On Herbrand's Theorem for Hybrid Logic*

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Abstract

The original version of Herbrand's theorem [8] for first-order logic provided the theoretical underpinning for automated theorem proving, by allowing a constructive method for associating with each first-order formula χ a sequence of quantifier-free formulas $\chi_1, \chi_2, \chi_3, \cdots$ so that χ has a first-order proof if and only if some χ_i is a tautology. Some other versions of Herbrand's theorem have been developed for classical logic, such as the one in [6], which states that a set of quantifier-free sentences is satisfiable if and only if it is propositionally satisfiable. The literature concerning versions of Herbrand's theorem proved in the context of non-classical logics is meager. We aim to investigate in this paper two versions of Herbrand's theorem for hybrid logic, which is an extension of modal logic that is expressive enough so as to allow identifying specific sates of the corresponding models, as well as describing the accessibility relation that connects these states, thus being completely suitable to deal with relational structures [3]. Our main results state that a set of satisfaction statements is satisfiable in a hybrid interpretation if and only if it is propositionally satisfiable.

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1 Introduction

Hybrid logics [3] are a breed of modal logics that provide appropriate syntax for referring to the associated possible-worlds semantics through the use of nominals. The latter, in particular, add to the modal description of relational structures the ability to refer to specific states. If modal logics have been successfully employed in specifying reactive systems, the hybrid component adds to them enough expressivity so as to refer to individual states and to reason about the system's local behavior at each of these states. Hybrid logics turn out thus to be strictly more expressive than their modal fragments. For example, irreflexivity ($i \to \neg\Diamond i$), asymmetry ($i \to \neg\Diamond\Diamond i$) or antisymmetry ($i \to \Box(\Diamond i \to i)$) are properties of the underlying transition structure which fail to be definable in standard modal logic (see [4]). Nonetheless, for the propositional case the satisfiability problem for hybrid logics is still decidable.

An important feature of hybrid logics that will play a central role in our approach is the fact that they allow for the specification of Robinson Diagrams [2]. Indeed, in these logics one may: (1) express equality between states named by i and j (note that $@_i j$ intends to affirm that the states named by i and j are identical, while $@_i \neg j$, being logically equivalent to $\neg @_i j$, intends to affirm that states i and j are distinct); (2) talk about accessibility between states through a modality (note that $@_i \Diamond j$ intends to affirm that the state named by j is a successor of the state named by i); (3) formulate satisfiability statements about a specific state (note that $@_i p$ intends to affirm that the proposition p is true at the state named by i, while $@_i \neg p$, being logically equivalent to $\neg @_i p$, intends to deny this). Consequently, within a hybrid logic one is able to completely describe the corresponding models using the rich underlying syntax.

Herbrand's theorem is a fundamental result of mathematical logic. It essentially allows a certain kind of reduction of first-order logic to propositional logic. While not aimed at providing an efficient procedure for (semi)-decidability, Herbrand-like theorems are ordinarily used as useful intermediate steps in proving that some theorem-proving resolution-based method works as intended. Several versions of Herbrand's theorem are now available for classical logic; here we present two versions for hybrid logics, using the concepts of satisfiability and propositional satisfiability, following the approach described in [6].

Outline of the paper. In Section 2 we start by recalling the basic hybrid logic. Theorem 2.13, our first Herbrand-like theorem, states that hybrid satisfiability is equivalent to propositional satisfiability for sets of satisfaction statements containing the equality axioms. In Section 3 we discuss the quantified hybrid logic — a logic less known than the basic hybrid logic. The strategy to establish a Herbrand-like theorem in this case follows the one for the classical first-order version, by making use of Skolemization to eliminate the existential quantifiers on world variables. The main result here is stated on Theorem 3.26. Section 4 wraps up with some pointers for future investigation.

2 The Case of the Basic Hybrid Logic

The simplest form of hybrid logic is based on the *basic hybrid language*, which adds nominals and the satisfaction operator to the language of propositional modal logic. This simple upgrade of the usual modal language carries great power in terms of expressivity.

Definition 2.1. Let $\mathcal{L} = \langle \text{Prop}, \text{Nom} \rangle$ be a *hybrid signature*, where Prop is a denumerable set of *propositional symbols* and Nom is a denumerable set of symbols disjoint from Prop. We use p, q, r and so on to refer to the elements in Prop. The elements in Nom are called *nominals* and we typically write them as i, j, k, and so on. The hybrid formulas over $\mathcal{L}$, which we denote by $\text{Form}_{@}(\mathcal{L})$, are defined by the following grammar:

$$\varphi ::= \ i \mid p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Diamond\varphi \mid @_i\varphi$$
$$\text{where } i \in \text{Nom and } p \in \text{Prop}.$$

The formulas with prefix @ are called *satisfaction statements*. The connectives $\vee$, $\rightarrow$, and $\Box$ are defined as usual. ◀

Definition 2.2. Let $\mathcal{L} = \langle \text{Prop}, \text{Nom} \rangle$ be a hybrid signature. A *hybrid structure* $\mathcal{M}$ *over* $\mathcal{L}$ is a tuple (W, R, N, V). Here, W is a non-empty set called *domain* whose elements are called *states* or *worlds*, $R \subseteq W \times W$ is called *accessibility relation*, $N : \text{Nom} \rightarrow W$ is a *hybrid nomination* and $V : \text{Prop} \rightarrow Pow(W)$ is a *hybrid valuation*. The pair $\langle W, R \rangle$ is called the *frame* underlying $\mathcal{M}$, and $\mathcal{M}$ is said to be a structure based on this frame. ◀

The satisfaction relation, which is defined next, is a generalization of Kripke-style satisfaction.

Definition 2.3. The *satisfaction relation* $\Vdash$ between a hybrid structure $\mathcal{M} = (W, R, N, V)$, a state $w \in W$, and a hybrid formula is recursively defined by:

- $\mathcal{M}, w \Vdash i$ iff $w = N(i)$;
- $\mathcal{M}, w \Vdash p$ iff $w \in V(p)$;
- $\mathcal{M}, w \Vdash \neg\varphi$ iff it is not the case that $\mathcal{M}, w \Vdash \varphi$;

- $\mathcal{M}, w \Vdash \varphi_1 \wedge \varphi_2$ iff $\mathcal{M}, w \Vdash \varphi_1$ and $\mathcal{M}, w \Vdash \varphi_2$;
- $\mathcal{M}, w \Vdash \Diamond\varphi$ iff $\exists w' \in W(wRw'$ and $\mathcal{M}, w' \Vdash \varphi)$;
- $\mathcal{M}, w \Vdash @_i\varphi$ iff $\mathcal{M}, w' \Vdash \varphi$, where $w' = N(i)$.

If $\mathcal{M}, w \Vdash \varphi$ we say that φ *is satisfied in* $\mathcal{M}$ *at* w. If φ is satisfied at all states in a structure $\mathcal{M}$, we write $\mathcal{M} \Vdash \varphi$. If φ is satisfied at all states in all structures based on a frame $\mathcal{F}$, then we say that φ *is valid on* $\mathcal{F}$ and we write $\mathcal{F} \Vdash \varphi$. If φ is valid on all frames, then we simply say that φ *is valid* and we write $\Vdash \varphi$. We say that a set Φ of hybrid formulas is *satisfiable* if there exists a model $\mathcal{M}$ and a world $w \in W$ such that $\mathcal{M}, w \Vdash \Phi$, i.e., $\mathcal{M}, w \Vdash \varphi$ for all $\varphi \in \Phi$. For $\Delta \subseteq \mathrm{Form}_@(\mathcal{L})$, we say that $\mathcal{M}$ *is a model of* Δ if $\mathcal{M} \Vdash \delta$ for all $\delta \in \Delta$. ◀

Definition 2.4. Let $\mathcal{L}$ be a hybrid signature. The set $\mathrm{At}(\mathcal{L})$ of *atomic satisfaction statements* (*atoms*, for short) *over* $\mathcal{L}$ is the set of $\mathcal{L}$-formulas of the forms $@_ip$, $@_i\Diamond j$, and $@_ij$ for $i, j \in \mathrm{Nom}$ and $p \in \mathrm{Prop}$. We use $\mathrm{BCAt}(\mathcal{L})$ to denote the set of all (finite) Boolean combinations of atomic satisfaction statements over $\mathcal{L}$, i.e., $\mathrm{BCAt}(\mathcal{L})$ is the smallest set containing $\mathrm{At}(\mathcal{L})$ and closed under $\wedge$ and $\neg$. ◀

Definition 2.5. An $\mathcal{L}$*-truth assignment* is a mapping $v : \mathrm{At}(\mathcal{L}) \to \{T, F\}$. Given an $\mathcal{L}$-truth assignment v, one may extend it to $\overline{v} : \mathrm{BCAt}(\mathcal{L}) \to \{T, F\}$ through the truth-functional interpretation of the propositional connectives. In order to simplify notation, given that this extension is unique, we will use v in order to refer both to an $\mathcal{L}$-truth assignment and to its extension $\overline{v}$. Let $\Phi \subseteq \mathrm{BCAt}(\mathcal{L})$. We say that Φ is *propositionally satisfiable* if there is an $\mathcal{L}$-truth assignment that simultaneously satisfies every member of Φ. We say that Φ is *propositionally unsatisfiable* if there is no such $\mathcal{L}$-truth assignment. ◀

We have now the basis to start investigating a first Herbrand-like theorem for hybrid logic:

Theorem 2.6. *Let* $\Phi \subseteq \mathrm{BCAt}(\mathcal{L})$. *If* Φ *is propositionally unsatisfiable then* Φ *is unsatisfiable.*

Proof. Suppose that Φ is satisfiable: then there is a model $\mathcal{M}$ and a world $w \in W$ such that $\mathcal{M}, w \Vdash \Phi$, i.e., $\mathcal{M}, w \Vdash \varphi$ for all $\varphi \in \Phi$.

Define $v^{\mathcal{M}} : \mathrm{At}(\mathcal{L}) \to \{T, F\}$ by setting $v^{\mathcal{M}}(\psi) = T$ iff $\mathcal{M}, w \Vdash \psi$.

Let us prove by induction on the structure of $\varphi \in \mathrm{BCAt}(\mathcal{L})$ that $v^{\mathcal{M}}(\varphi) = T$ iff $\mathcal{M}, w \Vdash \varphi$.

- If $\varphi \in \mathrm{At}(\mathcal{L})$, the result follows from the definition of $v^{\mathcal{M}}$.

- Suppose now, by Induction Hypothesis, (IH), that $\mathcal{M}, w \Vdash \varphi_i$ iff $v^{\mathcal{M}}(\varphi_i) = T$, for $i = 1, 2$.
  - If $\varphi = \varphi_1 \wedge \varphi_2$, then

$$\begin{array}{rcl} \mathcal{M}, w \Vdash \varphi & \text{iff} & \mathcal{M}, w \Vdash \varphi_1 \wedge \varphi_2 \\ & \text{iff} & \mathcal{M}, w \Vdash \varphi_1 \text{ and } \mathcal{M}, w \Vdash \varphi_2 \\ & \underset{\text{(IH)}}{\text{iff}} & v^{\mathcal{M}}(\varphi_1) = T \text{ and } v^{\mathcal{M}}(\varphi_2) = T \\ & \text{iff} & v^{\mathcal{M}}(\varphi_1 \wedge \varphi_2) = T \\ & \text{iff} & v^{\mathcal{M}}(\varphi) = T \end{array}$$

  - If $\varphi = \neg\psi$, then

$$\begin{array}{rcl} \mathcal{M}, w \Vdash \varphi & \text{iff} & \mathcal{M}, w \Vdash \neg\psi \\ & \text{iff} & \mathcal{M}, w \nVdash \psi \\ & \underset{\text{(IH)}}{\text{iff}} & v^{\mathcal{M}}(\psi) = F \\ & \text{iff} & v^{\mathcal{M}}(\neg\psi) = T \\ & \text{iff} & v^{\mathcal{M}}(\varphi) = T \end{array}$$

Since $\mathcal{M}, w \Vdash \Phi$, by assumption, we have that $v^{\mathcal{M}}(\varphi) = T$ for any $\varphi \in \Phi$. Therefore, Φ is propositionally satisfiable. ■

Example 2.7. Let $\mathcal{L} = \langle\{p, q\}, \{i, j\}\rangle$, and $\Phi = \{@_i p \vee @_i q, @_j \neg q, @_i j, @_i \Diamond j\}$.

The set Φ is satisfiable, as there is a model $\mathcal{M} = (W, R, N, V)$ such that $W = \{w\}$, $R = \{(i, i)\}$, $N(i) = N(j) = w$, $V(p) = \{w\}$ and $V(q) = \varnothing$, where $\mathcal{M}, w \Vdash \Phi$.

Define $v^{\mathcal{M}} : \text{At}(\mathcal{L}) \to \{T, F\}$ by setting $v^{\mathcal{M}}(\psi) = T$ iff $\mathcal{M}, w \Vdash \psi$. This implies that $v^{\mathcal{M}}(@_i p) = T$, $v^{\mathcal{M}}(@_i \Diamond j) = T$, $v^{\mathcal{M}}(@_i j) = T$ and for all other atomic satisfaction statements in $\mathcal{L}$, $v^{\mathcal{M}}$ assigns F. The extension of $v^{\mathcal{M}}$ to $\overline{v^{\mathcal{M}}}$ is straightforward. Thus Φ is propositionally satisfiable. ♦

The converse of the previous theorem is not true in general. Here is a counter-example:

Example 2.8. Let $\mathcal{L} = \langle\{p\}, \{i, j\}\rangle$, and $\Phi = \{@_i j, @_i p, @_j \neg p\}$.

Note that Φ is propositionally satisfiable: take $v^{\mathcal{M}} : \text{At}(\mathcal{L}) \to \{T, F\}$ to be such that $v^{\mathcal{M}}(@_i p) = T$, $v^{\mathcal{M}}(@_i j) = T$, and $v^{\mathcal{M}}$ assigns the value F to all other atomic satisfaction statements.

However, Φ is not satisfiable, as there is no model $\mathcal{M}$ such that $\mathcal{M}, w \Vdash \Phi$. Any model that satisfies the first formula in Φ has that $N(i) = N(j) = w$. From the second and the third formulas, one must have that $w \in V(p)$ and $w \notin V(p)$, respectively, which is a contradiction. ♦

As in the case of first-order logic with equality, the characteristic equality axioms need to be taken into consideration. In hybrid logic we do not have an explicit symbol of equality in the language; however, there are hybrid formulas that express the equality axioms over nominals in $\mathcal{L}$ (see [3]):

- *Reflexivity*: $@_i i$, for $i \in \mathrm{Nom}$;
- *Symmetry*: $@_i j \to @_j i$, for $i, j \in \mathrm{Nom}$;
- *Nom*: $(@_i \varphi \wedge @_i j) \to @_j \varphi$, for $i, j \in \mathrm{Nom}$ and $@_i \varphi$ an atomic satisfaction statement;
- *Bridge*: $(@_i \Diamond j \wedge @_j k) \to @_i \Diamond k$, for $i, j, k \in \mathrm{Nom}$.

The set of all equality axioms over the hybrid signature $\mathcal{L}$ is denoted by $\mathrm{Eq}(\mathcal{L})$. It is easy to check that these formulas are all valid in hybrid logic. Note that *Bridge* does not follow from the other axioms, as it is the only axiom where nominals are replaced in formula position.

Lemma 2.9. *Let $\mathcal{M}$ be a model and φ be a formula in* $\mathrm{BCAt}(\mathcal{L})$. *Then,*

$$\exists w \in W : \mathcal{M}, w \Vdash \varphi \text{ iff } \mathcal{M} \Vdash \varphi$$

Proof. We will check this result by induction on the structure of $\varphi \in \mathrm{BCAt}(\mathcal{L})$:
• For $\varphi = @_i \psi$ an atomic satisfaction statement:

$$\begin{array}{lll} \exists w \in W : \mathcal{M}, w \Vdash \varphi & \text{iff} & \exists w \in W : \mathcal{M}, w \Vdash @_i \psi \\ & \text{iff} & \mathcal{M}, w' \Vdash \psi, \text{ where } w' = N(i) \\ & \text{iff} & \mathcal{M} \Vdash @_i \psi \\ & \text{iff} & \mathcal{M} \Vdash \varphi \end{array}$$

• Suppose by (IH) that ψ and θ are such that the result holds. Then,
– For $\varphi = \neg\psi$:

$$\begin{array}{lll} \exists w \in W : \mathcal{M}, w \Vdash \varphi & \text{iff} & \exists w \in W : \mathcal{M}, w \Vdash \neg\psi \\ & \text{iff} & \exists w \in W : \mathcal{M}, w \nVdash \psi \\ & \underset{\text{(IH)}}{\text{iff}} & \mathcal{M} \nVdash \psi \\ & \text{iff} & \forall w \in W : \mathcal{M}, w \nVdash \psi \\ & \text{iff} & \forall w \in W : \mathcal{M}, w \Vdash \neg\psi \\ & \text{iff} & \mathcal{M} \Vdash \neg\psi \\ & \text{iff} & \mathcal{M} \Vdash \varphi \end{array}$$

– For $\varphi = \varphi_1 \wedge \varphi_2$:

For one implication:

$$
\begin{array}{rl}
\multicolumn{2}{l}{\exists w \in W : \mathcal{M}, w \Vdash \varphi} \\
\text{iff} & \exists w \in W : \mathcal{M}, w \Vdash \varphi_1 \wedge \varphi_2 \\
\text{iff} & \exists w \in W : (\mathcal{M}, w \Vdash \varphi_1 \text{ and } \mathcal{M}, w \Vdash \varphi_2) \\
\text{implies} & \exists w \in W : \mathcal{M}, w \Vdash \varphi_1 \text{ and } \exists w \in W : \mathcal{M}, w \Vdash \varphi_2 \\
\underset{\text{(IH)}}{\text{iff}} & \mathcal{M} \Vdash \varphi_1 \text{ and } \mathcal{M} \Vdash \varphi_2 \\
\text{iff} & \forall w \in W : \mathcal{M}, w \Vdash \varphi_1 \text{ and } \forall w \in W : \mathcal{M}, w \Vdash \varphi_2 \\
\text{iff} & \forall w \in W : \mathcal{M}, w \Vdash \varphi_1 \text{ and } \mathcal{M}, w \Vdash \varphi_2 \\
\text{iff} & \forall w \in W : \mathcal{M}, w \Vdash \varphi_1 \wedge \varphi_2 \\
\text{iff} & \mathcal{M} \Vdash \varphi_1 \wedge \varphi_2 \\
\text{iff} & \mathcal{M} \Vdash \varphi
\end{array}
$$

For the converse implication:

$$
\begin{array}{lcl}
\mathcal{M} \Vdash \varphi & \text{iff} & \mathcal{M} \Vdash \varphi_1 \wedge \varphi_2 \\
 & \text{iff} & \forall w \in W : \mathcal{M}, w \Vdash \varphi_1 \wedge \varphi_2 \\
 & \underset{\text{(given that } W \neq \varnothing)}{\text{implies}} & \exists w \in W : \mathcal{M}, w \Vdash \varphi_1 \wedge \varphi_2 \\
 & \text{iff} & \exists w \in W : \mathcal{M}, w \Vdash \varphi
\end{array}
$$

■

Let us consider next the binary relation $\sim$ defined on Nom by setting $i \sim j$ iff $v\,(@_i j) = T$.

Lemma 2.10. *The binary relation $\sim$ is an equivalence relation.*

Proof. [Reflexivity] is guaranteed by the homonymous axiom stated above, namely $@_i i$, for $i \in$ Nom. Once $\text{Eq}(\mathcal{L}) \subseteq \Phi$, then $v(@_i i) = T$ implies $i \sim i$.

[Symmetry] holds due to the fact that if $i \sim j$, then $v(@_i j) = T$, and given that $\text{Eq}(\mathcal{L}) \subseteq \Phi$, we have $v(@_i j \to @_j i) = T$, which implies that $v(@_j i) = T$. So, $j \sim i$.

[Transitivity] follows from *Symmetry* and the axiom *Nom*. Suppose $i \sim j$ and $j \sim k$. By [Symmetry] it follows that $j \sim i$ and $j \sim k$, thus $v(@_j i) = T$ and $v(@_j k) = T$. Once more, since $\text{Eq}(\mathcal{L}) \subseteq \Phi$, we have in particular that $v\,((@_j i \wedge @_j k) \to @_i k) = T$. We conclude that $v(@_i k) = T$, thus $i \sim k$. ■

The above result is crucial in proving Herbrand's Theorem for languages containing equality. Next we show that if for a set Φ of Boolean combinations of atomic satisfaction statements with equality there is a valuation v that assigns the value true to all atomic satisfaction statements in Φ, then there is a hybrid structure that satisfies the equality axioms and where Φ is satisfiable.

Theorem 2.11. *Assume* $\text{Eq}(\mathcal{L}) \subseteq \Phi \subseteq \text{BCAt}(\mathcal{L})$.
If Φ is unsatisfiable then Φ is propositionally unsatisfiable.

Proof. Suppose that Φ is propositionally satisfiable and let $v : \text{At}(\mathcal{L}) \to \{T, F\}$ be such that $v(\varphi) = T$ for any $\varphi \in \Phi$.

Let $W =$ Nom. We define the hybrid structure $\mathcal{M} = (W_v, R_v, N_v, V_v)$ such that:

- $W_v = W/\sim$;
- $[i]R_v[j]$ iff $v\,(@_i\Diamond j) = T$, for $i, j \in \text{Nom}$;
- $N_v(j) = [i]$ iff $v\,(@_i j) = T$, for $i, j \in \text{Nom}$; and
- $[i] \in V_v(p)$ iff $v\,(@_i p) = T$, for $i \in \text{Nom}$, $p \in \text{Prop}$.

Claim I. R_v is well-defined.

We want to prove that if $i \sim j$ and $k \sim l$, then $[i]R_v[k]$ implies $[j]R_v[l]$.

– Suppose that $i \sim j, k \sim l$ and $[i]R_v[k]$. By definition, we know that $[i]R_v[k]$ means that $v\,(@_i\Diamond k) = T$, and $i \sim j$ means that $v\,(@_i j) = T$. It follows that $v\,(@_i\Diamond k \wedge @_i j) = T$. The axiom *Nom* let us conclude then that $v\,(@_j\Diamond k) = T$. We also know that $k \sim l$ means that $v\,(@_k l) = T$. From the axiom *Bridge*, since $v\,(@_j\Diamond k \wedge @_k l) = T$, it follows that $v\,(@_j\Diamond l) = T$. Therefore, by definition, $[j]R_v[l]$.

Claim II. V_v is well-defined.

We want to prove that if $i \sim j$ then ($[i] \in V_v(p)$ iff $[j] \in V_v(p)$).

– Suppose that $i \sim j$ and $[i] \in V_v(p)$. By the definition of the equivalence relation $\sim$, $v(@_i j) = T$; and by the definition of V_v, $v(@_i p) = T$. Then $v(@_i p \wedge @_i j) = T$ and from *Nom* it follows that $v(@_j p) = T$. So, $[j] \in V_v(p)$. The converse direction is checked analogously in view of the symmetry of $\sim$.

All that is left to prove now is the satisfiability of Φ.

Claim III. For all $\varphi \in \text{BCAt}(\mathcal{L})$, $(\mathcal{M} \Vdash \varphi$ iff $v(\varphi) = T)$.

Below you should recall that for Boolean combinations of atomic satisfaction statements, satisfiability at one state is equivalent to satisfiability at all states, by Lemma 2.9.

- $\varphi = @_i p$

$$\begin{array}{lll} \mathcal{M} \Vdash @_i p & \text{iff} & \mathcal{M}, [i] \Vdash p \\ & \text{iff} & [i] \in V_v(p) \\ & \text{iff} & v(@_i p) = T \\ & \text{iff} & v(\varphi) = T \end{array}$$

- $\varphi = @_i\Diamond j$

$$\begin{array}{lll} \mathcal{M} \Vdash @_i\Diamond j & \text{iff} & \mathcal{M}, [i] \Vdash \Diamond j \\ & \text{iff} & \exists k :\ [i]R_v[k] \text{ and } \mathcal{M}, [k] \Vdash j \\ & \text{iff} & \exists k :\ [i]R_v[k] \text{ and } [k] = [j] \\ & \text{iff} & [i]R_v[j] \\ & \text{iff} & v(@_i\Diamond j) = T \\ & \text{iff} & v(\varphi) = T \end{array}$$

- $\varphi = @_i j$

$$\begin{array}{lll} \mathcal{M} \Vdash @_i j & \text{iff} & \mathcal{M}, [i] \Vdash j \\ & \text{iff} & [i] = [j] \\ & \text{iff} & v(@_i j) = T \\ & \text{iff} & v(\varphi) = T \end{array}$$

- By (IH), let φ_1, φ_2 be such that $\mathcal{M} \Vdash \varphi_i$ iff $v(\varphi_i) = T$, for $i = 1, 2$.

  This part is similar to Theorem 2.6, so we omit the details.

  – Given $\varphi = \varphi_1 \wedge \varphi_2$, note that

  $$\mathcal{M} \Vdash \varphi_1 \wedge \varphi_2 \quad \text{iff} \quad v(\varphi_1 \wedge \varphi_2) = T$$

  – Given $\varphi = \neg\varphi_1$, note that

  $$\mathcal{M} \Vdash \neg\varphi_1 \quad \text{iff} \quad v(\neg\varphi_1) = T$$

Thus, in particular, $\mathcal{M} \Vdash \Phi$, and this means that Φ is satisfiable. ■

We finish this section by generalizing the above results to compound satisfaction statements. Let φ be any satisfaction statement. The following rules allow us to rewrite φ by recursively applying the following rules in order to obtain a semantically equivalent formula $\varphi^\circ \in \mathrm{BCAt}(\mathcal{L}^*)$, where $\mathcal{L}^*$ is an expansion of $\mathcal{L}$ obtained by the addition of new nominals to the initial hybrid signature. Observe that such extension is possible since we considered Nom to be a denumerable set.

Rewrite Rules:

1. $@_i @_j \varphi \twoheadrightarrow @_j \varphi$
2. $@_i \neg\varphi \twoheadrightarrow \neg @_i \varphi$
3. $@_i (\varphi \wedge \psi) \twoheadrightarrow @_i \varphi \wedge @_i \psi$
4. $@_i \Diamond\varphi \twoheadrightarrow @_i \Diamond k \wedge @_k \varphi$, for k a fresh nominal

As the above rules successively decrease the complexity of satisfaction statements, it is clear that the associated rewrite system is terminating. In fact, by using the Knuth-Bendix completion algorithm it is easy to see that the rewrite system is also confluent. In this respect, it is worth noting that the formula $@_i @_j \Diamond\varphi$ may rewrite in two ways, namely as $@_j \Diamond k_1 \wedge @_{k_1} \varphi$ and as $@_j \Diamond k_2 \wedge @_{k_2} \varphi$. These are the same, however, modulo the introduced fresh nominals.Moreover, we should point out that Areces and Gorín, in [1], have investigated labeled resolution calculi for hybrid logics with inference rules similar to the above rewrite rules; namely our rules 1., 3. and 4. correspond to their $@$, $\wedge$ and $\langle r \rangle$ rules, respectively.

Example 2.12. Consider the formula $\varphi = @_i @_j \Diamond(p \wedge \neg q)$ in $\mathcal{L}$. It is clear that φ is not a Boolean combination of atomic satisfaction statements of $\mathcal{L}$.

Applying the rewrite rules yields that:

$$
\begin{aligned}
@_i @_j \Diamond(p \wedge \neg q) &\twoheadrightarrow @_j \Diamond(p \wedge \neg q) \\
&\twoheadrightarrow @_j \Diamond k \wedge @_k(p \wedge \neg q),\ k \text{ fresh} \\
&\twoheadrightarrow @_j \Diamond k \wedge (@_k p \wedge @_k \neg q) \\
&\twoheadrightarrow @_j \Diamond k \wedge (@_k p \wedge \neg @_k q)
\end{aligned}
$$

Thus $\varphi^\circ = @_j \Diamond k \wedge (@_k p \wedge \neg @_k q)$. Note that the new formula is in the hybrid signature $\mathcal{L}^*$ that expands $\mathcal{L}$ by the addition of the new nominal k. ⧫

Theorem 2.13 (Herbrand-like)**.** *Let* Φ *be a set of satisfaction statements such that* $\mathrm{Eq}(\mathcal{L}) \subseteq \Phi$*. Then* Φ *is propositionally unsatisfiable iff* Φ *is unsatisfiable.*

Proof. We exhaustively apply the previously introduced rules to the formulas of Φ and transform Φ into $\Phi^\circ := \{\varphi^\circ : \varphi \in \Phi\} \cup \mathrm{Eq}(\mathcal{L}^*)$. Note that Φ° is a subset of $\mathrm{BCAt}(\mathcal{L}^*)$, which contains the equality axioms in the expanded language, thus we may apply Theorems 2.6 and 2.11. ∎

3 The Case of Quantified Hybrid Logic

In this section we introduce a hybrid logic enriched with operators over world variables, typically written as s, t, u and so on, distinct from both nominals and propositional variables. We will also resort to an algebraic similarity type in order to allow function symbols. This logic, which we will call Algebraic Strong Priorean Logic, shares some similarities with the logic $\mathcal{HLCOV}(@, \forall, \exists)$ found in [9], namely in the use of quantifiers and functions, but it differs in the definition of terms; in particular, while $\mathcal{HLCOV}(@, \forall, \exists)$ allows for quantification over both state variables and functional terms, the Algebraic Strong Priorean Logic restricts quantifications to state variables.

Definition 3.1. An *algebraic similarity type* Σ is a tuple (F, σ) such that F is a non-empty set of function symbols, and σ assigns to each function symbol its arity. An algebraic similarity type together with a denumerable set of world variables, WVar, and a denumerable set of nominals, Nom, induces the set $\mathrm{Term}(\Sigma, \mathrm{WVar}, \mathrm{Nom})$ of Σ-terms, whose elements are the algebraic terms given by the grammar:

$$
t ::= \ i \mid s \mid f\big(t_1, \cdots, t_{\sigma(f)}\big)
$$

where $i \in \mathrm{Nom}, s \in \mathrm{WVar}$ and $f \in \mathrm{F}$. ◀

We may now introduce a powerful hybrid language, $\mathcal{H}(\Sigma, @, \forall)$, whose grammar is defined below:

Definition 3.2. A *hybrid similarity type* L is a tuple $(\mathrm{Prop}, \mathrm{Nom}, \mathrm{WVar})$, where Prop and Nom are as usual the set of propositional variables and the set of nominals of a hybrid signature, and WVar is a denumerable set of world variables. Let $\Sigma = \langle \mathrm{F}, \sigma \rangle$ be an algebraic similarity type. The well-formed formulas $\mathrm{Form}_{@,\forall}(L, \mathrm{Term}(\Sigma, \mathrm{WVar}, \mathrm{Nom}))$ over the hybrid similarity type L and the Σ-terms $\mathrm{Term}(\Sigma, \mathrm{WVar}, \mathrm{Nom})$ are defined by the following grammar:

$$\varphi ::= \ p \mid t \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Diamond\varphi \mid @_t\varphi \mid \forall s\, \varphi \mid \exists s\, \varphi$$
$$\text{where } p \in \mathrm{Prop},\ t \in \mathrm{Term}(\Sigma, \mathrm{WVar}, \mathrm{Nom}) \text{ and } s \in \mathrm{WVar}.$$

Note that @ can make use of Σ-terms, i.e., world variables and functional terms. The connectives $\vee$, $\rightarrow$, and $\Box$ are defined as usual. ◀

The earlier definition of a 'hybrid structure' is now upgraded as follows:

Definition 3.3. Let $L = \langle \mathrm{Prop}, \mathrm{Nom}, \mathrm{WVar} \rangle$ and $\Sigma = \langle \mathrm{F}, \sigma \rangle$ be, respectively, a hybrid and an algebraic similarity types. A *hybrid structure* $\mathcal{H}$ *over* $\langle L, \Sigma \rangle$ is a tuple $(W, R, \left(f^W\right)_{f \in \mathrm{F}}, N, V)$, where W, R, N and V are the domain, accessibility relation, hybrid nomination and valuation as introduced in Definition 2.2, and $\left(f^W\right)_{f \in \mathrm{F}}$ is a family containing for each $f \in \mathrm{F}$ an interpretation $f^W : W^{\sigma(f)} \rightarrow W$. ◀

As we need a mechanism for coping with the terms introduced in the above grammars, we consider now a *world assignment* $g : WVar \rightarrow W$. Two world assignments g and g' are called *s-variant* iff $g(u) = g'(u)$, for all $u \in \mathrm{WVar}$ such that $u \neq s$; in such case we write $g \overset{s}{\sim} g'$. We extend g to $\mathrm{Term}(\Sigma, WVar)$ in the following way:

$$\bar{g}(t) = \begin{cases} g(t), & \text{if } t \in \mathrm{WVar} \\ N(t), & \text{if } t \in \mathrm{Nom} \\ f^W\left(\bar{g}(t_1), \ldots, \bar{g}(t_{\sigma(f)})\right), & \text{if } t = f(t_1, \ldots, t_{\sigma(f)}),\ \text{for some } f \in F \end{cases}$$

In order to simplify notation, we will use g to denote both a world assignment and its extension.

The notion of satisfaction is now defined in the following way:

Definition 3.4. The *satisfaction relation* $\Vdash$ between a hybrid structure $\mathcal{H} = (W, R, N, V)$, a state $w \in W$, a world assignment g and a hybrid formula is recursively defined by:

- $\mathcal{H}, g, w \Vdash p$ iff $w \in V(p)$, for $p \in \mathrm{Prop}$;
- $\mathcal{H}, g, w \Vdash t$ iff $w = g(t)$, for $t \in \mathrm{Term}(\Sigma, \mathrm{WVar}, \mathrm{Nom})$;
- $\mathcal{H}, g, w \Vdash \neg\varphi$ iff it is not the case that $\mathcal{H}, g, w \Vdash \varphi$;
- $\mathcal{H}, g, w \Vdash \varphi_1 \wedge \varphi_2$ iff $\mathcal{H}, g, w \Vdash \varphi_1$ and $\mathcal{H}, g, w \Vdash \varphi_2$;
- $\mathcal{H}, g, w \Vdash \Diamond\varphi$ iff $\exists w' \in W(wRw'$ and $\mathcal{H}, g, w' \Vdash \varphi)$;

- $\mathcal{H}, g, w \Vdash @_t\varphi$ iff $\mathcal{H}, g, w' \Vdash \varphi$, where $w' = g(t)$, for $t \in \text{Term}(\Sigma, \text{WVar}, \text{Nom})$;
- $\mathcal{H}, g, w \Vdash \forall s\, \varphi$ iff $\mathcal{H}, g', w \Vdash \varphi$ for all g' such that $g' \overset{s}{\sim} g$;
- $\mathcal{H}, g, w \Vdash \exists s\, \varphi$ iff $\mathcal{H}, g', w \Vdash \varphi$ for some g' such that $g' \overset{s}{\sim} g$.

Here, $\mathcal{H}, g, w \Vdash \varphi$ is read as saying that φ *is satisfied at the state* w *in the hybrid structure* $\mathcal{H}$ *under the world assignment* g. ◀

We shall use the appellation *Algebraic Strong Priorean Logic* to refer to the logic induced by the above notion of satisfaction. It is worth pointing out that the Algebraic Strong Priorean Logic contains the logic of the hybrid language with a binder, as $\downarrow s.\varphi$ is expressible here by $\exists s\,(s \wedge \varphi)$. Such logic is very expressive. The algebraic structure over the set of worlds may be useful in several contexts. Here are some examples: on *trees*, one can consider a functional symbol for referring to the first common ancestor of two given nodes; on the *graph representations of maps*, one can consider a functional symbol for referring to an intermediate city that minimizes the distance between two other given cities; on *temporal frames*, one can consider functional symbols that allow pointing to a specific time after or before the current moment, or a function that allows one to say that something happens periodically.

Definition 3.5. A set Φ of formulas in $\text{Form}_{@,\forall}(L, \text{Term}(\Sigma, \text{WVar}, \text{Nom}))$ is said to be *satisfiable* if there exists a hybrid structure $\mathcal{H}$ over $\langle L, \Sigma\rangle$, a $w \in W$ and a world assignment g such that $\mathcal{H}, g, w \Vdash \varphi$ for all $\varphi \in \Phi$. We say that $\varphi \in \text{Form}_{@,\forall}(L, \text{Term}(\Sigma, \text{WVar}, \text{Nom}))$ is *satisfiable* if the singleton $\{\varphi\}$ is satisfiable. ◀

Definition 3.6. A *literal* in $\mathcal{H}\,(\Sigma, @, \forall)$ is a formula of the form: $@_a p$, $@_a\neg p$, $@_a\Diamond b$, $@_a\neg\Diamond b$, where $p \in \text{Prop}$, and $a, b \in \text{Term}(\Sigma, \text{WVar}, \text{Nom})$. ◀

Lemma 3.7 (Labelling)**.**
Let φ *be a formula in* $\text{Form}_{@,\forall}(L, \text{Term}(\Sigma, \text{WVar}, \text{Nom}))$. *Then*

$$\varphi \textit{ is satisfiable iff } @_i\varphi \textit{ is satisfiable},$$

where i *is a fresh nominal.*

Proof.

$$\begin{aligned}
\varphi \text{ is satisfiable} \quad &\text{iff} \quad \exists\mathcal{H}, \exists g, \exists w : \mathcal{H}, g, w \Vdash \varphi \\
&\text{iff} \quad \exists\tilde{\mathcal{H}}, \exists g, \exists w : \tilde{\mathcal{H}}, g, w \Vdash \varphi,\ w = \tilde{N}(i) \\
&\text{iff} \quad \exists\tilde{\mathcal{H}}, \exists g, \exists\tilde{w} : \tilde{\mathcal{H}}, g, \tilde{w} \Vdash @_i\varphi \\
&\text{iff} \quad @_i\varphi \text{ is satisfiable}
\end{aligned}$$

■

Our goal in what follows is to study the satisfiability of a formula in the Algebraic Strong Priorean Logic. Since the satisfiability problem of a formula φ is equivalent to the satisfiability problem of a formula $@_i\varphi$ — where i does not occur in φ — we will prove satisfiability of the latter. In order to do so, it will be convenient to rearrange formulas so that we end up with a formula in Prenex Conjunctive Normal Form, i.e., a formula in which quantifiers appear on the left, prefixing a quantifier-free part that is a conjunction of clauses, where clauses are disjunctions of literals.

Definition 3.8. A formula is said to be *rectified* if no world variable occurs both bound and free and if all quantifiers in the formula refer to different world variables. ◄

The renaming of bound world variables follows the same approach as in first-order logic, whose proof is standard:

Lemma 3.9. *It is always possible to perform a systematic renaming of bound (world) variables such that the result is a rectified formula, equivalent to the original one in the following way: if s occurs bounded in a formula φ and u does not occur at all, then φ is equivalent to the formula obtained by replacing all occurrences of s in the scope of a quantifier in φ with u.*

Given a formula φ as input, we will refer to the formula $\tilde{\varphi}$ produced by the above renaming procedure *the rectified version of φ*.

Definition 3.10. Let $s_1, \ldots, s_n$ be the world variables occurring free in φ. The *[rectified] existential closure of φ* is the formula which results from rectifying φ and then existentially bounding its free variables, i.e., it is the formula $\exists s_1 \ldots \exists s_n\, \tilde{\varphi}$, where $\tilde{\varphi}$ is the rectified version of φ. ◄

Lemma 3.11. *A formula φ and its existential closure ψ are equisatisfiable.*

Proof.

$$
\begin{aligned}
&\psi \text{ is satisfiable}\\
&\quad \text{iff} \quad \exists\mathcal{H}, \exists g, \exists w : \mathcal{H}, g, w \Vdash \exists s_1 \ldots \exists s_n\, \varphi\\
&\quad \text{iff} \quad \exists\mathcal{H}, \exists g, \exists w : \mathcal{H}, g_1, w \Vdash \exists s_2 \ldots \exists s_n\, \varphi, \text{ for some } g_1 \overset{s_1}{\sim} g\\
&\quad \text{iff} \quad \exists\mathcal{H}, \exists g, \exists w : \mathcal{H}, g_2, w \Vdash \exists s_3 \ldots \exists s_n\, \varphi, \text{ for some } g_2 \overset{s_2}{\sim} g_1 \overset{s_1}{\sim} g\\
&\quad \text{iff} \quad \cdots\\
&\quad \text{iff} \quad \exists\mathcal{H}, \exists g, \exists w : \mathcal{H}, g_n, w \Vdash \varphi, \text{ for some } g_n \overset{s_n}{\sim} g_{n-1} \overset{s_{n-1}}{\sim} \cdots \overset{s_2}{\sim} g_1 \overset{s_1}{\sim} g\\
&\quad \text{iff} \quad \exists\mathcal{H}, \exists g_n, \exists w : \mathcal{H}, g_n, w \Vdash \varphi\\
&\quad \text{iff} \quad \varphi \text{ is satisfiable}
\end{aligned}
$$

■

Let us apply the latter two results in the following examples:

Example 3.12. Let $\varphi_1 = @_i(\Diamond p \wedge \neg @_s p)$.

– This formula is rectified.

– The existential closure of φ_1 is the formula $\psi_1 = \exists s\, @_i(\Diamond p \wedge \neg @_s p)$.

It is easy to check that φ_1 and ψ_1 are equisatisfiable. ♦

Example 3.13. Let $\varphi_2 = @_i\, (\neg\, (\forall s @_s \neg p \wedge \exists s\, @_s p) \wedge @_s \neg p)$.

– This formula is not rectified.

The renaming of variables leads to $@_i\, (\neg\, (\forall t @_t \neg p \wedge \exists u @_u p) \wedge @_s \neg p)$, which is equivalent to φ_2.

– The (rectified) existential closure of φ_2 is the formula

$$\psi_2 = \exists s\, @_i\, (\neg\, (\forall t @_t \neg p \wedge \exists u @_u p) \wedge @_s \neg p).$$

The formulas φ_2 and ψ_2 are equisatifiable. ♦

Example 3.14. Let $\varphi_3 = @_i(\forall s \exists t @_s \Diamond t)$.

– This formula is rectified.

– Since φ_3 does not have free world variables, it coincides with its existential closure, ψ_3. ♦

The following theorem allows us to convert a formula into an equivalent formula in Prenex Conjunctive Normal Form.

Theorem 3.15. *Let $L = \langle \mathrm{Prop}, \mathrm{Nom}, \mathrm{WVar} \rangle$ be a hybrid similarity type, Σ be an algebraic similarity type, and $\mathcal{H}$ be a hybrid structure over $\langle L, \Sigma \rangle$. For each formula of the form $@_i\varphi$, where $\varphi \in \mathrm{Form}_{(\Sigma,@,\forall)}(L)$ and $i \in \mathrm{Nom}$ does not occur in φ, its existential closure ψ is equivalent to a formula in Prenex Conjunctive Normal Form.*

Proof. Let ψ be a formula in the conditions of the theorem.

Step 1: Use the double negation law, the De Morgan's laws, the duality equivalences $\forall s\, \varphi \equiv \neg \exists s\, \neg \varphi$ and $\Diamond \varphi \equiv \neg \Box \neg \varphi$, and the following rewrite rules until no further transformations apply.

$@_a(\theta_1 \wedge \theta_2) \twoheadrightarrow @_a\theta_1 \wedge @_a\theta_2$	$@_a(\theta_1 \vee \theta_2) \twoheadrightarrow @_a\theta_1 \vee @_a\theta_2$
$\neg @_a\theta \twoheadrightarrow @_a\neg\theta$	$@_a @_b \theta \twoheadrightarrow @_b\theta$
$@_a\Diamond\theta \twoheadrightarrow \exists u(@_a\Diamond u \wedge @_u\theta)$	$@_a \exists s\, \theta \twoheadrightarrow \exists s\, @_a\theta$
$@_a\Box\theta \twoheadrightarrow \forall u(@_a\Box\neg u \vee @_u\theta)$	$@_a \forall s \theta \twoheadrightarrow \forall s @_a\theta$

where $a, b \in \mathrm{Nom} \cup \mathrm{Term}(\Sigma, \mathrm{WVar}, \mathrm{Nom})$ and $u \in \mathrm{WVar}$ does not occur in ψ.

Step 2: Flush all quantifiers to the prefix position, as usual, and the result is a formula in Prenex Normal Form (since the variables added in **Step 1** are new, the formula remains rectified). Apply the associative and distributive laws as necessary in order to reach a formula in Prenex Conjunctive Normal Form.

Due to the rectified nature of the formulas over which the transformations have been applied, the resulting formulas are equivalent to the original ones. ■

We return to the previous examples and apply the latter result:

Example 3.16. Let $\psi_1 = \exists s\, @_i(\Diamond p \wedge \neg @_s p)$:
Step 1:

$$\begin{array}{rcl} \exists s\, @_i\,(\Diamond p \wedge \neg @_s p) & \twoheadrightarrow & \exists s\,(@_i \Diamond p \wedge @_i \neg @_s p) \\ & \twoheadrightarrow & \exists s\,(\exists u\,(@_i \Diamond u \wedge @_u p) \wedge @_i @_s \neg p) \\ & \twoheadrightarrow & \exists s\,(\exists u\,(@_i \Diamond u \wedge @_u p) \wedge @_s \neg p) \end{array}$$

Step 2: $\exists s \exists u\,(@_i \Diamond u \wedge @_u p \wedge @_s \neg p)$ ♦

Example 3.17. Let $\psi_2 = \exists s\, @_i\,(\neg\,(\forall t @_t \neg p \wedge \exists u @_u p) \wedge @_s \neg p)$.
Step 1:

$$\begin{array}{l} \exists s\, @_i\,(\neg\,(\forall t @_t \neg p \wedge \exists u @_u p) \wedge @_s \neg p) \\ \quad \twoheadrightarrow \quad \exists s\, @_i\,((\neg \forall t @_t \neg p \vee \neg \exists u @_u p) \wedge @_s \neg p) \\ \quad \twoheadrightarrow \quad \exists s\, @_i\,((\exists t \neg @_t \neg p \vee \forall u \neg @_u p) \wedge @_s \neg p) \\ \quad \twoheadrightarrow \quad \exists s\,(@_i\,(\exists t @_t \neg\neg p \vee \forall u @_u \neg p) \wedge @_i @_s \neg p) \\ \quad \twoheadrightarrow \quad \exists s\,((@_i \exists t @_t p \vee @_i \forall u @_u \neg p) \wedge @_s \neg p) \\ \quad \twoheadrightarrow \quad \exists s\,((\exists t @_i @_t p \vee \forall u @_i @_u \neg p) \wedge @_s \neg p) \\ \quad \twoheadrightarrow \quad \exists s\,((\exists t @_t p \vee \forall u @_u \neg p) \wedge @_s \neg p) \end{array}$$

Step 2: $\exists s \exists t \forall u\,((@_t p \vee @_u \neg p) \wedge @_s \neg p)$ ♦

Example 3.18. Let $\psi_3 = @_i(\forall s \exists t @_s \Diamond t)$.
Step 1:

$$\begin{array}{rcl} @_i(\forall s \exists t\, @_s \Diamond t) & \twoheadrightarrow & \forall s \exists t\,(@_i @_s \Diamond t) \\ & \twoheadrightarrow & \forall s \exists t\,(@_s \Diamond t) \end{array}$$

Step 2: $\forall s \exists t\,(@_s \Diamond t)$ ♦

Analogously to the corresponding construction in first-order logic, we can also resort to Skolemization in the Algebraic Strong Priorean Logic.

Lemma 3.19 (Skolemization in $\mathcal{H}\,(\Sigma, @, \forall)$)**.** *Let φ be a sentence of the form $\forall s_1 \ldots \forall s_n \exists s_{n+1}\, G(s_1, \ldots, s_n, s_{n+1})$ of $\mathcal{H}\,(\Sigma, @, \forall)$, where the existentially quantified variable s_{n+1} is preceded by n universally quantified variables. In case $n = 0$, augment the underlying hybrid similarity type with a new nominal c and form the sentence $G(c)$; otherwise, augment the underlying hybrid similarity type with a new n-ary function symbol f and form the sentence $\forall s_1, \ldots, s_n G(s_1, \ldots, s_n, f(s_1, \ldots, s_n))$. Let φ' denote this new* *sentence, formed after the appropriate augmentation of the language. Then, there is an extension $\mathcal{H}'$ of the model $\mathcal{H}$ such that:*

$$\mathcal{H}, g, w \Vdash \varphi \quad \textit{iff} \quad \mathcal{H}', g, w \Vdash \varphi'.$$

The (standard) proof of the latter result shows how to build the mentioned extension of the original model.

We now apply Skolemization to the previous examples.

Example 3.20. $\overline{\psi_1} = @_i \Diamond c_1 \wedge @_{c_1} p \wedge @_{c_2} \neg p$ ♦

Example 3.21. $\overline{\psi_2} = \forall u \left((@_{c_2} p \vee @_u \neg p) \wedge @_{c_1} \neg p \right)$ ♦

Example 3.22. $\overline{\psi_3} = \forall s \left(@_s \Diamond f(s) \right)$ ♦

Definition 3.23. A formula of $\mathcal{H}(\Sigma, @, \forall)$ is in *conjunctive Skolem form* if it is in Prenex Conjunctive Normal Form and its prefix contains only universal quantifiers. ◀

For a given formula φ, its Skolem Form is the result of applying labelling (Lemma 3.7), followed by the rectification and existential closure of the new formula (Lemma 3.11), then putting it in Prenex Conjunctive Normal Form (Theorem 3.15) and finally performing Skolemization (Lemma 3.19).

With conjunctive Skolem forms defined, we can state the following result:

Theorem 3.24. *A set Φ of formulas in $\mathcal{H}(\Sigma, @, \forall)$ is satisfiable iff the set of conjunctive Skolem forms of formulas in Φ is satisfiable.*

Proof. In view of Lemma 3.7, we know that the satisfiability of Φ is preserved when one considers the set $\{@_i\varphi \mid \varphi \in \Phi\}$, with i not occurring in any formula φ. Recall that such nominal is always possible to find, as we assumed Nom to be a denumerable set.

From Lemma 3.11, the satisfiability problem for $\{@_i\varphi \mid \varphi \in \Phi\}$ is the same as for $\{\overline{@_i\varphi} \mid \varphi \in \Phi\}$ where $\overline{@_i\varphi}$ represents the existential closure of $@_i\varphi$. This step is possible to accomplish since we also assumed WVar to be a denumerable set.

Furthermore, we can use the procedure employed in the proof of Theorem 3.15 in order to put formulas in Prenex Conjuntive Normal Form, and this is a procedure that strictly preserves the satisfiability of formulas. Thus we can deal with the satisfiability problem of $\{\mathrm{PCNF}\left(\overline{@_i\varphi}\right) \mid \varphi \in \Phi\}$ where $\mathrm{PCNF}(\psi)$ is the result of applying the steps in the proof of Theorem 3.15 to the formula ψ. Next we apply Skolemization to all formulas. Beware of the fact that the Skolem symbols introduced in each formula are to be disjoint. Let us call the resulting set $\tilde{\Phi}$. Clearly, by Lemma 3.19, the satisfiability problem for $\tilde{\Phi}$ is the same as for Φ. ■

The above relatively straightforward proof contrasts with proofs of the analogous result in first-order logic (see, e.g., [5]), which are often involved.

Definition 3.25. A *ground instance* of a sentence $\forall s_1 \ldots \forall s_n\, G(s_1, \ldots, s_n)$, with $G(s_1, \ldots, s_n)$ a quantifier-free formula of $\mathcal{H}(\Sigma, @, \forall)$, is a formula of the form $G(i_1, \ldots, i_n)$ which results from substituting all occurrences of $s_1, \ldots, s_n$ in G with nominals $i_1, \ldots, i_n$. ◀

Before presenting our Herbrand-like result for hybrid logic with quantifiers, we find it worth pointing out that hybrid logic can be translated into first-order logic with equality, and (a fragment of) first-order logic with equality can be translated back into (a fragment of) hybrid logic (cf. [3]). Both translations are truth-preserving. First-order logic is compact, which means that a set of first-order sentences is satisfiable if and only if every finite subset of it is satifiable. Furthermore, from our earlier Herbrand-like result (Theorem 2.6), we know that for a set of Boolean combination of atomic satisfaction statements, satisfiability implies propositional satisfiability.

Theorem 3.26 (Herbrand-like)**.** *Let L and Σ be, respectively, a hybrid and an algebraic similarity type, and let $\Phi \subseteq \mathrm{Form}_{@,\forall}(L, \mathrm{Term}(\Sigma, \mathrm{WVar}, \mathrm{Nom}))$. Then Φ is unsatisfiable iff some finite set Φ^* of ground instances of Skolem forms of $\Phi \cup \mathrm{Eq}(L)$ is propositionally unsatisfiable.*

Proof. By Theorem 3.24 the set Φ is unsatisfiable iff the set Ψ of conjunctive Skolem forms of formulas in Φ is unsatisfiable. So, in the present proof we will deal with Ψ.

Let us now prove the right-to-left direction of the theorem. First observe that, from Theorem 2.6, if a set Φ^* of ground instances of $\Psi \cup \mathrm{Eq}(L)$ is propositionally unsatisfiable then it is unsatisfiable. Furthermore, notice that a ground instance of a universal sentence τ is a logical consequence of τ. Therefore, if a set Φ^* of ground instances of $\Psi \cup \mathrm{Eq}(L)$ is unsatisfiable, then $\Psi \cup \mathrm{Eq}(L)$ is unsatisfiable, which yields that Ψ is unsatisfiable. It follows from the previous paragraph that Φ is unsatisfiable.

For the left-to-right direction of the theorem we prove the contrapositive: if every finite set of ground instances of Skolem forms of $\Phi \cup \mathrm{Eq}(L)$, i.e., ground instances of $\Psi \cup \mathrm{Eq}(L)$, is propositionally satisfiable, then Φ is satisfiable. Let Φ_0 be the set of all ground instances of $\Psi \cup \mathrm{Eq}(L)$. From the assumption that every finite subset of Φ_0 is propositionally satisfiable, it follows from compactness that the entire set Φ_0 is propositionally satisfiable. From Theorem 2.11, we conclude that Φ_0 is satisfiable. Thus $\Psi \cup \mathrm{Eq}(L)$ is satisfiable, from which Ψ is satisfiable, which finally implies that Φ is satisfiable. ■

4 Conclusion

We have proposed two versions of Herbrand's theorem in the context of hybrid logic, with a restriction to satisfaction statements, by making use of rules that rewrite each satisfaction statement as a Boolean combination of atomic satisfaction statements, and making use also of the fact that each model can be described by its diagram. We proved that a set of satisfaction statements is propositionally unsatisfiable if and only if it is unsatisfiable.

Formulas with quantifiers over objects constitute a challenge. In fact, allowing non-rigidity introduces a new set of problems: when dealing with non-rigid terms, i.e. terms that can designate different things at different possible worlds, the act of designation and the act of passing to an alternative world need not commute. For an example of how this has been dealt with elsewhere, it is worth to point out Fitting's version (cf. [7]) of Herbrand's theorem for the modal logic K with varying domains. Following the standard steps for Herbrand-like theorems, after going through Skolemization one gets non-rigid designators for some formulas and the above mentioned difficulty concerning non-commutativity ensues. In order to overcome this issue, Fitting resorted to the concepts of predicate abstraction and validity functional form. In short, if φ is a formula, then $\langle \lambda x.\varphi \rangle$ is a predicate abstraction that is to be applied to terms; loosely speaking, for $\langle \lambda x.\varphi \rangle(t)$ to be true at a world w, φ should be true in that world provided we take the value of x to be whatever the term t designates at w. The predicate abstraction mechanism does not have an important role to play in classical logic because all the classical connectives and quantifiers are 'transparent' to it. On the other hand, $\langle \lambda x.\Box\varphi \rangle(t)$ and $\Box\langle \lambda x.\varphi \rangle(t)$ may have very different meanings, from a semantical viewpoint. Fitting defines as modal Herbrand transform of a formula X the formula X' such that $X \to X'$ can be derived from a certain calculus that he presents. He later proves equivalence between the validity problem for a closed formula φ and for one of its modal Herbrand expansions, a notion built over that of modal Herbrand transforms. We are confident that within the hybrid scenario something similar is to be done: by adding just nominals and the satisfaction operator, and assuming that nominals are rigid, it would seem that @ is to behave as classical connectives and quantifiers do when interacting with the predicate abstraction mechanism, namely, that $\langle \lambda x.@_i\varphi \rangle(t)$ and $@_i\langle \lambda x.\varphi \rangle(t)$ are to share the same meaning. If the addition of nominals proves not to be worrisome, then updating the concept of modal Herbrand transform into hybrid Herbrand transform, after proper adjustments to the calculus proposed by Fitting in order to incorporate the hybrid machinery, should be rather trouble-free. The details need to be checked, of course, and we propose that as future work.

As in [1], we have here investigated a direct path towards the proofs of our main (Herbrand-like) results, without taking an indirect approach through first-order translations of the hybrid formulas. However, for a more straightforward comparison with the standard formulation of the Herbrand Theorem and its numerous applications, it might be worth exploring the connection of our present results concerning Hybrid Logic to the more long winded route going through its translation into classical first-order logic. For space reasons, though, we have to leave details of this reconnaissance to a future opportunity.

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References

[1] Carlos Areces and Daniel Gorín. Resolution with order and selection for hybrid logics. *Journal of Automated Reasoning*, 46(1):1–42, Jan 2011.

[2] Patrick Blackburn. Internalizing labelled deduction. *Journal of Logic and Computation*, 10(1):137–168, 2000.

[3] Patrick Blackburn. Representation, reasoning, and relational structures: A hybrid logic manifesto. *Logic Journal of the IGPL*, 8(3):339–365, 2000.

[4] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal logic.* Cambridge University Press, 2001.

[5] Alan Bundy. *The Computer Modelling of Mathematical Reasoning.* Academic Press Professional, Inc., San Diego, CA, USA, 1985.

[6] Stephen Cook and Phuong Nguyen. *Logical Foundations of Proof Complexity.* Cambridge University Press, New York, NY, USA, 1st edition, 2010.

[7] Melvin Fitting. A modal Herbrand theorem. *Fundamenta Informaticae*, 28(1-2):101–122, 1996.

[8] Jacques Herbrand. *Logical Writings.* Dordrecht, Holland, D. Reidel Pub. Co., 1971.

[9] Evangelos Tzanis. Algebraizing hybrid logic. Master's thesis, University of Amsterdam, Institute of Logic, Language and Computation, 2005.