + f_3 d\varphi^2$. For the line-element (8), one finds

$$\mathcal{A} = 2\pi \int_{r_H}^{R} dr \sqrt{f_0 f_1 f_3}\Big|_{\theta=0}\,. \tag{34}$$

Then the total mass-energy associated with the conical defect is [19]:

$$M_{(\mathrm{def})} = -P\mathcal{A}. \tag{35}$$

As expected, the (absolute) value of the conical excess δ decreases as α is increased (*i.e.* allowing for a larger $M_{(\phi)}$ contribution to the total mass). Therefore, for a BR set with fixed horizon and ring radii (r_H, R), a balanced configuration is achieved for a critical value of α. Further increasing α results in configurations with a conical excess $\delta > 0$, see Fig. 4 (left panel).

When considering instead a model with a fixed coupling constant $\alpha > 0$ and varying the size of the ring, this also results in the existence of a critical balanced configuration. The results for several values of α are shown in Fig. 4 (right panel). One can see that the (absolute value of the) total mass associated with the defect $M_{(\mathrm{def})}$ is always small as compared to the ADM mass M, while the mass associated with the scalar field $M_{(\phi)}$ takes negative values, and dominates over the horizon mass for $\delta > 0$.

The limit $R \to r_H$ of the solutions appears to be similar to the vacuum case, a BH solution with spherical horizon topology being approached (although this limit is difficult to study in our numerical scheme). Rather surprising, no arbitrarily large BRs were found for the cases investigated so far. Instead, as seen in Figs. 4, 5, the solutions stop to exist for a maximal value of δ, with a backbending and the occurrence of a secondary branch. However, clarifying the critical behaviour, together with a systematic investigation of the parameter space of solutions is beyond the purposes of this work.

[9] It is also possible to work with the conical singularity stretching towards the boundary, in which case the spacetime will not be asymptotically flat.

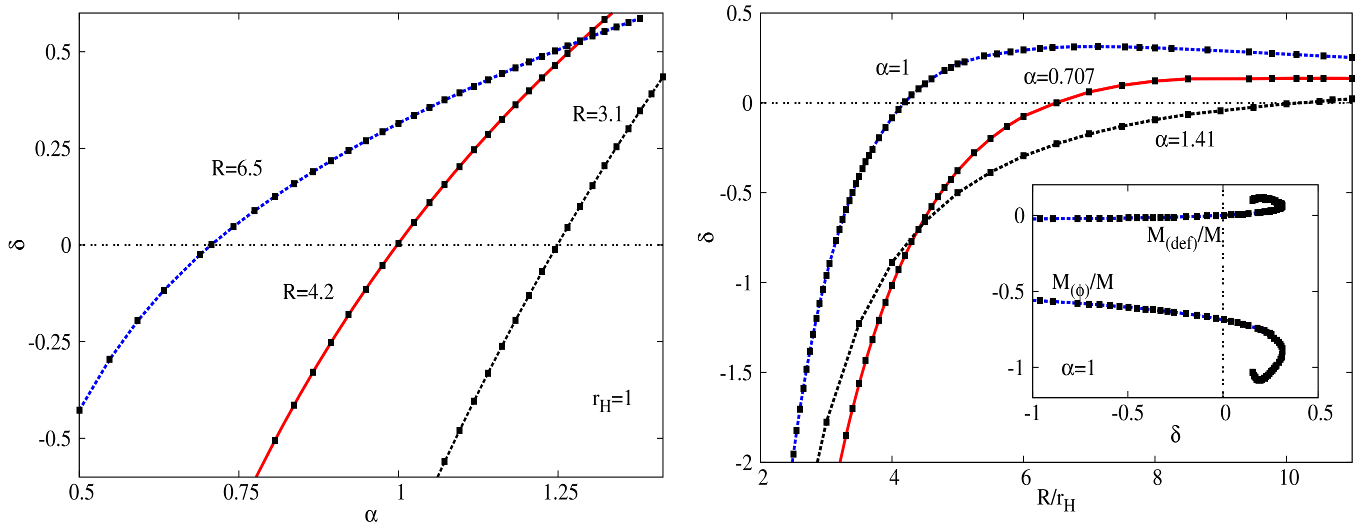

Fig. 4. The conical defect δ is shown as a function of the coupling parameter α and as a function of the ratio R/r_H (with R the radius of the ring and r_H the event horizon radius). In both cases, one notices the existence of balanced configurations ($\delta = 0$). The inset shows the ratio between the total mass associated with the conical defect $M_{(\mathrm{def})}$ and the ADM mass M as a function of δ (and the same for the mass $M_{(\phi)}$ stored in the scalar field).

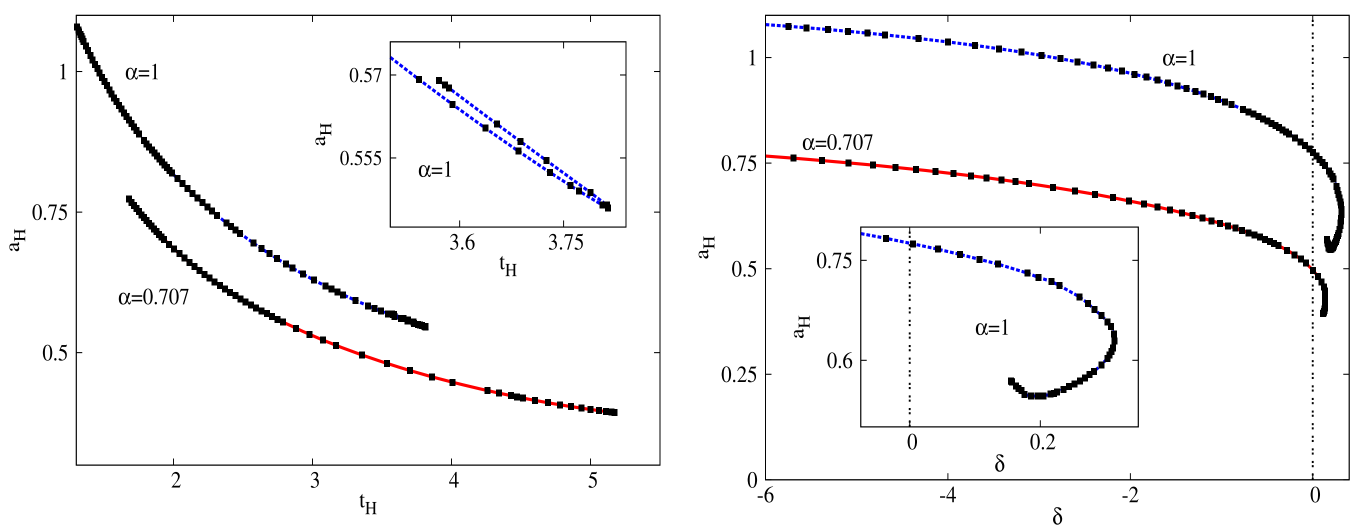

Fig. 5. The reduced area a_H is shown as a function of reduced temperature t_H and conical defect δ for given values of α.

4. Further remarks

The known five dimensional, static black rings (BRs) in a Minkowski spacetime background are plagued by conical singularities. As shown in this work, this pathology can be cured at the price of coupling Einstein gravity with a *'phantom'* scalar field. In such a model, when fixing the coupling constants, balanced solutions were shown to exist for critical radii of a BR.

The spinning, balanced, Emparan-Reall BRs are known to possess higher dimensional generalizations [24], [25] (although a closed form solution is still missing). Moreover, when increasing the number d of spacetime dimensions, a plethora of other black objects with various event horizon topologies are found (for a review, see [26]). While the unbalanced $d > 5$ BRs appear to be singular, (at least) the solutions with a $S^2 \times S^{d-4}$ horizon topology possess a well defined static limit, with conical singularities only [27], [28]. The results in this work suggest that these *ringoids* achieve balance when including a phantom field in the model.

Moreover, one can speculate that the same mechanism could allow for the existence of four dimensional BRs. The results of various theorems excluding a non-spherical topology of the horizon [29] would be circumvented for an exotic matter content violating the energy conditions (see [30] for some speculations in this direction).

In fact, following the approach in the Introduction, one can easily write a line element describing a four dimensional, asymptotically flat BH which is regular on and outside an horizon of $S^1 \times S^1$ topology. Although this geometry does not solve any obvious field theory model, it may give an idea about the properties of a four dimensional BR solution. For concreteness, let us consider the following metric:

$$ds^2 = \frac{R^2}{(x-y)^2}\left[\frac{dx^2}{1-x^2} + \frac{(1+\lambda x)^2}{H(x,y)}\left(\frac{1}{1+\lambda y}\frac{dy^2}{y^2-1} + \frac{y^2-1}{1-\lambda}d\varphi^2\right)\right] - \frac{1+\lambda y}{H(x,y)}dt^2\,, \tag{36}$$

where R, λ are free parameters (with $R > 0$ and $0 < \lambda < 1$), while x, y are toroidal coordinates, with $-\infty \le y \le -1$, $-1 \le x \le 1$, the asymptotic infinity being at $x \to y \to -1$. Also, $H(x, y)$ is a smooth, strictly positive function (with smooth derivatives as well), which controls the far field behaviour of the geometry. Then one can easily verify the absence of a conical singularity for the line-element (36), the periodicity of φ being 2π, as usual.

The line element (36) possesses an event horizon located at $y = -1/\lambda < -1$, the metric of its spatial cross-section being

$$d\sigma^2 = R^2 \left(\frac{\lambda^2}{(1+\lambda x)^2(1-x^2)} dx^2 + \frac{1+\lambda}{H(x,-1/\lambda)} d\varphi^2 \right). \tag{37}$$

This horizon has an $S^1 \times S^1$ topology, as results *e.g.* from the fact that its Euler characteristic vanishes. Also, the Hawking temperature and the event horizon area corresponding to the metric (36) are well defined, with

$$T_H = \frac{\sqrt{1-\lambda^2}}{4\pi R \lambda},$$
$$A_H = 2\pi R^2 \lambda \sqrt{1+\lambda} \int_{-1}^{1} dx \left((1+\lambda x) \sqrt{(1-x^2) H(x,-1/\lambda)} \right)^{-1}. \tag{38}$$

The line-element (36) has an associated energy-momentum tensor whose nonzero components (as found from the Einstein equations) are T_{xx}, T_{xy}, T_{yy}, $T_{\varphi\varphi}$ and T_{tt}, whose explicit form depend on the choice of $H(x, y)$. The simplest expression of this function compatible with regularity and the required asymptotic behaviour is

$$H(x, y) = (1-\lambda)\left(1+\nu\sqrt{x-y}\right), \tag{39}$$

with $\nu > 0$ a free parameter. Then the resulting line-element appears to be regular and free of pathologies on and outside the horizon. For example, the power series expansion of various quantities (like Kretschmann scalar, R and E_μ^ν) at $y = -1/\lambda$, $y = -1$ and $x = \pm 1$ is free of singularities. Also, smooth profiles are found when plotting the same quantities for various choices of the parameters λ, ν (with $R = 1$ without any loss of generality).

In the study of the far field expression of various quantities, we consider the following coordinate transformation

$$x = -\frac{r^2 - R^2}{\sqrt{(r^2-R^2)^2 + 4r^2R^2\cos^2\theta}},$$
$$y = -\frac{r^2 + R^2}{\sqrt{(r^2-R^2)^2 + 4r^2R^2\cos^2\theta}}, \tag{40}$$

with r, θ possessing (for large r) the usual interpretation, and $0 \le \theta \le \pi$. Then the Minkowski spacetime is recovered as $r \to \infty$, and one finds *e.g.*

$$g_{tt} = -1 + \frac{\sqrt{2}\nu R}{r} + O(1/r^2) + \ldots, \tag{41}$$

which implies an ADM mass $M = \frac{\nu R}{\sqrt{2}G} > 0$. However, one can easily show that, as expected, the energy density of the matter source, $\rho = -T_t^t = -E_t^t/(8\pi G)$, takes negative value for some region on and outside the horizon.

The basic results above hold as well for other choices of the function $H(x, y)$, and also for several generalizations of the line-element (36) we have considered. In all cases, we were not able to identify a field theory source for the energy-momentum tensor compatible with such metrics. However, (36) (or another version of it) could be useful as providing a *background* geometry in a numerical attempt to construct four dimensional BRs for a model with a matter source allowing for negative energy densities, in particular with a phantom scalar field.

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