



Product preservation and stable units for reflections into idempotent subvarieties

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Abstract. We give a necessary and sufficient condition for the preservation of finite products by a reflection of a variety of universal algebras into an idempotent subvariety. It is also shown that simple and semi-left-exact reflections into subvarieties of universal algebras are the same. It then follows that a reflection of a variety of universal algebras into an idempotent subvariety has stable units if and only if it is simple and the above-mentioned condition holds.

1 Introduction

Throughout this paper, by a *reflection* we mean a reflection $H \vdash I : \mathcal{C} \rightarrow \mathcal{M}$, with unit $\eta : 1_{\mathcal{C}} \rightarrow HI$, of a finitely complete category \mathcal{C} into a full subcategory \mathcal{M} of it, that is, a left-adjoint I of a full embedding $H : \mathcal{M} \rightarrow \mathcal{C}$ into a category with finite limits. T will always denote a terminal object of a given category \mathcal{C} ; for instance, if \mathcal{C} is a variety of universal algebras, then T is any one-element algebra in \mathcal{C} .

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Simple reflections, semi-left-exact reflections and reflections having stable units were originally introduced in [3], as reflections preserving certain pullbacks. An additional structure on a reflection was described in [10], involving a pullback-preserving functor $U : \mathcal{C} \rightarrow \mathcal{S}$ (in particular, \mathcal{S} can be the category of sets), allowing to simplify those preservation conditions by reducing them to the preservation of very special pullbacks.

The categorical version of monotone-light factorization for continuous maps of compact Hausdorff spaces was obtained in [2]. The results on the reflection of semigroups into semilattices obtained in [6] look similar to the results on the reflection of compact Hausdorff spaces into Stone spaces. In [10], it was shown that this is not similarity, but two special cases of the same ‘theory’.

In the same setting of [10], the present paper provides new results concerning the preservation of finite products, in case $\mathcal{S} = \mathbf{Set}$ and either \mathcal{C} or \mathcal{M} is an ‘idempotent’ category (see Definition 2.8). Then, it is possible to apply these results to the classification of reflections in the sense of [3].

In particular, we studied - with special care - reflections of varieties of universal algebras into subvarieties, where, to begin with, semi-left-exact and simple reflections are the same.

Now, we will give a brief account of the contents of this work. The reader may also find it helpful to check the two tables at the end of the paper, summarizing all the presented results.

In Section 3, we state a necessary and sufficient condition (see Proposition 3.3) for the preservation of the product of two objects by a reflection into an ‘idempotent’ subcategory, provided there exists a functor $U : \mathcal{C} \rightarrow \mathbf{Set}$ which preserves finite limits and reflects isomorphisms, and such that $U(\eta_C)$ is a surjection, for every unit morphism $\eta_C : C \rightarrow HI(C)$, $C \in \mathcal{C}$. It follows from Proposition 3.3 that:

- if \mathcal{C} is an ‘idempotent’ category (for instance, an idempotent variety of universal algebras), then finite products are preserved by the left-adjoint I (see Proposition 3.4);
- if \mathcal{C} is a variety of universal algebras and \mathcal{M} is one of its ‘idempotent’ subvarieties (see Definition 2.7), then I preserves finite products if and only if I preserves the product $F(x) \times F(x)$, here $F(x)$ stands for the free algebra on one generator in \mathcal{C} (see Proposition 3.8).

In Section 4 (see Proposition 4.1), we will show that a reflection is simple

if and only if it is semi-left-exact, provided its unit morphisms are effective descent morphisms (see the footnote in Section 4). In particular, this holds for a reflection of a variety of universal algebras into a subvariety.

It is a consequence of Propositions 3.8 and 4.1 that a reflection of a variety of universal algebras \mathcal{C} into an ‘idempotent’ subvariety \mathcal{M} , has stable units in the sense of [3] if and only if it is simple and I preserves the product $F(x) \times F(x)$ (see Proposition 4.4).

2 Preliminaries

THREE TYPES OF REFLECTIONS: SIMPLE, SEMI-LEFT-EXACT, AND HAVING STABLE UNITS

In this section we review the definition of *simple* and *semi-left-exact* reflections and reflections having *stable units* (notions introduced in [3]). One easily checks, from Definitions 2.1, 2.2 and 2.3 below, that if a reflection has stable units then it is semi-left-exact, which implies that it is simple.

Consider a reflection and let (E_I, M_I) be a prefactorization system as in [2, §3], that is,

$$E_I = (H(\text{mor}\mathcal{M}))^\uparrow, \quad M_I = (H(\text{mor}\mathcal{M}))^{\uparrow\downarrow},$$

where $H(\text{mor}\mathcal{M})$ stands for the class of all morphisms in \mathcal{C} which belong to the full subcategory \mathcal{M} , and the arrows correspond to the *diagonal fill-in* Galois connection (see [2, §2.1]).

A morphism $e : A \rightarrow B$ in \mathcal{C} belongs to E_I if and only if $I(e)$ is an isomorphism. Hence, if $e \in E_I$ and $e \circ f \in E_I$ then $f \in E_I$. In particular $\eta_C : C \rightarrow HI(C)$ lies in E_I , since \mathcal{M} is a full subcategory of \mathcal{C} .

Notice that every morphism in \mathcal{M} lies in M_I . Recall that the class M_I is pullback stable (see [2, §2]), and that (E_I, M_I) is a factorization system if, for some morphisms $e \in E_I$ and $m \in M_I$, $f = me$ for every morphism f in \mathcal{C} .

Definition 2.1. A reflection is called simple if $w \in E_I$ in every diagram of the form

$$\begin{array}{ccc}
 A & & \\
 \swarrow f & \searrow \eta_A & \\
 & B \times_{HI(B)} HI(A) & \longrightarrow HI(A) \\
 & \downarrow \pi_1 & \downarrow HI(f) \\
 & B & \xrightarrow{\eta_B} HI(B)
 \end{array}
 \tag{2.1}$$

where η_A and η_B are unit morphisms, and w is the unique morphism which makes the diagram commute.

Hence, (E_I, M_I) is a factorization system if the reflection is simple, since π_1 in diagram (2.1) is a pullback of a morphism in M_I , and so it is in M_I .

Definition 2.2. A reflection is semi-left-exact if $\pi_2 \in E_I$ in every pullback diagram of the form

$$\begin{array}{ccc}
 C \times_{HI(C)} M & \xrightarrow{\pi_2} & M \\
 \downarrow \pi_1 & & \downarrow g \\
 C & \xrightarrow{\eta_C} & HI(C)
 \end{array}$$

where η_C is a unit morphism and $M \in \mathcal{M}$.

A semi-left-exact reflection is also called admissible in categorical Galois theory (see [1]).

Definition 2.3. A reflection has stable units if the left-adjoint I preserves all pullback diagrams of the form

$$\begin{array}{ccc}
 C \times_{HI(C)} D & \xrightarrow{\pi_2} & D \\
 \downarrow \pi_1 & & \downarrow g \\
 C & \xrightarrow{\eta_C} & HI(C)
 \end{array}$$

where η_C is a unit morphism.

The following proposition is a well-known characterization of reflections having stable units.

Proposition 2.4. *A reflection has stable units if and only if the left-adjoint I preserves all pullback diagrams*

$$\begin{array}{ccc}
 A & \xrightarrow{g} & C \\
 \downarrow f & & \downarrow k \\
 B & \xrightarrow{h} & Z
 \end{array}$$

in \mathcal{C} for which $Z \in \mathcal{M}$.

THE GROUND STRUCTURE

We now define ‘*Ground Data*’. The definition of an idempotent variety of universal algebras will be given in Definition 2.7, and then will be generalized in Definition 2.8 to an abstract category.

Definition 2.5. A reflection is said to satisfy *Ground Data* if there is a functor $U : \mathcal{C} \rightarrow \mathbf{Set}$, from \mathcal{C} into the category of sets, such that the following three conditions hold:

- (i) U preserves finite limits;
- (ii) U reflects isomorphisms;
- (iii) $U(\eta_C) : U(C) \rightarrow UHI(C)$ is a surjection, for every unit morphism η_C , $C \in \mathcal{C}$.

Example 2.6. A reflection of a variety of universal algebras \mathcal{C} into a subvariety \mathcal{M} satisfies *Ground Data*:

- there exists a functor $U : \mathcal{C} \rightarrow \mathbf{Set}$, the functor that assigns to an algebra $C \in \mathcal{C}$ its underlying set $U(C)$, which reflects isomorphisms since these are just bijective homomorphisms in a variety of universal algebras;
- $U : \mathcal{C} \rightarrow \mathbf{Set}$ preserves finite limits, since it is a right adjoint of $F : \mathbf{Set} \rightarrow \mathcal{C}$, the functor that assigns to a set S the free algebra $F(S)$ generated by the elements of S ;
- every unit morphism $\eta_C : C \rightarrow HI(C)$ is a surjection in \mathbf{Set} , since η_C is the canonical projection of C into the quotient algebra C / \sim_C ; here \sim_C is the congruence generated by the ‘extra identities’ satisfied in the subvariety \mathcal{M} .

On the contrary, a forgetful functor $U : \mathcal{C} \rightarrow \mathbf{Set}$ from a category \mathcal{C} of topological spaces, does not reflect isomorphisms in general. For instance, in the category $\mathcal{C} = \mathbf{Top}$ of all topological spaces, consider the map $i : (X, \delta) \rightarrow (X, \tau)$, in which $X = \{a, b\}$, δ is the discrete topology, τ is the topology $\{\emptyset, X\}$, and $U(i)$ is the identity function on the set X .

There still are cases for which a forgetful functor from a category of topological spaces reflects isomorphisms, such as for the category of compact Hausdorff spaces $\mathcal{C} = \mathbf{CompHaus}$. In fact, it is well known that $U : \mathbf{CompHaus} \rightarrow \mathbf{Set}$ is monadic.

In every case, either varieties of universal algebras or the two above-mentioned categories of topological spaces, the forgetful functor U preserves finite limits, since U is always a right adjoint.¹

Definition 2.7. We say that a variety of universal algebras \mathcal{C} is idempotent if any of the following three equivalent conditions hold:

- (i) every one-element subset $\{x\}$ of any $C \in \mathcal{C}$ is a subalgebra;
- (ii) the free algebra $F(x)$ on one generator² in \mathcal{C} is a singleton;
- (iii) $x = \theta(x, \dots, x)$, for every n -ary operation θ on any $C \in \mathcal{C}$, with $x \in C$ and $n \in \mathbb{N}_0$.

Definition 2.8. Let \mathcal{C} be a category with terminal object, such that there exists a functor $U : \mathcal{C} \rightarrow \mathbf{Set}$, from \mathcal{C} into sets, which preserves terminal objects. The category \mathcal{C} is called idempotent with respect to the functor $U : \mathcal{C} \rightarrow \mathbf{Set}$, when

$$U_{T,A} : \mathcal{C}(T, A) \rightarrow \mathbf{Set}(U(T), U(A)),$$

the restriction of U to the hom-set $\mathcal{C}(T, A)$, is a surjection for every object $A \in \mathcal{C}$, with T a terminal object in \mathcal{C} .

The category \mathcal{C} will just be called idempotent when the functor U is the obvious one, as in the case in which \mathcal{C} is a variety of universal algebras, or if it is the larger category in a reflection satisfying *Ground Data*.

¹If \mathcal{C} is determined by a class of algebras of the same type closed under products and subalgebras, then (i) and (ii) in Definition 2.5 hold, and provided \mathcal{C} is also closed under isomorphisms then U is a right adjoint, as it is well known. The following results concerning varieties could be easily adapted to this more general context (remember that a variety of universal algebras is just such a category \mathcal{C} of algebras closed under homomorphic images).

²A more precise notation would be $F(\{x\})$.

Example 2.9. A category of topological spaces \mathcal{C} with terminal object is idempotent (with respect to the forgetful functor into sets), since every map from a singleton into any topological space is a continuous map in such a category.

SEMI-LEFT-EXACTNESS AND STABLE UNITS VIA
CONNECTED COMPONENTS

In Definition 2.10 and Lemmas 2.11 and 2.12 below, the reflection satisfies *Ground Data* and the subcategory \mathcal{M} is idempotent (with respect to the composite functor UH in Definition 2.5). These two Lemmas were proved in [10].

Definition 2.10. Consider any morphism $\mu : T \rightarrow HI(C)$ from a terminal object into $HI(C)$, for some $C \in \mathcal{C}$. The *connected component* associated to the morphism μ is the pullback C_μ in the following pullback square:

$$\begin{array}{ccc} C_\mu & \longrightarrow & T \\ \downarrow & & \downarrow \mu \\ C & \xrightarrow{\eta_C} & HI(C) \end{array}$$

Lemma 2.11. *The reflection is semi-left-exact if and only if $HI(C_\mu) \cong T$, for every connected component C_μ .*

Lemma 2.12. *The reflection has stable units if and only if $HI(C_\mu \times D_\nu) \cong T$, for every product of any pair of connected components C_μ and D_ν .*

3 Preservation of finite products by reflections into subvarieties

In this section, we begin by recalling a known lemma in the category of sets. This lemma (Lemma 3.1) will be used in the proof of Lemma 3.2, which states a sufficient condition for the preservation of finite products by a reflection satisfying *Ground Data*.

In Proposition 3.3, we state necessary and sufficient conditions for the preservation of finite products by a reflection into an idempotent subcategory, provided it satisfies *Ground Data*.

In Proposition 3.4, it is shown that finite products are preserved if the reflection satisfies *Ground Data* and \mathcal{C} is idempotent. Note that just asking the full subcategory \mathcal{M} to be idempotent would not be enough. See Example 4.6, where, in the reflection of M -sets into the idempotent category of sets, there is no preservation of finite products if the non-trivial monoid M satisfies the cancellation law.

In Lemma 3.7, we state a sufficient condition for the preservation of finite products by a reflection into a subvariety of universal algebras.

Finally, in Proposition 3.8, we state a necessary and sufficient condition for the preservation of finite products by a reflection into an idempotent subvariety of universal algebras.

Lemma 3.1. *Consider the following commutative diagram in \mathbf{Set} , where α_D , $\alpha_{D \times E}$, and α_E are surjections, and the bottom line is a product diagram:*

$$\begin{array}{ccccc}
 B & \xleftarrow{f} & A & \xrightarrow{g} & C \\
 \alpha_D \uparrow & & \uparrow \alpha_{D \times E} & & \uparrow \alpha_E \\
 D & \xleftarrow{pr_1} & D \times E & \xrightarrow{pr_2} & E
 \end{array}$$

The following two conditions are equivalent:

(i) *for every $e \in E$ the map $\Gamma_e : D \rightarrow A$, $d \mapsto \alpha_{D \times E}(d, e)$ (in the left-hand commutative diagram), factorizes through α_D , and for every $d \in D$ the map $\Gamma_d : E \rightarrow A$, $e \mapsto \alpha_{D \times E}(d, e)$ (in the right-hand commutative diagram), factorizes through α_E :*

$$\begin{array}{ccc}
 D & \xrightarrow{\Gamma_e} & A \\
 d \searrow & & \uparrow \alpha_{D \times E} \\
 & & D \times E \\
 & \searrow (d, e) &
 \end{array}
 \qquad
 \begin{array}{ccc}
 E & \xrightarrow{\Gamma_d} & A \\
 e \searrow & & \uparrow \alpha_{D \times E} \\
 & & D \times E \\
 & \searrow (d, e) &
 \end{array}$$

(ii) *the maps f and g are jointly monic.*

Lemma 3.2. *Consider a reflection satisfying *Ground Data*, with unit $\eta : 1_{\mathcal{C}} \rightarrow HI$. The left-adjoint I preserves the product $Q \times R$ if the following conditions hold:*

(i) for each r fixed in $U(R)$, there exists a morphism $\gamma_r : Q \rightarrow HI(Q \times R)$, such that

$$U(\gamma_r)(d) = U(\eta_{Q \times R})(d, r), \text{ for all } d \in U(Q);$$

(ii) for each q fixed in $U(Q)$, there exists a morphism $\gamma_q : Q \rightarrow HI(Q \times R)$, such that

$$U(\gamma_q)(e) = U(\eta_{Q \times R})(q, e), \text{ for all } e \in U(R).$$

Proof. Since $\eta_Q : Q \rightarrow HI(Q)$ is universal from Q to H , it induces a morphism $\beta : I(Q) \rightarrow I(Q \times R)$, such that the following diagram commutes:

$$\begin{array}{ccc} Q & \xrightarrow{\eta_Q} & HI(Q) \\ & \searrow \gamma_r & \downarrow H(\beta) \\ & & HI(Q \times R) \end{array} \quad (3.1)$$

Applying the functor U to Diagram (3.1), one concludes that $U(\gamma_r)$ factorizes through the surjective map $U(\eta_Q)$. By analogous arguments, one can also conclude that $U(\gamma_q)$ factorizes through the surjective map $U(\eta_R)$. Now, consider the following:

$$\begin{array}{ccccc} HI(Q) & \xleftarrow{p_1} & HI(Q) \times HI(R) & \xrightarrow{p_2} & HI(R) \\ \uparrow 1_{HI(Q)} & & \uparrow \langle HI(\pi_1), HI(\pi_2) \rangle & & \uparrow 1_{HI(R)} \\ HI(Q) & \xleftarrow{HI(\pi_1)} & HI(Q \times R) & \xrightarrow{HI(\pi_2)} & HI(R) \\ \uparrow \eta_Q & & \uparrow \eta_{Q \times R} & & \uparrow \eta_R \\ Q & \xleftarrow{\pi_1} & Q \times R & \xrightarrow{\pi_2} & R \end{array} \quad (3.2)$$

where the upper and bottom lines are both product diagrams. Applying the finite limits preserving functor U to Diagram (3.2) and Lemma 3.1, $UHI(\pi_1)$ and $UHI(\pi_2)$ are jointly monic. So $\langle UHI(\pi_1), UHI(\pi_2) \rangle$ is an

injective map. On the other hand, since $U(\eta_Q)$ and $U(\eta_R)$ are surjective maps, $U(\eta_Q) \times U(\eta_R) = \langle UHI(\pi_1), UHI(\pi_2) \rangle \circ U(\eta_{Q \times R})$ is also a surjective map. Finally, since U reflects isomorphisms, $I(Q \times R) \cong I(Q) \times I(R)$. \square

Proposition 3.3. *Consider a reflection into an idempotent subcategory \mathcal{M} , satisfying Ground Data, with unit $\eta : 1_{\mathcal{C}} \rightarrow HI$. The left-adjoint I preserves the product $Q \times R$ if and only if the following conditions hold:*

(i) *for each r fixed in $U(R)$, there exists a morphism $\gamma_r : Q \rightarrow HI(Q \times R)$, such that*

$$U(\gamma_r)(d) = U(\eta_{Q \times R})(d, r), \text{ for all } d \in U(Q);$$

(ii) *for each q fixed in $U(Q)$, there exists a morphism $\gamma_q : Q \rightarrow HI(Q \times R)$, such that*

$$U(\gamma_q)(e) = U(\eta_{Q \times R})(q, e), \text{ for all } e \in U(R).$$

Proof. If the product $Q \times R$ is preserved by the reflector I , then $w = \langle HI(\pi_1), HI(\pi_2) \rangle$ in Diagram (3.2) is an isomorphism in \mathcal{M} . Now consider for $r \in U(R)$ fixed the following morphism in \mathcal{C} :

$$\gamma_r : Q \xrightarrow{\langle id_Q, ! \rangle} Q \times R \xrightarrow{\langle \eta_Q, h_r \rangle} HI(Q) \times HI(R) \xrightarrow{w^{-1}} HI(Q \times R), \quad (3.3)$$

where $h_r : T \rightarrow HI(R)$ is a morphism in \mathcal{M} , such that $U(h_r) = f_r : \{r\} \rightarrow UHI(R)$; $r \mapsto U(\eta_R)(r)$. It is easy to check that $U(\gamma_r)(q) = U(\eta_{Q \times R})(q, r)$, for all $q \in U(Q)$. There exists analogously a morphism $\gamma_q : R \rightarrow HI(Q \times R)$ in \mathcal{C} , for any fixed $q \in U(Q)$, such that $U(\gamma_q)(r) = U(\eta_{Q \times R})(q, r)$, for all $r \in U(R)$. The converse follows from Lemma 3.2. \square

Proposition 3.4. *Consider a reflection of an idempotent category \mathcal{C} , satisfying Ground Data. The left-adjoint I preserves finite products.*

Proof. Let Q and R be objects of \mathcal{C} . For every $r \in U(R)$, consider the inclusion map f_r of $\{r\}$ into $U(R)$. Since, by hypothesis, there exists a morphism $f : T \rightarrow R$ such that $U(f) = f_r$, and since \mathcal{C} has finite products, there is a morphism $1_Q \times f : Q \times T \rightarrow Q \times R$, such that $U(1_Q \times f) \cong 1_{U(Q)} \times f_r : U(Q) \times \{r\} \rightarrow U(Q) \times U(R)$, as in the following product diagram:

$$\begin{array}{ccccc}
Q & \xleftarrow{\pi_1} & Q \times R & \xrightarrow{\pi_2} & R \\
1_Q \uparrow & & \uparrow 1_Q \times f & & \uparrow f \\
Q & \xleftarrow{\pi_Q} & Q \times T & \xrightarrow{\pi_T} & T \\
& \searrow 1_Q & \uparrow \langle 1_Q, ! \rangle & \nearrow ! & \\
& & Q & &
\end{array} \tag{3.4}$$

Since T is a terminal object, there exists a unique morphism $! : Q \rightarrow T$ such that $U(!) : U(Q) \rightarrow \{r\}$ is the unique map from $U(Q)$ to $\{r\}$. Then, there exists a morphism $\langle 1_Q, ! \rangle : Q \rightarrow Q \times T$ (see Diagram 3.4). Therefore, there exists a morphism

$\gamma_r = \eta_{Q \times R} \circ (1_Q \times f) \circ \langle 1_Q, ! \rangle : Q \rightarrow Q \times T \rightarrow Q \times R \rightarrow HI(Q \times R)$, such that

$$\begin{aligned}
U(\gamma_r) &= U(\eta_{Q \times R} \circ (1_Q \times f) \circ \langle 1_Q, ! \rangle) \\
&= U(\eta_{Q \times R}) \circ (1_{U(Q)} \times f_r) \circ \langle 1_{U(Q)}, U(!) \rangle \\
&= U(\eta_{Q \times R}) \circ \langle 1_{U(Q)}, f_r \circ U(!) \rangle.
\end{aligned}$$

Hence, $U(\gamma_r)(a) = U(\eta_{Q \times R})(a, r)$, for all $a \in U(Q)$, with $r \in U(R)$. One can construct, for every $q \in U(Q)$, by analogous arguments, a morphism

$\gamma_q = \eta_{Q \times R} \circ (g \times 1_R) \circ \langle !, 1_R \rangle : R \rightarrow Q \times T \rightarrow Q \times R \rightarrow HI(Q \times R)$, such that

$$\begin{aligned}
U(\gamma_q) &= U(\eta_{Q \times R} \circ (g \times 1_R) \circ \langle !, 1_R \rangle) \\
&= U(\eta_{Q \times R}) \circ (f_q \times 1_{U(R)}) \circ \langle U(!), 1_{U(R)} \rangle \\
&= U(\eta_{Q \times R}) \circ \langle f_q \circ U(!), 1_{U(R)} \rangle,
\end{aligned}$$

where $f_q : \{q\} \rightarrow U(Q)$ is the inclusion map. Hence, $U(\gamma_q)(b) = U(\eta_{Q \times R})(q, b)$, for all $b \in U(R)$, with $q \in U(Q)$. Finally, by Lemma 3.2, one concludes that the left-adjoint I preserves the product $Q \times R$. \square

Corollary 3.5. *Consider a reflection of an idempotent variety of universal algebras into a subvariety. The left-adjoint preserves finite products.*

Example 3.6. Any reflection of an idempotent variety of magmas into one of its subvarieties preserves finite products. The reflection of the idempotent variety of quandles into its subvariety of sets preserves finite products (although it is not semi-left-exact; see [4] and check Corollary 4.5 below.

In the following, we will see that the sufficient condition for the preservation of finite products in Lemma 3.2 holds for a reflection into a subvariety of universal algebras, provided $I(F(x) \times F(x)) = T$.

Lemma 3.7. *Consider a reflection of a variety \mathcal{C} of universal algebras into a subvariety, and let $F(x)$ be the free algebra on one generator in \mathcal{C} . If $I(F(x) \times F(x)) = T$ then the left-adjoint I preserves finite products.*

Proof. Let Q and R be objects in \mathcal{C} , and $U : \mathcal{C} \rightarrow \mathbf{Set}$ be the forgetful functor into sets. The maps

$$q : \{x\} \rightarrow U(Q) \quad \text{and} \quad r : \{x\} \rightarrow U(R),$$

$$x \mapsto q \quad \quad \quad x \mapsto r$$

extend uniquely and respectively to the homomorphisms $h_q : F(x) \rightarrow Q$ and $h_r : F(x) \rightarrow R$, because the inclusion map $\{x\} \subset UF(x)$ is universal from $\{x\}$ to U . Hence, for any $(q, r) \in Q \times R$, there exists a unique homomorphism $h_q \times h_r$ which makes the following product diagram commute:

$$\begin{array}{ccccc} Q & \xleftarrow{\pi_Q} & Q \times R & \xrightarrow{\pi_R} & R \\ h_q \uparrow & & \uparrow h_q \times h_r & & \uparrow h_r \\ F(x) & \xleftarrow{\quad} & F(x) \times F(x) & \xrightarrow{\quad} & F(x) \end{array} .$$

Since $\eta : 1_{\mathcal{C}} \rightarrow HI$ is a natural transformation, the following is a commutative diagram:

$$\begin{array}{ccc} F(x) \times F(x) & \xrightarrow{\eta_{F(x) \times F(x)}} & HI(F(x) \times F(x)) \\ h_q \times h_r \downarrow & & \downarrow HI(h_q \times h_r) \\ Q \times R & \xrightarrow{\eta_{Q \times R}} & HI(Q \times R) \end{array}$$

and so, as $I(F(x) \times F(x)) = T$, the following condition holds:

$$(h_q(w_1), h_r(w_2)) \sim_{Q \times R} (h_q(w_3), h_r(w_4)), \quad (3.5)$$

for all $q \in Q$ and $r \in R$, and for all w_1, w_2, w_3 and $w_4 \in F(x)$, where $\sim_{Q \times R}$ is the congruence associated to the surjective homomorphism $\eta_{Q \times R} :$

$Q \times R \rightarrow HI(Q \times R)$. We will prove next that the map

$$\lambda_r : Q \rightarrow HI(Q \times R), \quad q \mapsto [(q, r)]_{\sim_{Q \times R}}$$

is a homomorphism, for every $r \in R$. Let θ be an operation on \mathcal{C} , of arity $n \in \mathbb{N}_0$, and let $q_1, \dots, q_n \in Q$. Since $\theta_Q(q_1, \dots, q_n) = q = h_q(x)$, for some $q \in Q$, and $r = h_r(x)$, then

$$\begin{aligned} \lambda_r(\theta_Q(q_1, \dots, q_n)) &= [(h_q(x), h_r(x))]_{\sim_{Q \times R}} \\ &= [(h_q(x), h_r(\theta_{F(x)}(x, \dots, x)))]_{\sim_{Q \times R}} \text{ by (3.5)} \\ &= [(\theta_Q(q_1, \dots, q_n), \theta_R(r, \dots, r))]_{\sim_{Q \times R}} \\ &= [\theta_{Q \times R}((q_1, r), \dots, (q_n, r))]_{\sim_{Q \times R}} \\ &= \theta_{HI(Q \times R)}([(q_1, r)]_{\sim_{Q \times R}}, \dots, [(q_n, r)]_{\sim_{Q \times R}}) \\ &= \theta_{HI(Q \times R)}(\lambda_r(q_1), \dots, \lambda_r(q_n)). \end{aligned}$$

Hence, there is a homomorphism $\lambda_r : Q \rightarrow HI(Q \times R)$, such that $U(\lambda_r)(q) = U(\eta_{Q \times R})(q, r)$, with $q \in Q$, for every $r \in R$. By analogous arguments we would conclude that, for every $q \in Q$, there exists a homomorphism $\lambda_q : R \rightarrow HI(Q \times R)$, such that $U(\lambda_q)(r) = U(\eta_{Q \times R})(q, r)$, $r \in R$. Finally, by Lemma 3.2, the left-adjoint I preserves the product $Q \times R$. \square

Proposition 3.8. *If $H \vdash I : \mathcal{C} \rightarrow \mathcal{M}$ is a reflection of a variety of universal algebras into an idempotent subvariety, then the following conditions are equivalent:*

- (i) I preserves finite products;
- (ii) I preserves the product $F(x) \times F(x)$;
- (iii) $I(F(x) \times F(x)) = T$.

Proof. If I preserves finite products, then, in particular, I preserves the product $F(x) \times F(x)$. If I preserves the product $F(x) \times F(x)$, that is, $I(F(x) \times F(x)) \cong I(F(x)) \times I(F(x))$, then $I(F(x) \times F(x)) = T$, since \mathcal{M} is idempotent, $T = I(F(x))$ is the free algebra on one generator in \mathcal{M} . If $I(F(x) \times F(x)) = T$, then I preserves finite products by Lemma 3.7. \square

Example 3.9. Consider the reflection of the variety of power associative magmas into its subvariety of semilattices (or, more generally, into any variety of bands). Recall that a magma is power-associative if any sub-magma generated by one of its elements is associative. The free power

associative magma on one-element set $F(x)$ is isomorphic to the commutative semigroup $(\mathbb{N}, +)$ of positive integer numbers. Therefore, $F(x) \times F(x) \cong \mathbb{N} \times \mathbb{N}$. On the other hand, $\mathbb{N} \times \mathbb{N}$ is semilattice indecomposable since it is archimedean (see [6] or [9]), that is, $I(\mathbb{N} \times \mathbb{N}) = T$. Hence, the reflector I preserves finite products by Proposition 3.8.

4 Classifying reflections

In this section, we show that a reflection is semi-left-exact if and only if it is simple, provided its unit morphisms are effective descent morphisms³ in \mathcal{C} . This holds for every reflection of a variety of universal algebras into one of its subvarieties. Finally, we characterize the reflections into idempotent subvarieties of universal algebras having stable units.

Proposition 4.1. *Consider a reflection in which every unit morphism is an effective descent morphism in \mathcal{C} . Then, the reflection is simple if and only if it is semi-left-exact.*

Proof. $H \dashv I$ is semi-left-exact, by Definition 2.2, if and only if $\pi_2 \in E_I$, in every pullback square of the following form:

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & H(X) \\ \pi_1 \downarrow & & \downarrow g \\ B & \xrightarrow{\eta_B} & HI(B) \end{array} .$$

Consider the following commutative diagrams:

$$\begin{array}{ccccc} P & \xrightarrow{\pi_2} & H(X) & \xrightarrow{\eta_{H(X)}} & HIH(X) \cong H(X) \\ \pi_1 \downarrow & & \downarrow g & (i) & \downarrow HI(g) \cong g \\ B & \xrightarrow{\eta_B} & HI(B) & \xrightarrow{\eta_{HI(B)}} & HIHI(B) \cong HI(B) \end{array} \quad (4.1)$$

³A morphism $p : E \rightarrow B$ in \mathcal{C} is an effective descent morphism when the functor “pullback along p ” $p^* : \mathcal{C}/B \rightarrow \mathcal{C}/E$ is monadic.

$$\begin{array}{ccccc}
 P & \xrightarrow{\eta_P} & HI(P) & \xrightarrow{HI(\pi_2)} & HIH(X) \cong H(X) \\
 \pi_1 \downarrow & & \downarrow HI(\pi_1) & & \downarrow HI(g) \cong g \\
 (ii) & & & & (4.2) \\
 B & \xrightarrow{\eta_B} & HI(B) & \xrightarrow{HI(\eta_B)} & HIHI(B) \cong HI(B)
 \end{array}$$

First note that $g, \pi_1 \in M_I$, because $g \in \mathcal{M}$ (see [2, §3]) and the class of morphisms M_I is stable for pullbacks. Since the reflection is simple, the square (i) is a pullback, and so the outside square of Diagram (4.1) is a pullback. As $HI(\pi_2) \circ \eta_P = \eta_{H(X)} \circ \pi_2$ and $HI(\eta_B) \circ \eta_B = \eta_{HI(B)} \circ \eta_B$, the outside square of Diagram (4.2) is also a pullback. According to Lemma 4.6 in [2], since the outside square of Diagram (4.2) is a pullback, (ii) is a pullback (because the reflection is simple) and η_B is an effective descent morphism in \mathcal{C} , then (iii) is a pullback, too. On the other hand, $HI(\eta_B)$ is an isomorphism, because $H \vdash I : \mathcal{C} \rightarrow \mathcal{M}$ is a reflection into a full subcategory. Hence, $HI(\pi_2)$ is also an isomorphism. Therefore, $\pi_2 \in E_I$. \square

Last Proposition 4.1 applies to any reflection of a variety of universal algebras into one of its subvarieties, as stated in the following proposition.

Proposition 4.2. *The reflection of a variety of universal algebras into a subvariety is semi-left-exact if and only if it is simple.*

Proof. It is well known that a variety of universal algebras is an exact category. In an exact category the effective descent morphisms are just the regular epimorphisms (see [2, §4.7]). On the other hand, it is easy to check that, in a variety of universal algebras the regular epimorphisms are just the surjective homomorphisms. Hence, the unit morphisms of a reflection of a variety of universal algebras into one of its subvarieties, are always effective descent morphisms, since they are surjective homomorphisms. Therefore, by Proposition 4.1, simple and semi-left-exact reflections of varieties of universal algebras are the same. \square

The following lemma will be used in Proposition 4.4, which characterizes the property of having stable units for reflections into idempotent subvarieties of universal algebras.

Lemma 4.3. *Consider a reflection into an idempotent subcategory, satisfying Ground Data, and such that the left-adjoint preserves finite products. Then, the reflection has stable units if and only if it is semi-left-exact.*

Proof. The proof follows straightforwardly from Lemmas 2.11 and 2.12. \square

Proposition 4.4. *Consider a reflection of a variety of universal algebras \mathcal{C} into an idempotent subvariety. The following two conditions are equivalent, where $F(x)$ is the free algebra on one generator in \mathcal{C} :*

- (i) *the reflection is simple and the left-adjoint preserves the product $F(x) \times F(x)$;*
- (ii) *the reflection has stable units.*

Proof. If the left-adjoint I preserves the product $F(x) \times F(x)$ then, by Proposition 3.8, I preserves finite products. On the other hand, by Proposition 4.2, the reflection is semi-left-exact. Hence, by Lemma 2.12 and Lemma 2.11 the reflection has stable units. Conversely, a product $Q \times R$ in \mathcal{C} is just the pullback $Q \times_T R$, where T is a terminal object in \mathcal{C} . This pullback $Q \times_T R$ is preserved by I according to Proposition 2.4, provided $T \in \mathcal{M}$. In fact, one can assume, without loss of generality, that $T \in \mathcal{M}$, since T and $HI(T)$ are isomorphic. \square

The next corollary follows straightforwardly from Corollary 3.5 and Proposition 4.4.

Corollary 4.5. *Consider a reflection of an idempotent variety of universal algebras into one of its subvarieties. The reflection has stable units if and only if it is simple.*

Example 4.6. Let S be a set and M a monoid with unit 1_M ($1_M m = m = m 1_M$, for every $m \in M$). An M -set whose underlying set is S (an object in the category $M\text{-Set}$, which is a variety of universal algebras) is an algebra with only unary operations $m(s) = ms$ (one operation for each element of the monoid), such that $1_M s = s$ and $m'(ms) = (m'm)s$ for all $m, m' \in M$, $s \in S$. Every set $S \in \mathbf{Set}$ can be seen as an M -set, provided we state $ms = s$, for all $m \in M$, $s \in S$.

We present now an example of application of Proposition 4.4 to the reflection $H \vdash I : M\text{-Set} \rightarrow \mathbf{Set}$ of M -sets into its idempotent subvariety

of sets, associated to the congruence generated, on every $S \in M\text{-Set}$, by $ms = s$, for all $m \in M$, $s \in S$.

It is well known that this reflection of $M\text{-Set}$ into Set is semi-left-exact (that is, simple; see Proposition 4.2), which follows from more general results (see [1, §6.2]). Here, we study the preservation of finite products and the stable units property, when M satisfies the cancellation law ($ca = cb \Rightarrow a = b$, for any a, b and $c \in M$)⁴ and when M has a zero element 0_M ($0_M m = 0_M = m 0_M$, for every $m \in M$).

A congruence on an M -set S contains $R = \{(s, ms) \in S \times S \mid s \in S, m \in M\}$ if and only if it contains $R^* = \{(ms, m's) \in S \times S \mid s \in S, m, m' \in M\}$, by symmetry and transitivity. Therefore, R and R^* generate the same congruence.

Let C_S be a subset of $S \times S$ consisting of those elements $(a, b) \in S$ for which there exist $z_0, \dots, z_n \in S$ where $n \in \mathbb{N}$ satisfying the following:

$$a = z_0 \wedge b = z_n \wedge (z_i, z_{i+1}) \in R^*, i \in \{0, \dots, n-1\} \quad (4.3)$$

Since C_S is the transitive closure of R^* , C_S is contained in every congruence that contains R . In fact, C_S is the congruence generated by R , because

- $\{(a, a) \mid a \in S\} \subseteq C_S$,
- C_S is obviously symmetric and transitive, and
- C_S respects the (unary) operations on S , since if there exists a finite sequence between a and b as in (4.3), then there exists a finite sequence between ma and mb as in (4.3), for all $m \in M$.

As usual, $a \sim_S b$ will state that a and b are related, which is equivalent to $(a, b) \in C_S$, and the class of equivalence of (a, b) in C_S will be denoted by $[(a, b)]_{\sim_S}$, for any elements $a, b \in S$.

Finally, note that $F(x) = M$, $F(x) \times F(x) = M \times M$ and clearly $I(F(x)) = T$, in $M\text{-Set}$.

WHEN M SATISFIES THE CANCELLATION LAW

⁴Left-cancellation law, to be more precise, since M -sets were defined above as left monoid actions.

We are going to show that, if $M(\neq T)$ is a cancellative monoid then $I(F(x) \times F(x)) \neq T$. It is clear that, $I(F(x) \times F(x)) = T$ if and only if $(1_M, 1_M) \sim_{M \times M} (m, m')$ for all $m, m' \in M$. Then, according to (4.3), $(1_M, 1_M) \sim_{M \times M} (m, m')$ if and only if there exists a finite sequence

$$(1_M, 1_M) = (m_0, m'_0), (m_1, m'_1), \dots, (m_i, m'_i), (m_{i+1}, m'_{i+1}), \dots, (m_n, m'_n) = (m, m'), \quad (4.4)$$

such that, for every pair $((m_i, m'_i), (m_{i+1}, m'_{i+1}))$, $(m_i, m'_i) = c(a, b)$ and $(m_{i+1}, m'_{i+1}) = d(a, b)$, for some $a, b, c, d \in M$. That is, for each $i \in \{0, 1, \dots, n-1\}$, there exist $a, b, c, d \in M$, such that $ca = m_i$, $cb = m'_i$, $da = m_{i+1}$ and $db = m'_{i+1}$.

Let $M \neq \{1_M\}$ be a cancellative monoid. It can be easily checked that $[(1_M, 1_M)]_{\sim_{M \times M}} \neq [(m, m')]_{\sim_{M \times M}}$ for $m \neq m'$. We will prove it by induction on the length of the finite sequence (4.4), as follows.

Let $n = 1$. Then, there exist $a, b, c, d \in M$, such that $c(a, b) = (1_M, 1_M)$ and $d(a, b) = (m, m')$, that is, $ca = 1_M$, $cb = 1_M$, $da = m$ and $db = m'$. Since M satisfies the cancellation law, $ca = cb$ implies $a = b$, and therefore $m = m'$. Now, suppose that, for any sequence (4.4) of length n between $(1_M, 1_M)$ and (m, m') , for every pair $((m_i, m'_i), (m_{i+1}, m'_{i+1}))$ we must have $m_i = m'_i$, $m_{i+1} = m'_{i+1}$, with $0 \leq i \leq n-1$. Then, for a sequence as in (4.4) of length $n+1$ between $(1_M, 1_M)$ and (m, m') , we have

$$(1_M, 1_M), (m_1, m_1), \dots, (m_n, m_n), (m, m'),$$

such that there exist $a, b, c, d \in M$ with $c(a, b) = (m_n, m_n)$ and $d(a, b) = (m, m')$, that is, $ca = m_{n-1}$, $cb = m_{n-1}$, $da = m$, and $db = m'$. By the cancellation law, $a = b$, and therefore $m = m'$. Hence, $I(F(x) \times F(x)) \neq T$. Thus, if the monoid $M(\neq T)$ satisfies the cancellation law this reflection does not have stable units, according to Proposition 4.4, although it is semi-left-exact (simple, see Proposition 4.2).

WHEN M HAS A ZERO ELEMENT

We will see that $I(F(x) \times F(x)) = T$, provided M has a zero element 0_M .

$I(F(x) \times F(x)) = T$ if and only if $(0_M, 0_M) \sim_{M \times M} (m, m')$, for all $m, m' \in M$.

According to (4.3), $(0_M, 0_M) \sim_{M \times M} (m, m')$ if there exist a, b, c , and $d \in M$, such that $c(a, b) = (0_M, 0_M)$ and $d(a, b) = (m, m')$, that is, $ca = 0_M$,

$cb = 0_M$, $da = m$, and $db = m'$. This condition is satisfied for every pair (m, m') by taking $c = 0_M$, $a = m$, $b = m'$, and $d = 1_M$.

Hence, if the monoid M has a zero element then the reflection $H \vdash I : M\text{-Set} \rightarrow \mathbf{Set}$ has stable units (see Proposition 4.4).

Finite product preservation
for reflections $\mathbf{I} : \mathcal{C} \rightarrow \mathcal{M}$ with unit $\eta : 1_{\mathcal{C}} \rightarrow HI$,
satisfying *Ground Data*, into idempotent subcategories \mathcal{M}

\mathcal{M} is an idempotent subcategory of \mathcal{C}	\mathbf{I} preserves $\mathbf{Q} \times \mathbf{R}$ if and only if for each $\mathbf{r} \in \mathbf{U}(\mathbf{R})$, there exists $\gamma_{\mathbf{r}} : \mathbf{Q} \rightarrow \mathbf{HI}(\mathbf{Q} \times \mathbf{R})$, such that $\mathbf{U}(\gamma_{\mathbf{r}})(\mathbf{d}) = \mathbf{U}(\eta_{\mathbf{Q} \times \mathbf{R}})(\mathbf{d}, \mathbf{r})$, for all $\mathbf{d} \in \mathbf{U}(\mathbf{Q})$; and for each $\mathbf{q} \in \mathbf{U}(\mathbf{Q})$, there exists $\gamma_{\mathbf{q}} : \mathbf{Q} \rightarrow \mathbf{HI}(\mathbf{Q} \times \mathbf{R})$, such that $\mathbf{U}(\gamma_{\mathbf{q}})(\mathbf{e}) = \mathbf{U}(\eta_{\mathbf{Q} \times \mathbf{R}})(\mathbf{q}, \mathbf{e})$, for all $\mathbf{e} \in \mathbf{U}(\mathbf{R})$	Pr. 3.3
\mathcal{C} is an idempotent category	\mathbf{I} preserves finite products	Pr. 3.4
\mathcal{M} is an idempotent subvariety of a variety \mathcal{C} of universal algebras	\mathbf{I} preserves finite products if and only if $\mathbf{I}(\mathbf{F}(\mathbf{x}) \times \mathbf{F}(\mathbf{x})) = \mathbf{T}$	Pr. 3.8
\mathcal{C} is an idempotent variety of universal algebras	\mathbf{I} preserves finite products	Cor. 3.5

**Stable units for reflections I: $\mathcal{C} \rightarrow \mathcal{M}$
into idempotent subvarieties of universal algebras**

\mathcal{M} is an idempotent subvariety of \mathcal{C}	<i>the reflection has stable units if and only if it is simple and I preserves $\mathbf{F}(\mathbf{x}) \times \mathbf{F}(\mathbf{x})$</i>	<i>Pr. 4.2 and Pr. 4.4</i>
\mathcal{C} is an idempotent variety	<i>the reflection has stable units if and only if it is simple</i>	<i>Pr. 4.2 and Cor. 4.5</i>

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