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# Analysis of fractional integro-differential equations of thermistor type

**Abstract:** We survey methods and results of fractional differential equations in which an unknown function is under the operation of integration and/or differentiation of fractional order. As an illustrative example, we review results as regards fractional integral and differential equations of thermistor type. Several nonlocal problems are considered: problems concerned with Riemann–Liouville, Caputo, and time-scale fractional operators. The existence and uniqueness of positive solutions are obtained through suitable fixed-point theorems in proper Banach spaces. Additionally, existence and continuation theorems are given, ensuring global existence.

**Keywords:** Integral and differential equations, fractional operators, nonlocal thermistor problems, time scales, dynamic equations, positive solutions, local and global existence, fixed-point theorems

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## 1 Introduction

Fractional calculus covers a wide range of classical fields in mathematics and its applications, such as Abel's integral equation, viscoelasticity, analysis of feedback amplifiers, capacitor theory, fractances, electric conductance, mathematical biology and optimal control [4]. In particular, Abel integral equations have been well studied, with many publications devoted to its applications in different fields. One can say that Abel's integral equations, of the first and second kind, are the most celebrated integral equations of fractional order [45]. The former investigations on such equations are due to Niels Henrik Abel himself, for the first kind [67], and to Hille and Tamarkin for the second kind [42]. Abel was led to his equations studying the *tautochrone* problem (from the Greek *tauto*, meaning same, and *chrono*, meaning time), that is, the problem of determining the shape of a curve for which the time taken by an object sliding with-

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out friction in uniform gravity to its lowest point is independent of its starting point. The curve is a cycloid, simultaneously the *tautochrone* and the *brachistochrone* curve, which brings together the subjects of fractional calculus and the calculus of variations [45, 58].

Fractional integral equations occur in many situations where physical measurements are to be evaluated, e. g., in evaluation of spectroscopic measurements of cylindrical gas discharges, the study of the solar or a planetary atmosphere, the investigation of star densities in a globular cluster, the inversion of travel times of seismic waves for determination of terrestrial sub-surface structure, and spherical stereology [33, 34]. For detailed descriptions and analysis of such integral equations of fractional order, we refer the reader to the books of Gorenflo and Vessella [36] and Craig and Brown [22]. See also [32]. Another field, in which fractional integral equations with general weakly singular kernels appear naturally, is inverse boundary value problems in partial differential equations, in particular parabolic ones [35]. Here we are mainly interested in questions involving existence and uniqueness of the solutions of fractional integral equations.

The existence and uniqueness of the solutions for FDEs have been intensely studied by many mathematicians [41, 47, 52, 60]. However, most available results were concerned with the existence–uniqueness of the solutions for FDEs on a finite interval. Since continuation theorems for FDEs are not well developed, results about global existence-uniqueness of the solution of FDEs on the half axis $[0, +\infty)$, by using directly the results from local existence, have only recently flourished [12, 54]. Here we address such issues. To motivate our study, we can mention two types of electrical circuits, which are related with fractional calculus. Circuits of the first type consist of capacitors and resistors, which are described by conventional (integer-order) models, but for which the circuit itself, as a whole, may have non-integer-order properties, becoming a fractance device, that is, an electrical element that exhibits fractional-order impedance properties. Circuits of the second type may consist of resistors and capacitors, both modeled in the classical sense, and fractances. In particular, we can consider thermistor-type problems, which are highly nonlinear and mathematically challenging [72].

Inspired by modern developments of the study of thermistors, where fractional partial differential equations have a crucial role to play, we consider here mathematical models and tools that serve as prototypes for other integral problems. It turns out that the available computational methods for such mathematical problems are not theoretically sound, in the sense they rely on results of local existence. Here we review the recent theory of global existence for nonlocal fractional problems of thermistor type [70, 74, 76, 77]. We begin, in Section 2, with an historical account of the theory of FDEs dealing with Cauchy-type problems and their reduction to Volterra integral equations. Section 3 contains some of the main tools in the area: we recall a necessary and sufficient condition for a subspace of continuous functions to be precompact; Schauder's fixed-point theorem; and a useful generalization of Gronwall's inequality. In Section 4, our main concern is existence and uniqueness of the solution to a fractional-order nonlocal Riemann–Liouville thermistor problem of the form

$$\begin{gathered} D^{2\alpha}u(t) = \frac{\lambda f(u(t))}{(\int_0^T f(u(x))\,dx)^2} + h(t)\,, \quad t \in (0,T)\,, \\ I^{\beta}u(t)|_{t=0} = 0, \quad \forall \beta \in (0,1], \end{gathered} \tag{1}$$

under suitable conditions on $f$ and $h$ (see Theorem 1). We also establish the boundedness of $u$ (Theorem 2). Here the constant $\lambda$ is a positive dimensionless real parameter. The unknown function $u$ may be interpreted as the temperature generated by an electric current flowing through a conductor [51, 83]. We assume that $T$ is a fixed positive real and $\alpha > 0$ a parameter describing the order of the FD. In the case $\alpha = 1$ and $h \equiv 0$, (1) becomes the one-dimensional nonlocal steady state thermistor problem; the values of $0 < \alpha < \frac{1}{2}$ correspond to intermediate processes. In Section 5, a more general Caputo thermistor problem (8) on the half axis $[0, +\infty)$ is considered, instead of the bounded interval $[0, T)$ of (1). One of the main difficulties lies in handling the nonlocal term

$$\frac{\lambda f(t, u(t))}{(\int_0^t f(x, u(x))\,dx)^2}$$

of problem (8), where, in contrast with problem (1), function $f$ depends on both time and the unknown function $u$. We are concerned with local existence on a finite interval (Theorem 3), as well as results of continuation (Theorem 4) and global existence (Theorems 5 and 6) via Schauder's fixed-point theorem. Finally, in Section 6, we consider fractional integral and differential equations on time scales, that is, on arbitrary nonempty closed subsets of the real numbers. The investigation of dynamic equations on time scales allows one to unify and extend the theories of difference and differential equations into a single theory [13]. A time scale is a model of time, and the theory has found important applications in several contexts that require simultaneous modeling of discrete and continuous data. Its usefulness appears in many different areas, and the reader interested in applications is refereed to [3, 5, 20, 59, 63] and the references therein. The idea to join the two subjects of FC and the calculus on time scales, on a single theory, was born with the work in [14–16] and is subject to strong current research since 2011: see, e. g., [17, 19, 62, 63, 77]. Using Schauder's fixed-point theorem, we obtain existence and uniqueness results of positive solutions for a fractional Riemann–Liouville nonlocal thermistor problem on arbitrary nonempty closed subsets of the real numbers (Theorems 8 and 9). We end with Section 7, presenting our conclusions.

## 2 Historical account

Abel's integral equations of first and second kinds can be formulated, respectively, as

$$f(x) = \int_0^x \frac{k(x,s)g(s)}{(x-s)^\alpha} ds, \quad 0 < \alpha < 1, \quad 0 \leq x \leq b, \tag{2}$$

and

$$f(x) = \int_0^x \frac{k(x,s)g(s)}{(x-s)^\alpha} ds + g(x), \quad 0 < \alpha < 1, \quad 0 \leq x \leq b, \tag{3}$$

where $g$ is the unknown function to be found, $f$ is a well behaved function, and $k$ is the kernel. These celebrated equations appear frequently in many physical and engineering problems, like semi-conductors, heat conduction, metallurgy and chemical reactions [32, 36]. The special case $\alpha = 1/2$ arises often. If the kernel is given by $k(x,s) = \frac{1}{\Gamma(1-\alpha)}$, then (2) is a fractional integral equation of order $1-\alpha$ [45]. This problem is a generalization of the tautochrone problem, and it is related with the birth of the fractional calculus of variations [10]. For solving integral equations (2)–(3) of Abel type, several approaches are possible, e. g., using transformation techniques [38], orthogonal polynomials [61], integral operators [81], fractional calculus [30, 80], Bessel functions [78], wavelets [11], methods based on semigroups [43, 44], as well as many other techniques [55, 65].

FDEs, in which an unknown function is contained under the operation of a FD, have a long history, enriched by the intensive development of the theory of fractional calculus and their applications in the last decades. Fractional ordinary differential equations have the following form:

$$F(x, y(x), D^{\alpha_1} y(x), D^{\alpha_2} y(x), \ldots, D^{\alpha_n} y(x)) = g(x),$$

where $F(x, y_1, y_2, \ldots, y_n)$ and $g(x)$ are given functions, $D^{\alpha_k}$ are fractional differentiation operators of real order $\alpha_k > 0$, $k = 1, 2, \ldots, n$ [68]. For example, one can consider nonlinear differential equations of the form

$$D^\alpha y(x) = f(x, y(x)) \tag{4}$$

with real $\alpha > 0$ or complex $\alpha$, $\mathrm{Re}(\alpha) > 0$. Similarly to the investigation of ordinary differential equations, the methods for FDEs are essentially based on the study of equivalent Volterra integral equations. Equations (2)–(3) are examples of Volterra integral equations of the first and second kinds, respectively. Several authors have developed methods to deal with fractional integro-differential equations and construct solutions for ordinary and partial differential equations, with the general goal of obtaining a unified theory of special functions [49, 68]. In [35], analytical solutions to some linear operators of fractional integration and fractional differentiation are obtained, using Laplace transforms. The 1918 note of O'Shaughnessy and Post [64], is one of the first references to develop a method for solving the equation

$$(D^{\frac{1}{2}}y)(x) = \frac{y}{x}$$

with the Riemann–Liouville derivative. Fifteen years later, in 1933, Fujiwara considered the FDE

$$(D_+^{\alpha}y)(x) = \left(\frac{\alpha}{x}\right)^{\alpha} y(x)$$

using the Hadamard FD of order $\alpha > 0$ [28]. Provided $f(x, y)$ is bounded in a special domain and satisfies a Lipschitz condition with respect to $y$, Pitcher and Sewell have shown in 1938 how the nonlinear FDE

$$(D_{a+}^{\alpha}y)(x) = f(x, y(x)), \quad 0 < \alpha < 1, \quad a \in \mathbb{R}, \tag{5}$$

in the sense of Riemann–Liouville, can be reduced to a Volterra integral equation, proving existence and uniqueness of a continuous solution to (5) [66]. In 1965, Al-Bassam considered the nonlinear Cauchy-type problem of fractional order

$$\begin{aligned} (D_{a+}^{\alpha}y)(x) &= f(x, y(x)), \quad 0 < \alpha < 1, \quad a \in \mathbb{R}, \\ (D_{a+}^{\alpha-1}y)(x)/(x = a) &= (I_{a+}^{1-\alpha}y)(x)/(x = a) = b_1, \quad b_1 \in \mathbb{R}. \end{aligned} \tag{6}$$

Similarly as before, he applied the method of successive approximations to the equivalent reduced Volterra integral equation and the contraction mapping method, establishing existence of a unique solution [9]. In 1978, Al-Abedeen and Arora considered the problem

$$(D_{a+}^{\alpha}y)(x) = f(x, y(x)), \quad (I_{a+}^{1-\alpha}y)(c) = y_0, \quad a < c < b, \quad y_0 \in \mathbb{R},$$

with $0 < \alpha \leq 1$, and they proved an existence and uniqueness result for the corresponding Volterra nonlinear integral equation [8]. On the basis of Picard method and Schauder's fixed-point theorem, Tazali obtained in 1982 two local existence results of a continuous solution to (6) [82]. Interestingly, more general existence and uniqueness results than the ones of [82], also obtained using a fixed-point theorem and equivalent nonlinear integral formulations, have been published in 1977 by Leskovskiĭ [53]. Similar results, on the basis of fixed-point theorems and integral equations, were derived by Semenchuk in 1982 [69]. In 1988 [27], El-Sayed examined problem (4) on a finite interval where the FD is considered in the sense of Gelfand and Shilov [29]. Hadid, in 1995, used a fixed-point theorem to prove existence of a solution to the corresponding integral equation of (6) [37]. Using the contraction mapping method on a complete metric space, Hayek et al. established in 1999 existence and uniqueness of a continuous solution to the Cauchy-type problem described by the following system of FDEs:

$$(D_{0+}^{\alpha}y)(x) = f(x, y(x)), \quad y(a) = b, \quad 0 < \alpha \leq 1, \quad a > 0, \quad b \in \mathbb{R}^n;$$

see [39]. In 2000, Kilbas, Bonilla and Trujillo studied the Cauchy-type problem

$$\begin{aligned}
&(D_{a+}^{\alpha}y)(x) = f(x, y(x)), \quad n-1 < \alpha \leq n, \quad n = -[-\alpha], \\
&(D_{a+}^{\alpha-k)}y)(x)/(x = a) = b_k, \quad b_k \in \mathbb{R}, \quad k = 1, 2, \ldots, n,
\end{aligned}$$

with complex $\alpha$, $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$, on a finite interval $[a, b]$ [46]. By using the method of successive expansions and Avery–Henderson and Leggett–Williams multiple fixed-point theorems on cones, they proved existence of multiple positive solutions for the corresponding nonlinear Volterra integral equation [46]. See also [24, 48]. It should be noted, however, that in some function spaces the proof of equivalence between solutions of Cauchy-type problems for FDEs and corresponding reduced Volterra integral equations constitute a major difficulty [21]. For more recent results, we refer the reader to [2]. The techniques are, however, similar to the ones already mentioned; details are below.

# 3 Fundamental results

Let $C([0, T])$ be the space of all continuous functions on $[0, T]$. The following three auxiliary lemmas are particularly useful for our purposes.

**Lemma 1** (See [54]). *Let $M$ be a subset of $C([0, T])$. Then $M$ is precompact if and only if the following conditions hold:*
1. *$\{u(t) : u \in M\}$ is uniformly bounded,*
2. *$\{u(t) : u \in M\}$ is equicontinuous on $[0, T]$.*

**Lemma 2** (Schauder's fixed-point theorem [23]). *Let $U$ be a closed bounded convex subset of a Banach space $X$. If $T : U \to U$ is completely continuous, then $T$ has a fixed point in $U$.*

We also recall the following version of Gronwall's lemma.

**Lemma 3** (Generalized Gronwall's inequality [40, 84]). *Let $v : [0, b] \to [0, +\infty)$ be a real function and $w(\cdot)$ be a nonnegative, locally integrable function on $[0, b]$. Suppose that there exist $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + a \int_0^t \frac{v(s)}{(t-s)^{\alpha}} ds.$$

*Then there exists a constant $k = k(\alpha)$ such that*

$$v(t) \leq w(t) + ka \int_0^t \frac{w(s)}{(t-s)^\alpha} ds$$

*for $t \in [0, b]$.*

# 4 Nonlocal Riemann–Liouville problem

Let $0 < \alpha < \frac{1}{2}$ and $X = (C([0,T]), \|\cdot\|)$. For $x \in C([0,T])$, define the norm

$$\|x\| = \sup_{t \in [0,T]} \{e^{-Nt} |x(t)|\},$$

which is equivalent to the standard supremum norm for $f \in C([0,T])$ [1]. The use of this norm is technical and allows us to simplify the integral calculus. By $L^1([0,T], \mathbb{R})$, we denote the set of Lebesgue integrable functions on $[0,T]$. We consider problem (1) with the following assumptions:

($H_1$) $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lipschitz continuous function with Lipschitz constant $L_f$ such that $c_1 \leq f(u) \leq c_2$, with $c_1$, $c_2$ two positive constants;

($H_2$) $h$ is continuous on $(0,T)$ with $h \in L^\infty(0,T)$.

Our first result asserts existence of a unique solution to (1) on $C(\mathbb{R}^+)$ of the form

$$\begin{aligned} u(t) &= I^{2\alpha} \left\{ \frac{\lambda f(u)}{(\int_0^T f(u)\,dx)^2} + h(t) \right\} \\ &= \int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)} \left\{ \frac{\lambda f(u)}{(\int_0^T f(u)\,dx)^2} + h(s) \right\} ds. \end{aligned} \tag{7}$$

## 4.1 Existence and uniqueness

We begin proving equivalence between (1) and (7) on the space $C(\mathbb{R}^+)$. This restriction of the space of functions allows one to exclude from the proof a stationary function with Riemann–Liouville derivative of order $2\alpha$ equal to $d \cdot t^{2\alpha-1}$, $d \in \mathbb{R}$, which belongs to the space $C_{1-2\alpha}[0,T]$ of continuous weighted functions.

**Lemma 4.** *Suppose that $\alpha \in (0, \frac{1}{2})$. Then the nonlocal problem* (1) *is equivalent to the integral equation* (7) *on the space $C(\mathbb{R}^+)$.*

**Theorem 1.** *Let $f$ and $h$ satisfy hypotheses* ($H_1$) *and* ($H_2$). *Then there exists a unique solution $u \in X$ of* (1) *for all* $0 < \lambda < \frac{N^{2\alpha}}{L_f(\frac{1}{(c_1T)^2} + \frac{2c_2^2T}{(c_1T)^4} e^{NT})}$.

*Proof.* Let $F : X \to X$ be defined by

$$Fu = I^{2\alpha}\left\{\frac{\lambda f(u)}{(\int_0^T f(u)\,dx)^2} + h(t)\right\}.$$

For a well chosen $\lambda > 0$ we can prove that the map $F : X \to X$ is a contraction and it has a fixed point $u = Fu$. Hence, there exists a unique $u \in X$ that is the solution to the integral equation (7). The result follows from Lemma 4. □

## 4.2 Boundedness

We now show that the assumption that electrical conductivity $f(u)$ is bounded (hypothesis ($H_1$)) allows one to assert the boundedness of $u$.

**Theorem 2.** *Under hypotheses* ($H_1$) *and* ($H_2$) *and* $\lambda > 0$, *if* $u$ *is the solution of* (7), *then*

$$\|u\| \le \frac{(\frac{\lambda}{(c_1 T)^2} f(0) + h_\infty)}{N^{2\alpha}} e^{\frac{\lambda L_f}{(c_1 T N^\alpha)^2}}.$$

For more details on the subject see [74].

# 5 Nonlocal Caputo thermistor problem

Now, our main goal consists to prove global existence of the solutions for a fractional Caputo nonlocal thermistor problem. Precisely, we consider the following fractional-order initial value problem:

$$\begin{aligned} {}^C D_{0+}^{2\alpha} u(t) &= \frac{\lambda f(t, u(t))}{(\int_0^t f(x, u(x))\,dx)^2}, \quad t \in (0, \infty), \\ u(t)|_{t=0} &= u_0, \end{aligned} \tag{8}$$

where ${}^C D_{0+}^{2\alpha}$ is the fractional Caputo derivative operator of order $2\alpha$ with $0 < \alpha < \frac{1}{2}$ a real parameter. We shall assume the following hypotheses:

($H_1$) $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a Lipschitz continuous function with Lipschitz constant $L_f$ with respect to the second variable such that $c_1 \le f(s, u) \le c_2$ with $c_1$ and $c_2$ two positive constants;

($H_2$) there exists a positive constant $M$ such that $f(s, u) \le M s^2$;

($H_3$) $|f(s, u) - f(s, v)| \le s^2 |u - v|$ or, in a more general manner, there exists a constant $\omega \ge 2$ such that $|f(s, u) - f(s, v)| \le s^\omega |u - v|$.

## 5.1 Local existence theorem

In this subsection, a local existence theorem of the solutions for (8) is obtained by applying Schauder's fixed-point theorem. In order to transform (8) into a fixed-point problem, we give in the following lemma an equivalent integral form of (8).

**Lemma 5.** *Suppose that* $(\mathrm{H}_1)$–$(\mathrm{H}_3)$ *holds. Then the initial value problem* (8) *is equivalent to*

$$u(t) = u_0 + \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{f(s,u(s))}{(\int_0^t f(x,u)\,dx)^2} ds. \tag{9}$$

*Proof.* It is a simple exercise to see that $u$ is a solution of the integral equation (9) if and only if it is also a solution of the IVP (8). ☐

**Theorem 3.** *Suppose that conditions* $(\mathrm{H}_1)$–$(\mathrm{H}_3)$ *are verified. Then* (8) *has at least one solution* $u \in C[0,h]$ *for some* $T \geq h > 0$.

*Proof.* Let

$$E = \Big\{u \in C[0,T] : \|u-u_0\|_{C[0,T]} = \sup_{0\leq t\leq T} |u-u_0| \leq b\Big\},$$

where $b$ is a positive constant. Further, put

$$D_h = \{u : u \in C[0,h],\ \|u-u_0\|_{C[0,h]} \leq b\},$$

where

$$h = \min\left\{\left(b\left(\frac{\lambda M}{\Gamma(2\alpha+1)c_1^2}\right)^{-1}\right)^{\frac{1}{2\alpha}}, T\right\}$$

and $0 < \alpha < \frac{1}{2}$. It is clear that $h \leq T$. Note also that $D_h$ is a nonempty, bounded, closed, and convex subset of $C[0,h]$. In order to apply Schauder's fixed-point theorem, we define the following operator $A$:

$$(Au)(t) = u_0 + \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{f(s,u(s))}{(\int_0^t f(x,u)\,dx)^2} ds, \quad t \in [0,h]. \tag{10}$$

It is clear that all solutions of (8) are fixed points of (10). Then, by assumptions $(\mathrm{H}_1)$ and $(\mathrm{H}_2)$, it follows that $AD_h \subset D_h$. Our next step, in order to prove Theorem 3, consists to use the following two technical lemmas.

**Lemma 6.** *The operator $A$ is continuous.*

**Lemma 7.** *The operator $AD_h$ is continuous.*

One can prove that $\{(Au)(t) : u \in D_h\}$ is equicontinuous. Taking into account that $AD_h \subset D_h$, we infer that $AD_h$ is precompact. This implies that $A$ is completely continuous. As a consequence of Schauder's fixed-point theorem and Lemma 5, we conclude that problem (8) has a local solution. This completes the proof of Theorem 3. □

## 5.2 Continuation results

Now we give a continuation theorem for the fractional Caputo nonlocal thermistor problem (8). First, we present the definition of a noncontinuable solution.

**Definition 1** (See [50]). Let $u(t)$ on $(0, \beta)$ and $\tilde{u}(t)$ on $(0, \tilde{\beta})$ be both solutions of (8). If $\beta < \tilde{\beta}$ and $u(t) = \tilde{u}(t)$ for $t \in (0, \beta)$, then we say that $\tilde{u}(t)$ can be continued to $(0, \tilde{\beta})$. A solution $u(t)$ is noncontinuable if it has no continuation. The existing interval of the noncontinuable solution $u(t)$ is called the maximum existing interval of $u(t)$.

**Theorem 4.** *Assume that conditions* $(\mathrm{H}_1)$–$(\mathrm{H}_3)$ *are satisfied. Then* $u = u(t)$, $t \in (0, \beta)$, *is noncontinuable if and if only for some* $\eta \in (0, \frac{\beta}{2})$ *and any bounded closed subset* $S \subset [\eta, +\infty) \times \mathbb{R}$ *there exists a* $t^* \in [\eta, \beta)$ *such that* $(t^*, u(t^*)) \notin S$.

*Proof.* Suppose that there exists a compact subset $S \subset [\eta, +\infty) \times \mathbb{R}$ such that

$$\{(t, u(t)) : t \in [\eta, \beta)\} \subset S.$$

Compactness of $S$ implies $\beta < +\infty$. The proof follows from Lemmas 8 and 9. □

**Lemma 8.** *The limit* $\lim_{t \to \beta^-} u(t)$ *exists.*

*Proof.* The proof is based on the Cauchy convergence criterion. □

The second step of the proof of Theorem 4 consists to show the following result.

**Lemma 9.** *The function* $u(t)$ *is continuable.*

*Proof.* As $S$ is a closed subset, we can say that $(\beta, u^*) \in S$. Define $u(\beta) = u^*$. Hence, $u(t) \in C[0, \beta]$. Then we define the operator $K$ by

$$(Kv)(t) = u_1 + \frac{\lambda}{\Gamma(2\alpha)} \int_{\beta}^{t} (t-s)^{2\alpha-1} \frac{f(s, v(s))}{(\int_0^t f(x, v)\, dx)^2} ds,$$

where

$$u_1 = u_0 + \frac{\lambda}{\Gamma(2\alpha)} \int_{0}^{\beta} (t-s)^{2\alpha-1} \frac{f(s, v(s))}{(\int_0^t f(x, v)\, dx)^2} ds,$$

$v \in C([\beta, \beta+1])$, $t \in [\beta, \beta+1]$. Set

$$E_b = \left\{(t, v) : \beta \le t \le \beta + 1, |v| \le \max_{\beta \le t \le \beta+1} |u_1(t)| + b\right\}$$

and

$$E_h = \left\{ v \in C[\beta, \beta+1] : \max_{t\in[\beta,\beta+h]} \left| v(t) - u_1(t) \right| \le b, v(\beta) = u_1(\beta) \right\},$$

where $h = \min\{(b(\frac{\lambda M}{\Gamma(2\alpha+1)c_1^2})^{-1})^{\frac{1}{2\alpha}}, 1\}$. Similarly to the proof of Theorem 3, we show that $K$ is completely continuous on $E_b$, which shows that the operator $K$ is continuous. We show that $KE_h$ is equicontinuous. Consequently, $K$ is completely continuous. Then Schauder's fixed-point theorem can be applied to see that the operator $K$ has a fixed point $\tilde{u}(t) \in E_h$. On other words, we have

$$\begin{aligned} \tilde{u}(t) &= u_1 + \frac{\lambda}{\Gamma(2\alpha)} \int_\beta^t (t-s)^{2\alpha-1} \frac{f(s,\tilde{u}(s))}{(\int_0^t f(x,\tilde{u}(x))\,dx)^2} ds \\ &= u_0 + \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{f(s,\tilde{u}(s))}{(\int_0^t f(x,\tilde{u}(x))\,dx)^2} ds, \end{aligned}$$

$t \in [\beta, \beta+h]$, where

$$\tilde{u}(t) = \begin{cases} u(t), & t \in (0,\beta], \\ \tilde{u}(t), & t \in [\beta, \beta+h]. \end{cases}$$

It follows that $\tilde{u}(t) \in C[0, \beta+h]$ and

$$\tilde{u}(t) = u_0 + \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{f(s,\tilde{u}(s))}{(\int_0^t f(x,\tilde{u}(x))\,dx)^2} ds.$$

Therefore, according to Lemma 5, $\tilde{u}(t)$ is a solution of (8) on $(0, \beta+h]$. This is absurd because $u(t)$ is noncontinuable. This completes the proof of Lemma 9. □

Similarly to Theorem 1, uniqueness of the solution to problem (8) is derived from the proof of Theorem 4 for a well chosen $\lambda$.

## 5.3 Global existence of the solutions

Now we provide two sets of sufficient conditions for the existence of a global solution for (8) (Theorems 5 and 6). We begin with an auxiliary lemma.

**Lemma 10.** *Suppose that conditions* $(\mathrm{H}_1)$–$(\mathrm{H}_3)$ *hold. Let* $u(t)$ *be a solution of* (8) *on* $(0,\beta)$. *If* $u(t)$ *is bounded on* $[\tau,\beta)$ *for some* $\tau > 0$, *then* $\beta = +\infty$.

*Proof.* The proof follows immediately from the results of Subsection 5.2. □

**Theorem 5.** *Suppose that conditions* $(\mathrm{H}_1)$–$(\mathrm{H}_3)$ *hold. Then* (8) *has a solution in* $C([0,+\infty))$.

*Proof.* The existence of a local solution $u(t)$ of (8) is ensured thanks to Theorem 3. We already know, by Lemma 5, that $u(t)$ is also a solution to the integral equation

$$u(t) = u_0 + \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{f(s,u(s))}{(\int_0^t f(x,u(x))\,dx)^2} ds.$$

Suppose that the existing interval of the noncontinuable solution $u(t)$ is $(0,\beta)$, $\beta < +\infty$. Then

$$\begin{aligned} |u(t)| &= \left| u_0 + \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{f(s,u(s))}{(\int_0^t f(x,u(x))\,dx)^2} ds \right| \\ &\le |u_0| + \frac{\lambda}{\Gamma(2\alpha)} \frac{1}{(c_1 t)^2} \int_0^t (t-s)^{2\alpha-1} |f(s,u(s))| ds \\ &\le |u_0| + \frac{\lambda}{\Gamma(2\alpha)} \frac{1}{c_1^2} \int_0^t \frac{|u(s)|}{(t-s)^{1-2\alpha}} ds. \end{aligned}$$

By Lemma 5, there exists a constant $k(\alpha)$ such that, for $t \in (0,\beta)$, we have

$$|u(t)| \le |u_0| + k|u_0| \frac{\lambda}{\Gamma(2\alpha)} \frac{1}{c_1^2} \int_0^t (t-s)^{2\alpha-1} ds,$$

which is bounded on $(0,\beta)$. Thus, by Lemma 10, problem (8) has a solution $u(t)$ on $(0,+\infty)$. □

Next we give another sufficient condition ensuring global existence for (8).

**Theorem 6.** *Suppose that there exist positive constants* $c_3$, $c_4$ *and* $c_5$ *such that* $c_3 \le |f(s,x)| \le c_4|x| + c_5$. *Then* (8) *has a solution in* $C([0,+\infty))$.

*Proof.* Suppose that the maximum existing interval of $u(t)$ is $(0,\beta)$, $\beta < +\infty$. We claim that $u(t)$ is bounded on $[\tau,\beta)$ for any $\tau \in (0,\beta)$. Indeed, we have

$$\begin{aligned} |u(t)| &= \left| u_0 + \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{f(s,u(s))}{(\int_0^t f(x,u(x))\,dx)^2} ds \right| \\ &\le |u_0| + \frac{\lambda}{\Gamma(2\alpha)} \frac{c_3}{(c_1\tau)^2} \int_0^t (t-s)^{2\alpha-1} ds + \frac{\lambda}{\Gamma(2\alpha)} \frac{c_2}{(c_1\tau)^2} \int_0^t \frac{|u(s)|}{(t-s)^{1-2\alpha}} ds. \end{aligned}$$

If we take

$$w(t) = |u_0| + \frac{\lambda}{\Gamma(2\alpha)} \frac{c_3}{(c_1\tau)^2} \int_0^t (t-s)^{2\alpha-1} ds,$$

which is bounded, and

$$a = \frac{\lambda c_2}{\Gamma(2\alpha)} \frac{1}{(c_1\beta)^2},$$

it follows, according with Lemma 3, that $v(t) = |u(t)|$ is bounded. Thus, by Lemma 10, problem (8) has a solution $u(t)$ on $(0, +\infty)$. □

For more details on the subject, see [70, 71, 74, 75, 79].

# 6 Fractional problems on arbitrary time scales

Throughout the remainder of this chapter, we denote by $\mathbb{T}$ a time scale, which is a nonempty closed subset of $\mathbb{R}$ with its inherited topology. For convenience, we make the blanket assumption that $t_0$ and $T$ are points in $\mathbb{T}$. Our main concern is to prove existence and uniqueness of the solution to a fractional-order nonlocal thermistor problem of the form

$$\begin{aligned} {}^{\mathbb{T}}D^{2\alpha}_{t_{0+}} u(t) &= \frac{\lambda f(u(t))}{(\int_{t_0}^T f(u(x)) \bigtriangleup x)^2}, \quad t \in (t_0, T), \\ {}^{\mathbb{T}}I^{\beta}_{t_{0+}} u(t_0) &= 0, \quad \forall \beta \in (0,1), \end{aligned} \tag{11}$$

under suitable conditions on $f$ as described below. We assume that $\alpha \in (0,1)$ is a parameter describing the order of the FD; ${}^{\mathbb{T}}D^{2\alpha}_{t_{0+}}$ is the left Riemann–Liouville FD operator of order $2\alpha$ on $\mathbb{T}$; ${}^{\mathbb{T}}I^{\beta}_{t_{0+}}$ is the left Riemann–Liouville FI operator of order $\beta$ defined on $\mathbb{T}$ by [19] (see Section 6.1, where these definitions and main properties of the fractional operators on time scales are recalled). As before, $u$ may be interpreted as the temperature inside the conductor and $f(u)$ the electrical conductivity of the material.

In the literature, many existence results for dynamic equations on time scales are available [25, 26]. In recent years, there has also been significant interest in the use of FDEs in mathematical modeling [6, 56, 85]. However, much of the work published to date has been concerned with it separately by the time-scale community and by the fractional one. Results on fractional dynamic equations on time scales are scarce [7]. Here we give existence and uniqueness results for the fractional-order nonlocal thermistor problem on time scales (11), putting together time-scale and fractional domains. According with [57, 62, 63], this is quite appropriate from the point of view of practical applications. Our main aim is to prove existence of the solutions for (11) using a fixed-point theorem and, consequently, uniqueness (see Theorems 8 and 9). For more details see [77].

## 6.1 Fractional calculus on time scales

We deal with the notions of Riemann–Liouville FIs and FDs on time scales, the so called BHT fractional calculus on time scales [62]. For local approaches, we refer the reader to [17, 18]. Here we are interested in nonlocal operators, which are the ones who make sense with respect to thermistor-type problems [72, 73]. Although we restrict ourselves to the delta approach on time scales, similar results are trivially obtained for the nabla fractional case [31].

**Definition 2** (Riemann–Liouville FI on time scales [19]). Let $\mathbb{T}$ be a time scale and $[a,b]$ an interval of $\mathbb{T}$. Then the left fractional integral on time scales of order $0 < \alpha < 1$ of a function $g : \mathbb{T} \to \mathbb{R}$ is defined by

$$^{\mathbb{T}}I^{\alpha}_{a+}g(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s) \,\triangle s,$$

where $\Gamma$ is the Euler gamma function.

The left Riemann–Liouville FD operator of order $\alpha$ on time scales is then defined using Definition 2 of FI.

**Definition 3** (Riemann–Liouville FD on time scales [19]). Let $\mathbb{T}$ be a time scale, $[a,b]$ an interval of $\mathbb{T}$, and $\alpha \in (0,1)$. Then the left Riemann–Liouville FD on time scales of order $\alpha$ of a function $g : \mathbb{T} \to \mathbb{R}$ is defined by

$$^{\mathbb{T}}D^{\alpha}_{a+}g(t) = \left( \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} g(s) \,\triangle s \right)^{\triangle}.$$

**Remark 1.** If $\mathbb{T} = \mathbb{R}$, then we obtain from Definitions 2 and 3, respectively, the usual left Rieman–Liouville FI and FD.

**Proposition 1** (See [19]). *Let $\mathbb{T}$ be a time scale, $g : \mathbb{T} \to \mathbb{R}$ and $0 < \alpha < 1$. Then*

$$^{\mathbb{T}}D^{\alpha}_{a+}g = \triangle \circ {}^{\mathbb{T}}I^{1-\alpha}_{a+}g.$$

**Proposition 2** (See [19]). *If $\alpha > 0$ and $g \in C([a,b])$, then*

$$^{\mathbb{T}}D^{\alpha}_{a+} \circ {}^{\mathbb{T}}I^{\alpha}_{a+}g = g.$$

**Proposition 3** (See [19]). *Let $g \in C([a,b])$, $0 < \alpha < 1$. If $^{\mathbb{T}}I^{1-\alpha}_{a+}u(a) = 0$, then*

$$^{\mathbb{T}}I^{\alpha}_{a+} \circ {}^{\mathbb{T}}D^{\alpha}_{a+}g = g.$$

**Theorem 7** (See [19]). *Let $g \in C([a,b])$, $\alpha > 0$, and ${}^{\mathbb{T}}_{a}I^{\alpha}_{t}([a,b])$ be the space of functions that can be represented by the Riemann–Liouville $\triangle$-integral of order $\alpha$ of some $C([a,b])$-function. Then*

$$g \in {}^{\mathbb{T}}_{a}I^{\alpha}_{t}([a,b])$$

*if and only if*

$$ {}^{\mathbb{T}}I_{a+}^{1-\alpha} g \in C^1([a,b]) $$

*and*

$$ {}^{\mathbb{T}}I_{a+}^{1-\alpha} g(a) = 0. $$

The following result of the calculus on time scales is also useful.

**Proposition 4** (See [7]). *Let $\mathbb{T}$ be a time scale and $g$ an increasing continuous function on the time-scale interval $[a,b]$. If $G$ is the extension of $g$ to the real interval $[a,b]$ defined by*

$$ G(s) := \begin{cases} g(s) & \text{if } s \in \mathbb{T}, \\ g(t) & \text{if } s \in (t, \sigma(t)) \notin \mathbb{T}, \end{cases} $$

*then*

$$ \int_a^b g(t) \triangle t \leq \int_a^b G(t) dt, $$

*where $\sigma : \mathbb{T} \to \mathbb{T}$ is the forward jump operator of $\mathbb{T}$ defined by*

$$ \sigma(t) := \inf\{s \in \mathbb{T} : s > t\}. $$

## 6.2 Existence

We begin by giving an integral representation to our problem (11). Note that the case $0 < \alpha < \frac{1}{2}$ is coherent with our fractional operators with $2\alpha - 1 < 0$.

**Lemma 11.** *Let $0 < \alpha < \frac{1}{2}$. Problem (11) is equivalent to*

$$ u(t) = \frac{\lambda}{\Gamma(2\alpha)} \int_{t_0}^{t} (t-s)^{2\alpha-1} \frac{f(u(s))}{(\int_{t_0}^{T} f(u) \triangle x)^2} \triangle s. \tag{12} $$

*Proof.* We have

$$ \begin{aligned} {}^{\mathbb{T}}D_{t_{0+}}^{2\alpha} u(t) &= \frac{\lambda}{\Gamma(2\alpha)} \left( \int_{t_0}^{t} (t-s)^{2\alpha-1} \frac{f(u(s))}{(\int_{t_0}^{T} f(u) \triangle x)^2} \triangle s \right)^{\triangle} \\ &= \left({}^{\mathbb{T}}I_{t_{0+}}^{1-2\alpha} u(t)\right)^{\triangle} = (\triangle \circ {}^{\mathbb{T}}I_{t_{0+}}^{1-2\alpha}) u(t). \end{aligned} $$

The result follows from Proposition 3: ${}^{\mathbb{T}}I_{t_{0+}}^{2\alpha} \circ ({}^{\mathbb{T}}D_{t_{0+}}^{2\alpha}(u)) = u$. □

For the sake of simplicity, we take $t_0 = 0$. It is easy to see that (11) has a solution $u = u(t)$ if and only if $u$ is a fixed point of the operator $K : X \to X$ defined by

$$Ku(t) = \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{f(u(s))}{(\int_0^T f(u) \triangle x)^2} \triangle s. \tag{13}$$

To prove existence of the solution, we begin by showing that the operator $K$ defined by (13) verifies the conditions of Schauder's fixed-point theorem.

**Lemma 12.** *The operator $K$ is continuous.*

**Lemma 13.** *The operator $K$ sends bounded sets into bounded sets on $\mathbb{C}([0,T],\mathbb{R})$.*

**Lemma 14.** *The operator $K$ sends bounded sets into equicontinuous sets of $\mathbb{C}(I,\mathbb{R})$.*

It follows by Schauder's fixed-point theorem that (11) has a solution on $I$. We have just proved Theorem 8.

**Theorem 8** (Existence of the solution). *Let $0 < \alpha < \frac{1}{2}$ and $f$ satisfies hypothesis* $(\mathrm{H}_1)$. *Then there exists a solution $u \in X$ of* (11) *for all $\lambda > 0$.*

## 6.3 Uniqueness

We now derive uniqueness of the solution to problem (11).

**Theorem 9** (Uniqueness of the solution). *Let $0 < \alpha < \frac{1}{2}$ and $f$ be a function satisfying the hypothesis* $(\mathrm{H}_1)$. *If*

$$0 < \lambda < \left( \frac{T^{2\alpha} L_f}{(c_1 T)^2 \Gamma(2\alpha+1)} + \frac{2c_2^2 T^{2(\alpha+1)} L_f}{(c_1 T)^4 \Gamma(2\alpha+1)} \right)^{-1},$$

*then the solution predicted by Theorem* 8 *is unique.*

The map $K : X \to X$ is a contraction. It follows by the Banach principle that it has a fixed point $u = Fu$. Hence, there exists a unique $u \in X$ solution of (12).

# 7 Conclusion

We gave a survey on methods and results based on the reduction of FDEs to integral equations. As an example, we have reviewed the main results and proof techniques of [71, 74, 77]. The employed mathematical techniques are quite general and effective, and they can be used to cover a wide class of integral equations of fractional order. We trust the theoretical results here reported will have a positive impact on the development of computer methods for integral equations of fractional order.

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