STRING C-GROUP REPRESENTATIONS OF ALTERNATING GROUPS

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Abstract. We prove that for any integer $n \geq 12$, and for every r in the interval $[3, \ldots, \lfloor (n-1)/2 \rfloor]$, the group A_n has a string C-group representation of rank r, and hence that the only alternating group whose set of such ranks is not an interval is A_{11}.

Keywords: abstract regular polytopes, Coxeter groups, alternating groups, string C-groups.
2000 Math Subj. Class: 52B11, 20D06.

1. Introduction

String C-group representations have gained much attention in recent years as they are in one-to-one correspondence with abstract regular polytopes. More precisely, given an abstract regular polytope and a base flag of the polytope, one can construct a string C-group representation whose group G is the automorphism group of the polytope that is generated by the set of involutory automorphisms sending the base flag to its adjacent flags [32, Section 2E]. Hence the study of string C-group representations has interest not only for group theory, but also for geometry.

Classifications of string C-group representations received a big impetus thanks to experimental work of Leemans and Vauthier [31] and also Hartley [20]. These were pushed further for instance in [21, 27, 15, 11]. The results obtained in [31] quickly led to the determination of the highest rank of a string C-group representation of Suzuki groups [26]. Other families of almost simple groups were then investigated: the almost simple groups with socle $\mathrm{PSL}(2, q)$ [28, 29, 14], groups $\mathrm{PSL}(3, q)$ and $\mathrm{PGL}(3, q)$ [5], groups $\mathrm{PSL}(4, q)$ [3], small Ree groups [30], orthogonal and symplectic groups in characteristic 2, and finally, symmetric groups [16] and alternating groups [17, 18]. In particular, only the last four families gave rise to string C-group representations of arbitrary large rank. In [2], it is shown that, for all integers $m \geq 2$, and all integers $k \geq 2$, the orthogonal groups $\mathrm{O}^{\pm}(2m, \mathbb{F}_{2^k})$ act on abstract regular polytopes of rank $2m$, and the symplectic groups $\mathrm{Sp}(2m, \mathbb{F}_{2^k})$ act on abstract regular polytopes of rank $2m + 1$. A symmetric group S_n is known to have string C-group representations of highest rank $n - 1$ [6] and an alternating group A_n is known to have string C-group representations of highest rank $\lfloor \frac{n-1}{2} \rfloor$ when $n \geq 12$ [8]. It is worth noting that not only almost simple groups have been investigated. For instance, Cameron, Fernandes, Leemans and Mixer determined the maximal rank of a string C-group representation of a transitive permutation group in [7]. Conder determined in [12] the smallest string C-group representations

Group	Set of ranks
A_5	$\{3\}$
A_6	$\emptyset$
A_7	$\emptyset$
A_8	$\emptyset$
A_9	$\{3,4\}$
A_{10}	$\{3,4,5\}$
A_{11}	$\{3,6\}$
A_{12}	$\{3,4,5\}$

Table 1. Set of ranks for small alternating groups.

of rank r. It turns out that when r is at least 9, all such groups are 2-groups. Further studies on string C-group representations of 2-groups are available for instance in [23, 24].

The authors looked at the symmetric groups in [16] and proved three important facts. Firstly, when $n \geq 5$, the $(n-1)$-simplex is, up to isomorphism, the unique string C-group representation of S_n with rank $n-1$. Secondly, they showed that when $n \geq 7$, there is also, up to isomorphism, a unique string C-group representation of rank $n-2$. And finally, they showed that for every $n \geq 4$, and for every integer r in the interval $[3, \ldots n-1]$, a symmetric group S_n has at least one string C-group representation of rank r. Therefore, the symmetric groups have no gaps in their set of ranks. The first and second theorems have been extended in [19] where the authors of this paper, together with Mark Mixer, classified string C-group representations of rank $n-3$ (for $n \geq 9$) and $n-4$ (for $n \geq 11$) of the symmetric group S_n.

Also with Mixer, the authors produced in [17, 18] string C-group representations of rank $\lfloor (n-1)/2 \rfloor$ of the alternating groups, with $n \geq 12$. In the process of obtaining these results, they computed all string C-group representations of A_n with $n \leq 12$. They found that the set of ranks for the alternating groups of small degree were as given in Table 1. The case $n = 11$ turned out to be special in the sense that it was the only example encountered so far of a group whose set of ranks presented gaps. In this paper, we prove a similar result as the third theorem of [16]. Our main result is stated as follows.

Theorem 1.1. *For $n \geq 12$ and for every $3 \leq r \leq \lfloor (n-1)/2 \rfloor$, the group A_n has at least one string C-group representation of rank r.*

This theorem shows indeed that the case $n = 11$ is special among the alternating groups. The main tool in the proof of our main theorem is to find good permutation representation graphs that turn out to be CPR graphs, for every rank $3 \leq r \leq \lfloor (n-1)/2 \rfloor$ once n is fixed. We use a proof similar to that of the third theorem of [16] to tackle most cases and are just left dealing with finding string C-group representations of ranks four and five for A_n when n is even, and ranks four, five and six, when $n \equiv 3 \mod 4$.

The paper is organised as follows. In Section 2, we recall the basic definitions about string C-groups. In Section 3, we recall the definitions of permutation representation graphs and CPR-graphs and give some results that will be useful in proving Theorem 1.1. In Section 4, we prove Theorem 1.1. In Section 5, we give some final remarks.

As to notation for groups, we denote a cyclic group of order n by C_n, a dihedral group of degree n and order $2n$ by D_n, and by p^n an elementary abelian group of order p^n. Also, if G is a permutation group, the group G^+ is the subgroup of G generated by the even permutations in G, and if $G^+ = G$ (so that all elements of G are even) then we call G an *even permutation group*.

2. String C-groups

An abstract polytope is a combinatorial object which generalizes a classical convex polytope in Euclidean space. When the automorphism group of an abstract polytope acts regularly on its set of flags, the polytope is called *regular*, and in that case, its automorphism group admits a string C-group representation. Additionally, each abstract regular polytope can be constructed from a string C-group representation, and thus abstract regular polytopes and string C-groups representations are basically the same objects. For more details on the subject see [32, Section 2E].

A *Coxeter group* is a group with generators $\rho_0, \ldots, \rho_{r-1}$ and presentation

$$\langle \rho_i \mid (\rho_i\rho_j)^{m_{i,j}} = \varepsilon \text{ for all } i,\, j \in \{0, \ldots, r-1\}\rangle$$

where ε is the identity element of the group, each $m_{i,j}$ is a positive integer or infinity, $m_{i,i} = 1$, and $m_{i,j} = m_{j,i} > 1$ for $i \neq j$. It follows from the definition, that a Coxeter group satisfies the next condition called the *intersection property*.

$$\forall J, K \subseteq \{0, \ldots, r-1\}, \langle \rho_j \mid j \in J\rangle \cap \langle \rho_k \mid k \in K\rangle = \langle \rho_j \mid j \in J \cap K\rangle$$

A Coxeter group G can be represented by a *Coxeter diagram* $\mathcal{D}$. This Coxeter diagram $\mathcal{D}$ is a labelled graph which represents the set of relations of G. More precisely, the vertices of the graph correspond to the generators ρ_i of G, and for each i and j, an edge with label $m_{i,j}$ joins the ith and the jth vertices; conventionally, edges of label 2 are omitted. By a *string (Coxeter) diagram* we mean a Coxeter diagram with each connected component linear. A Coxeter group with a string diagram is called a *string Coxeter group*.

More generally, we define a *string group generated by involutions*, or *sggi* for short, as a pair (G, S) where G is a group, $S := \{\rho_0, \ldots, \rho_{r-1}\}$ is a finite set of involutions of G that generate G and that satisfy the following property, called the *commuting property*.

$$\forall i, j \in \{0, \ldots, r-1\},\ |i-j| > 1 \Rightarrow (\rho_i\rho_j)^2 = 1$$

Finally, a *string C-group representation* of a group G is a pair (G, S) that is a sggi and that satisfies the intersection property. In this case the underlying "Coxeter" diagram for (G, S) is a string diagram. The *(Schläfli) type* of (G, S) is $\{p_1, \ldots, p_{r-1}\}$ where p_i is the order of $\rho_{i-1}\rho_i$, $i \in \{1, \ldots, r-1\}$, and the *rank* of a string C-group representation (or of a sggi) (G, S) is the size of S. When the context is clear, we sometimes do not specify the set of generators S and we talk about a string C-group G instead of a string C-group representation (G, S).

The *set of ranks* of a group G is the largest set of integers I such that for each $r \in I$, there exists at least one string C-group representation of G with rank r.

Let $\Gamma := (G, S)$ be a sggi with $S := \{\rho_0, \ldots, \rho_{r-1}\}$. We denote by G_I with $I \subseteq \{0, \ldots, r-1\}$ the subgroup of G generated by the involutions with indices that are not in I and let $\Gamma_I := (G_I, \{\rho_j : j \notin I\})$; it follows from the definition that if Γ is a string C-group representation of G, each Γ_I is itself a string C-group representation of G_I. Also, for $i, j \in \{0, \ldots, r-1\}$, we denote $G_i = \langle \rho_j \mid j \neq i \rangle$ and $G_{i,j} := (G_i)_j$. The following two results show that when Γ_0 and Γ_{r-1} are string C-group representations, the intersection property for (G, S) is verified by checking only one condition.

Proposition 2.1. [32, Proposition 2E16] *Let $\Gamma := (G, S)$ be a sggi with $S := \{\rho_0, \ldots, \rho_{r-1}\}$. Suppose that Γ_0 and Γ_{r-1} are string C-group representations. If $G_0 \cap G_{r-1} = G_{0,r-1}$, then Γ is a string C-group representation of G.*

We point out that the inclusion $G_0 \cap G_{r-1} \geq G_{0,r-1}$ is immediate, and thus we only need to check that $G_0 \cap G_{r-1} \leq G_{0,r-1}$. The following proposition makes it even simpler to check if a pair (G, S) is a string C-group representation when $G_{0,r-1}$ is a maximal subgroup of either G_0 or G_{r-1} (or both).

Proposition 2.2. [17, Lemma 2.2] *Let $\Gamma = (G, S)$ be a sggi with $S := \{\rho_0, \ldots, \rho_{r-1}\}$ and $G := \langle S \rangle$. Suppose that Γ_0 and Γ_{r-1}, are string C-group representations of G_0 and G_{r-1} respectively. If $\rho_{r-1} \notin G_{r-1}$ and $G_{0,r-1}$ is maximal in G_0, then Γ is a string C-group representation of G.*

3. PERMUTATION REPRESENTATION GRAPHS AND CPR GRAPHS

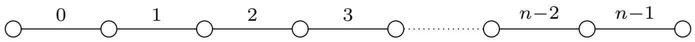
Let G be a group of permutations acting on a set $\{1, \ldots, n\}$. Let $S := \{\rho_0, \ldots, \rho_{r-1}\}$ be a set of r involutions of G that generate G. We define the *permutation representation graph* $\mathcal{G}$ of G, as the r-edge-labeled multigraph with n vertices and with an i-edge $\{a, b\}$ whenever $a\rho_i = b$ with $a \neq b$.

The pair (G, S) is a sggi if and only if $\mathcal{G}$ satisfies the following properties:

1. The graph induced by edges of label i is a matching;
2. Each connected component of the graph induced by edges of labels i and j, for $|i - j| \geq 2$, is a single vertex, a single edge, a double edge, or a square with alternating labels.

When (G, S) is a string C-group representation, the permutation representation graph $\mathcal{G}$ is called a *CPR graph*, as defined in [33]. In rank 3, there are a couple of known results to determine if a 3-edge-labeled multigraph is a CPR graph. For higher ranks, no such arguments were accomplished.

One simple example of a CPR graph is the one corresponding to the $(n-1)$-simplex as follows:



In [16], for each rank $3 \leq r \leq n-2$, a string C-group representation of rank r of S_n was found. In [17], the authors constructed a string C-group representation of rank $r \geq 4$ of A_n for some n. This is summarized in the following two theorems, and the associated CPR graphs are given in Table 2.

Theorem 3.1. [16, Theorem 3] *For $n \geq 5$ and $3 \leq r \leq n-2$, there is a string C-group representation of rank r and type $\{n-r+2, 6, 3^{r-3}\}$ of S_n.*

Theorem 3.2. [17, Theorem 1.1] *For each rank $k \geq 3$, there is a string C-group representation of rank k of A_n for some n. In particular, for each even rank $r \geq 4$, there is a string C-group representation of A_{2r+1} of type $\{10, 3^{r-2}\}$, and for each odd rank $q \geq 5$, there is a string C-group representation of A_{2q+3} of type $\{10, 3^{q-4}, 6, 4\}$.*

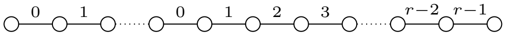
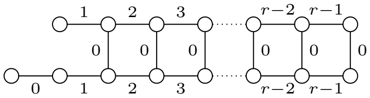
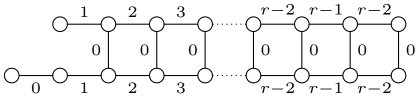
Group	Schläfli Type	CPR Graph
S_n $(3\leq r\leq n-2)$	$\{n-r+2,\, 6,\, 3^{r-3}\}$	
A_{2r+1} (r even and ≥ 4)	$\{10, 3^{r-2}\}$	
A_{2r+3} (r odd and ≥ 5)	$\{10, 3^{r-4}, 6, 4\}$	

TABLE 2. String C-group representations of S_n and A_n

Permutation representation graphs are a very useful tool for the construction of string groups generated by involutions. We will use them in the proof of our main theorem.

The term sesqui-extension was first introduced in [17]. Let us recall its meaning. Let $\Phi = \langle \alpha_0, \ldots, \alpha_{d-1}\rangle$ be a sggi, and let τ be an involution in a supergroup of Φ such that $\tau \notin \Phi$ and τ centralizes Φ. For fixed k, we define the group $\Phi^* = \langle \alpha_i\tau^{\eta_i} \,|\, i \in \{0, \ldots, d-1\}\rangle$ where $\eta_i = 1$ if $i = k$ and 0 otherwise, and call this the *sesqui-extension* of Φ with respect to α_k and τ. In particular, a permutation representation graph having two connected components, one of which is a single k-edge and the other contains at least one k-edge, represents a sesqui-extention of a group (the group corresponding to the biggest component) with respect to the generator k.

Proposition 3.3. [18, Proposition 5.4] *If $\Phi = \langle \alpha_i \,|\, i = 0, \ldots, d-1\rangle$ and $\Phi^* = \langle \alpha_i\tau^{\eta_i} \,|\, i \in \{0, \ldots, d-1\}\rangle$ is a sesqui-extension of Φ with respect to α_k, then $(\Phi, \{\alpha_i \,|\, i = 0, \ldots, d-1\})$ is a string C-group representation if and only if $(\Phi^*, \{\alpha_i\tau^{\eta_i} \,|\, i \in \{0, \ldots, d-1\}\})$ is a string C-group representation. Moreover one of the following situations occur.*

1. *$\tau \in \Phi^*$, in which case Φ^* is isomorphic to $\Phi \times \langle\tau\rangle \cong \Phi \times C_2$; or*
2. *$\tau \notin \Phi^*$, in which case Φ^* is isomorphic to Φ.*

Sesqui-extensions will be used later to check the intersection condition on the permutation representations of the groups of our main theorem.

We also apply the techniques used in the proof of Theorem 3.1 based on a construction of Hartley and Leemans available in [22]. The key of the proof of Theorem 3.1 was to start from the CPR graph of the $(n-1)$-simplex with generators $\rho_1, \ldots, \rho_{n-1}$ where ρ_i is the transposition $(i, i+1)$ in S_n. Let $d = n-1$. At each step, we start with a string C-group representation of rank d and generators $\rho_1, \ldots, \rho_d$. We replace ρ_{d-2} by $\rho_{d-2}\rho_d$ and we drop ρ_d. As proved in [16], we get in this way a new string C-group representation with generators $\rho_1, \ldots, \rho_{d-1}$. We can repeat this until $d = 3$. We give in Table 3 an example of this process for S_7.

In order to prove that the permutation groups of our main theorem are isomorphic to alternating groups we use the following results.

Theorem 3.4. [25] *Let G be a primitive permutation group of finite degree n, containing a cycle of prime length fixing at least three points. Then $G \geq A_n$.*

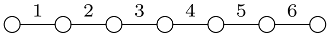
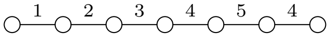
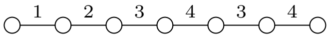
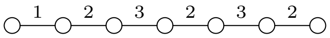
Generators	CPR graph	Schläfli type
(1,2),(2,3),(3,4),(4,5),(5,6),(6,7)		{3,3,3,3,3}
(1,2),(2,3),(3,4),(4,5)(6,7),(5,6)		{3,3,6,4}
(1,2),(2,3),(3,4)(5,6),(4,5)(6,7)		{3,6,5}
(1,2),(2,3)(4,5)(6,7),(3,4)(5,6)		{6,6}

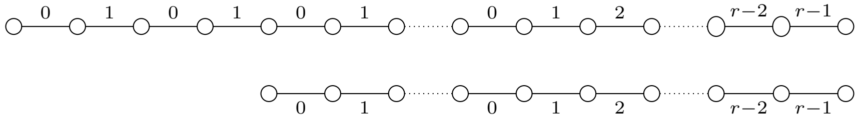
TABLE 3. The induction process used on S_7

Proposition 3.5. [18, Proposition 3.3] *Let* $G = \langle \rho_0, \ldots, \rho_{r-1} \rangle$ *be a transitive permutation group acting on the points* $\{1, \ldots, n\}$ *with* $n \geq 5$*, and let* $G^* = \langle \rho_0, \ldots, \rho_{r-1}, \rho_r, \rho_{r+1} \rangle$*, where*

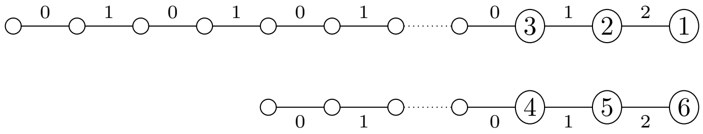
$$\rho_r = (i, n+1)(n+2, n+3) \text{ for some } i \in \{1, \ldots, n\}$$
$$\rho_{r+1} = (n+1, n+2)(n+3, n+4).$$

Then $G^* = A_{n+4}$ *or* S_{n+4}*, depending on whether or not* G *is even.*

Proposition 3.6. *The following graph, with* $n \geq 8$ *vertices,* n *even and* $r \in \{3, \ldots, \frac{n-2}{2}\}$*, is a CPR graph for* $(S_{\frac{n-4}{2}} \times S_{\frac{n+4}{2}})^+$.



Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation given by the graph of this proposition. Let us first consider $r = 3$.



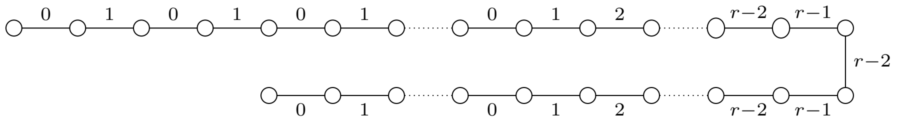
We see that Γ_0 and Γ_2 are string C-group representations and as $G_0 \cap G_2 = G_{0,2} \cong C_2$, Γ is itself a string C-group representation by Proposition 2.1.

Let us prove that G is isomorphic to $(S_{\frac{n-4}{2}} \times S_{\frac{n+4}{2}})^+$. We first prove that G contains the 3-cycles $(1, 2, 3)$ and $(4, 5, 6)$ (the vertices of the above graph on the right). Let l be the least integer such that $(\rho_0\rho_1)^l$ fixes all the vertices of the component of the graph on the bottom. We see that $(\rho_1\rho_2)^2 = (1, 2, 3)(4, 5, 6)$. The latter element conjugated by $(\rho_0\rho_1)^l$ is equal to $\alpha = (a, b, c)(4, 5, 6)$ with $\{a, b, c\} \cap \{1, 2, 3\} = \{1\}$. Hence $(\alpha(\rho_1\rho_2)^2)^5 = (4, 6, 5)$ and $(1, 2, 3) = (4, 6, 5)(\rho_1\rho_2)^2$.

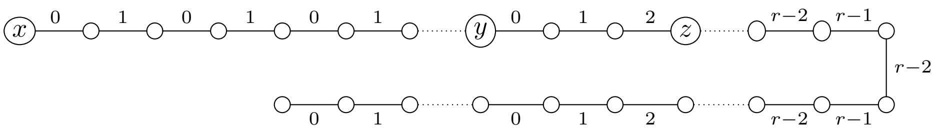
Now by transitivity in each of the two components of the graph we find that G has a subgroup isomorphic to $A_{\frac{n-4}{2}} \times A_{\frac{n+4}{2}}$. As in addition $\rho_2 \notin A_{\frac{n-4}{2}} \times A_{\frac{n+4}{2}}$ and G is a group of even permutations, the group G is isomorphic to $(S_{\frac{n-4}{2}} \times S_{\frac{n+4}{2}})^+$.

Now let $r > 3$. We may assume by induction that Γ_{r-1} is a string C-group representation and G_{r-1} is isomorphic to $(S_{\frac{n-6}{2}} \times S_{\frac{n+2}{2}})^+$. In addition Γ_0 is a string C-group representation with group G_0 isomorphic to S_{r-1}. By the intersection of the orbits of G_0 and G_{r-1} we conclude that $G_0 \cap G_{r-1}$ and $G_{0,r-1}$ are both isomorphic to S_{r-2}. Therefore Γ is a string C-group representation of G. Moreover it is clear that G is isomorphic to $(S_{\frac{n-4}{2}} \times S_{\frac{n+4}{2}})^+$. □

Proposition 3.7. *The following graph, with $n \geq 10$ vertices, n even and $r \in \{5, \dots, \frac{n-2}{2}\}$, is a CPR graph for S_n.*

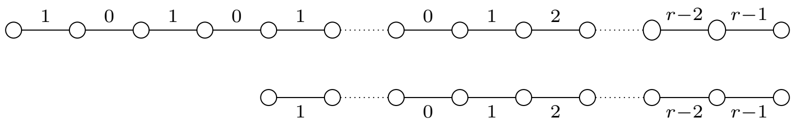


Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation given by the graph of this proposition. The permutation representation graph is connected, hence G is transitive. Let x be the first point on the left of the graph. The stabilizer of x has at most the same orbits as G_0. Consider the vertices y and z as in the following graph.



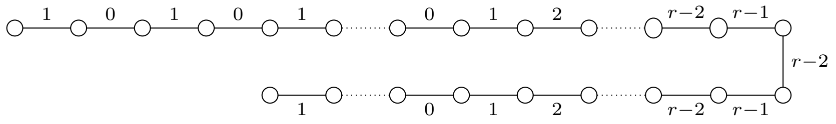
We see that $y\rho_2^{\rho_1\rho_0} = z$ and $\rho_2^{\rho_1\rho_0}$ fixes x. More generally the appropriate conjugations of ρ_2 by powers of $\rho_0\rho_1$ fuse the orbits of G_0 while fixing x. Hence G is 2-transitive and therefore primitive. Moreover, it contains a 3-cycle (explicitly given in the proof of Proposition 3.6) and an odd permutation. Hence, by Theorem 3.4, it is isomorphic to S_{n-1}. By Proposition 3.3 and [18, Table 2] we may conclude that Γ_0 is a string C-group representation of the group $C_2 \times (C_2 \wr S_{r-1})$. By Proposition 3.6, the sggi Γ_{r-1} is a string C-group representation of $(S_{\frac{n-6}{2}} \times S_{\frac{n+2}{2}})^+$. From the intersection of the orbits of G_0 and G_{r-1} we also conclude that $G_0 \cap G_{r-1} = G_{0,r-1} \cong C_2 \times (S_{\frac{n-7}{2}} \times S_{\frac{n+1}{2}})^+$. Hence Γ is a string C-group representation. ☐

Proposition 3.8. *The following graph, with $n \geq 10$ vertices, n even and $r \in \{3, \dots, \frac{n-2}{2}\}$, is a CPR graph for $(S_{\frac{n-4}{2}} \times S_{\frac{n+4}{2}})^+$.*



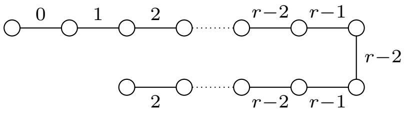
Proof. Similar to that of Proposition 3.6. ☐

Proposition 3.9. *The following graph, with $n \geq 12$ vertices, n even and $r \in \{5, \dots, \frac{n-2}{2}\}$, is a CPR graph for S_n.*

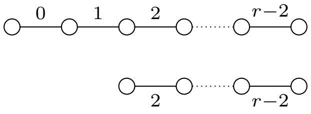


Proof. Similar to that of Proposition 3.7. ☐

Proposition 3.10. *The following graph, with $n \geq 8$ vertices, n even and $r = n/2$, is a CPR graph for S_n.*

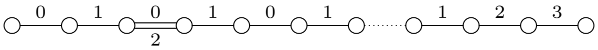


Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation given by the graph of this proposition. Removing the 0-edge from the graph we get a CPR graph for a symmetric group of degree $n-1$ (see Table 2 of [18]). Hence Γ_0 is a string C-group representation. Now consider the sggi $\Phi := (H, T)$ with the following permutation representation graph.



For $r = 4$, Φ is a string C-group representation with H isomorphic to $C_2 \times S_4$. Assume by induction that Φ_{r-2} is a string C-group representation with H_{r-2} isomorphic to $S_{r-1} \times S_{r-3}$. As Φ_0 is a string C-group representation and $H_0 \cap H_{r-2} \leq S_{r-2} \times S_{r-3} \cong H_{0,r-2}$, Φ is a string C-group representation. Moreover H is isomorphic to $S_{r-1} \times S_{r-3}$. Now by Proposition 3.3 the sggi Γ_{r-1} is a string C-group representation and G_{r-1} is isomorphic to $C_2 \times S_{r-1} \times S_{r-3}$. By the intersection of the orbits of G_0 and G_{r-1} we find that $G_0 \cap G_{r-1} = G_{0,r-1}$ Hence Γ is a string C-group representation. As G_0 is isomorphic to S_{n-1} and stabilizes the first vertex on the left, we conclude that G is isomorphic to S_n. □

Proposition 3.11. *The following graph with n vertices, $n \equiv 3 \mod 4$ and $n \geq 11$, is a CPR graph for S_n.*



Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation given by the graph of this proposition. The group G_3 is an even transitive group containing a 3-cycle, namely $(\rho_1\rho_2)^4$, and the stabilizer of a point in G_3 is transitive on the remaining points. Hence by Theorem 3.4 the group G_3 is isomorphic to A_{n-1}. Consequently G is isomorphic to S_n. Moreover as G_3 is a simple group generated by three independent involution, the sggi Γ_3 is string C-group representation. It is also easy to check that Γ_0 is string C-group representation and that $G_3 \cap G_0 = G_{0,3}$, as it is sufficient to consider the case $n = 11$. Hence Γ is a string C-group representation and G is isomorphic to S_n as wanted. □

4. PROOF OF THEOREM 1.1

For each $n \geq 12$, the group A_n has at least one string C-group representation of rank three. Indeed, we can rely on [9, 10] which covers all but a small number of small cases that can be easily dealt with MAGMA [1], or [34]. Hence we have to construct examples of rank 4 and above. Also, the case where $n = 12$ is done in [17], hence we may assume $n > 12$.

We divide the rest of the proof is a series of theorems depending on the values of n and r as described in Table 4. Theorem 4.1 comes from [18], and we use it in

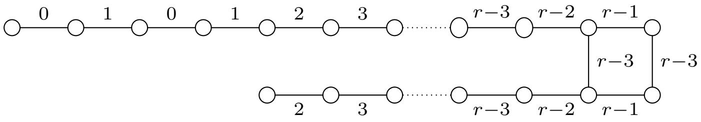
n	r	Reference
n even	$6 \leq r \leq (n-2)/2$	Theorem 4.2
$n \equiv 0 \mod 4$	$r = 5$	Theorem 4.6
	$r = 4$	Theorem 4.5
$n \equiv 2 \mod 4$	$r = 5$	Theorem 4.4
	$r = 4$	Theorem 4.3
$n \equiv 1 \mod 4$	$4 \leq r \leq (n-1)/2$	Theorem 4.7
$n \equiv 3 \mod 4$	$r = (n-1)/2$	Theorem 4.8
	$7 \leq r < (n-1)/2$ and r odd	Theorem 4.9
	$r = (n-1)/2 - 1$	Theorem 4.10
	$8 \leq r < (n-1)/2$ and r even	Theorem 4.11
	$r = 4$	Theorem 4.12
	$r = 5$	Theorems 4.13 and 4.15
	$r = 6$	Theorem 4.14

TABLE 4. The structure of the proof depending on n and r

Theorem 4.2 to construct string C-group representations of rank $6 \leq r \leq (n-2)/2$ for n even.

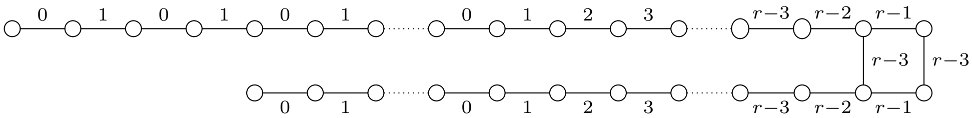
4.1. **The even case.** We will construct a family of CPR graphs of even ranks "reducing" the rank of a CPR graph having highest possible rank. Let us consider the graph given in the following theorem.

Theorem 4.1. [18] *If* $n \geq 14$ *is even and* $r = \frac{n-2}{2} \geq 6$*, then the following graph is a CPR graph for* A_n*.*

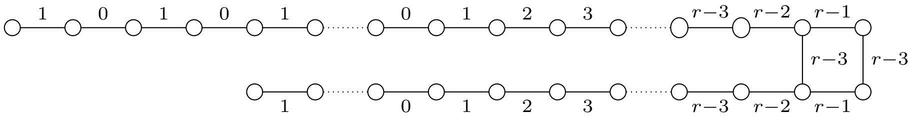


Moreover the corresponding string C-group representation has type $\{5, 6, 3^{r-6}, 6, 6, 3\}$*.*

Theorem 4.2. *If* n *is an even integer,* $n \geq 14$ *and* $6 \leq r \leq \frac{n-2}{2}$*, then the group* A_n *admits a string C-group representation of rank* r*, with Schläfli type* $\{LCM(4+i, i), 6, 3^{r-6}, 6, 6, 3\}$ *where* $i = (n-2)/2 - r + 1$*, and with the following CPR graph*

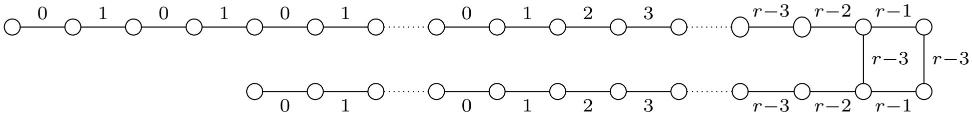


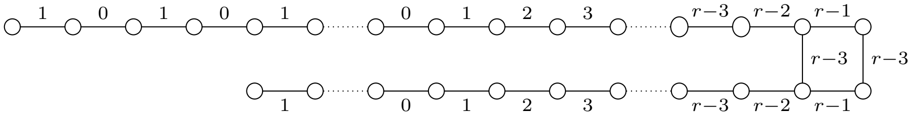
for ($n \equiv 2 \mod 4$ *and* $n - r$ *even) or* ($n \equiv 0 \mod 4$ *and* $n - r$ *odd) and the following CPR-graph*



for ($n \equiv 2 \mod 4$ *and* $n - r$ *odd) or* ($n \equiv 0 \mod 4$ *and* $n - r$ *even).*

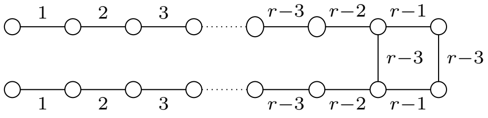
Proof. From the graph of Theorem 4.1 we construct a family of graphs with n vertices and $r \in \{6, \dots, \frac{n-2}{2}\}$ adding, on the top and on the bottom of the graph, two sequences of edges, of the same size, with alternate labels 0 and 1. So we have the following two possibilities.





Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. The statement holds for $n = 14$ and $r = 6$ by Theorem 4.1. Assume $n > 14$.

The involution ρ_1 can be decomposed as $\rho_1 = \tau\alpha_1$ where α_1 is the restriction of ρ_1 to the biggest G_0-orbit and τ is the restriction of ρ_1 to the union of G_0-orbits of size 2. The following CPR graph has group isomorphic to $(2^r : S_r)^+$ as shown in [18, Lemma 6.6]. It is exactly the graph we obtain by replacing ρ_1 by α_1 and forgetting about the points fixed by G_0.



We find that $\alpha_1 = \rho_2\rho_1\rho_2\rho_1\rho_2 \in G_0$, then also $\tau \in G_0$ and therefore by Proposition 3.3, G_0 is a sesqui-extension of the group $(2^r : S_r)^+$ and G_0 is isomorphic to $C_2 \times (2^r : S_r)^+ \cong 2^r : S_r$ as $\tau \in G_0$. Moreover, Γ_0 is a string C-group representation.

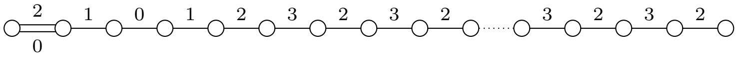
We use a similar argument to prove that Γ_{r-1} is a string C-group, starting from the CPR graph given in Proposition 3.7 when ($n \equiv 2 \mod 4$ and $n - r$ even) or ($n \equiv 0 \mod 4$ and $n - r$ odd), and from the CPR graph given in Proposition 3.9 when ($n \equiv 2 \mod 4$ and $n - r$ odd) or ($n \equiv 0 \mod 4$ and $n - r$ even). In that case, however, since the restriction of $\rho_{r-2}\rho_{r-3}$ to the biggest orbit of G_{r-1} is an element of even order, $G_{r-1} \cong S_{n-2}$. Since A_n acts primitively on the set of unordered pairs of points, the stabilizer in A_n of a fixed pair is maximal in A_n, and such stabilizers have precisely the structure of G_{r-1}. As G_{r-1} is a maximal subgroup of A_n and $\rho_{r-1} \notin G_{r-1}$, it follows that G is isomorphic to A_n. Let us now prove that $G_{0,r-1} = G_0 \cap G_{r-1}$. The orbits of $G_0 \cap G_{r-1}$ have to be suborbits of G_0 and of G_{r-1}, hence $G_0 \cap G_{r-1} \leq (C_2 \times (2^{r-1} : S_{r-1}) \times C_2)^+ \cong G_{0,r-1}$. Hence, by Proposition 2.1, Γ is a string C-group representation of A_n.

Let $i = (n-2)/2 - r + 1$. Then it is easy to see from the CPR-graph that the Schläfli type of the string C-group representation of A_n of rank r obtained by this construction is $\{LCM(4+i, i), 6, 3^{r-6}, 6, 6, 3\}$. The first entry of the symbol comes from the fact that there are 0-1-components on the upper side of the graph and on the lower side of the graph and the upper one has 4 more vertices than the lower one. □

It remains to construct examples in rank 4 and 5 for n even. We split the discussion in two cases, namely the case where $n \equiv 0 \mod 4$ and the case where $n \equiv 2 \mod 4$.

Theorem 4.3. *If $n \equiv 2 \mod 4$ with $n \geq 10$, then the group A_n admits a string C-group representation of rank 4, with Schläfli type $\{5, 6, n-4\}$, with the following CPR-graph.*

(F_1)

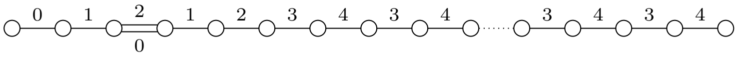


Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. In this case G_3 is a sesqui-extension of a string C-group representation of A_5, hence by Proposition 3.3, $G_3 \cong C_2 \times A_5$ and Γ_3 is a string C-group representation of rank 3. Moreover, $G_{0,3}$ is isomorphic to $C_2 \times D_3 \cong D_6$ and therefore $G_{0,3}$ is maximal in G_3. So, by Proposition 2.2, it remains to prove that Γ_0 is also a string C-group representation. Now, $\Gamma_{0,3}$ and $\Gamma_{0,1}$ are obviously string C-group representations of dihedral groups. The group $G_{0,1,3}$ is a cyclic group of order 2 and the subgroups $G_{0,3}$ and $G_{0,1}$ will have the same intersection no matter what the value of n is. We can thus assume $n = 10$ and check by hand or using MAGMA that $G_0 \cap G_3 = G_{0,3}$. Hence Γ_0 is a string C-group representation. This concludes the proof that a sggi with permutation representation graph (F_1) is a string C-group representation. It remains to show that the four generators generate A_n. The element $\rho_0\rho_1$ is a 5-cycle and G is primitive, as for instance ρ_0 cannot preserve any block system. Hence, by Theorem 3.4, G is isomorphic to A_n.

The Schläfli type is obvious from the permutation representation graph. □

Theorem 4.4. *If $n \equiv 2 \mod 4$ with $n \geq 10$, then the group A_n admits a string C-group representation of rank 5, with Schläfli type $\{5, 5, 6, n-5\}$, with the following CPR-graph.*

(F_2)

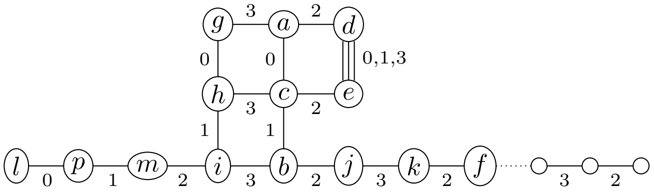


Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. In this case, G_4 is a sesqui-extension of a group isomorphic to $(S_7 \times C_2)^+ \cong S_7$ whose CPR graph is given in Table 2 of [18]. Hence Γ_4 is a string C-group representation. By Proposition 3.5 the group G_0 is isomorphic to A_{n-1}. The subgroup $G_{0,4}$ is isomorphic to S_6, in addition $G_{0,1,4} \cong D_6$ and $G_{0,1} \cong S_{n-4}$. Increasing n will not change the intersection between $G_{0,1}$ and $G_{0,4}$. Hence we can check with MAGMA that $G_{0,1} \cap G_{0,4} = G_{0,1,4}$ for $n = 10$. Thus $\Gamma_{0,1}$ is a string C-group representation and so is Γ_0 and so is Γ, as $G_0 \cong A_{n-1}$ and G is transitive. Moreover G is isomorphic to A_n since it is transitive on n points and the stabilizer of a point in G contains $G_0 \cong A_{n-1}$.

The Schläfli type is obvious from the permutation representation graph. □

Theorem 4.5. *If $n \equiv 0 \mod 4$ with $n \geq 16$, then the group A_n admits a string C-group representation of rank 4, with Schläfli type $\{3, 12, LCM(n-8, 6)\}$, with the following CPR-graph.*

(F_3)

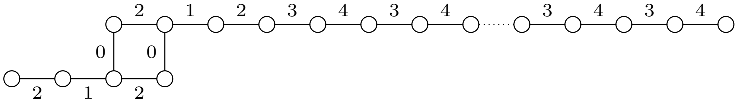


Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. In this case, G_3 is isomorphic to $2^2 : S_3 \times S_3$ and $G_{0,3}$ is isomorphic to D_{12} no matter what the value of n is, thanks to the shape of the graph. Observe that the left connected component of the graph, obtained when removing the 3-edges, gives the CPR graph of the octahedron. Thus it can easily be checked with Magma that Γ_3 is a string C-group representation with type $\{3, 12\}$. The group G_0 is transitive on $n-1$ points, namely all vertices of the graph except l. Moreover, the stabilizer of l and p in G has at most two more orbits thanks to the connected components of the permutation representation graph obtained by removing edges labelled 0 and 1. The element $(\rho_1\rho_2\rho_3\rho_2)^3$ moves point i to point d while fixing both l and p. Hence G_0 is 2-transitive on $n-1$ vertices (all but l). Therefore G_0 is primitive on these points. Now the element $(\rho_1\rho_2\rho_3\rho_2) = (l)(p, j, m)(i, e, g, d, h)(a, c, f, b)\ldots$ has the property that the cycles we did not write are transpositions. Indeed, ρ_1 does not do anything on these points and so the action on these points is given by $\rho_2\rho_3\rho_2 = \rho_3^{\rho_2}$ which is an involution. Hence $(\rho_1\rho_2\rho_3\rho_2)^{12} \in G_0$ is a 5-cycle fixing more than three points. By Theorem 3.4, we can therefore conclude that G_0 is isomorphic to A_{n-1}. As G_0 is a simple group, since it is generated by three involutions (namely ρ_1, ρ_2, ρ_3), two of which commute, Γ_0 is a string C-group representation by [13, Theorem 4.1]. It remains to check that $G_{0,3} = G_0 \cap G_3$ to prove that these graphs give indeed string C-group representations. This can be checked with Magma for $n = 12$ and the result can be extended for any n.

The Schläfli type is obvious from the permutation representation graph. □

Theorem 4.6. *If* $n \equiv 0 \mod 4$ *with* $n \geq 12$*, then the group* A_n *admits a string C-group representation of rank* 5*, with Schläfli type* $\{3, 4, 6, n-7\}$*, with the following CPR-graph.*

(F_4)

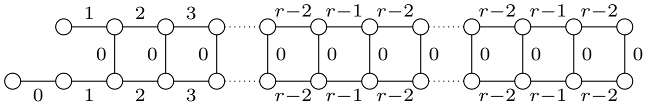


Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. In this case, G_4 is a sesqui-extension of the group of a string C-group representation of S_9, that can be found for instance in the atlas [31]. The sggi $\Gamma_{0,1}$ is a string C-group representation of S_{n-6} and $G_{0,4}$ is isomorphic to $S_5 \times D_4$. Now $\rho_2\rho_3$ has order 6, so $G_{0,1,4}$ is isomorphic to D_6 and it is obvious from the permutation representation graph that $G_{0,4} \cap G_{0,1} = G_{0,1,4}$ and $G_{0,4} \cap G_{1,4} = G_{0,1,4}$. Hence Γ_0 and Γ_4 are string C-group representations by Proposition 2.1. As $G_0 \cap G_4$ must have orbits that are suborbits of those of G_0 and of those of G_4, we readily see that $G_0 \cap G_4 = G_{0,4}$. This concludes the proof that every graph of shape (F_4) gives a string C-group representation. As G is a primitive group generated by even permutations and $(\rho_2\rho_3)^2$ is a 3-cycle, we see that G is isomorphic to A_n by Theorem 3.4.

The Schläfli type is obvious from the permutation representation graph. □

4.2. **The odd case.**

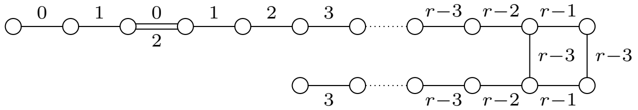
Theorem 4.7. *If* n *and* r *are integers with* $n \geq 13$*,* $n \equiv 1 \mod 4$ *and* $4 \leq r \leq (n-1)/2$*, then the group* A_n *admits a string C-group representation of rank* r*, with Schläfli type* $\{10, 3^{\frac{n-1}{2}-2}\}$ *when* $r = \frac{n-1}{2}$ *and* $\{10, 3^{r-4}, 6, \frac{n-1}{2} - r + 3\}$ *when* $r < \frac{n-1}{2}$*, and with the following CPR graph.*



Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. Clearly G is a group of even permutations and it must be primitive as ρ_0 cannot preserve a non-trivial block system. Let us prove that G is isomorphic to A_n. We see that $(\rho_0\rho_1)^2$ is a 5-cycle, hence by Theorem 3.4, the group G is isomorphic to A_n. It remains to prove that Γ satisfies the intersection property. We know that for $n = 13$, the sggi Γ is a string C-group representation of rank 6 and Schläfli type $\{10, 3, 3, 3, 3\}$. It can be checked with MAGMA that Γ is also a string C-group representation for $n = 13$ and $r \in \{4, 5\}$. By induction we may assume that G_{r-1} is a sesqui-extension of the group of a string C-group representation. Hence by Proposition 3.3, the sggi Γ_{r-1} satisfies the intersection property. By the first line of Table 2, it is easy to see that Γ_0 is a string C-group representation. Finally, $G_{0,r-1} = G_0 \cap G_{r-1} \cong S_{r-1} \times C_2$. By Proposition 2.1, we conclude that Γ is a string C-group representation. Using this technique, we have just constructed string C-group representations of rank r for every $4 \leq r \leq \frac{n-1}{2}$. Their Schläfli types are $\{10, 3^{\frac{n-1}{2}-2}\}$ when $r = \frac{n-1}{2}$ and $\{10, 3^{r-4}, 6, \frac{n-1}{2} - r + 3\}$ when $r < \frac{n-1}{2}$. □

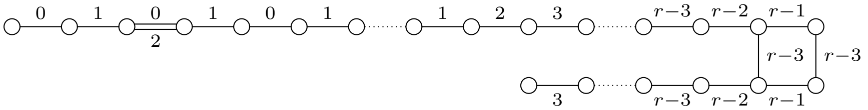
The following theorem gives the string C-group representations of rank $r = (n-1)/2$ in the case where $n \equiv 3 \mod 4$.

Theorem 4.8. [18] *If* n *and* r *are integers with* $n \geq 15$*,* $n \equiv 3 \mod 4$ *and* $r = (n-1)/2$*, then the group* A_n *admits a string C-group representation of rank* r*, with Schläfli type* $\{5, 5, 6, 3^{r-7}, 6, 6, 3\}$*, and with the following CPR graph.*



From these examples, we construct examples of the same rank but for groups of degree $n + 4k$ where k is an integer, by adding a sequence of alternating 0- and 1-edges of length $4k$ between the first and the second 2-edge (counting from the left).

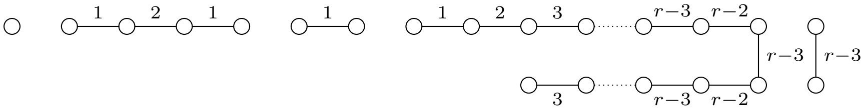
Theorem 4.9. *If* n *and* r *are integers with* $n \geq 15$*,* $n \equiv 3 \mod 4$ *and* $7 \leq r < (n-1)/2$*,* r *odd, then the group* A_n *admits a string C-group representation of rank* r*, with Schläfli type* $\{n - 2(r-2), 12, 6, 3^{r-7}, 6, 6, 3\}$*, and with the following CPR graph.*



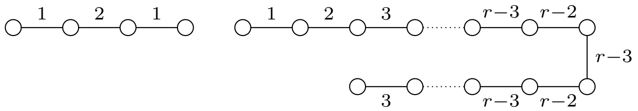
Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. The group G_0 is acting as $S_{2(r-1)}$ on the orbit of size $2(r-1)$ and as D_4 on the orbit of size 4, making G_0 isomorphic to $A_{2(r-1)} : D_4$. Observe that G_0 has a structure that only depends on the rank, not on the degree of G.

The group $G_{0,r-1}$ is isomorphic to $S_{2(r-2)} : D_4$. It is a maximal subgroup of G_0. Hence $G_0 \cap G_{r-1} = G_{0,r-1}$.

Let us now prove that Γ_0 and Γ_{r-1} are string C-group representations. We start with Γ_0. The group $G_{0,1}$ is the same (up to removing the fixed points) as the one of Theorem 4.8. Hence Γ_0 is a string C-group representation. The sggi $\Gamma_{0,r-1}$ has the following permutation representation graph, where there might be more than one 1-edge disconnected from the rest of the graph.



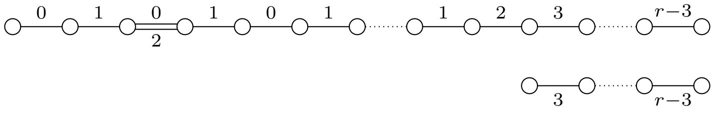
If we prove that the sggi corresponding to the following permutation representation graph is a string C-group representation, we may then apply Proposition 3.3 in order to show that $\Gamma_{0,r-1}$ is also a string C-group representation.



Let us call $\Phi := (H, T)$ the sggi having this permutation representation graph. By Proposition 3.10 the connected component on the right of the graph above gives a string C-group representation. By Proposition 3.3 the graph that we obtain from the graph pictured above by removing the 2-edge on the left is a CPR graph. Since removing the 2-edge on the left does not change the order of the group H_1, by [32, Proposition 2E17] we find that Φ is a string C-group representation. Hence Γ_0 is a string C-group representation.

Let us now prove that Γ_{r-1} is a string C-group representation.

The group $G_{r-2,r-1}$ is a sesqui-extension of the group K of the sggi $\Psi := (K, U)$ having the following permutation representation graph.



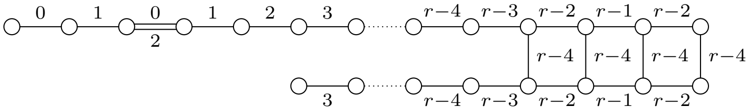
Let a and b be the sizes of the connected components of the graph above. For $r = 6$, K is a sesqui-extension of the group of the string C-group representation of Proposition 3.11, hence by Proposition 3.3, K is isomorphic to $S_a \cong (S_a \times 2)^+$. By induction we may assume that Ψ_{r-3} is a string C-group representation and K_{r-3} is isomorphic to $(S_{a-1} \times S_{b-1})^+$. As Ψ_0 is a string C-group representation and $K_0 \cap K_{r-3} = K_{0,r-3}$ we find that Ψ is itself a string C-group representation. Moreover K is clearly isomorphic to $(S_a \times S_b)^+$. With this, using Proposition 3.3, we see that $\Gamma_{r-2,r-1}$ is a string C-group representation. Finally $G_{0,r-1} \cap G_{r-2,r-1} \leq (D_4 \times S_{2(r-3)} \times 2)^+ \cong G_{0,r-2,r-1}$.

Hence we have proved that Γ_{r-1} is a string C-group representation and therefore G itself is a string C-group.

It is easy to see from the permutation representation graph in the theorem that the Schläfli type of the string C-group representation of rank r of A_n obtained by this construction is $\{n-2(r-2), 12, 6, 3^{r-7}, 6, 6, 3\}$. □

The previous two theorems enable us to construct examples of all possible odd ranks at least 7 for A_n with $n \equiv 3 \mod 4$ and $n \geq 15$. We now construct an example of rank $(n-3)/2$ for A_n from the example of rank $(n-1)/2$, that we will use to construct all examples of even rank at least 8.

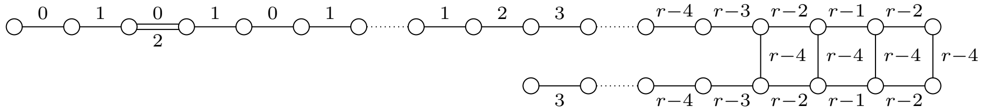
Theorem 4.10. *If n and r are integers are such that $n \geq 19$, $n \equiv 3 \mod 4$ and $r = (n-1)/2-1$, then the group A_n admits a string C-group representation of rank r, with Schläfli type $\{5, 5, 6, 3^{r-8}, 6, 6, 6, 4\}$, and with the following CPR graph.*



Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. The group G_{r-1} is a sesqui-extension of the group given in Theorem 4.8. Hence Γ_{r-1} is a string C-group representation. The sggi Γ_0 can be proved to be a string C-group representation using similar techniques to those the proof of the previous theorem. The fact that $G_0 \cap G_{r-1} = G_{0,r-1}$ follows from the fact that G_{r-1} is a sesqui-extension of the group given in Theorem 4.8 and the orbits of the respective subgroups. □

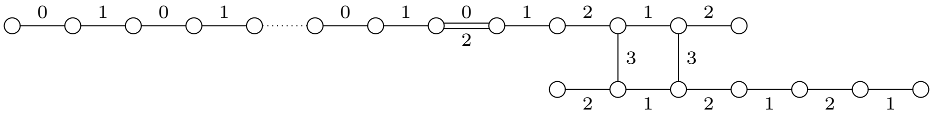
As in the case of odd ranks, from these examples we construct examples of the same rank but for groups of degree $n + 4k$ where k is an integer, by adding a sequence of alternating 0- and 1-edges of length $4k$ between the 1-edge and the second 2-edge (counting from the left).

Theorem 4.11. *If n and r are integers such that $n \equiv 3 \mod 4$, $n \geq 19$ and $8 \leq r < (n-1)/2-1$, r even, then the group A_n admits a string C-group representation of rank r, with Schläfli type $\{n-2(r-1), 12, 6, 3^{r-8}, 6, 6, 6, 4\}$, and with the following CPR graph.*



There are two ways to prove this theorem, either by a proof similar to that of Theorem 4.9 or by a proof similar to that of Theorem 4.10. We leave the details to the interested reader.

Theorem 4.12. *If $n \equiv 3 \mod 4$ with $n \geq 15$, then the group A_n admits a string C-group representation of rank 4, with Schläfli type $\{10, 7, 4\}$ for $n = 15$ and $\{2(n-10), 14, 4\}$ for $n > 15$, with the following CPR-graph.*

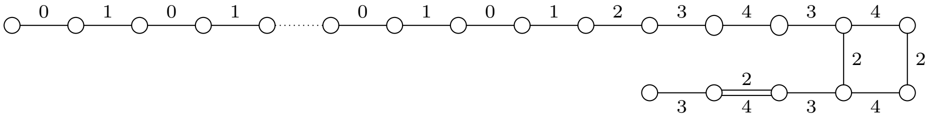


Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. The group G_0 is isomorphic to $2^6 : A_7 : C_2$ for $n = 15$ and $2^6 : A_7 : C_2 \times C_2$ for $n \geq 19$, no matter how big n is. It can easily be checked with Magma that Γ_0 is a string C-group representation for $n = 15$ and $n = 19$ and since adding more points to the graph will not change the structure of G_0, we can conclude that Γ_0 is a string C-group representation for every $n \geq 15$. The group G_3 acts as S_{n-7} on the vertices of the top of the graph and acts as D_7 on the remaining vertices, and is a subgroup of $(A_{n-7} \times D_7)^+$. We can thus conclude that G_3 is $A_{n-7} \times D_7$. The group $G_{0,3}$ is isomorphic to D_7 for $n = 15$ and $C_2 \times D_7$ when $n \geq 19$ (as there are extra 1-edges in the graph). The group $G_{2,3}$ is isomorphic to $D_{(n-10)}$. It is obvious from the permutation representation graph that $G_{0,3} \cap G_{2,3}$ is isomorphic to C_2. Hence, by Proposition 2.1, the sggi Γ_3 is a string C-group representation. Now, the intersection $G_0 \cap G_3 = G_{0,3}$ need only to be checked in the cases $n \in \{15, 19\}$, which can be done with Magma. Hence, again, by Proposition 2.1, we see that Γ is a string C-group representation.

It remains to show that G is isomorphic to A_n. The structure of G_3 shows that the action of G_3 on the $(n-7)$ vertices at the top of the graph is A_{n-7}. Hence there exists a cycle of order 3 in G_0 acting on those vertices. This cycle necessarily fixes the 7 other vertices, so it is a cycle of G. Moreover, that action is $(n-9)$-transitive on the top vertices. Hence the stabilizer, in G, of the leftmost vertex of the graph must be transitive on the remaining vertices and G is 2-transitive, therefore primitive. Then, by Theorem 3.4, we can conclude that $G \geq A_n$. Since all generators of G are even permutations, we conclude that G is isomorphic to A_n.

The Schläfli type follows immediately from the permutation representation graph. □

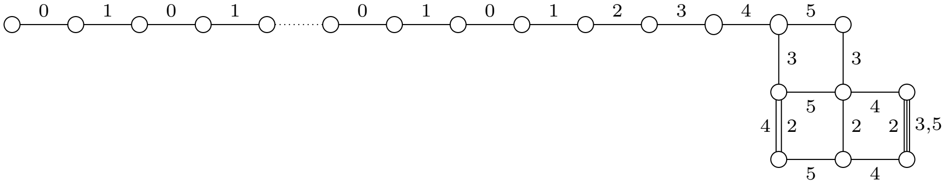
Theorem 4.13. *If $n \equiv 3 \mod 4$ with $n \geq 15$, then the group A_n admits a string C-group representation of rank 5, with Schläfli type $\{n-10, 6, 6, 5\}$, with the following CPR-graph.*



Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. The group G_0 is isomorphic to S_{12} no matter how large n is. One can easily check with Magma that the permutation representation graph corresponding to Γ_0 is a CPR graph. The group $G_{0,4}$ is isomorphic to $2^3 : S_3 \times S_3$ no matter how large n is. $G_{3,4}$ is isomorphic to S_{n-9} by Theorem 3.4, as it contains a cycle of length 3, namely $(\rho_1\rho_2)^2$ and is obviously 2-transitive on $n-9$ vertices. Moreover, by [13, Theorem 4.1], $\Gamma_{3,4}$ is a string C-group representation as it is generated by three involutions, two of which commute. The group $G_{0,3,4}$ is isomorphic to D_6.

Looking at the respective orbits of $G_{0,4}$ and $G_{3,4}$ we can conclude that $G_{0,4} \cap G_{3,4} = G_{034}$ and therefore Γ_4 is a string C-group representation. Moreover, one can check that the group G_4 is isomorphic to $A_{n-8} \times C_2 : S_3$ but this is not needed to finish the proof. Now, it is easy to check with MAGMA that $G_0 \cap G_4 = G_{0,4}$ for $n = 15$ and this intersection does not depend on the degree of G. Therefore, by Proposition 2.1, we may conclude that Γ is a string C-group representation with the given permutation representation graph. A similar argument as in the proof of Theorem 4.12 shows that G is isomorphic to A_n. The Schläfli type follows immediately from the permutation representation graph. □

Theorem 4.14. *If* $n \equiv 3 \mod 4$ *with* $n \geq 15$*, then the group* A_n *admits a string C-group representation of rank* 6*, with Schläfli type* $\{n-10, 6, 3, 5, 3\}$*, with the following CPR-graph.*

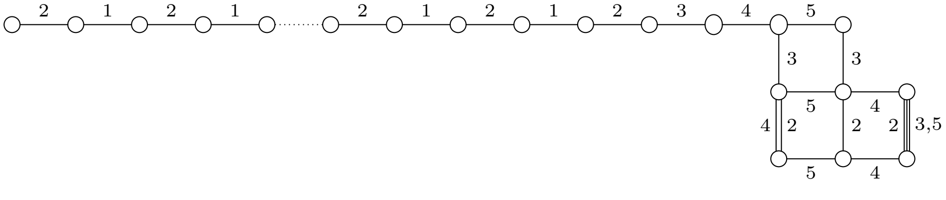


Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. The group G_0 is isomorphic to S_{12} no matter how big n is. One can easily check with MAGMA that the permutation representation graph corresponding to Γ_0 is a CPR graph. We have $G_{0,5} \cong S_7 \times A_5$ no matter how big n is. Here $G_{3,4,5} \cong S_{n-9}$ as proven in the previous theorem (for G_{34} in the previous theorem is the same group as $G_{3,4,5}$ here). Similarly, we have $G_{0,4,5} \cong 2^2 : S_3 \times S_3$. As $G_{3,4,5} \cap G_{0,4,5} = G_{0,3,4,5}$ independently on how big n is, we can conclude by Proposition 2.1 that $\Gamma_{4,5}$ is a string C-group representation. Similarly, as $G_{0,5} \cap G_{4,5} = G_{0,4,5}$ no matter how big n is, we can conclude by Proposition 2.1 that Γ_5 is a string C-group representation. Finally, as $G_0 \cap G_5 = G_{0,5}$ no matter how big n is, we conclude that Γ is a string C-group representation.

It remains to show that G is isomorphic to A_n. Similar arguments as in the proof of the previous two theorems lead to that conclusion. The Schläfli type follows immediately from the permutation representation graph. □

Observe that this last family of string C-group representations of rank 6 gives, using the same general construction we used in Theorems 4.2 and 4.7, a family of string C-groups of rank 5 with Schläfli type $\{n-10, 6, 5, 3\}$.

Theorem 4.15. *If* $n \equiv 3 \mod 4$ *with* $n \geq 15$*, then the group* A_n *admits a string C-group representation of rank* 5*, with Schläfli type* $\{n-9, 6, 5, 3\}$*, with the following CPR-graph.*



We leave the proof of this last theorem to the interested reader as it is very similar to the previous proofs.

5. Concluding remarks

Mark Mixer mentioned a similar result in 2015 at the AMS Fall Eastern Sectional Meeting in Rutgers (talk 1115-20-283).

The techniques we developed in this paper inspired Brooksbank and the second author to develop a general rank reduction technique, now available in [4].

6. Acknowledgements

The authors thank Mark Mixer for observing that there was a mistake somewhere in the case $n \equiv 3 \mod 4$ in a previous version of this paper. They also thank two anonymous referees for numerous comments that improved a previous version of this paper. This research was supported by the Portuguese Foundation for Science and Technology (FCT- Fundação para a Ciência e a Tecnologia), through CIDMA - Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2019 (CIDMA).

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