Separation set distance for 2D Convolutional Codes*

José I. Iglesias[1], Diego Napp[2], Carmen Perea[3], Raquel Pinto[2] and Rita Simões[2§]

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Abstract

In this paper we address the problem of extending the well-known notion of column distance of one-dimensional (1D) convolutional codes to the context of multidimensional (nD) convolutional codes. In particular, we treat the 2D case and propose a new and more general notion than the one previously introduced in this context. We derive upper bounds on the distances that lead to the novel notion of Maximum Separation Set Distance Profile 2D convolutional codes. This notion naturally extends the notion of Maximum Distance Profile 1D convolutional code. Characterizations in terms of the sliding parity-check matrices are presented.

1 Introduction

Convolutional codes appeared as a generalization of linear codes. The main assumption was that each codeword wasn't treated independently, but rather as part of a sequence, and that encoding or decoding of one of them would affect the previous or next ones. Hence the whole sequence was taken as the main object of study, and each constant block was assigned the power of an indeterminate corresponding to its position in the sequence. Therefore, the sequences of vectors were represented by polynomial vectors, and generator matrices were allowed to have polynomial components instead of just constants. A convolutional code is then defined as the submodule over the ring of polynomials in one variable given by the image of one such polynomial generator matrix. The natural question

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†1 José I. Iglesias is with Department of Mathematics and IUFFyM at University of Salamanca, Pl Merced, 1, 37008 Salamanca, Spain joseig@usal.es

‡2 Diego Napp, Raquel Pinto and Rita Simões are with the Department of Mathematics of University of Aveiro, Campus de Santiago, 3810-193 Aveiro, Portugal diego@ua.pt, raquel@ua.pt,rita.simoes@ua.pt

§3 Carmen Perea is with Center of Operations Research University Institute of Miguel Hernández University, Avinguda de la Universidad, 03202 Elche, Alacant, Spain perea@umh.es

then arises of whether one could make a similar generalization step by adding further variables.

Two-dimensional (2D) convolutional codes are the generalization of classical convolutional codes by considering polynomial vectors and matrices on 2 variables. In this sense, usual convolutional codes are 1D convolutional codes. 2D convolutional codes have potential applications in communications systems where data have a two dimensional nature, such as images, videos, etc.

While 1D convolutional codes have been rather well-understood, the literature about 2D convolutional codes is limited. Algebraic aspects and fundamental issues were first presented in [11] and later in [12] and [7]. C. Charoenlarpnopparut and N. K. Bose explored in [13] the relation between the Gröbner bases and nD convolutional codes and in [9] the internal properties of their input-state-output representations were investigated.

Interestingly, there has been little prior work on distances of 2D convolutional codes, most of it on the notion of (free) distance [2] and [3]. The fundamental notion of column distance of 1D convolutional codes, introduced in [4], was first extended to the 2D context in [10]. However, in that preliminary work the notion of column distance was restricted to consider the particular subset of codewords with non zero constant term, i.e. $v(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} v_{i,j} z_1^i z_2^j$ with $v(0,0) = v_{0,0} \neq 0$. This was a technical restriction in order to facilitate the analysis and to obtain simpler extensions of the existing results in the 1D case.

Here we get rid of that assumption and introduce the notion of column distance of 2D convolutional codes in a more natural way. We present upper bounds on these distances and provide characterizations in terms of the properties of the sliding parity-check matrices of the code. These results allow to introduce the notion of Maximum Separation Set Distance Profile 2D convolutional code which can be considered as the 2D analog of the well-known class of Maximum Distance Profile (MDP) convolutional codes [4].

2 Preliminaries

2.1 2D convolutional codes

In this paper, we consider convolutional codes constituted by finite support two dimensional sequences defined in $\mathbb{F}^n$, where $\mathbb{F}$ is a finite field,

$$\begin{array}{rccl} v: & \mathbb{N}^2 & \rightarrow & \mathbb{F}^n \\ & (i,j) & \mapsto & v_{i,j} \end{array}$$

These sequences can be represented as polynomial vectors in two variables

$$v(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} v_{i,j} z_1^i z_2^j \in \mathbb{F}[z_1, z_2]^n$$

where $\mathbb{F}[z_1, z_2]$ denotes the ring of polynomials in two variables.

Definition 1 *A* 2D convolutional code of rate k/n *over* $\mathbb{F}$ *is a free* $\mathbb{F}[z_1, z_2]$*-submodule of* $\mathbb{F}[z_1, z_2]^n$ *of rank* k*.*

Any polynomial matrix $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n \times k}$ whose columns constitute a basis of $\mathcal{C}$, i.e., such that

$$\begin{aligned} \mathcal{C} &= \mathrm{im}_{\mathbb{F}[z_1,z_2]}\, G(z_1, z_2) \\ &= \{v(z_1, z_2) \in \mathbb{F}^n[z_1, z_2] : v(z_1, z_2) = G(z_1, z_2)u(z_1, z_2), \\ &\qquad u(z_1, z_2) \in \mathbb{F}^k[z_1, z_2]\}, \end{aligned}$$

is called an *encoder* of $\mathcal{C}$. The vector $u(z_1, z_2)$ is called the *information sequence*, which produces the corresponding codeword $v(z_1, z_2) = G(z_1, z_2)u(z_1, z_2)$.

Two full column rank matrices $G(z_1, z_2), \overline{G}(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n \times k}$ are *equivalent encoders* if they generate the same 2D convolutional code, i.e., $\mathrm{im}_{\mathbb{F}[z_1,z_2]}\, G(z_1, z_2) = \mathrm{im}_{\mathbb{F}[z_1,z_2]}\, \overline{G}(z_1, z_2)$, which happens if and only if there exists a unimodular[1] matrix $U(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{k \times k}$ such that $G(z_1, z_2)U(z_1, z_2) = \overline{G}(z_1, z_2)$ [6].

A matrix $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n \times k}$, with $n \geq k$, is called *right factor prime* (rFP) if for every factorization

$$G(z_1, z_2) = \overline{G}(z_1, z_2)V(z_1, z_2),$$

with $V(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{k \times k}$, $V(z_1, z_2)$ is unimodular. An immediate consequence of the above definition is that right factor prime matrices are always full column rank.

As unimodular matrices preserve the primeness properties of the encoders of a code we have that if a 2D convolutional code $\mathcal{C}$ admits a rFP encoder then all its encoders are rFP. Moreover, if an encoder $G(z_1, z_2)$ of $\mathcal{C}$ is such that $G(0, 0)$ is full column rank, the same happens for all the encoders of $\mathcal{C}$.

Definition 2 *A 2D convolutional code that admits rFP encoders is called noncatastrophic and a 2D convolutional code that admits an encoder $G(z_1, z_2)$ such that $G(0, 0)$ is full column rank is called a delay-free 2D convolutional code.*

A noncatastrophic code admits a kernel representation as stated in the following proposition.

Proposition 1 *[11] Let $\mathcal{C}$ be a 2D convolutional code of rate k/n. Then there exists a matrix $H(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{(n-k) \times k}$ such that*

$$\begin{aligned} \mathcal{C} &= \ker_{\mathbb{F}[z_1,z_2]}\, H(z_1, z_2) \\ &= \{v(z_1, z_2) \in \mathbb{F}[z_1, z_2]^n : H(z_1, z_2)v(z_1, z_2) = 0\} \end{aligned}$$

if and only if $\mathcal{C}$ is noncatastrophic.

Definition 3 *Let $\mathcal{C}$ be a noncatastrophic 2D convolutional code $\mathcal{C}$ of rate k/n. A matrix $H(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{(n-k) \times k}$ such that $\mathcal{C} = \ker_{\mathbb{F}[z_1,z_2]} H(z_1, z_2)$, is called a parity-check matrix (or a syndrome former) of $\mathcal{C}$.*

[1] A matrix $U(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{k \times k}$ is *unimodular* if it has a polynomial inverse or equivalently if $\det U(z_1, z_2) \in \mathbb{F} \setminus \{0\}$.

In this paper we will consider noncatastrophic delay-free convolutional codes.

Codes with good distance are more robust to error corruption. The degree of a 1D convolutional code is one of the parameters of the generalized Singleton bound on the distance of these codes. To define similar notions for 2D convolutional codes, we need to consider first the usual notion of *(total) degree* of a polynomial matrix

$$G(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} G(i,j) z_1^i z_2^j \in \mathbb{F}[z_1, z_2]^{n\times k},$$

with $G(i,j) \in \mathbb{F}^{n\times k}$, defined as

$$\deg(G(z_1, z_2)) = \max\{i + j : G(i,j) \neq 0\}.$$

We can define the (total) degree of a polynomial vector (or just of a polynomial) in the same way.

We define the degree of a 2D convolutional code in a similar way as it is defined for 1D convolutional codes.

Definition 4 *[3] Let $\mathcal{C}$ be a 2D convolutional code, $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n\times k}$ an encoder of $\mathcal{C}$ and ν_i the column degree of the ith column of $G(z_1, z_2)$, i.e, the maximum degree of the entries of the ith column of $G(z_1, z_2)$. The external degree of $G(z_1, z_2)$, denoted by $\delta_e(G(z_1, z_2))$, is defined as*

$$\delta_e\left(G(z_1, z_2)\right) = \sum_{i=1}^{k} \nu_i$$

and the degree of $\mathcal{C}$, denoted by δ, is defined as the minimum of the external degrees of all the encoders of $\mathcal{C}$.

Next we define the notion of distance of a 2D convolutional code as in [12].

Definition 5 *The distance of a 2D convolutional code $\mathcal{C}$ is defined as*

$$\mathrm{dist}(\mathcal{C}) = \min\{\mathrm{wt}(v) : v \in \mathcal{C} \setminus \{0\}\}.$$

with

$$\mathrm{wt}(v) = \sum_{(i,j)\in\mathbb{N}^2} \mathrm{wt}(v_{i,j}),$$

where $v(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} v_{i,j} z_1^i z_2^j$, $v_{ij} \in \mathbb{F}^n$, and $\mathrm{wt}(v_{i,j})$ is the number of nonzero entries of $v_{i,j}$.

Theorem 1 *[3] Let $\mathcal{C}$ be a 2D convolutional code of rate k/n and degree δ; then*

$$\mathrm{dist}(\mathcal{C}) \leq \frac{n}{2}\left(\left\lfloor\frac{\delta}{k}\right\rfloor + 1\right)\left(\left\lfloor\frac{\delta}{k}\right\rfloor + 2\right) - k\left(\left\lfloor\frac{\delta}{k}\right\rfloor + 1\right) + \delta + 1$$

This upper bound is called the 2*D generalized Singleton bound.*

2.2 Separations sets

Let us call *past* and *future* of a point $(i,j) \in \mathbb{Z}^2$ the sets

$$\begin{aligned} \mathcal{P}_{(i,j)} &= \left\{(h,k) \in \mathbb{Z}^2 : h \leq i \text{ and } k \leq j\right\} \\ \mathcal{F}_{(i,j)} &= \left\{(h,k) \in \mathbb{Z}^2 : i \leq h \text{ and } j \leq k\right\} \end{aligned}$$

respectively. The sets

$$\mathcal{C}_l = \left\{(i,j) \in \mathbb{Z}^2 : i+j = l\right\}, \qquad \text{for } l = 0, 1, \ldots \tag{1}$$

are called *separation sets*.

Thus, given a separation set $\mathcal{C}_l$, $\mathbb{Z}^2$ is partitioned in two subsets

$$\mathcal{P}_{\mathcal{C}_l} = \bigcup_{(i,j)\in\mathcal{C}_l} \mathcal{P}_{(i,j)} \qquad \text{and} \qquad \mathcal{F}_{\mathcal{C}_l} = \bigcup_{(i,j)\in\mathcal{C}_l} \mathcal{F}_{(i,j)}$$

Given a word

$$v(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} v_{i,j} z_1^i z_2^j \in \mathbb{F}^n[z_1, z_2],$$

the *support of* v is defined as

$$S^v = \left\{(i,j) \in \mathbb{N}^2 : v_{i,j} \neq 0\right\}.$$

We call $\mathcal{C}_{l_0}$ the *initial separation set* of v if

$$l_0 = \min\left\{l \in \{0, 1, \ldots\} : \mathcal{C}_l \cap S^v \neq \emptyset\right\}$$

and l_0 the *initial index of* v.

3 Separation set distance for 2D codes

3.1 Separation set distance of a 2D convolutional code

In this section we define the sequence of separation set distances of a 2D convolutional code. As their counterparts on 1D convolutional codes, the column distances, they measure the “spread” of the codewords truncated at a certain degree.

Let us define the *truncation of length* l of a word

$$v(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} v_{i,j} z_1^i z_2^j, \quad v_{ij} \in \mathbb{F}^n,$$

to the interval $[h, h+l]$, for $l, h \in \mathbb{N}$, as

$$v_{[h,h+l]} = \sum_{h \leq i+j \leq h+l} v_{i,j} z_1^i z_2^j.$$

Definition 6 *The l-th* separation set distance *of the 2D convolutional code $\mathcal{C}$, d_l^s, $l \in \{0,1,\dots\}$ is defined as*

$$d_l^s = \min\{\mathrm{wt}\left(v_{[l_0,l_0+l]}\right) : v \in \mathcal{C}\setminus\{0\} \textit{ and } l_0 \textit{ is the initial index of } v\}.$$

Let $G(z_1,z_2) = \sum_{(i,j)\in\mathbb{N}^2} G_{i,j} z_1^i z_2^j$, $G_{i,j} \in \mathbb{F}^{n\times k}$, be an encoder of $\mathcal{C}$, and let $v(z_1,z_2) = \sum_{(i,j)\in\mathbb{N}^2} v_{i,j} z_1^i z_2^j \in \mathcal{C}$, $v_{i,j} \in \mathbb{F}^n$ and $u(z_1,z_2) = \sum_{(i,j)\in\mathbb{N}^2} u_{i,j} z_1^i z_2^j \in \mathbb{F}^k[z_1,z_2]$, $u_{i,j} \in \mathbb{F}^k$, be such that $v(z_1,z_2) = G(z_1,z_2)u(z_1,z_2)$. Then

$$\left(v_{0,0} \mid v_{1,0} \quad v_{0,1} \mid v_{2,0} \quad v_{1,1} \quad v_{0,2} \mid \dots \right)^T,$$

can be obtained by multiplying the sliding matrix

$$G^s = \left(\begin{array}{c|cc|ccc|c}
G_{0,0} & & & & & & \\ \hline
G_{1,0} & G_{0,0} & & & & & \\
G_{0,1} & & G_{0,0} & & & & \\ \hline
G_{2,0} & G_{1,0} & & G_{0,0} & & & \\
G_{1,1} & G_{0,1} & G_{1,0} & & G_{0,0} & & \\
G_{0,2} & & G_{0,1} & & & G_{0,0} & \\ \hline
\vdots & & \vdots & & \vdots & & \ddots
\end{array}\right) \tag{2}$$

by the vector

$$\left(u_{0,0} \mid u_{1,0} \quad u_{0,1} \mid u_{2,0} \quad u_{1,1} \quad u_{0,2} \mid \dots \right)^T,$$

Note that since we assume that $G_{0,0}$ is full column rank, if $(i,j) \in \mathbb{N}^2$ is such that $S^v \cap \mathcal{P}_{(i,j)} \setminus \{(i,j)\} = \emptyset$, i.e., $v(z_1,z_2)$ is zero on the "strict past" of (i,j), the same happens for $u(z_1,z_2)$, i.e., $S^u \cap \mathcal{P}_{(i,j)} \setminus \{(i,j)\} = \emptyset$. Moreover, if l_0 is the initial index of v then l_0 is also the initial index of u and

$$\mathcal{C}_{l_0} \cap S^u = \mathcal{C}_{l_0} \cap S^v$$

Then we have that

$$\begin{aligned} d_l^s = \min\Big\{ & \mathrm{wt}\left(G^s_{l_{0|_l}} \begin{pmatrix} u_{l_0} & u_{l_0+1} & \cdots & u_{l_0+l}\end{pmatrix}^T\right) : \\ & u_{l_0+i} = \begin{pmatrix} u_{l_0+i,0} & u_{l_0+i-1,1} & \cdots & u_{0,l_0+i}\end{pmatrix} \in \mathbb{F}^{1\times k(l_0+i+1)}, \\ & i = 0,\dots,l \text{ and } u_{l_0} \neq 0, l_0 \in \mathbb{N}_0 \Big\}. \end{aligned}$$

where

$$G^s_{l_{0|_l}} = \left(\begin{array}{c|c|c|c}
\mathcal{G}_{0,l_0+1} & & & \\ \hline
\mathcal{G}_{1,l_0+1} & \mathcal{G}_{0,l_0+2} & & \\ \hline
\vdots & \vdots & \ddots & \\ \hline
\mathcal{G}_{l,l_0+1} & \mathcal{G}_{l-1,l_0+2} & & \mathcal{G}_{0,l_0+l+1}
\end{array}\right)$$

with

$$\mathcal{G}_{h,t} = \begin{pmatrix} G_{h,0} & & & \\ G_{h-1,1} & G_{h,0} & & \\ \vdots & \vdots & \ddots & \\ G_{0,h} & G_{1,h-1} & & G_{h,0} \\ & G_{0,h} & & \vdots \\ & & \ddots & \\ & & & G_{0,h} \end{pmatrix} \in \mathbb{F}^{(h+t)n \times tk}$$

If $H(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} H_{i,j} z_1^i z_2^j \in \mathbb{F}^{(n-k)\times n}$ is a parity-check matrix of $\mathcal{C}$, then for $v(z_1, z_2) \in \mathbb{F}^n[z_1, z_2]$ it follows that $v(z_1, z_2) \in \mathcal{C}$ if and only if $H(z_1, z_2)v(z_1, z_2) = 0$, or equivalently, if we consider the parity-check matrix in the sliding matrix form

$$H^s = \left(\begin{array}{c|cc|ccc|c} H_{0,0} & & & & & & \\ \hline H_{1,0} & H_{0,0} & & & & & \\ H_{0,1} & & H_{0,0} & & & & \\ \hline H_{2,0} & H_{1,0} & & H_{0,0} & & & \\ H_{1,1} & H_{0,1} & H_{1,0} & & H_{0,0} & & \\ H_{0,2} & & H_{0,1} & & & H_{0,0} & \\ \hline \vdots & \vdots & & \vdots & & & \ddots \end{array}\right) \tag{3}$$

if and only if $H^s \begin{pmatrix} v_{0,0} & v_{1,0} & v_{0,1} & \cdots \end{pmatrix}^T = 0$. Then

$$\begin{aligned} d_l^s = \min \Big\{ & \mathrm{wt}\left(v_{[l_0, l_0+l]}\right) : H^s_{l_{0|_l}} \begin{pmatrix} v_{l_0} & \cdots & v_{l_0+l} \end{pmatrix}^T = 0 : \\ & v_{l_0+i} = \begin{pmatrix} v_{l_0+i,0} & v_{l_0+i-1,1} & \cdots & v_{0,l_0+i} \end{pmatrix} \in \mathbb{F}^{1\times n(l_0+i+1)}, \\ & i = 0, \ldots, l \text{ and } l_0 \text{ is the initial index of } v \Big\} \end{aligned}$$

where $H^s_{l_{0|_l}}$ is defined analogously as $G^s_{l_{0|_l}}$.

Proposition 2 *Let $\mathcal{C}$ be a 2D convolutional code. Then*

1. *The separation set distances is a nondecreasing sequence, i.e., $d_0^s \leq d_1^s \leq d_2^s \leq \cdots$.*

2. *The sequence of separation set distances converges in a finite number of steps to the distance of the code, i.e., there exists $l \geq 0$ such that $d_l^s = \mathrm{dist}(\mathcal{C})$.*

Proof 1 *For the first statement note that if $v(z_1, z_2) \in \mathcal{C}$ is a codeword such that*

$$d_r^s = \mathrm{wt}\left(v_{[l_0, l_0+r]}\right),$$

where l_0 is the initial index of $v(z_1, z_2)$, then

$$\begin{aligned} d_r^s &= \mathrm{wt}(v_{[l_0,l_0+r]}) \\ &= \sum_{(i,j)\in\mathcal{C}_{l_0}\cup\cdots\cup\mathcal{C}_{l_0+r}} \mathrm{wt}(v_{ij}) \\ &\geq \sum_{(i,j)\in\mathcal{C}_{l_0}\cup\cdots\cup\mathcal{C}_{l_0+r-1}} \mathrm{wt}(v_{ij}) \\ &= \mathrm{wt}(v_{[l_0,l_0+r-1]}) \\ &\geq d_{r-1}^s. \end{aligned}$$

With respect to the second one, let $v(z_1, z_2) \in \mathcal{C}$ be a codeword such that $\mathrm{wt}(v(z_1, z_2)) = \mathrm{dist}(\mathcal{C})$. Without loss of generality, after a proper shifting via multiplication by appropriate $z_1^{-i}z_2^{-j}$ we may assume that

$$S^v \cap \{(i,0) : i \in \mathbb{N}\} \neq \emptyset \qquad \textit{and} \qquad S^v \cap \{(0,j) : j \in \mathbb{N}\} \neq \emptyset.$$

Let $r = \min\{k \in \mathbb{N} : S^v \cap \mathcal{F}_{\mathcal{C}_k}\} = \emptyset$. Then $v(z_1, z_2) = v(z_1, z_2)_{[l_0,k]}$ and

$$d_{free} = \mathrm{wt}(v(z_1, z_2)) = \mathrm{wt}\left(v(z_1, z_2)_{[l_0,k]}\right) \geq d_r^s$$

for all $r \geq k - l_0$.

Hence, the non decreasing sequence of separation set distances is upper-bounded, and therefore it has a limit d_∞^s which is reached for a certain l. Let us assume that $d_\infty^s < d_{free}$, i.e. there is a codeword $\tilde{v}(z_1, z_2)$ such that

$$\mathrm{wt}(\tilde{v}_{[l_0,l_0+k]}) = \mathrm{wt}(\tilde{v}_{[l_0,l_0+k+1]}) = \ldots = d_\infty^s < d_{free},$$

where l_0 is the initial index of $\tilde{v}(z_1, z_2)$, then $\tilde{v} = \tilde{v}_{[l_0,l_0k]} \in \mathcal{C}$, and

$$\mathrm{wt}(\tilde{v}) = d_\infty^s < d_{free}$$

from where we get a contradiction.

Let $v(z_1, z_2) \in \mathcal{C}$ and $u(z_1, z_2) \in \mathbb{F}^k[z_1, z_2]$ be such that $v(z_1, z_2) = G(z_1, z_2)u(z_1, z_2)$ and let l_0 be the initial index of $v(z_1, z_2)$ and $u(z_1, z_2)$. If $S^{v_{[l_0,l_0+l]}} \cap \{(i, 0) : i \in \mathbb{N}\} = \emptyset$ then $S^{u_{[l_0,l_0+l]}} \cap \{(i, 0) : i \in \mathbb{N}\} = \emptyset$ and

$$\tilde{v}(z_1, z_2) = G(z_1, z_2)z_2^{-1}u(z_1, z_2)_{[l_0,l_0+l]} \in \mathcal{C}$$

and is such that

$$\mathrm{wt}(\tilde{v}_{[l_0-1,l_0-1+l]}) = \mathrm{wt}(v_{[l_0,l_0+l]})$$

In the same way, if $S^{v_{[l_0,l_0+l]}} \cap \{(0, j) : j \in \mathbb{N}\} = \emptyset$ then also $S^{u_{[l_0,l_0+l]}} \cap \{(0, j) : j \in \mathbb{N}\} = \emptyset$ and

$$\mathrm{wt}(\hat{v}_{[l_0-1,l_0-1+l]}) = \mathrm{wt}(v_{[l_0,l_0+l]})$$

where $\hat{v}(z_1, z_2) = G(z_1, z_2)z_1^{-1}u(z_1, z_2)_{[l_0,l_0+l]}$.

These considerations imply the following proposition.

Proposition 3 *Let $d \in \mathbb{N}$. Then $d_j^s = d$, for $j \in \mathbb{N}$, if and only if the following conditions are satisfied:*

1. *all the $d-1$ columns of $H^s_{l_{0|_j}}$ in which*

   (a) *one column has index in*

   $$\begin{aligned} &\{1,\ldots,n\} \cup \{(l_0+1)n+1,\ldots,(l_0+2)n\} \cup \cdots \cup \\ &\cup \cdots \cup \{[jl_0+1+2+\cdots+j]n+1,\ldots \\ &\ldots,[jl_0+1+2+\cdots+(j+1)]n\} \end{aligned}$$

   (b) *one column has index in*

   $$\begin{aligned} &\{l_0n+1,\ldots,(l_0+1)n\} \cup \\ &\cup \{(2l_0+2)n+1,\ldots,(2l_0+3)n\} \cup \\ &\cup \{(3l_0+5)n+1,\ldots,(3l_0+6)n\} \cup \cdots \cup \\ &\cup \cdots \cup \{[(j+1)l_0+1+2+\cdots+j]n+1,\ldots \\ &\ldots[(j+1)l_0+1+2+\cdots+(j+1)]n\} \end{aligned}$$

   (c) *one column has index in $\{1,\ldots,(l_0+1)n\}$*

   are linearly independent, for all $l_0 \in \mathbb{N}$.

2. *there existe $l_0 \in \mathbb{N}$ such that there exist d columns of $H^s_{l_{0|_j}}$ that satisfy the conditions a), b) and c) of 2), that are linearly dependent.*

The following proposition gives an upper bound on the separation set distances of a 2D convolutional code, and shows that if an l-th separation set distance achieves the upper bound, then all the formers ones are also optimal.

Proposition 4 *Let $\mathcal{C}$ be a 2D convolutional code of rate k/n. Then*

1. $d_l^s \leq \dfrac{(n-k)(l+2)(l+1)}{2} + 1$*, for all $l \in \mathbb{N}$;*

2. *if* $d_l^s = \dfrac{(n-k)(l+2)(l+1)}{2} + 1$*, for some l, then*

   $$d_{l'}^s = \frac{(n-k)(l'+2)(l'+1)}{2} + 1$$

   for all $0 \leq l' < l$.

Proof 2 *We can give an upperbound by bounding the minimum weight of a subset of the codewords. In particular we consider those codewords $v(z_1, z_2) =$*

$\sum_{(i,j)\in\mathbb{N}^2} v_{i,j} z_1 z_2^j$. *Coefficientwise, we are computing the minimum weight of vectors of the form*

$$\left(\begin{array}{c|cc|ccc|c} G_{0,0} & & & & & & \\ \hline G_{1,0} & G_{0,0} & & & & & \\ G_{0,1} & & G_{0,0} & & & & \\ \hline G_{2,0} & G_{1,0} & & G_{0,0} & & & \\ G_{1,1} & G_{0,1} & G_{1,0} & & G_{0,0} & & \\ G_{0,2} & & G_{0,1} & & & G_{0,0} & \\ \hline \vdots & \vdots & & \ddots & & & \ddots \end{array}\right) \begin{pmatrix} u_{0,0} \\ u_{1,0} \\ u_{0,1} \\ u_{2,0} \\ u_{1,1} \\ u_{0,2} \\ \vdots \end{pmatrix} \tag{4}$$

where $u_{0,0} \neq 0$.

Clearly, there exists $u_{0,0}$ *such that* $\mathrm{wt}(\mathrm{G}_{0,0}\mathrm{u}_{0,0}) \leq \mathrm{n} - \mathrm{k} + 1$. *As* $G_{0,0}$ *is full column rank, there always exists a vector* $u_{1,0}$ *such that*

$$\mathrm{wt}(\mathrm{v}_{1,0}) = \mathrm{wt}(\mathrm{G}_{1,0}\mathrm{u}_{0,0} + \mathrm{G}_{0,0}\mathrm{u}_{1,0}) \leq \mathrm{n} - \mathrm{k},$$

i.e., we can always select $u_{1,0} \in \mathbb{F}^k$ *to "erasure" the rank of* $G_{0,0}$ *coordinates from* $G_{1,0}u_{0,0}$*, and in this case is the rank of* $G_{0,0}$ *is* $n-k$. *This together, with the weight of the previous block* $v_{0,0} = G_{0,0}u_{0,0}$ *yields* $\mathrm{wt}(\mathrm{v}_{0,0}) + \mathrm{wt}(\mathrm{v}_{1,0}) = \mathrm{n} - \mathrm{k} + 1 + \mathrm{n} - \mathrm{k} = 2(\mathrm{n} - \mathrm{k}) + 1$. *Exactly the same arguments can be used for the following blocks in (). Summing up these weights yields the desired result.*

Proof 3 *First observe that the separation set distance is equal to* d *if and only if none of the first* n *columns of* $H^s_{l_{o|_l}}$ *is contained in the span of any other* $d-2$ *columns and one of the first* n *columns of* $H^s_{l_{o|_l}}$ *is in the span of some other* $d-1$ *columns of that matrix. Clearly, to prove the result we just need to show that it holds for* $i = j-1$. *Note that it holds that*

$$H^s_j = \left(\begin{array}{c|ccc} H^s_{j-1} & & & \\ \hline & G_{0,0} & & \\ & & \ddots & \\ & & & G_{0,0} \end{array}\right). \tag{5}$$

Assume that one of the first columns of H^s_{j-1} *is in the span of some other* $(n-k)\frac{(j+1)(j)}{2} - 1$ *columns. If so and as* $\begin{pmatrix} G_{0,0} & & \\ & \ddots & \\ & & G_{0,0} \end{pmatrix}$ *has full row rank, then it follows that one of the first columns of* H^s_j *is in the span of some other* $(n-k)\frac{(j+1)(j)}{2} - 1 + (n-k)j = (n-k)\frac{(j+2)(j+1)}{2} - 1$ *columns of* H^s_j *which is a contradiction with optimality of* d^s_j. *This concludes the proof.*

3.2 Maximum Separation Set Distance Profile 2D convolutional code

We are now in position to introduce for the first time the novel definition of Maximum Separation Set Distance Profile in the context of 2D convolutional codes. This notion is the analog of Maximum Distance Profile of 1D convolutional codes and express the idea that these codes possess the maximum separation set distance that is possible to achieve at all instances.

Definition 7 *A* $2D$ *convolutional code with rate* k/n *and degree* δ *is said to have* Maximum Separation Set Distance Profile *if*

$$d_l^s = \frac{(n-k)(l+1)(l+2)}{2} + 1$$

for $l \leq L$ *where*

$$\begin{aligned} L &= \max\Big\{ l \in \mathbb{N} : \frac{(n-k)(l+1)(l+2)}{2} + 1 \leq s, \ \textit{where} \\ & s = \frac{n}{2}\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 2\right) - k\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + 1 \Big\} \end{aligned}$$

Note that from Proposition 4, a 2D convolutional code has Maximum Separation Set Distance Profile if and only if

$$d_L^s = \frac{(n-k)(L+1)(L+2)}{2} + 1.$$

The existence of Maximum Separation Set Distance Profile 2D convolutional codes of any rate and degree must be investigated. Next, we will consider the particular case of $k = 1$.

A 2D convolutional code of rate $1/n$ and degree δ has Maximum Separation Set Distance Profile if

$$d_l^s = \frac{(n-1)(l+1)(l+2)}{2} + 1$$

for $l \leq L$ where

$$L := \delta + \left\lfloor \frac{\sqrt{(2\delta+3)^2(n-1) + 4\delta(\delta+3)}}{2\sqrt{n-1}} - \frac{2\delta+3}{2} \right\rfloor.$$

Let us give a construction of a 2D convolutional code of rate $1/n$ and degree $\delta = 1$ with Maximum Separation Set Distance Profile. In this case $L = 1$. Let $X \in \mathbb{F}^{n \times 2}$ be a superregular matrix, i.e., a matrix with all minors of any order different from zero. Construction of superregular matrices can be founded in [1]. Write $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$, with $X_i \in \mathbb{F}^n$, $i = 1, 2$, let

$$G(z_1, z_2) = X_1 + X_2 z_1 + X_2 z_2$$

and consider $\mathcal{C}$ the 2D convolutional code with encoder $G(z_1, z_2)$. It is easy to see that $\mathcal{C}$ is a noncatastrophic delay-free 2D convolutional code with degree 1. Let us see that $\mathcal{C}$ is a Maximum Separation Set Distance Profile code. For that we need to show that for all nonzero $v(z_1, z_2) \in \mathcal{C}$, $\mathrm{wt}\left(v_{[l_0,l_0+1]}\right) \geq 3n-2$, where l_0 in the initial index of $v(z_1, z_2)$.

Let $v(z_1, z_2) = G(z_1, z_2)u(z_1, z_2)$ and write

$$v(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} v_{ij} z_1^i z_2^j, \quad v_{ij} \in \mathbb{F}^n,$$

and

$$u(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} u_{ij} z_1^i z_2^j, \quad u_{ij} \in \mathbb{F}.$$

If $l_0 = 0$, then $u_{00} \neq 0$ and therefore $v_{00} = X_1 u_{00}$ has weight n and $v_{10} = X \begin{bmatrix} u_{10} \\ u_{00} \end{bmatrix}$ and $v_{01} = X \begin{bmatrix} u_{01} \\ u_{00} \end{bmatrix}$ have weight greater than or equal to $n-1$ because X is superregular and therefore $\mathrm{wt}\left(v_{[l_0,l_0+1]}\right) \geq 3n-2$.

In case $l_0 > 0$, let us consider two cases: $u_{l_0,0} \neq 0$ or $u_{0,l_0} \neq 0$ and $u_{l_0,0} = u_{0,l_0} = 0$.

1. Let us assume that that $u_{l_0,0} \neq 0$. Then $v_{l_0,0} = X_1 u_{l_0,0}$ has weight n and $v_{l_0+1,0} = X \begin{bmatrix} u_{l_0+1,0} \\ u_{l_0,0} \end{bmatrix}$ has weight greater or equal than $n-1$. Since

$$\mathrm{wt}\left(v_{0,l_0} z_2^{l_0} + v_{0,l_0+1} z_2^{l_0+1}\right) \geq n$$

   it follows that $\mathrm{wt}(v) \geq 3n-2$. The same reasoning can be done if $u_{0,l_0} \neq 0$.

2. If $u_{l_0,0} = u_{0,l_0} = 0$, then

$$u_{l_0+1,0} \neq 0 \quad \text{and} \quad u_{0,l_0+1} \neq 0$$

   which implies that

$$v_{l_0+1,0} = X_1 u_{l_0+1,0} \quad \text{and} \quad v_{0,l_0+1} = X_1 u_{0,l_0+1}$$

   have weight n. Moreover, there exists t such that $u_{l_0-t,t} \neq 0$ and therefore $v_{l_0-t,t} = X_1 u_{l_0-t,t}$ has weight n, which implies that $\mathrm{wt}\left(v_{[l_0,l_0+1]}\right) \geq 3n-2$.

Further research must be done to obtain constructions of Maximum Separation Set Distance Profile 2D convolutional code of degree $\delta \geq 2$. We believe that other type of constructions must be considered.

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