On the topological charge of $SO(2)$ gauged Skyrmions in $2+1$ and $3+1$ dimensions

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A B S T R A C T

The question of the dependence of the topological charge q of a gauged Skyrmion, on the gauge field, is studied quantitatively. Two examples, both gauged with $SO(2)$ are studied and contrasted: **i)** The $O(3)$ model in $2+1$ dimensions, and **ii)** The $O(4)$ model in $3+1$ dimensions. In case **i)**, where the (usual) Chern–Simons (CS) term is present, the value of q changes sign, going through zero. This evolution is tracked by a parameter characterising the solutions in the given theory. In case **ii)**, in which dimensions no CS density is available, the evolution of q is not observed.

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1. Introduction

The topological charge q of gauged Skyrmions presents peculiarities that are absent in gauged Higgs solitons. While in the latter case this quantity always equals the winding number of the Higgs field, in the case of Skyrmions the value of q in general may depart from the winding number, or the "baryon number".

The winding number of a Skyrme system is a topological charge [1], which provides the static energy with a lower bound. What we refer to as the topological charge of a gauged Skyrme system, is merely the number which gives the lower bound on the static energy. This contrasts with (gauged) Higgs systems where the topological charge encoded in the Higgs (and gauge) field, also supplies the energy lower bound [2,3]. While in Higgs models this topological charge is the magnetic monopole charge [4], by contrast the magnetic flux of gauged Skyrme models in space dimensions $D \geq 3$ vanishes [5], except in $D = 2$,[1] and that only in the presence of Chern–Simons dynamics.

The topological charge for the $SO(2)$ gauged $O(4)$ (Skyrme) model in $3+1$ dimensions is defined in Ref. [7] and in [8,9], and, for the $SO(2)$ gauged $O(3)$ (planar Skyrme) model in $2+1$ dimen-

sions in Ref. [10]. The topological charges for the $SO(D)$ gauged $O(D+1)$ sigma model on $\mathbb{R}^D$ ($D = 2, 3, 4$) were defined in [11], and the soliton of the $SO(3)$ gauged $O(4)$ sigma model on $\mathbb{R}^3$ was first reported in [5]. More detailed studies of the latter solitons were given in [12], while solitons of the $SO(4)$ gauged $O(5)$ sigma model on $\mathbb{R}^4$ were reported in [13]. The most comprehensive and recent definitions of the generic topological charges of gauged Skyrmions are given in Appendix **B** of Ref. [14].

With the exception of the work in Ref. [7], where attention was focused on the decay of the baryon number,[2] the works of [8–11, 5,13] are concerned with the construction of topologically stable solitons. Here, we are concerned with the $SO(2)$ gauged $O(3)$ and $O(4)$ sigma models in $2+1$ and $3+1$ dimensions, respectively. But unlike in the works of Refs. [10,8,9], our focus here will be the evolution of the topological charge, *e.g.*, its possible decay.[3]

In odd dimensional spacetimes, where the Chern–Simons (CS) term is defined, one finds that the effect of this CS dynamics results in the mass/energy of the static solitons both to increase and to decrease with increasing global charge (electric charge and

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[1] See for example [6].

[2] Well known works in this direction are, *e.g.*, [16,17], employ instead $SU(2)$ gauging that does not result in topologically stable solitons.
[3] $SO(2)$ gauged Skyrmions in $2+1$ dimensions have been studied extensively (see Refs. [10,6,18–20] and more recently in [21,22]). The $SO(2)$ gauged Skyrmions in $3+1$ dimensions have been quantitatively studied mainly in Refs. [8,9] (see also the recent analytical results in Ref. [23]).

spin). This was clearly demonstrated in Section **4** of Ref. [6] in $2 + 1$ dimensions, where the effects of Chern–Simons dynamics were studied also in other $2 + 1$ dimensional models. This evolution of the mass/energy of the static solitons was tracked by a parameter characterising the solution in the *given* theory, which is strictly due to the CS dynamics. In the present work we show that this same dynamical effect results in the evolution of the value of the topological charge q, changing sign and passing through zero, in the given theory. Such a solution cannot be found in the absence of the CS term [6]. This is the main result of the first part of the present work, namely the study of the $SO(2)$ gauged $O(3)$ model, with emphasis on the issue of topological charge q. (A detailed study of generic $SO(2)$ gauged Skyrmions is given in [15].)

In even dimensional spacetimes however, no CS density is defined. There is nonetheless a class of Skyrme–CS (SCS) densities satisfying the properties of the usual CS density in all dimensions, including even dimensions. These are introduced in Ref. [14]. It happens that in the present example this quantity vanishes under the imposition of appropriate symmetries, and in the absence of any Chern–Simons type dynamics we are limited to studying the model considered in Refs. [8,9]. There, it was found that the mass/energy of the soliton of the gauged system increases monotonically with increasing electric (global) charge. Since no parameter characterising the solutions is available, the coupling strength of the Maxwell field, λ_0 in (11) below, was varied in [8]. This clearly changes the theory being considered, but since in any given theory the value of both the energy and the electric charge Q_e depend on λ_0, this enabled [8] the tracking of the energy with increasing Q_e. Here by contrast, we consider the dependence of the topological charge on λ_0 and show that q remains constant.

This paper is organised as follows. In Section **2** below we present our quantitative results on the $SO(2)$ gauged Skyrmion in $2 + 1$ dimensions, and in Section **3** our results on the $SO(2)$ gauged Skyrmion in $3 + 1$ dimensions. In Section **4** we summarise our results and point out to further developments.

2. $SO(2)$ gauged Skyrmions in $2 + 1$ dimensions

Ever since Schroers' construction of solitons [10] of the $U(1)$ gauged planar Skyrme model, there has been considerable interest [4] in this area. The dynamics of the Abelian field in Ref. [10] was described by the Maxwell density, subsequently the latter was replaced by the Chern–Simons (CS) density [18–20]. More recently, the effect of Chern–Simons dynamics on the combined Maxwell–CS $O(3)$ sigma model was studied in Ref. [6]. What was found there (in Section **4**) is that the mass/energy of the soliton decreased with increasing global charges – electric charge and angular momentum – in some ranges of the parameters. This effect is absent, in the absence of the CS term in the Lagrangian, and is strictly a result of the Chern–Simons dynamics.

In the present note, we pursue further consequences of the presence of the Chern–Simons term. The new effect in question is the evolution of the topological charge due to the gauging of the $O(3)$ sigma model. While this is not strictly a dynamical effect, it is again predicated on the presence of a CS term in the Lagrangian. The precise mechanism of this is the role that the CS term plays in establishing the topological lower bound (Belavin inequalities). This, and other details will be exposed in a future extended work.

The Skyrme model on $\mathbb{R}^2$ is described by the scalar $\phi^a = (\phi^\alpha, \phi^3)$, $|\phi^a|^2 = 1$, and the $SO(2)$ gauging prescription is[5]

$$D_\mu \phi^\alpha = \partial_\mu \phi^\alpha + A_\mu (\varepsilon \phi)^\alpha , \quad (\varepsilon \phi)^\alpha = \varepsilon^{\alpha\beta} \phi^\beta , \tag{1}$$

$$D_\mu \phi^3 = \partial_\mu \phi^3 , \quad \mu = i, 0, \ i = 1, 2, \tag{2}$$

A_μ being the $SO(2)$ connection.

The topological charge density for the $SO(2)$ gauged Skyrme model is [11], [14], [15]

$$\varrho = \varrho_0 + 2\varepsilon_{ij}\partial_i[(\phi^3 - v)A_j], \tag{3}$$

independently of whether the dynamics is Maxwell or CS. In (3), ϱ_0 is the winding number density whose volume integral is the "baryon number", A_i, $i = 1, 2$, are the magnetic components of the Abelian connection, and v is a real constant.

What is crucial here is that in the Maxwell gauged case the Belavin inequalities, which must be satisfied by the requirement of topological stability, force the constant v to be equal to one, $v = 1$, and as a consequence of the boundary condition $\phi^3(\infty) = 1$, the contribution of the Abelian field to the integral of ϱ disappears. What remains is the integral of ϱ_0, namely, the "baryon number". The situation is starkly different when the dynamics of the Abelian field is described by the Chern–Simons term. In that case, it follows from the Bogomol'nyi analysis, that the constant v in (3) can take any value, including $v \neq 1$, as a consequence of which the topological charge will depart from the "baryon number", quantitatively. It is our aim here to track the evolution of the topological charge, due to this mechanism.

To illustrate this mechanism briefly, consider the choice of potential $V \equiv V[\phi^a]$ in the Lagrangian. In the case of Maxwell dynamics, this potential is fixed by the constraints of the Belavin inequalities to be

$$V_{\mathrm{M}} = \frac{1}{2}\lambda(1 - \phi^3)^2 , \tag{4}$$

consistently with the boundary value $\phi^3(\infty) = 1$, when $V_{\mathrm{M}}[\phi^3(\infty)] = 0$.

The corresponding Bogomol'nyi analysis in the case of Chern–Simons dynamics, which was presented in [20] and in Appendix C of [15], leads to the potential

$$V_{\mathrm{CS}} = \frac{1}{32}\lambda\left(\frac{\eta}{\kappa}\right)^2 |\phi^\alpha|^2 (\phi^3 - v)^2 , \quad \lambda > 0 . \tag{5}$$

(In (4)–(5), λ, η and κ are real constants.) The striking property of the potential (5) is that the (finite energy) condition $V_{\mathrm{CS}}[\phi^a(\infty)] = 0$ can be satisfied for any value of the real constant v, since at spatial infinity

$$|\phi^3|^2 \to 1 \quad \Rightarrow \quad |\phi^\alpha|^2 \to 0 ; \quad \alpha = 1, 2. \tag{6}$$

This is precisely the situation where the topological charge will depart from the winding number n, namely from the "baryon number". In practice, $v = 0$ is the most convenient choice.

The Chern–Simons–Skyrme (CS–S) model studied in Ref. [20], can be augmented by the Maxwell term without invalidating the topological inequalities, to yield the Maxwell–CS–S model described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \kappa\varepsilon^{\lambda\mu\nu}A_\lambda F_{\mu\nu} - \frac{1}{8}\tau|D_{[\mu}\phi^a D_{\nu]}\phi^b|^2$$
$$+ \frac{1}{2}\eta^2|D_\mu\phi^a|^2 - \eta^4 V_{\mathrm{CS}}[\phi^a], \tag{7}$$

considered here.

The Lagrangian (7) is not the minimal one that follows from the Belavin inequalities. It has been further augmented by the quartic kinetic Skyrme term with coupling τ. The reason for this is

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[4] See Ref. [6] and citations there.

[5] The prescription for gauging the $O(D + 1)$ Skyrme scalar ϕ^a, $a = 1, 2, \ldots, D+1$, on $\mathbb{R}^D$ with gauge group $SO(N)$, $2 \leq N \leq D$, is summarised most recently in Appendix **B** of Ref. [14].
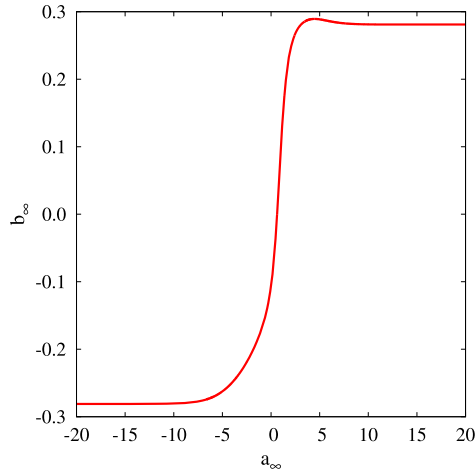
Fig. 1. b_∞ vs. a_∞ for vortices with $n = 1$, $\lambda = 1.6$, $\eta = 1$, $\kappa = 1$, and $\tau = 1$.

our requirement that after gauge decoupling, the system support Skyrmions with winding number n.

To analyse the system (7) numerically, we subject it to azimuthal symmetry

$$\phi^\alpha = \sin f(r)\, n^\alpha, \quad \phi^3 = \cos f(r), \quad n^\alpha = \begin{pmatrix} \cos n\theta \\ \sin n\theta \end{pmatrix}, \qquad (8)$$

$$A_i = \left(\frac{a(r) - n}{r} \right)(\varepsilon \hat{x})_i, \quad A_0 = b(r), \qquad (9)$$

where θ is the azimuthal angle and n is the winding number of the Skyrme scalar.

We have analysed the reduced one dimensional Lagrangian descended from (7) subject to (8), (9), numerically. Also, subject to the above Ansatz, the volume integral of ϱ (as given by (3)), namely the topological charge, can be readily evaluated as[6]

$$q = -\frac{1}{8\pi} \int \varrho\, d^2 x = \frac{1}{2}(n + a_\infty), \qquad (10)$$

where $a_\infty = a(\infty)$.

We see from relation (10) that the value of the topological charge q depends on the sign and magnitude of a_∞. In particular, when a_∞ is negative ($a_\infty < 0$) the value of q will be diminished, resulting in the annihilation of q when $a_\infty = -n$. This mechanism enables the tracking of the evolution of the topological charge by means of the parameter a_∞, characterising the solutions in a *given* theory/model.

The spectrum of the asymptotic value a_∞ of magnetic function $a(r)$ results from the spectrum of the asymptotic value b_∞ of the electric function $b(r)$, as seen in Fig. 1. This mechanism was encountered in Ref. [6], and technically it results from the fact that the equation of motion of $b(r)$ can be solved for arbitrary values of the asymptotic quantity b_∞.

This is a property of theories featuring a Chern–Simons (CS) density, which combines both the magnetic and the electric fields. In $2 + 1$ dimensions, in particular, it is well known that the electric flux is actually proportional to the *vortex number*, in other words the (topological) magnetic flux. Thus the mechanism in question is strictly predicated by the presence of a CS term, $\kappa \neq 0$.

We have constructed such solutions with $\kappa \neq 0$, the relation of the asymptotic value of the electric function, b_∞, to the magnetic

counterpart, a_∞, being displayed in Fig. 1. We know already from our results in Ref. [6] that due to this effect, the dependence of the mass/energy E on the global charges is non-standard, namely the dependence on the electric charge Q_e and the angular momentum J displayed respectively (left and right panels) in Fig. 2.

Here, we analyse further the relation of the topological charge q and the energy E. In Fig. 3 (left panel) we represent the energy per unit winding number n versus the topological charge for three values of n for the same parameters as in Fig. 1. For small values of the coupling strength τ of the (quartic) Skyrme kinetic term, the minimum of the energy occurs at values of the topological charge around the winding number. However, when higher values of τ are considered, the minimum of the energy occurs at values clearly different from the winding number. This is shown in Fig. 3 (right panel), where the minimum of the curve for $n = 1$, $\tau = 10$ is located at $q \approx 0.556$. Concerning the stability of these solutions, one would be tempted to state that most stable solution would correspond to that with least energy, which according to Fig. 3 (right panel) does not possess integer topological charge, in general.

From Fig. 3. (right panel), we see that in the absence of the Skyrme term ($\tau = 0$) the energetically favoured configurations are those with topological charge equal to the winding number. But when the Skyrme term is introduced, with increasing values of the coupling strength τ, it appears that the energetically favoured solutions have topological charge progressively smaller than the winding number, the latter being the topological charge of the Skyrmion in the gauge decoupling limit.

3. $SO(2)$ gauged Skyrmions in $3 + 1$ dimensions

The stationary solutions of this system were studied in Refs. [8] and [9]. In [8], it was seen that the mass of the electrically charged Skyrmion was larger than that of the electrically neutral one, while in [9] it was shown that these Skyrmions spin. That the mass of the electrically charged Skyrmion is always larger than the electrically neutral one was seen in [8] by observing that these energies both depend on the Maxwell coupling λ_0 (in (11) below), thus plotting both these energies *versus* λ_0 illustrates this fact, as seen in Fig. 4 (left panel).

That the mass increases monotonically with increasing electric charge seen in [8] is not surprising, since we know from our work in Ref. [6] that to reverse this trend can be achieved only by the introduction of Chern–Simons (CS) dynamics, and, in $3 + 1$ dimensions no (usual) CS density is defined. While there is a class of densities introduced in Ref. [14] which share the properties of Chern–Simons that can be defined in even dimensional spacetimes, the particular one that applies here, defined in terms of the Maxwell field and the $O(6)$ Skyrme scalar, actually vanishes when subjected to axial symmetry. Hence, in the absence of any Chern–Simons dynamics, we are back to the study in Refs. [8,9].

What we have done here, additionally to tracking the increase of mass with increasing electric charge and angular momentum in Refs. [8,9], is to study the effect of the electromagnetic field on the topological charge.[7]

The studied $SO(2)$ gauged Skyrmion system is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}\lambda_0 |F_{\mu\nu}|^2 + \frac{1}{2}\lambda_1 |D_\mu \phi^a|^2$$

$$\quad - \frac{1}{4}\lambda_2 |D_{[\mu}\phi^a D_{\nu]}\phi^b|^2 + \lambda_3 V(\phi^a), \qquad (11)$$

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[6] Subject to the Ansatz (8)–(9), the electric charge Q_e is $Q_e = 8\pi\kappa\,(n - a_\infty)$, while the total mass/energy E and angular momentum J are computed as integrals of the T_{00} and T_{i0} components of the energy momentum tensor [6].

[7] Strictly speaking this is the deformation of the topological charge by the gauge field, that provides a lower bound on the energy of the gauged system.
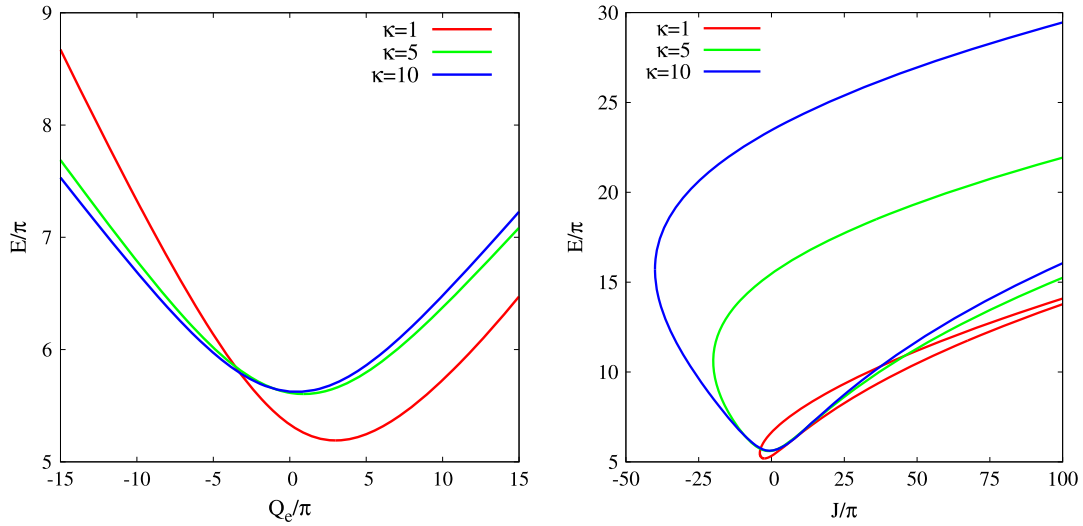
Fig. 2. *Left panel:* Energy E *vs.* electric charge Q_e for vortices with $n = 1$, $\lambda = 1.6$, $\eta = 1$, $\tau = 1$, and several values of κ. *Right panel:* Energy E *vs.* angular momentum J for vortices with $n = 1$, $\lambda = 1.6$, $\eta = 1$, $\tau = 1$, and several values of κ.
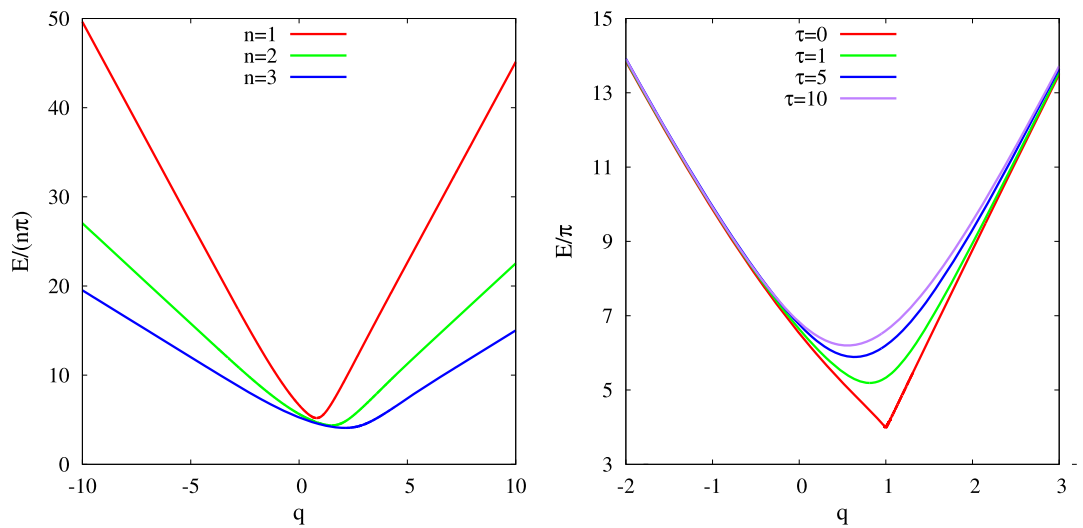

Fig. 3. *Left panel:* Energy E *vs.* topological charge q for vortices with $n = 1, 2, 3$, $\lambda = 1.6$, $\eta = 1$, $\kappa = 1$, and $\tau = 1$. *Right panel:* Same for vortices with $n = 1$, $\lambda = 1.6$, $\eta = 1$, $\kappa = 1$, and several values of τ.
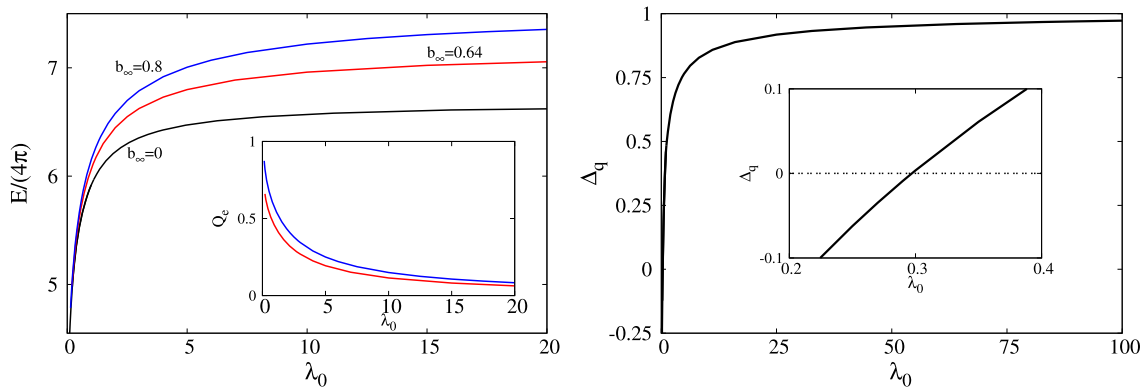

Fig. 4. *Left panel:* The energy E and the electric charge Q_e are shown *vs.* the coupling constant λ_0 for SO(2) gauged Skyrmions in $3 + 1$ dimensions, with $n = 1$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$, and several values of b_∞. *Right panel:* The relative difference $\Delta_q = (q_G - q_W)/(q_G + q_W)$ of the individual contributions to the topological charge $q = q_G + q_W = 1$ is shown *vs.* the coupling constant λ_0 for the same solutions.

where $\phi^a = (\phi^\alpha, \phi^A)$; $\alpha = 1, 2$; $A = 3, 4$, with $|\phi^a|^2 = 1$, Skyrme scalars of the $O(4)$ sigma model, with $V = 1 - \phi^4$ the usual Skyrme potential. λ_i $(i = 0, 1, 2, 3)$, in (11) are real numbers parameterising the coupling constants of the model. Also, the gauging prescriptions for ϕ^a is

$$D_\mu \phi^\alpha = \partial_\mu \phi^\alpha + A_\mu (\varepsilon \phi)^\alpha , \qquad D_\mu \phi^A = \partial_\mu \phi^A ,$$
$$\alpha = 1, 2 , \quad A = 3, 4. \tag{12}$$

In the ungauged case, the topological charge density of the Skyrmions is

$$\varrho_0 = \epsilon_{ijk} \epsilon_{abcd} \partial_i \phi^a \partial_j \phi^b \partial_k \phi^c \phi^d . \tag{13}$$

The natural generalization of this expression in the presence of gauge fields is found by replacing partial derivatives with gauge derivatives,

$$\varrho_G = \epsilon_{ijk} \epsilon_{abcd} D_i \phi^a D_j \phi^b D_k \phi^c \phi^d, \tag{14}$$

which, however, does not lead to a conserved charge.

The conserved, gauge invariant generalization of (13) is the sum of two terms[8]

$$\varrho = \varrho_0 + \partial_i \Omega_i, \ \text{with} \ \Omega_i = 3\epsilon_{ijk} \varepsilon^{AB} A_j \partial_k \phi^A \phi^B, \tag{15}$$

which can be written alternatively in a manifestly gauge invariant form

$$\varrho = \varrho_G + W, \ \text{with} \ W = \frac{3}{2} \epsilon_{ijk} F_{ij} (\varepsilon^{AB} A_j \phi^B D_k \phi^A). \tag{16}$$

As such, the topological charge q consists of the sum of two terms, encoding the contribution of ϱ_G and W in (16)

$$q = -\frac{1}{2\pi^2} \int \varrho \, d^3x = q_G + q_W, \tag{17}$$

with q_G the *bare* charge (as found by integrating directly the density (14)), and q_W a *compensating* contribution (from (16)). Moreover, as seen from (15) the topological charge is still an integer, as long as there are no singularities and the surface terms from Ω_i vanish.

Imposing axial symmetry on the Maxwell connection $A_\mu = (A_i, A_z, A_0)$, we have the Ansatz

$$A_i = -\left(\frac{a+n}{\rho}\right) (\varepsilon \hat{x})_i , \quad A_z = 0 , \quad A_0 = -b, \tag{18}$$

where $a = a(\rho, z)$, $b = b(\rho, z)$, and, $\rho^2 = |x_i|^2$, $i = 1, 2$.

The solutions constructed are typified by the asymptotic values

$$A_i = -\left(\frac{a_\infty + n}{\rho}\right) (\varepsilon \hat{x})_i , \quad A_0 = -b_\infty + \frac{Q}{\sqrt{\rho^2 + z^2}} , \tag{19}$$

where Q is the electric charge and b_∞ is a free constant characterising the given solution (the electrostatic potential). Also, in contrast with the solutions in $2 + 1$ dimensions constructed in the previous Section, there is no freedom in the asymptotic value of the magnetic potential, with $a_\infty = -n$ for all solutions. Again we are using the notation $a_\infty = a(\infty)$ and $b_\infty = b(\infty)$.

The corresponding Ansatz for the $O(4)$ scalar ϕ^a is

$$\phi^\alpha = R \, n^\alpha , \qquad \phi^3 = S , \qquad \phi^4 = T , \tag{20}$$

where R, S and T are functions of ρ and z, subject to the constraint $R^2 + S^2 + T^2 = 1$. Also, n^α is the unit vector in the subplane (x^1, x^2), with vorticity n.

Subject to symmetry (18)–(20), the topological charge density (15) (or (16)) reduces to the two dimensional density

$$\varrho = \frac{1}{\rho} \Big\{ 2nR \left(R \, \partial_{[\rho} S \, \partial_{z]} T + S \, \partial_{[\rho} T \, \partial_{z]} R + T \, \partial_{[\rho} R \, \partial_{z]} S \right)$$
$$- \left(S \, \partial_{[\rho} a \, \partial_{z]} T - T \, \partial_{[\rho} a \, \partial_{z]} S \right) - 2(a+n)\partial_{[\rho} S \, \partial_{z]} T \Big\}, \tag{21}$$

which, as expected, is a *curl*. In principle, the value of the integral of (21) could be different from the integral of ϱ_0, since it encodes a contribution of the gauge potential. However, in contrast with the $2 + 1$ dimensional system endowed with Chern–Simons dynamics,[9] studied in the previous Section, this contribution vanishes for regular configurations, and the integral of (21) is still equal to the baryon number, $q = n$, for *any* value of the varying the coupling constants, in particular for any λ_0. At the same time, the contribution of the gauge invariant terms q_G and q_W to the topological charge is model dependent, as confirmed by our numerical results.[10]

The dependence of the mass/energy E and of the electric charge Q_e of the spinning Skyrmions on λ_0 is displayed in Fig. 4 (left panel). As expected, for fixed coupling constants λ_i and winding number n, E increases with increasing electrostatic potential b_∞ for all values of λ_0. In the gauge decoupling limit $\lambda_0 \to \infty$, these are the uncharged, spinning Skyrmions characterised by b_∞, found in Ref. [26].

The difference between the 'bare' charge q_G and the 'compensating' one q_W is displayed in Fig. 4 (right panel) as a function of λ_0. We see that the influence of the gauge field is to cause this quantity to have positive, zero and negative values in distinct theories. For large λ_0, the contribution of q_W is negligible, with $q_G \to q$. However, q_G decreases with λ_0, with q_W dominating for small λ_0. We note that for very small values of λ_0, when the difference between q_G and q_W is vanishing (or even becomes negative, as seen from Fig. 4 (right panel)), the energy of the Skyrmion (as seen in Fig. 4 (left panel)), is finite.

4. Summary

We have studied two $SO(2)$ gauged Skyrme systems, the $O(3)$ and $O(4)$ sigma models in $2+1$ and $3+1$ dimensions, respectively. In the first case, the system can be endowed with Chern–Simons dynamics, while in the second case, not. The main message here is to highlight the contrast when Chern–Simons dynamics is present or absent.

When Chern–Simons dynamics is present, two related phenomena are observed. **i)** The mass of the soliton can both increase and decrease with increasing global charge (electric or spin), and **ii)** The topological charge can evolve through positive to zero to negative values, for various solutions characterised by a_∞, the asymptotic value of the magnetic field defining the topological charge. This evolution is contingent on the dependence of b_∞, the asymptotic value of the electric field, this relation being dependent on

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[8] This definition is that used earlier in Refs. [7] and [8].

[9] In $2 + 1$ dimensions, a_∞ was related to b_∞, which provided a lever to pass from one solution to another, resulting in the evolution of q, while $a_\infty = -n$ in $3 + 1$ dimensions.

[10] Different from the previous work [8,9], the sigma-model constraint $R^2 + S^2 + T^2 = 1$ is imposed here by using the Lagrange multiplier method, as explained e.g. in [24,25]. The solutions with $b_\infty \neq 0$ are rotating [9], possessing an angular momentum $J = 4\pi\lambda_0 Q_e$. The mass/energy E and the angular momentum J are computed from the components T_{00} and T_{i0}, respectively, of the energy momentum tensor (which includes also the contribution of the Maxwell field).

the intertwining of the electric and magnetic fields in the definition of the Chern–Simons density. This is clearly demonstrated in the $2+1$ dimensional system studied here, for the $SO(2)$ gauged Skyrmion, in a model with both Maxwell and Chern–Simons dynamics, and both with and without a quartic kinetic Skyrme term.

In the particular model studied here, the energetically favoured configurations appear to occur for smaller values of the topological charge, with progressively stronger coupling τ of the Skyrme term. Indeed, when $\tau = 0$ the minimum energy configurations coincide with the winding number n (see Fig. 3), while for very large values of τ the energetically favoured configuration tends to the $n/2$ (see quantitative details in [15]).

In the $3+1$ dimensional system studied, where there is no Chern–Simons dynamics, these phenomena are absent. The mass of the soliton increases monotonically with increasing the electric charge, and the topological charge is fixed. Moreover, changing the theory by varying a coupling constant does not lead to changing values of the topological charge. We suggest that the phenomena of non-standard mass-electric charge/spin[11] and evolving topological charge observed in the $2+1$ dimensional example are absent in the $3+1$ dimensional model here, because of the absence of Chern–Simons dynamics in the latter case.

The challenge is to supply an example of a gauged Skyrme model in $3+1$ dimensions, where a Chern–Simons term is defined. Such examples are considered in Ref. [14]. In the case of $SO(2)$ gauging considered in this work, such a Skyrme–CS term was considered, but it turned out that that term vanished under symmetry imposition. The challenge thus is to consider gauging with a higher (appropriate) gauge group, but this is a subject of another investigation.

Acknowledgements

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[11] This phenomenon is observed also in (gauged) Higgs theories, where the Chern–Simons (CS) dynamics is supplied by Higgs–CS [14] (HCS) terms, reported in Refs. [27,28].