Pascal Trapezoids Emerging from Hypercomplex Polynomial Sequences

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Abstract

The construction of two different representations of special Appell polynomials in $(n+1)$ real variables with values in a Clifford algebra suggested to explore the relation between the respective coefficients. Properties of sequences resulting from such relation and an interesting trapezoidal array of their elements are pointed out.

2010 Mathematics Subject Classifications : 11B83, 05A10, 30G35
Keywords: special sequences, binomial coefficients, Pascal trapezoids, hypercomplex polynomials

Introduction

In this paper we focus on polynomial sequences in $(n+1)$ real variables with values in the real vector space of paravectors in the corresponding Clifford algebra $\mathcal{C}\ell_{0,n}$. We start by introducing some basics of that algebra. The reader can find more details in [4].

Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis of the real Euclidean vector space $\mathbb{R}^n$ endowed with a product according to the multiplication rules

$$e_ie_j + e_je_i = -2\delta_{ij},\ i,j = 1,\cdots,n,$$

where δ_{ij} is the Kronecker symbol. This non-commutative product generates the associative 2^n-dimensional Clifford algebra $\mathcal{C}\ell_{0,n}$ over $\mathbb{R}$. The elements z of $\mathcal{C}\ell_{0,n}$, called *hypercomplex numbers*, are of the form $z = \sum_A z_Ae_A$, where $z_A \in \mathbb{R}$ and the basis $\{e_A : A \subseteq \{1,\cdots,n\}\}$ is formed by $e_A = e_{h_1}e_{h_2}\cdots e_{h_r}$, $1 \leq h_1 < \cdots < h_r \leq n,\ e_\emptyset = e_0 = 1$.

The vector space $\mathbb{R}^{n+1}$ is embedded in $\mathcal{C}\ell_{0,n}$ by the identification of the real $(n+1)-$tuple $(x_0, x_1, \cdots, x_n)$ with the paravector

$$x = x_0 + \underline{x} = x_0 + x_1e_1 + \cdots + x_ne_n \in \mathcal{A}_n := \text{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset \mathcal{C}\ell_{0,n}.$$

The conjugate of $x \in \mathcal{A}_n$ is given by $\bar{x} = x_0 - \underline{x}$. The so-called scalar part x_0 and the vector part $\underline{x}$ of x can be written in the form $x_0 = (x+\bar{x})/2$ and $\underline{x} = (x-\bar{x})/2$, respectively. The norm of x is given by $|x| = (x\bar{x})^{\frac{1}{2}} = (x_0^2+x_1^2+\cdots+x_n^2)^{\frac{1}{2}}$. Consequently, the inverse of each non-zero x is $x^{-1} = \bar{x}|x|^{-2}$.

We consider $\mathcal{C}\ell_{0,n}$-valued functions defined as mappings

$$f : \Omega \subset \mathbb{R}^{n+1} \cong \mathcal{A}_n \longmapsto \mathcal{C}\ell_{0,n}$$

such that $f(x) = \sum_A f_A(x)e_A,\ f_A(x) \in \mathbb{R}$ and Ω is an open subset of $\mathbb{R}^{n+1},\ n \geq 1$.

The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$ is defined by $\bar{\partial} := \frac{1}{2}(\partial_0 + \partial_{\underline{x}})$, with $\partial_0 := \frac{\partial}{\partial x_0}$ and $\partial_{\underline{x}} := \sum_{k=1}^{n} e_k \frac{\partial}{\partial x_k}$. Its conjugate, also called the hypercomplex differential operator, is denoted by $\partial := \frac{1}{2}(\partial_0 - \partial_{\underline{x}})$.

The analogue of a holomorphic function is now a $\mathcal{C}^1$-function f that is a solution of the differential equation $\bar{\partial} f = 0$ (resp.$f\bar{\partial} = 0$) and is called *left monogenic* (resp. *right monogenic*).

The concept of hypercomplex differentiability as a generalization of complex differentiability reads as follows. A function f defined in Ω is hypercomplex differentiable if and only if it has a uniquely defined areolar derivative f' in each point of Ω (for details, see [10]). A hypercomplex differentiable function f is real differentiable and consequently f' is given by $f' = \partial f = \frac{1}{2}(\partial_0 - \partial_{\underline{x}})f$. Since a hypercomplex differentiable function f is monogenic, it follows that $f' = \partial_0 f = -\partial_{\underline{x}} f$ (see [9]).

Noting that $\bar{\partial} x = \frac{1-n}{2}$, it is clear that the identity function $f(x) = x$ (and its integer powers) belongs to the class of monogenic functions only if $n = 1$, i.e., in the complex case. Thus, the construction of polynomials which behave with respect to the derivative like simple powers of $x \in \mathcal{A}_n$ is a problem of its own interest. Applying Appell's ideas [3], authors of this paper started a systematic study on Appell sequences in the framework of Hypercomplex Function Theory ([5, 8, 11]).

There are two natural representations of $\mathcal{A}_n$-valued homogeneous polynomials, one by using $(x, \bar{x})$ and the other by using $(x_0, \underline{x})$. Both representations involve coefficients whose relation leads to sequences of nonnegative integers. The main goal of the present paper is to emphasize properties of those sequences and their relation with Pascal's like triangles.

Main Results

We focus on the following two representations of $\mathcal{A}_n$-valued homogeneous monogenic polynomials $\mathcal{P}_k^n(x)$ introduced in [7, 12]:

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} T_s^k(n) x^{k-s} \bar{x}^s \tag{1}$$

and

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} \binom{k}{s} c_s(n)\, x_0^{k-s}\, \underline{x}^s, \tag{2}$$

where the coefficients $T_s^k(n)$ and $c_s(n)$ take the form

$$T_s^k(n) = \binom{k}{s} \frac{\left(\frac{n+1}{2}\right)_{k-s} \left(\frac{n-1}{2}\right)_s}{(n)_k}$$

($(a)_r$ denotes the Pochhammer symbol, i.e., $(a)_0 := 1, (a)_r := \prod_{t=0}^{r-1}(a+t),\ r \geq 1$), and

$$c_s(n) = \begin{cases} \frac{s!!(n-2)!!}{(n+s-1)!!}, & \text{if } s \text{ is odd} \\ c_{s-1}(n), & \text{if } s \text{ is even} \end{cases},$$

respectively.

The sequences $(\mathcal{P}_k^n(x))_{k \geq 0}$ are generalized Appell sequences, with $\mathcal{P}_0^n(x) = 1$, according to the following definition ([7]).

Definition 1. *A sequence of $\mathcal{A}_n$-valued monogenic polynomials $(\mathcal{Q}_k(x))_{k\geq 0}$ is called a generalized Appell sequence, if $\mathcal{Q}_k(x)$ is of exact degree k, for each $k = 0, 1, \ldots$, and $\partial\mathcal{Q}_k(x) = k\mathcal{Q}_{k-1}(x),\ k = 1, 2, \ldots$.*

The relation between the representations (1) and (2) is intrinsically linked to the one of respective coefficients. In order to point out such relation we start by considering the $(k+1)-$dimensional vectors

$$\mathbf{T}_k(n) = \begin{bmatrix} T_0^k(n) & T_1^k(n) & \ldots & T_{k-1}^k(n) & T_k^k(n) \end{bmatrix}^T$$

and

$$\mathbf{C}_k(n) = \begin{bmatrix} c_0(n) & c_1(n) & \ldots & c_{k-1}(n) & c_k(n) \end{bmatrix}^T.$$

Each component of $\mathbf{T}_k(n)$ can be written as linear combination of the components of $\mathbf{C}_k(n)$ as follows:

$$T_s^k(n) = \tfrac{1}{2^k}\binom{k}{s}\sum_{j=0}^{k}\sigma_{s,j}^k c_j(n), \quad k = 0, 1, \cdots;\ s = 0, \cdots, k \tag{3}$$

where

$$\sigma_{s,j}^k = \sum_{m=0}^{s}(-1)^m\binom{s}{m}\binom{k-s}{j-m}. \tag{4}$$

(cf. [6, Thm. 6]).

Reciprocally, each component of $\mathbf{C}_k(n)$ can also be written in the following way:

$$c_{k-i}(n) = \frac{1}{\binom{k}{i}}\sum_{s=0}^{k}(-1)^s\sigma_{s,i}^k T_s^k(n), \quad k = 0, 1, \cdots;\ i = 0, 1, \ldots, k \tag{5}$$

(cf. [6, Thm. 7]).

The transformation from $\mathbf{C}_k(n)$ to $\mathbf{T}_k(n)$ can be derived in matrix form. Define the diagonal matrix $D_k = \mathrm{diag}[\binom{k}{0}\ \binom{k}{1}\ \cdots\ \binom{k}{k}]$ and the matrix $S_k := \left[s_{ij}^k\right]_{i,j=1}^{k+1}$ such that $s_{ij}^k = \sigma_{i-1,j-1}^k$. Thus, (3) can be written in the form

$$\mathbf{T}_k(n) = N_k\mathbf{C}_k(n),$$

where $N_k = \frac{1}{2^k}D_kS_k$. Analogously, the matrix form of (5) is obtained as

$$\mathbf{C}_k(n) = \tilde{N}_k\mathbf{T}_k(n),$$

where $\tilde{N}_k = D_k^{-1}\tilde{S}_kD$, $D = \mathrm{diag}[1\ \ -1\ \cdots\ (-1)^k]$ and the entries of $\tilde{S}_k$ are given by $\tilde{s}_{ij}^k = \sigma_{j-1,k-i+1}^k$.

We observe that the connection between $\mathbf{C}_k(n)$ and $\mathbf{T}_k(n)$ relies indeed on the nonnegative integers (4).

The importance of the Pascal's triangle in issues related to Appell polynomials has already been studied. For instance, it appears in the matrix representation of real and hypercomplex Appell polynomials. For details we refer to [1, 2] and references therein. That relation led us to believe that the integers (4) would also be somehow linked to triangles of that type.

Let i, j, k be arbitrary nonnegative integers such that $j \leq k$. For each fixed i, we arrange the numbers $\sigma_{i,j}^k$ in a triangle with rows k and ordered from $j = 0$ to $j = k$ (see Table 1).

Table 1: Triangles associated with $\sigma_{i,j}^{k}$

$$\begin{array}{c}
\sigma_{i,0}^{0} \\
\sigma_{i,0}^{1} \quad \sigma_{i,1}^{1} \\
\sigma_{i,0}^{2} \quad \sigma_{i,1}^{2} \quad \sigma_{i,2}^{2} \\
\sigma_{i,0}^{3} \quad \sigma_{i,1}^{3} \quad \sigma_{i,2}^{3} \quad \sigma_{i,3}^{3} \\
\sigma_{i,0}^{4} \quad \sigma_{i,1}^{4} \quad \sigma_{i,2}^{4} \quad \sigma_{i,3}^{4} \quad \sigma_{i,4}^{4} \\
\cdot^{\cdot^{\cdot}} \quad \cdot^{\cdot^{\cdot}} \qquad\qquad \ddots \quad \ddots \quad \ddots
\end{array}$$

For each fixed value of i, the coeficients $\sigma_{i,j}^{k}$ satisfy the recurrence relation

$$\sigma_{i,j+1}^{(k+1)} = \sigma_{i,j}^{k} + \sigma_{i,j+1}^{k}, \ (0 \leq j \leq k-1, k \geq i) \tag{6}$$

with boundary conditions

$$\sigma_{i,0}^{k} = 1, \qquad \sigma_{i,k}^{k} = (-1)^{i}, \ (k \geq i) \tag{7}$$

and initial values

$$\sigma_{i,j}^{(i)} = \binom{i}{j}(-1)^{j}, \ j = 1, \ldots, i-1 \tag{8}$$

(cf. [6, Thm. 10]).

Notice that when $i = 0$, formulae (6)-(7)-(8) coincide with the Pascal recurrence $\binom{k+1}{j+1} = \binom{k}{j} + \binom{k}{j+1}$, for all integers $k, j : 0 \leq j \leq k-1$, with initial/boundary values $\binom{k}{0} = \binom{k}{k} = 1$. As is well known, this formula allows the construction of the Pascal triangle. Now, the general formulae (6)-(7)-(8) permit, for each fixed i, the construction of *Pascal trapezoids*, i.e., each triangle (Table 1) encloses a trapezoidal substructure in itself (see Table 2) for the cases $i = 1$ and $i = 2$).

Table 2: Triangles associated with $\sigma_{i,j}^{k}$, $i = 1, 2$

$$\begin{array}{c}
0 \\
1 \quad -1 \\
1 \quad 0 \quad -1 \\
1 \quad 1 \quad -1 \quad -1 \\
1 \quad 2 \quad 0 \quad -2 \quad -1 \\
1 \quad 3 \quad 2 \quad -2 \quad -3 \quad -1 \\
1 \quad 4 \quad 5 \quad 0 \quad -5 \quad -4 \quad -1 \\
1 \quad 5 \quad 9 \quad 5 \quad -5 \quad -9 \quad -5 \quad -1 \\
\cdot^{\cdot^{\cdot}} \quad \cdot^{\cdot^{\cdot}} \quad \cdot^{\cdot^{\cdot}} \quad \cdot^{\cdot^{\cdot}} \qquad \ddots \quad \ddots \quad \ddots \ \ddots \\
\\
i=1
\end{array}
\qquad
\begin{array}{c}
2 \\
-2 \quad -2 \\
1 \quad -2 \quad 1 \\
1 \quad -1 \quad -1 \quad 1 \\
1 \quad 0 \quad -2 \quad 0 \quad 1 \\
1 \quad 1 \quad -2 \quad -2 \quad 1 \quad 1 \\
1 \quad 2 \quad -1 \quad -4 \quad -1 \quad 2 \quad 1 \\
1 \quad 3 \quad 1 \quad -5 \quad -5 \quad 1 \quad 3 \quad 1 \\
\cdot^{\cdot^{\cdot}} \quad \cdot^{\cdot^{\cdot}} \quad \cdot^{\cdot^{\cdot}} \quad \cdot^{\cdot^{\cdot}} \qquad \ddots \quad \ddots \quad \ddots \ \ddots \\
\\
i=2
\end{array}$$

More details on this subject can be seen in [6].

Conclusion

Two particular representations of generalized Appell polynomials in the hypercomplex context lead to special integer sequences satisfying a Pascal recurrence. The construction of Pascal trapezoids resulting from that recurrence has been examined.

Trapezoids presented in this paper, but not obtained as emerging from hypercomplex Appell polynomials, occure also in the expansion of $(1-x)^i(1+x)^{k-i}$. Indeed, $(1-x)^i(1+x)^{k-i} = \sum_{j=0}^{k} \sigma_{i,j}^{k}\, x^j$.

Acknowledgements

The work of the second author was supported by Portuguese funds through the CMAT - Centre of Mathematics and FCT within the Project UID/MAT/00013/2013. The work of the other authors was supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology ("FCT-Fundação para a Ciência e Tecnologia"), within project PEst-OE/MAT/UI4106/2013.

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