Vietoris' number sequence and its generalizations through hypercomplex function theory

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Abstract

The so-called Vietoris' number sequence is a sequence of rational numbers that appeared for the first time in a celebrated theorem by Vietoris (1958) about the positivity of certain trigonometric sums with important applications in harmonic analysis (Askey/Steinig, 1974) and in the theory of stable holomorphic functions (Ruscheweyh/ Salinas, 2004). In the context of hypercomplex function theory those numbers appear as coefficients of special homogeneous polynomials in $\mathbb{R}^3$ whose generalization to an arbitrary dimension n lead to a n-parameter generalized Vietoris' number sequence that characterizes hypercomplex Appell polynomials in $\mathbb{R}^n$.

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Introduction

The Vietoris' number sequence $\mathcal{S}$ is the following sequence of rational numbers

$$1,\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{3}{8},\ \tfrac{3}{8},\ \tfrac{5}{16},\ \tfrac{5}{16}, \tfrac{35}{128}, \tfrac{35}{128}, \tfrac{63}{256}, \tfrac{63}{256}, \tfrac{231}{1024}, \tfrac{231}{1024}, \ldots. \tag{1}$$

which by means of the *generalized central binomial coefficient* $\binom{k}{\lfloor \frac{k}{2} \rfloor}$ can be written in compact form (cf. [3]) as $\mathcal{S} = (c_k)_{k\geq 0}$, where

$$c_k = \frac{1}{2^k}\binom{k}{\lfloor \frac{k}{2} \rfloor} = \frac{\left(\frac{1}{2}\right)_{\lfloor \frac{k+1}{2} \rfloor}}{(1)_{\lfloor \frac{k+1}{2} \rfloor}}. \tag{2}$$

Here, as usual, $\lfloor \cdot \rfloor$ denotes the floor function and $(\cdot)_k$ is the raising factorial in the classical form of the Pochhammer symbol.

Seemingly this sequence appeared, for the first time, in the context of positive trigonometric sums in a celebrated paper of L. Vietoris [11]. Askey's version [2, p. 5] of Vietoris' theorem is the following:

Theorem 1 (L. Vietoris)**.**

$$\sum_{k=1}^{n} a_k \sin k\theta > 0,\ \ 0 < \theta < \pi,\ \ \textit{and}\ \sum_{k=0}^{n} a_k \cos k\theta > 0,\ \ 0 \leq \theta < \pi,$$

where

$$a_{2k} = a_{2k+1} = \frac{(\frac{1}{2})_k}{k!}, \qquad k = 0, 1, \ldots. \tag{3}$$

We call attention to the fact that because of (3), the coefficients in the *sine* sum are exactly the elements of $\mathcal{S}$ in (2) or, explicitly, in (1). Compared with the traditional way of defining the coefficient sequence by (3), the use of the properties of the generalized central binomial coefficient allows a unique representation (2) with consecutively running index k.

In the context of hypercomplex function theory, the sequence $\mathcal{S}$ characterizes a special homogeneous polynomial sequence that can be considered in higher dimensions as the counterpart of the sequence of holomorphic powers.

In the sequel we will use the following basic concepts and notations. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ endowed with a non-commutative product according to the multiplication rules

$$e_ie_j + e_je_i = -2\delta_{ij},\ i,j = 1,2,\ldots,n,$$

where δ_{ij} is the Kronecker symbol. This generates the associative 2^n-dimensional Clifford algebra $\mathcal{C}\ell_{0,n}$ over $\mathbb{R}$, whose elements are of the form $a = \sum_A a_A e_A,\ a_A \in \mathbb{R}$, with $A \subseteq \{1,\cdots,n\}$, $e_A = e_{l_1}e_{l_2}\cdots e_{l_r}$, where $1 \le l_1 < \cdots < l_r \le n$ and $e_\emptyset = e_0 = 1$. In general, the vector space $\mathbb{R}^{n+1}$ is embedded in $\mathcal{C}\ell_{0,n}$ by identifying the element $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ with the element (paravector)

$$x = x_0 + \sum_{k=1}^{n} e_k x_k = x_0 + \underline{x} \in \mathcal{A}_n := \mathrm{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset \mathcal{C}\ell_{0,n}.$$

Its conjugate is $\bar{x} = x_0 - \underline{x}$ and the norm of x is given by $|x| = (x\bar{x})^{1/2} = (\bar{x}x)^{1/2} = \left(\sum_{k=0}^{n} x_k^2\right)^{1/2}$.

We consider $\mathcal{C}\ell_{0,n}$-valued functions defined as mappings

$$f : \Omega \subset \mathbb{R}^{n+1} \cong \mathcal{A}_n \longmapsto \mathcal{C}\ell_{0,n}$$

such that $f(x) = \sum_A f_A(x)e_A,\ f_A(x) \in \mathbb{R}$ and Ω is an open subset of $\mathbb{R}^{n+1},\ n \ge 1$.

The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$ is $\overline{\partial} := \frac{1}{2}(\partial_0 + \partial_{\underline{x}})$, with $\partial_0 := \frac{\partial}{\partial x_0}$, and $\partial_{\underline{x}} := \sum_{k=1}^{n} e_k \frac{\partial}{\partial x_k}$. A C^1-function f is called *left (right) monogenic*, or simply *monogenic* in $\mathbb{R}^{n+1}$ if it is a solution of the differential equation $\overline{\partial} f = 0$ $(f\overline{\partial} = 0)$.

Notice that the operator $\partial := \frac{1}{2}(\partial_0 - \partial_{\underline{x}})$ is the conjugate generalized Cauchy-Riemann operator and acts as derivative of a monogenic function (cf.[9]). Therefore the hypercomplex derivative of a monogenic function f can be calculated as $\partial f = \frac{1}{2}(\partial_0 - \partial_{\underline{x}})f = \partial_0 f$, i.e. in the same way as the complex derivative of a holomorphic function.

Main Results

In the center of our attention is the sequence of paravector-valued monogenic polynomials $(\mathcal{P}_k^n)_{n\in\mathbb{N}}$ such that

$$\partial\mathcal{P}_k^n(x) = k\,\mathcal{P}_{k-1}^n(x), \qquad x \in \mathcal{A}_n, \qquad k = 1,2,\ldots. \tag{4}$$

Choosing as initial value $\mathcal{P}_0^n = 1$, the recurrence (4) together with the requirement of monogeinity,

$$\overline{\partial}\mathcal{P}_k^n(x) = 0, \qquad x \in \mathcal{A}_n,$$

lead to the explicit representation

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} \binom{k}{s} c_s(n)\, x_0^{k-s}\, \underline{x}^s, \qquad x \in \mathcal{A}_n, \tag{5}$$

where

$$c_s(n) := \frac{\left(\frac{1}{2}\right)_{\lfloor \frac{k+1}{2} \rfloor}}{\left(\frac{n}{2}\right)_{\lfloor \frac{k+1}{2} \rfloor}},\ s = 0, \ldots, k,\ k = 1, 2, \ldots. \tag{6}$$

See [4, 7, 8] for details.

The equality (4) for monogenic homogeneous polynomials generalizes to higher dimensions the classical concept of Appell polynomials (cf.[1]). We remark that the initial value of a Clifford algebra-valued Appell polynomial sequence can be a real, a Clifford number or a vector-valued monogenic polynomial of a fixed degree (a *monogenic constant*) (see [6, 10]).

Taking into account that for $n = 2$, (6) gives exactly the rational numbers (2) that constitute the Vietoris' number sequence $\mathcal{S}$, the coefficients sequence $(\mathcal{S}(n))_{n\in\mathbb{N}}$ characterizing the Appell polynomials (5) and whose general term is given by (6) is a n-parameter generalization of $\mathcal{S}$.

Moreover, the representation (6) in terms of quotients of numbers represented by the Pochhammer symbol suggests the use of the well known Gauss' hypergeometric function

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{+\infty} \frac{(a)_k\,(b)_k}{(c)_k} \frac{z^k}{k!},\quad |z| < 1,\quad a, b \in \mathbb{C},\quad c \in \mathbb{C} \setminus \mathbb{Z}_0^-$$

to derive a generating function of the generalized Vietoris' sequence $(\mathcal{S}(n))_{n\in\mathbb{N}}$. In fact, in [5] the following result was obtained.

Theorem 2. *Let $G(.,n)$ be the following real-valued function depending on a parameter $n \in \mathbb{N}$:*

$$G(t;n) = \begin{cases} \frac{1}{t}\left[(1+t)\,{}_2F_1(\frac{1}{2}, 1; \frac{n}{2}; t^2) - 1\right], & \text{if } t \in]-1, 0[\cup]0, 1[\\ 1, & \text{if } t = 0. \end{cases}$$

Then, for any fixed $n \in \mathbb{N}$, $G(.,n)$ is a one-parameter generating function of the sequence $\mathcal{S}(n)$.

It is clear that we can obtain a closed formula for the generating function of the sequence $(\mathcal{S}(n))_{n\in\mathbb{N}}$ as long as a closed formula for the corresponding hypergeometric series is known. As examples we list some cases where closed formulae can be easily obtained:

1. $n = 1$

   In this case, $c_k(1) = 1$ $(k \geq 0)$ and the corresponding generating function is given by

   $$G(t;1) = \frac{1}{t}\left[(1+t)\,{}_2F_1(\tfrac{1}{2}, 1; \tfrac{1}{2}; t^2) - 1\right] = \frac{1}{1-t},$$

   because ${}_2F_1(\frac{1}{2}, 1; \frac{1}{2}; t^2)$ reduces to the geometric function.

2. $n = 2$

   Recalling (2), we have $c_{2k}(2) = c_{2k-1}(2) = \frac{\left(\frac{1}{2}\right)_k}{k!}$ and

   $$G(t;2) = \frac{1}{t}\left[(1+t)\,{}_2F_1(\tfrac{1}{2}, 1; 1; t^2) - 1\right] = \frac{\sqrt{1+t} - \sqrt{1-t}}{t\sqrt{1-t}}.$$

3. $n = 3$

   The generalized Vietoris' numbers are $c_{2k}(3) = c_{2k-1}(3) = \frac{1}{2k+1}$ and the corresponding generating function is given by

   $$G(t;3) = \frac{1}{t}\left[(1+t)\,{}_2F_1(\tfrac{1}{2}, 1; \tfrac{3}{2}; t^2) - 1\right] = \frac{1}{t}\left(\frac{t+1}{t}\ln\sqrt{\frac{1+t}{1-t}} - 1\right).$$

4. $n = 4$

   In this case, $c_{2k}(4) = c_{2k-1}(4) = \frac{\left(\frac{1}{2}\right)_k}{(k+1)!}$ and

$$G(t;4) = \frac{1}{t}\left[(1+t)\,{}_2F_1(\tfrac{1}{2},1;2;t^2) - 1\right] = \frac{2t+1-\sqrt{1-t^2}}{t(1+\sqrt{1-t^2})}.$$

Conclusion

By providing a link between the Vietoris' number sequence and hypercomplex Appell polynomials, we were able to define one-parameter generalized Vietoris' number sequences and obtain their generating functions.

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