Application of the hypercomplex fractional integro-differential operators to the fractional Stokes equation∗

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Abstract

We present a generalization of several results of the classical continuous Clifford function theory to the context of fractional Clifford analysis. The aim of this paper is to show how the fractional integro-differential hypercomplex operator calculus can be applied to a concrete fractional Stokes problem in arbitrary dimensions which has been attracting recent interest (cf. [1,6]).

1 Basics on fractional calculus and special functions

For $a, b \in \mathbb{R}$ with $a < b$ and $\alpha > 0$, the left and right Riemann-Liouville fractional integrals $I_{a+}^{\alpha}$ and $I_{b-}^{\alpha}$ of order $\alpha$ are defined by (see [5])

$$\left(I_{a+}^{\alpha}f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{f(t)(x-t)^{1-\alpha}}{t^{1-\alpha}} dt, \quad x > a,$$

$$\left(I_{b-}^{\alpha}f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} \frac{f(t)(t-x)^{1-\alpha}}{(t-x)^{1-\alpha}} dt, \quad x < b. \quad (1)$$

By $RLD_{a+}^{\alpha}$ and $RLD_{b-}^{\alpha}$, we denote the left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$ on $[a, b] \subset \mathbb{R}$ (see [5]):

$$\left(RLD_{a+}^{\alpha}f\right)(x) = \left(D^{m}I_{a+}^{m-\alpha}f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^{m}}{dx^{m}} \int_{a}^{x} \frac{f(t)(x-t)^{\alpha-m+1}}{(x-t)^{\alpha}} dt, \quad x > a \quad (2)$$

$$\left(RLD_{b-}^{\alpha}f\right)(x) = (-1)^{m} \left(D^{m}I_{b-}^{m-\alpha}f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^{m}}{dx^{m}} \int_{x}^{b} \frac{f(t)(t-x)^{\alpha-m+1}}{(t-x)^{\alpha}} dt, \quad x < b. \quad (3)$$

Here, \( m = [\alpha] + 1 \) and \([\alpha]\) means the integer part of \( \alpha \). The symbols \( CD_{a+}^\alpha \) and \( CD_{b-}^\alpha \) denote the left (resp. right) Caputo fractional derivative of order \( \alpha > 0 \):

\[
( CD_{a+}^\alpha f)(x) = \left(I_{a+}^{\alpha - m}D^m f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f(\tau)(t-x)^{m-\alpha-1}}{(x-\tau)^{m-\alpha}dt}, \quad x > a
\]

\[
( CD_{b-}^\alpha f)(x) = (-1)^m \left(I_{b-}^{\alpha - m}D^m f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b \frac{f(\tau)(t-x)^{m-\alpha-1}}{(t-\tau)^{m-\alpha}dt}, \quad x < b.
\]

We denote by \( I_{a+}^\alpha (L_i) \) the class of functions \( f \) that are represented by the fractional integral \( \int_0^x \) of a summable function, that is \( f = I_{a+}^\alpha \varphi \), with \( \varphi \in L_i(a,b) \). The space \( AC^m([a,b]) \) contains all functions that are continuously differentiable over \([a,b] \) up to the order \( m-1 \) and \( f^{(m-1)} \) is supposed to be absolutely continuous over \([a,b] \).

To explicitly describe the integral kernels that are used in the sequel we need to introduce the two-parameter Mittag-Leffler function \( E_{\mu,\nu}(z) \) (cf [3]) as \( E_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k+\nu)} \), \( \mu > 0, \nu \in \mathbb{R}, z \in \mathbb{C} \). Let us now turn to the treatment of the higher dimensional setting. We consider bounded open rectangular domains in \( \mathbb{R}^n \) of the form \( \Omega = \prod_{i=1}^n [a_i, b_i] \). Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \), with \( \alpha_i \in [0,1], i = 1, \ldots, n \). The \( n \)-parameter fractional Laplace operators \( RL\Delta_{\alpha+}^\alpha \) and \( C\Delta_{\alpha+}^\alpha \) are going to be used in the sequel to treat the fractional Stokes problem. For \( \alpha = 1 \) the partial derivatives \( \frac{\partial}{\partial x_i} \) and \( \frac{\partial}{\partial x_i}^{\alpha_i} \) are the left Riemann-Liouville and Caputo fractional derivatives (2) and (3) of orders \( 1+\alpha_i \) and \( \frac{1+\alpha_i}{2} \), with respect to the variable \( x_i \in [a_i, b_i] \). Under certain conditions we have that \( RL\Delta_{\alpha+}^\alpha = -RLD_{a+}^\alpha RL\Delta_{a+}^\alpha \) and \( C\Delta_{\alpha+}^\alpha = -C\Delta_{a+}^\alpha C\Delta_{a+}^\alpha \) (see [2]). Due to the nature of the eigenfunctions and the fundamental solution of these operators we additionally need to consider the variable \( \tilde{x} = (x_2, \ldots, x_n) \in \tilde{\Omega} = \prod_{i=2}^n [a_i, b_i] \), and the fractional Laplace and Dirac operators acting on \( \tilde{x} \) defined by:

\[
RL\Delta_{\alpha+}^\alpha = \sum_{i=2}^{n} RL D_{\alpha+}^\alpha, \quad C\Delta_{\alpha+}^\alpha = \sum_{i=2}^{n} C\Delta_{\alpha+}^\alpha, \quad RL\Delta_{a+}^\alpha = \sum_{i=2}^{n} RL D_{\alpha+}^\alpha, \quad C\Delta_{a+}^\alpha = \sum_{i=2}^{n} C\Delta_{a+}^\alpha.
\]

Next recalling from [2] we know that a family of fundamental solutions of the fractional Dirac operator \( C\Delta_{a+}^\alpha \) can be represented in the way \( C\Delta_{a+}^\alpha(x) = \sum_{i=1}^n \epsilon_i (C\Delta_{a+}^\alpha, i)(x) \), where

\[
(C\Delta_{a+}^\alpha)_{1}(x) = \left( x_1 - a_1 \right)^{1+\alpha_i} E_{1+\alpha_i, 1}^{1-\alpha_i} \left( -(x_1 - a_1)^{1+\alpha_i} C\Delta_{a+}^\alpha \right) g_0(\tilde{x}) + \left( x_1 - a_1 \right)^{1-\alpha_i} E_{1+\alpha_i, 2}^{1+\alpha_i} \left(-(x_1 - a_1)^{1+\alpha_i} C\Delta_{a+}^\alpha \right) g_1(\tilde{x}),
\]

and for \( i = 2, \ldots, n \)

\[
(C\Delta_{a+}^\alpha)_{i}(x) = \left( E_{1+\alpha_i, 1} \left( -(x_1 - a_1)^{1+\alpha_i} C\Delta_{a+}^\alpha \right) a_i^{1+\alpha_i} \right) g_0(\tilde{x}) + \left( E_{1+\alpha_i, 2} \left( -(x_1 - a_1)^{1+\alpha_i} C\Delta_{a+}^\alpha \right) a_i^{1+\alpha_i} \right) g_1(\tilde{x}),
\]

with \( g_0(\tilde{x}) = v(a_1, \tilde{x}) \) and \( g_1(\tilde{x}) = v_{' x_1}(a_1, \tilde{x}) \). The functions \( v \) and \( v_{' x_1} \) are defined in Corollary 3.5 of [2].

2 Fractional Hypercomplex Integral Operators

In this section we recall the definitions and the main properties of the fractional versions of the Teodorescu and Cauchy-Bitsadze operators that are going to be used in the sequel to treat the fractional Stokes problem. For all the detailed proofs and calculations we refer to our paper [2]. First we recall the following fractional Stokes formula
Theorem 2.1 Let \( f, g \in C_{\ell, n}(\Omega) \cap AC^1(\Omega) \cap AC(\Omega) \) Then we have
\[
\int_{\Omega} \left[ -(f \ C D_{\alpha}^0) (x) g(x) + f(x) \left( R L D_{\alpha}^0 \right) (x) g(x) \right] \, dx = \int_{\partial \Omega} f(x) \, d\sigma(x) \, (I_{\alpha}^0 g)(x),
\]
where \( d\sigma(x) = n(x) \, d\Omega \), with \( n(x) \) being the outward pointing unit normal vector at \( x \in \partial \Omega \), where \( d\Omega \) is the classical surface element, and where \( dx \) represents the \( n \)-dimensional volume element.

Replacing \( f \) by \( C G_{\alpha}^0 (x - y) \) in (10) we now may obtain the following fractional Borel-Pompeiu formula (a detailed proof is presented in [2]).

Theorem 2.2 Let \( g \in C_{\ell, n}(\Omega) \cap AC^1(\Omega) \cap AC(\Omega) \). Then the following fractional Borel-Pompeiu formula holds
\[
- \int_{\Omega} C G_{\alpha}^0 (x + a - y) \left( R L D_{\alpha}^0 ight) (y) g(y) \, dy + \int_{\partial \Omega} C G_{\alpha}^0 (x + a - y) \, d\sigma(y) \, (I_{\alpha}^0 g)(y) = g(x).
\]

From (11) we may introduce the following definition.

Definition 2.3 Let \( g \in AC^1(\Omega) \). Then the linear integral operators
\[
(T^\alpha g)(x) = - \int_{\Omega} C G_{\alpha}^0 (x + a - y) g(y) \, dy, \quad (F^\alpha g)(x) = \int_{\partial \Omega} C G_{\alpha}^0 (x + a - y) \, d\sigma(y) \, (I_{\alpha}^0 g)(y)
\]
are called the fractional Teodorescu and Cauchy- Bitsadze operator, respectively.

The previous definition allows us to rewrite (11) in the alternative form \( (T^\alpha R L D_{\alpha}^0 g)(x) + (F^\alpha g)(x) = g(x) \), with \( x \in \Omega \). For the regularity and mapping properties of \( (12) \) we refer to [2]. Again, in [2] we proved the following result:

Theorem 2.4 The fractional operator \( T^\alpha \) is the right inverse of \( C D_{\alpha}^0 \), i.e., for \( g \in L_\alpha(\Omega) \), with \( \alpha \in \left[ 1, \frac{2}{1 - \alpha} \right] \) and \( \alpha^* = \min_{1 \leq i \leq n} \{ \alpha_i \} \), we have \( (C D_{\alpha}^0 \, T^\alpha g)(x) = g(x) \).

All these tools in hand allow us to obtain the following Hodge-type decomposition which is our key tool to treat boundary value problems related to the fractional Dirac operator, such as presented with a small example in the next section (see [2] for a detailed proof):

Theorem 2.5 Let \( q = \frac{2p}{2 - (1 - \alpha^*) p} \), \( p \in \left[ 1, \frac{2}{1 - \alpha^*} \right] \), and \( \alpha^* = \min_{1 \leq i \leq n} \{ \alpha_i \} \). The space \( L_\alpha(\Omega) \) admits the following direct decomposition
\[
L_\alpha(\Omega) = L_\alpha(\Omega) \cap \ker \left( C D_{\alpha}^0 \right) \oplus C D_{\alpha}^0 \left( W_{\alpha+}^{0, p}(\Omega) \right),
\]
where \( W_{\alpha+}^{0, p}(\Omega) \) is the space of functions \( g \in W_{\alpha+}^{0, p}(\Omega) \) such that \( \text{tr}(g) = 0 \). Moreover, we can define the following projectors
\[
P^\alpha : L_\alpha(\Omega) \to L_\alpha(\Omega) \cap \ker \left( C D_{\alpha}^0 \right), \quad Q^\alpha : L_\alpha(\Omega) \to C D_{\alpha}^0 \left( W_{\alpha+}^{0, p}(\Omega) \right).
\]

In the previous theorem the fractional Sobolev space \( W_{\alpha+}^{0, p}(\Omega) \) has the following norm:
\[
\| f \|_{W_{\alpha+}^{0, p}(\Omega)} := \| f \|_{L_\alpha(\Omega)} + \sum_{k=1}^{n} \left\| \partial_{x_k}^{1 + \alpha_k} f \right\|_{L_p(\Omega)},
\]
where \( \cdot \) is the usual \( L_p \)-norm in \( \Omega \), and \( \alpha = (\alpha_1, \ldots, \alpha_n) \), with \( \alpha_k \in [0, 1], k = 1, \ldots, n \).

Remark 2.6 We would like to remark that our results coincide with the classical ones presented in [4] when considering the limit case of \( \alpha = (1, \ldots, 1) \). However, we can notice differences in the fractional setting, for instance in the expression of the fundamental solution and in the function spaces considered.
3 A fractional Stokes problem

Recently fractional versions of the Stokes problem have attracted a fast growing interest (see for instance [6]). The application of the version of the Laplacian allows us to model sub-diffusion problems of (in our case incompressible) flows. The following system describes the simplest model of Stokes equation with sub (resp. super) dissipation. In the Riemann-Liouville case (the Caputo case is treated analogously) it has the form

\[-\text{RL}^{\alpha} \Delta_a^\alpha u + \text{grad}^\alpha p = F \quad \text{in } \Omega\]
\[\text{div}^\alpha u = 0 \quad \text{in } \Omega\]
\[u = 0 \quad \text{on } \partial \Omega\]

Here again we suppose that \( \Omega \) is a rectangular domain, \( F \) is given, \( p \) is the unknown pressure of the flow and \( u \) its unknown velocity. As in the continuous case treated in [4], the hypercomplex fractional calculus that we proposed in the previous section, now allows us to set up closed solution formulas for \( u \) and \( p \). To proceed in this direction, remember that following [2] the fractional Laplacian can be split in the form

\[-\text{RL}^{\alpha} \Delta_a^\alpha a + u + \text{grad}^\alpha p = F \quad \text{in } \Omega\]

If we now apply the fractional \( T^\alpha \)-operator from the left to this equation we get

\[T^\alpha \text{RL}^{\alpha} u - F^\alpha \text{RL}^{\alpha} u + p - F^\alpha p = T^\alpha F\]

Application of the projector \( Q^\alpha \) arising the fractional version of the Hodge decomposition then leads to

\[Q^\alpha \text{RL}^{\alpha} u - Q^\alpha F^\alpha \text{RL}^{\alpha} u + Q^\alpha p - Q^\alpha F^\alpha p = T^\alpha F\]

Since \( F^\alpha \text{RL}^{\alpha} u \) and \( F^\alpha p \) are in the kernel of \( \text{RL}^{\alpha} \), we get \( Q^\alpha F^\alpha \text{RL}^{\alpha} u = 0 \) and \( Q^\alpha F^\alpha p = 0 \) so that our original equation simplifies to

\[Q^\alpha \text{RL}^{\alpha} u + Q^\alpha p = Q^\alpha T^\alpha F\]

Next we apply once more \( T^\alpha \) to the left of the equation and use that \( Q^\alpha \text{RL}^{\alpha} = \text{RL}^{\alpha} \), so that the latter equation is equivalent to

\[T^\alpha \text{RL}^{\alpha} u + T^\alpha Q^\alpha p = T^\alpha Q^\alpha T^\alpha F\]

which in turn equals

\[u - F^\alpha u + T^\alpha Q^\alpha p = T^\alpha Q^\alpha T^\alpha F,\]

so that we finally get the following formula for the velocity of the flow

\[u = T^\alpha Q^\alpha T^\alpha F - T^\alpha Q^\alpha p\]

The pressure then can be determined by the equation

\[\text{Sc}(Q^\alpha p) = \text{Sc}(T^\alpha Q^\alpha T^\alpha F)\]

resulting from the second equation.
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