

Plane sets invisible in finitely many directions

Alexander Plakhov*

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Abstract

We consider the problem of mirror invisibility for plane sets. Given a circle and a finite number of unit vectors (defining the directions of invisibility) such that the angles between them are commensurable with π , for any $\varepsilon > 0$ there exists a set invisible in the chosen directions that contains the circle and is contained in its ε -neighborhood. This set is the disjoint union of infinitely many domains with piecewise smooth boundary.

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1 Introduction

Nowadays, the problem of invisibility and its practical realization attracts a lot of attention of mathematical, physical, and technological communities. Some approaches to constructing invisibility devices are based on rapidly growing metamaterial technologies [2, 3, 12, 13, 14]. A very interesting approach, in the framework of geometric optics, is related to creating a transparent refracting (with variable refraction index) coating for objects to be hidden [5, 6].

Here we deal with billiard invisibility, when a mirrored object (a body) is represented by a bounded domain or by a disjoint union of (finitely or countably many) domains in Euclidean space, and propagation of light rays is represented by motion of billiard particles outside the body.

*Center for R&D in Mathematics and Applications, Department of Mathematics, University of Aveiro, Portugal and Institute for Information Transmission Problems, Moscow, Russia.

In what follows we consider billiard trajectories $x(t)$ parameterized by time t and representing the motion, uniform between consecutive reflections, of a billiard particle. We concentrate on trajectories that are defined for all $t \in \mathbb{R}$ and have a finite number of reflections.

Take a bounded plane set $B \subset \mathbb{R}^2$ and consider the billiard in its complement $B^c = \mathbb{R}^2 \setminus B$. Slightly abusing the language, we say that a parameterized billiard trajectory $x(t)$, $t \in \mathbb{R}$, is *invisible*, if the initial and final (semi-infinite) segments of the trajectory lie on the same straight line and are passed in the same direction. In other words, let $x(t) = b + vt$ for t sufficiently small; the trajectory is invisible if for t sufficiently large and for a certain constant $\tau \in \mathbb{R}$ we have $x(t) = b + v(t - \tau)$. The corresponding straight line l is called to be *associated* with the invisible trajectory (see Fig. 1).

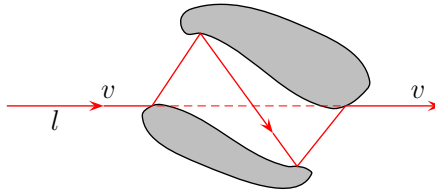


Figure 1: An invisible trajectory and the associated straight line l .

Let $v \in S^1$. We say that B is *invisible in the direction of v* (or *along the vector v*), if almost all¹ straight lines with the director vector v are associated with invisible trajectories. Obviously, B is invisible along v if and only if it is invisible along $-v$. An example of a set invisible in a direction is provided in Fig. 2; for more details see [1].

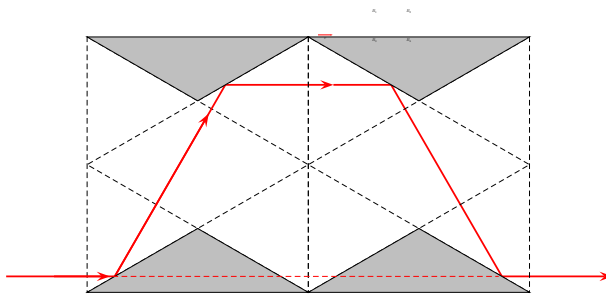


Figure 2: The union of shaded triangles $B = B_1 \cup B_2 \cup B_3 \cup B_4$ is invisible in the direction of v .

Likewise, we say that B is *invisible from a point $O \in \mathbb{R}^2$* , if almost all straight lines through O are associated with invisible trajectories and, additionally, O lies outside the

¹with respect to the natural one-dimensional Lebesgue measure

convex hull of B . An example of a set invisible in a direction is given in Fig. 3; for more details of this construction see [11] or [7].

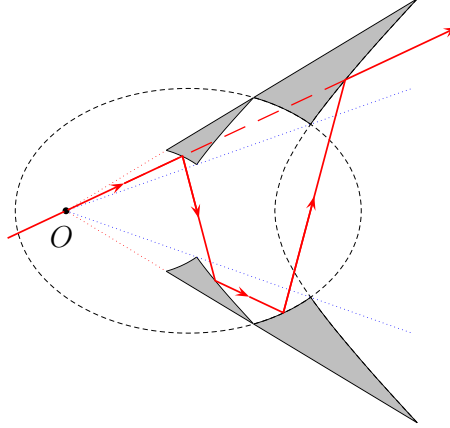


Figure 3: The union of four shaded curvilinear triangles is invisible from the point O .

All the definitions given can be generalized to higher dimensions.

Invisibility in a direction can be treated as invisibility from an infinitely distant point.

In general, the problem of billiard invisibility remains open. We conjecture that

- (a) given a finite set of directions and a finite set of points, there is a body (a finite or countable union of bounded domains) invisible in these directions and from these points;
- (b) the set of invisible trajectories for each body has measure zero.

Conjecture (b) is closely connected with the famous Ivrii conjecture stating that the set of periodic billiard trajectories in a bounded domain has measure zero [4].

An interesting (and probably even more difficult than conjectures (a) and (b)) problem concerns the maximal dimension of the set of invisible trajectories.

Up to the moment the following is known.

- For any two unit vectors in \mathbb{R}^2 there exists a plane body invisible along these vectors [9].
- For any three mutually orthogonal unit vectors in \mathbb{R}^3 there exists a 3D body invisible along these vectors [9].
- For any two points in \mathbb{R}^n ($n \geq 2$) there exists an n -dimensional body invisible from these points [10].
- There does not exist a body in \mathbb{R}^n ($n \geq 2$) invisible in all directions or, equivalently, from all points [8].

The aim of this paper is to prove the following theorem.

Theorem 1. *Consider a finite set of unit vectors in \mathbb{R}^2 such that the angle between any two vectors is commensurable with π , and let $D_1 \subset D_2 \in \mathbb{R}^2$ be two different concentric circles. Then there is a set B invisible along these vectors such that $D_1 \subset B \subset D_2$. (Here B is a countably connected set, with each connected component being a domain with piecewise smooth boundary.)*

We believe that the technical condition on the angles could be eventually omitted and that D_1 and D_2 could be taken to be arbitrary bounded convex bodies (with $D_1 \subset D_2$ and $\partial D_1 \cap \partial D_2 = \emptyset$). We also believe that the statement of the theorem generalizes to higher dimensions.

The rest of the paper is devoted to the proof of this theorem.

2 Reformulation of Theorem 1 and the main idea of the proof

Take a natural n so that π/n is a common multiplier of all angles between the directions of invisibility. We can take n so large that a regular $2n$ -gon P_1 circumscribed about D_1 is contained in D_2 . Let us choose P_1 in such a way that the direction of invisibility is the external bisector of an angle of P_1 (and therefore all directions of invisibility are external bisectors of angles of P_1).

Extending if necessary the set of directions of invisibility, one can assume that it is composed of the external bisectors of *all* vertices of P_1 . That is, we have the set of $2n$ unit vectors with the angle π/n between nearest vectors (see Fig. 4).

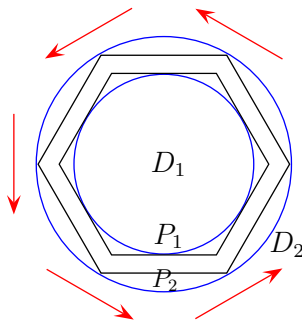


Figure 4: The regular hexagons P_1 and P_2 circumscribed about and inscribed in the concentric circles, D_1 and D_2 , and the 6 directions of invisibility.

The angles between a side of P_1 and the $2n$ vectors take the values $\pi/(2n)$, $3\pi/(2n)$, \dots , $(4n-1)\pi/(2n)$.

Consider the $2n$ -gon P_2 similar to P_1 inscribed in D_2 . Obviously, P_2 contains P_1 . Now Theorem 1 can be stated in the following equivalent form.

Theorem 2. *Consider two similar regular $2n$ -gons $P_1 \subset P_2$ and the $2n$ unit vectors parallel to the external bisectors of the angles of P_1 . Then there exists a set B invisible along these vectors such that $P_1 \subset B \subset P_2$.*

Let us first explain the main idea of the proof. The key element of the underlying construction involves a collection of $n + 1$ curves, a big one and n small ones. We have n parallel flows of particles incident on the big curve. Each incident flow (say, the i th one) is reflected first from the big curve and then from the corresponding (the i th) small one, and turns into another parallel flow, with the width much smaller than that of the original one. That is, the n incident parallel flows after two reflections are transformed into n compressed parallel flows, and the compression ratio can be chosen to be arbitrarily small (see Fig. 5).

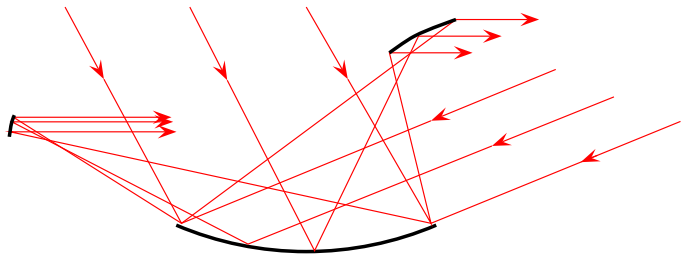


Figure 5: The system of a big curve and two small ones transforms two incident parallel flows into two horizontal parallel flows of smaller width.

Note in passing that a single incident parallel flow ($n = 1$) can easily be compressed using two arcs of confocal parabolas, but in the general case of $n \geq 2$ incident flows, parabolas are useless. Instead, we take the big curve to be a circular arc, and carefully design the n small curves.

To make an object invisible, we shield it from the incident flows by adding several collections of curves described above. Each incident flow is reflected by several big curves (circular arcs). The portion of the flow reflected by each arc is then transformed into a parallel flow of smaller width, which goes around the body and is then transformed back into the same portion of the original flow (see Fig. 6).

Unfortunately, the added objects (collections of curves) are themselves not invisible. In other words, the resulting body remains visible, but its "visibility" is significantly reduced. This observation suggests an idea to determine an infinite procedure. The first step of the procedure has just been done. At the second step, we make the added objects

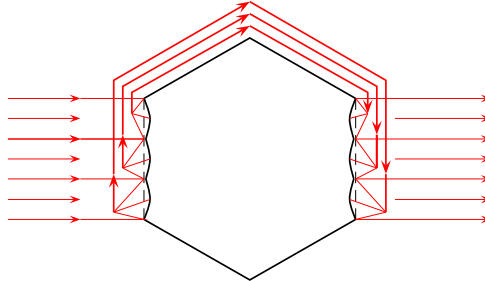


Figure 6: An attempt to create an invisible body. Only a part of an incident flow is shown, and only two sides of the polygon (the original body) are substituted by hollows in the picture.

invisible by adding some more collections of curves (of the second order), and so on. As a result of infinitely many steps we obtain an (infinitely connected) body which is completely invisible.

Of course the devil is in the details, and the rest of the paper is devoted to a rigorous implementation of the above idea. Some details of this simple scheme will be further modified; for instance, we shall actually perform *double compression* of incident flows. (For this purpose, the number of small curves will be triplicated.)

First we prove two lemmas.

3 Two lemmas on flow compression

In what follows, by a *light ray* we mean a directed segment of a billiard trajectory between two consecutive reflections. In particular, an *incident light ray* is a directed half-line representing the part of a billiard trajectory before the first reflection.

Consider a vertical flow of light rays incident on the graph of a C^3 function $y = \phi(x)$ defined on a segment $[-x_0, x_0]$ such that $\phi''(x) > 0$. (Thus, ϕ is strictly convex.) Note that the reflected flow forms a caustic, and the light ray reflected by the graph at $(x, \phi(x))$ will then touch the caustic at the point $L_x(1)$, where

$$L_x(\tau) = (x, \phi(x)) + \frac{\tau}{2\phi''(x)}(-2\phi'(x), 1 - \phi'^2(x)),$$

and in particular (using the shorthand notation $\phi(0) = \phi_0$, $\phi'(0) = \phi'_0$, $\phi''(0) = \phi''_0$), the light ray reflected at the point $(0, \phi_0)$ will touch the caustic at the point $L_0(1) =$

$$(0, \phi_0) + \frac{1}{2\phi_0'}(-2\phi_0', 1 - \phi_0'^2).$$

Indeed, the tangency point is also the point of intersection of two infinitesimally close reflected rays

$$(x, \phi(x)) + t(-2\phi'(x), 1 - \phi'^2(x)) \text{ and } (x+dx, \phi(x+dx)) + (t+dt)(-2\phi'(x+dx), 1 - \phi'^2(x+dx)).$$

Equating these two expressions and neglecting terms of higher orders, one obtains

$$0 = (dx, \phi'(x)dx) + t(-2\phi''(x)dx, -2\phi'(x)\phi''(x)dx) + dt(-2\phi'(x), 1 - \phi'^2(x)),$$

and hence,

$$(1 - 2t\phi''(x))dx V_1 + dt V_2 = 0,$$

where $V_1 = (1, \phi'(x))$ and $V_2 = (-2\phi'(x), 1 - \phi'^2(x))$. Since V_1 and V_2 are not collinear, we conclude that $dt = 0$ and $t = 1/(2\phi''(x))$, and therefore the tangency point is $L_x(1)$.

Consider the part of the graph corresponding to $-\varepsilon \leq x \leq \varepsilon$, where $\varepsilon > 0$ is a small parameter. The length of the corresponding caustic is $O(\varepsilon)$, and the maximum angle between tangent lines at different points of the caustic is also $O(\varepsilon)$. Therefore the width of the reflected flow behind the caustic at a distance $O(\varepsilon)$ from the caustic is $O(\varepsilon^2)$. Thus, one can expect that a curve crossing this flow (behind the caustic at a distance $O(\varepsilon)$ from it) in a required direction has the length $O(\varepsilon^2)$.

Formalizing these ideas, we get the following lemma.

Lemma 1. *There is a real value $c_0 > 0$ such that for all $l > c_0$, in an $O(\varepsilon^2)$ -neighborhood of the point $L_0(1+l\varepsilon)$ there is a smooth curve S of length $O(\varepsilon^2)$ such that a vertical flow of particles with the x -coordinate $-\varepsilon \leq x \leq \varepsilon$, after two reflections from the graph of ϕ and from this curve is transformed into a parallel flow of width $O(\varepsilon^2)$ in the direction $(1, \phi_0')$ (see Fig. 7). Given a constant $c > c_0$, the above estimates $O(\varepsilon^2)$, $\varepsilon \rightarrow 0$ are uniform in the interval $l \in [c_0, c]$. The second reflection induces a smooth one-to-one correspondence between incident light rays and points of the curve.*

Let us comment this lemma. The crossing curve should not intersect the caustic; otherwise it forms an undesirable cusp at the point of intersection. By choosing an appropriate $c_0 > 0$, one ensures that the curve is situated behind the caustic at a safe distance, so that non-intersection is guaranteed. By fixing $c > c_0$, one ensures that this distance is at most $O(\varepsilon)$.

In the particular case when ϕ is a polynomial of the second order (and therefore its graph is an arc of a parabola with vertical axis), the caustic degenerates to a point — the focus of the parabola. In this case the width of the reflected flow is proportional to the distance from the focus to the second reflecting curve (which is also an arc of a parabola) and can be made arbitrarily small.

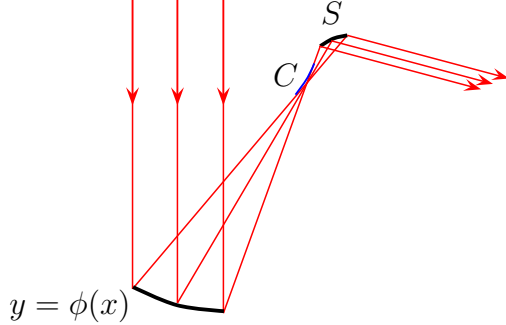


Figure 7: The vertical flow is reflected by the graph $y = \phi(x)$. The reflected flow touches the caustic C ; then it is reflected once again by a smaller curve S and is transformed into another (thinner) parallel flow.

Proof. The particle reflected at $(x, \phi(x))$ further moves with the velocity $(1 + \phi'^2(x))^{-1}(-2\phi'(x), 1 - \phi'^2(x))$ and spends the time $t(1 + \phi'^2(x))$ to reach the point

$$W(x, t) = (x, \phi(x)) + t(-2\phi'(x), 1 - \phi'^2(x)).$$

The partial derivatives of W are $W'_x = (1 - 2t\phi''(x))(1, \phi'(x))$ and $W'_t = (-2\phi'(x), 1 - \phi'^2(x))$, and their vector product is

$$W'_x \times W'_t = (1 - 2t\phi''(x))(1 + \phi'^2(x)).$$

The condition $W'_x \times W'_t = 0$ means that the vectors W'_x and W'_t are linearly dependent and is equivalent to the equation $t = 1/(2\phi''(x))$ determining the caustic. Outside the caustic the map $(x, t) \mapsto W(x, t)$ is a local diffeomorphism.

For the sake of simplicity of notation, define

$$\tau_1(x) = \frac{(-2\phi'(x), 1 - \phi'^2(x))}{1 + \phi'^2(x)}, \quad \tau_2(x) = \frac{(1, \phi'(x))}{\sqrt{1 + \phi'^2(x)}}, \quad \tau_2^0 = \tau_2(0) = \frac{(1, \phi'_0)}{\sqrt{1 + \phi_0'^2}}.$$

We are looking for a reflecting curve in the form $W(x, t(x))$, $x \in [-\varepsilon, \varepsilon]$. The condition stating that the vector τ_1 , after a reflection from the curve at the point $W(x, t(x))$ becomes the vector τ_2^0 , means that the tangent vector to the curve

$$\frac{d}{dx}W(x, t(x)) = W'_x + W'_t t'(x) = (1 - 2t\phi''(x))\sqrt{1 + \phi'^2(x)}\tau_2(x) + t'(x)(1 + \phi'^2(x))\tau_1(x) \quad (1)$$

(a) is parallel to the sum $\tau_1(x) + \tau_2^0$ and (b) does not vanish.

The vector τ_2^0 can be decomposed in the basis $\tau_1(x), \tau_2(x)$ as follows:

$$\tau_2^0 = \alpha(x)\tau_1(x) + (1 + \beta(x))\tau_2(x),$$

where $\alpha(x)$ and $\beta(x)$ are smooth function satisfying $\alpha(0) = 0 = \beta(0)$. Then the above conditions (a) and (b) take, respectively, the form of the ODE for the unknown function $t = t(x)$

$$t' = \frac{1 + \alpha(x)}{1 + \beta(x)} \frac{1 - 2\phi''(x)t}{\sqrt{1 + \phi'^2(x)}}, \quad x \in [-\varepsilon, \varepsilon] \quad (2)$$

and of the inequality

$$t \neq \frac{1}{2\phi''(x)}. \quad (3)$$

The solution of the (linear inhomogeneous of the first order) differential equation (2) exists for all initial conditions $t(-\varepsilon)$, provided that $1 + \beta(x) \neq 0$ in $[-\varepsilon, \varepsilon]$. It remains to check inequality (3).

Take a value $c_1 > 0$ such that for $\varepsilon > 0$ sufficiently small and for $x \in [-\varepsilon, \varepsilon]$ holds

$$\frac{\phi_0''}{1 + c_1\varepsilon} < \phi''(x) < \phi_0''(1 + c_1\varepsilon)$$

(here we use that ϕ is in the class C^3), and take arbitrary c and c_0 such that $c_1 < c_0 < c$. Additionally, for ε sufficiently small and for $x \in [-\varepsilon, \varepsilon]$ we have

$$0 < \frac{1 + \alpha(x)}{1 + \beta(x)} < 2 \quad \text{and} \quad \sqrt{1 + \phi'^2(x)} < 2\sqrt{1 + \phi_0'^2}.$$

Choose an arbitrary l satisfying $c_0 \leq l \leq c$ and take the initial value of our ODE (2)

$$t(-\varepsilon) = \frac{1 + l\varepsilon}{2\phi_0''}.$$

Then for ε sufficiently small we have

$$1 - 2\phi''(-\varepsilon)t(-\varepsilon) < 1 - \phi''(-\varepsilon) \frac{1 + c_1\varepsilon}{\phi_0''} < 0.$$

Let us prove by *reductio ad absurdum* that

$$1 - 2\phi''(x)t(x) < 0 \quad \text{for all } x \in [-\varepsilon, \varepsilon], \quad (4)$$

and therefore, inequality (3) is always satisfied. Assume that (4) is violated for some x and take $x^* = \inf\{x \in [-\varepsilon, \varepsilon] : 1 - 2\phi''(x)t(x) \geq 0\}$. We have $-\varepsilon < x^* \leq \varepsilon$ and

$$1 - 2\phi''(x^*)t(x^*) = 0. \quad (5)$$

For $x < x^*$ one has $t'(x) < 0$, hence $t(x) \leq t(-\varepsilon) < (1 + c\varepsilon)/(2\phi_0'')$. Therefore

$$t'(x) > 2[1 - 2\phi''(x)t] > 2 \left[1 - 2\phi_0''(1 + c_1\varepsilon) \frac{1 + c\varepsilon}{2\phi_0''} \right] > -2\varepsilon(2c + c^2\varepsilon),$$

and so, for ε sufficiently small

$$t(x^*) \geq t(-\varepsilon) + 2\varepsilon \inf_{x \in [-\varepsilon, x^*]} t'(x) \geq \frac{1 + c_0\varepsilon}{2\phi_0''} - 4\varepsilon^2(2c + c^2\varepsilon) > \frac{1 + c_1\varepsilon}{2\phi_0''} > \frac{1}{2\phi''(x^*)}.$$

These inequalities contradict equation (5); thus, (4) is proved and condition (b) is verified.

Now, using the inequalities $-\varepsilon(2c + c^2\varepsilon) < 1 - 2\phi''(x)t(x) < 0$ and $-2\varepsilon(2c + c^2\varepsilon) < t'(x) < 0$ and equation (1), one obtains

$$\left| \frac{d}{dx} W(x, t(x)) \right| \leq |1 - 2\phi''(x)t(x)| \sqrt{1 + \phi'^2(x)} + |t'(x)|(1 + \phi'^2(x)) \leq 10\varepsilon(2c + c^2\varepsilon)(1 + \phi_0'^2),$$

and therefore,

$$\frac{d}{dx} W(x, t(x)) = O(\varepsilon).$$

Thus, the length of the curve $W(x, t(x))$, $x \in [-\varepsilon, \varepsilon]$ is $O(\varepsilon^2)$ (more precisely, it does not exceed $20\varepsilon^2(2c + c^2\varepsilon)(1 + \phi_0'^2)$).

We have $t(0) = t(-\varepsilon) + O(\varepsilon^2) = (1 + l\varepsilon)/(2\phi_0'') + O(\varepsilon^2)$, and so, the middle point $W(0, t(0))$ of the curve is

$$W(0, t(0)) = (0, \phi_0) + \left(\frac{1 + l\varepsilon}{2\phi_0''} + O(\varepsilon^2) \right) (-2\phi_0', 1 - \phi_0'^2) = L_0(1 + l\varepsilon) + O(\varepsilon^2);$$

thus, the entire curve is contained in an $O(\varepsilon^2)$ -neighborhood of $L_0(1 + l\varepsilon)$.

For ε sufficiently small the light rays of the flow, after the reflection from $\text{graph}(\phi)$, are not parallel to any of the tangent lines of the curve, and therefore intersect the curve only once. The light ray corresponding to a value x , $x \in [-\varepsilon, \varepsilon]$ is reflected by the point $W(x, t(x))$ of the curve. Thus, the mapping $x \mapsto W(x, t(x))$ establishes a smooth one-to-one mapping between the incident rays and points of the curve. \square

Now apply Lemma 1 to the case where an inclined parallel flow is reflected by a circular hollow. Let α ($0 < \alpha < \pi$) be the angle of inclination (that is, the velocity of the incident flow be $(\cos \alpha, -\sin \alpha)$) and the hollow be of the form

$$y = r - \sqrt{r^2 - x^2}, \quad -\varepsilon \leq x \leq \varepsilon. \quad (6)$$

Both incident and reflected light rays can be viewed as chords of a circle of radius r . The reflected light ray touches the caustic, and the tangency point divides the corresponding chord in the ratio 1 : 3 (from the point of reflection). This can be easily seen from the following geometric consideration (see Fig. 8).

Two chords corresponding to reflected light rays are resting on two circular arcs with the angular measures related as 1 : 3. Two triangles formed by these chords are similar, with the ratio of their opposite (resting on the corresponding arcs) sides going to 1 : 3 when one of the chords approaches the other one. Hence, the ratio of lengths of the parts of each chord divided by the intersection point has the same limit.

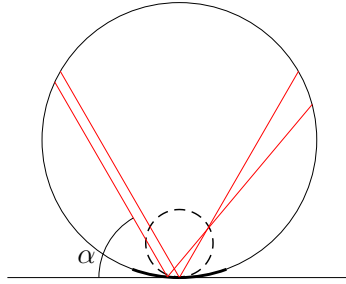


Figure 8: Two nearby parallel light rays reflected by a circular arc.

Consider the incident light ray hitting the hollow at the lower point (center of the hollow). The length of the corresponding chord is $2r \sin \alpha$. Therefore the distance from the point of reflection to the point where the reflected ray touches the caustic is $\frac{1}{2}r \sin \alpha$. This point lies on the circumference of radius $r/4$ (dashed in Fig. 8) touching the original circumference at the lower point. Parameterize the reflected ray by

$$R(\tau) = R_\alpha(\tau) = \frac{\tau}{2} r \sin \alpha (\cos \alpha, \sin \alpha); \quad (7)$$

the point of tangency $R(1)$ of this ray to the caustic lies on the aforementioned circumference of radius $r/4$.

The following lemma is a direct consequence of Lemma 1.

Lemma 2. Fix r and α , and fix the horizontal vector $v_0 = (1, 0)$ or $(-1, 0)$. There is a real value $c_0 = c_0(r, \alpha) > 0$ such that for all $l > c_0$, in an $O(\varepsilon^2)$ -neighborhood of the point $R(1 + l\varepsilon)$ there exists a smooth curve of length $O(\varepsilon^2)$ such that the flow falling at the angle α on the hollow (6), after two reflections, first from the hollow and then from the curve, turns into a horizontal flow of width $O(\varepsilon^2)$ in the direction v_0 . The tangent to the curve at each point has an inclination angle $\alpha/2 + O(\varepsilon)$ with respect to the horizontal line, if $v_0 = (1, 0)$, and $\alpha/2 - \pi/2 + O(\varepsilon)$, if $v_0 = (-1, 0)$. For any constant $c > c_0$, the above estimates $O(\varepsilon^2)$ are uniform in $l \in [c_0, c]$.

Notice that this curve (let it be called the *second reflecting curve*) is situated at the height

$$\frac{r}{2} \sin^2 \alpha + O(\varepsilon) \quad (8)$$

above the horizontal line $y = -r$ (tangent to the hollow at its lower point). We will use this formula later in Section 5 when we construct a multilevel system of reflecting sets.

Proof. Suppose that $v_0 = (1, 0)$. Take the orthogonal coordinate system $O\xi\eta$ so that O coincides with the midpoint of the arc (6) and the axis $O\eta$ is parallel and counter-directional to the incident flow. We have $x = \sin \alpha \xi - \cos \alpha \eta$, $y = \cos \alpha \xi + \sin \alpha \eta$. In this system the function ϕ takes the form

$$\eta = \phi(\xi) = r \sin \alpha - \sqrt{r^2 - (\xi - r \cos \alpha)^2}.$$

Direct calculations show that $\phi(0) = 0$, $\phi'(0) = -\cot \alpha$, $\phi''(0) = 1/(r \sin^3 \alpha)$, and therefore, the half-line $L_0(\tau)$, $\tau \geq 0$, in the $\xi\eta$ -coordinate system has the form

$$L_0(\tau) = \frac{\tau}{2\phi''(0)} (-2\phi'(0), 1 - \phi'^2(0)) = \frac{\tau}{2} r \sin \alpha (\sin 2\alpha, -\cos 2\alpha).$$

In the original xy -coordinate system this half-line (with the same parametrization) takes the form $R(\tau)$ and the vector $(1, \phi'(0))$ takes the form $\frac{1}{\sin \alpha}(1, 0)$.

Applying Lemma 1 in the $\xi\eta$ -coordinate system and then passing to the original xy -coordinate system, one comes to the claim of Lemma 2. In particular, since at each point of the curve there is a reflection, and the light ray has the slope $\alpha + O(\varepsilon)$ before and is horizontal after the reflection, we conclude that the tangent to the curve at this point has the slope $\alpha/2 + O(\varepsilon)$.

Note that the second reflecting curve corresponding to the angle of incidence α and $v_0 = (-1, 0)$ is symmetric to the curve corresponding to $\pi - \alpha$ and $v_0 = (1, 0)$, and $R_\alpha(1 + l\varepsilon)$ is symmetric to $R_{\pi-\alpha}(1 + l\varepsilon)$, with respect to the vertical line $x = 0$. This implies the claim of the lemma for $v_0 = (-1, 0)$. \square

4 Constructing a system of reflecting sets on a fixed level

Now we are going to incorporate the second reflecting curve into a set of a very special form called *s-set* ("s" from side). Take again a circular hollow and a parallel flow incident on it at an angle α ($0 < \alpha < \pi$). Consider the coordinate system such that the hollow has the form (6), and correspondingly the flow velocity is $(\cos \alpha, -\sin \alpha)$.

We first define an auxiliary object called *s-triangle*; it is an isosceles triangle with the horizontal base and the angle $\pi/(2n)$ at the base, and with the apex turned downward. The length of the base is $\varepsilon^{3/2}$ (therefore it is much greater than ε^2 but much smaller than ε). In Figs. 9 and 10, ABC is an s-triangle. Let MN be its midline with the midpoint P , and let Q be the midpoint of the base AC . We additionally require that the midpoint of PQ coincides with the point $R_\alpha(1 + l\varepsilon)$. Thus, the s-triangle corresponding to a given incident flow and to a given hollow is uniquely defined by the parameters ε , r , α , and l , and will further be denoted as $\Delta_{\varepsilon,r,\alpha,l}$.

By Lemma 2, the second reflecting curve corresponding to the parameter l lies in an $O(\varepsilon^2)$ -neighborhood of $R_\alpha(1 + l\varepsilon)$, and therefore belongs to the trapezoid $ACNM$. Its slope is positive, if the final direction of the flow is $(1, 0)$, and negative otherwise. Consider the figure bounded below by the curve and two horizontal segments and above by a part of the broken line $MACN$. The horizontal segments join the endpoints of the curve with the segments AM and CN . In Fig. 9 two examples of such a figure, with small and large α , and with the final direction $v_0 = (1, 0)$ "to the right" are displayed.

Let us now define the third and fourth reflecting curves. They are arcs of confocal parabolas with the horizontal axis of symmetry. The lower endpoint of the third curve is N , and the upper endpoint lies on the horizontal segment to the right of the second curve. The fourth curve is contained in the triangle MNB and is homothetic to the third curve with the homothety center at the common focus and with a small homothety ratio (see Fig. 10). This pair of curves serves for further compression of the flow; the horizontal flow of width $O(\varepsilon^2)$, after two reflections from these curves, turns into another flow in the same direction (hereafter referred to as the *thin flow*) of even smaller width, and the compression ratio coincides with the homothety ratio and can be made arbitrarily small (see Fig. 10).

Consider two domains: the former (larger) one is the part of the s-triangle ABC bounded by the base AC ; a part of the segment AM ; the segment CN ; the horizontal segment joining the side AB with the lower endpoint of the second reflecting curve; the second and third reflecting curves; and the horizontal segment joining their upper endpoints. The latter (smaller) domain is the curvilinear triangle bounded by the fourth

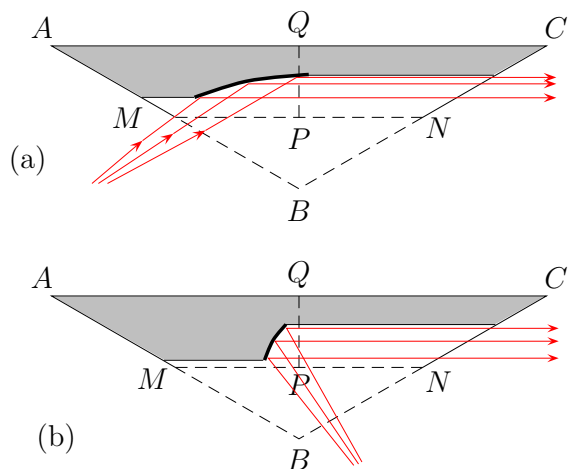


Figure 9: Constructing an s -set: the initial stage. The second reflecting curves (shown in bold) correspond to the cases where (a) α is small and (b) α is close to π . In both cases, the flow behind the caustic is reflected by the curve and turns into a horizontal flow of width $O(\varepsilon^2)$ directed to the right.

reflecting curve and vertical and horizontal line segments. The union of these domains is called to be an s -set related to the incident flow and to the hollow under consideration, and ABC is called to be the s -triangle associated with this s -set.

Parameterize the segment BP by B_τ , $\tau \in [0, 1]$, so as $B_0 = B$ and $B_1 = P$. From the above description we deduce that for any interval $(\tau_1, \tau_2) \subset [0, 1]$, the resulting thin horizontal flow can be made to pass between the points B_{τ_1} and B_{τ_2} (that is, at any height and with arbitrarily small width). Let the common focus of the parabolas be the midpoint of the interval $B_{\tau_1}B_{\tau_2}$, and let the second arc of parabola (the fourth reflecting curve) be chosen to be the maximal one, provided that the resulting thin flow goes between B_{τ_1} and B_{τ_2} and the smaller domain is contained in the s -triangle. Thus, the resulting s -set is uniquely determined by the parameters ε , r , α , l , $v_0 = (\pm 1, 0)$, and (τ_1, τ_2) .

The above conclusions can be summarized in the following lemma.

Lemma 3. *Take an interval $(\tau_1, \tau_2) \subset [0, 1]$ and fix $v_0 \in \{(1, 0), (-1, 0)\}$. There is a real value $c_0 = c_0(r, \alpha) > 0$ such that for all $l > c_0$, in an $O(\varepsilon^2)$ -neighborhood of the point $R_\alpha(1 + l\varepsilon)$ there is an s -set such that the flow falling at the angle α on the hollow (6), after a reflection from the hollow and three reflections from the s -set turns into a thin horizontal flow in the direction v_0 contained between the points B_{τ_1} and B_{τ_2} (see Fig. 11). For any fixed $c > c_0$ the above estimate $O(\varepsilon^2)$ is uniform in $l \in [c_0, c]$. The s -triangle $\Delta_{\varepsilon, r, \alpha, l}$ associated with the s -set does not depend on the choice of τ_1 , τ_2 , and v_0 .*

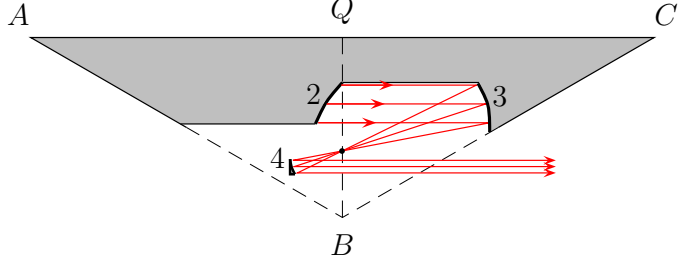


Figure 10: Constructing an s-set: the final stage. Reflections of the flow from the third and fourth curves (arcs of parabolas) are shown. The second, third, and fourth reflecting curves are labeled by "2", "3", and "4". The s-set is the union of two domains; the former one is colored light gray and the latter one is the tiny spot near the mark "4".

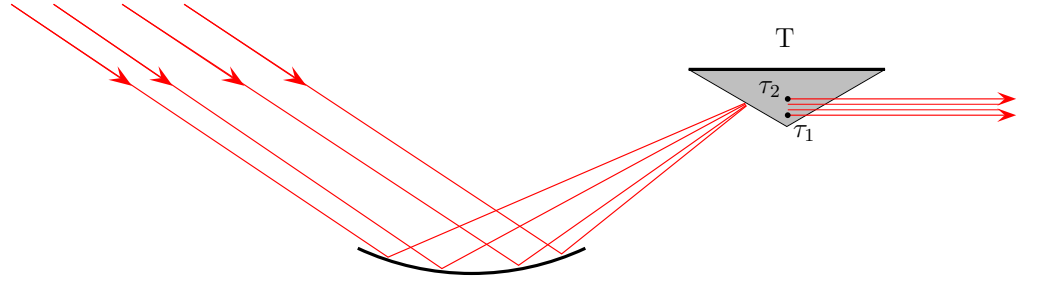


Figure 11: A general scheme of reflection from an s-set. Only the s-triangle T containing this set is shown, and three reflections from the s-set are not seen here. The points B_{τ_1} and B_{τ_2} are substituted with τ_1 and τ_2 .

Now fix r and take two complementary angles of incidence α and $\pi - \alpha$ ($0 < \alpha < \pi/2$). Consider two parallel flows incident on the infinite periodic sequence of hollows

$$y = -\sqrt{r^2 - (x - 2m\varepsilon)^2}, \quad 2m\varepsilon - \varepsilon \leq x \leq 2m\varepsilon + \varepsilon, \quad m \in \mathbb{Z} \quad (9)$$

at the angles α and $\pi - \alpha$. Fix the parameter l ; then we have two periodic sequences of s-triangles at the same height $R_\alpha(1 + l\varepsilon) = R_\alpha(1) + O(\varepsilon)$,

$$m_1(2\varepsilon, 0) + \Delta_{\varepsilon, r, \alpha, l}, \quad m_1 \in \mathbb{Z} \quad \text{and} \quad m_2(2\varepsilon, 0) + \Delta_{\varepsilon, r, \pi - \alpha, l}, \quad m_2 \in \mathbb{Z}.$$

Let us show that for a certain $l = l(\varepsilon)$ these triangles do not overlap.

Introduce the *axes* of the portions of the two flows reflected by the m th hollow. These axes are, by definition, the half-lines

$$m(2\varepsilon, 0) + R_\alpha(\tau), \tau > 0 \quad \text{and} \quad m(2\varepsilon, 0) + R_{\pi-\alpha}(\tau), \tau > 0.$$

The point of intersection, $m_1(2\varepsilon, 0) + R_\alpha(1 + l\varepsilon) = m_2(2\varepsilon, 0) + R_{\pi-\alpha}(1 + l\varepsilon)$, of two axes related to different flows and different hollows corresponds to the value l (identical for the two axes)

$$l = l_m(\varepsilon) = \frac{2m}{r \cos \alpha \sin \alpha} - \frac{1}{\varepsilon}, \quad \text{with } m = m_2 - m_1 \in \mathbb{Z}$$

(see Fig. 12).

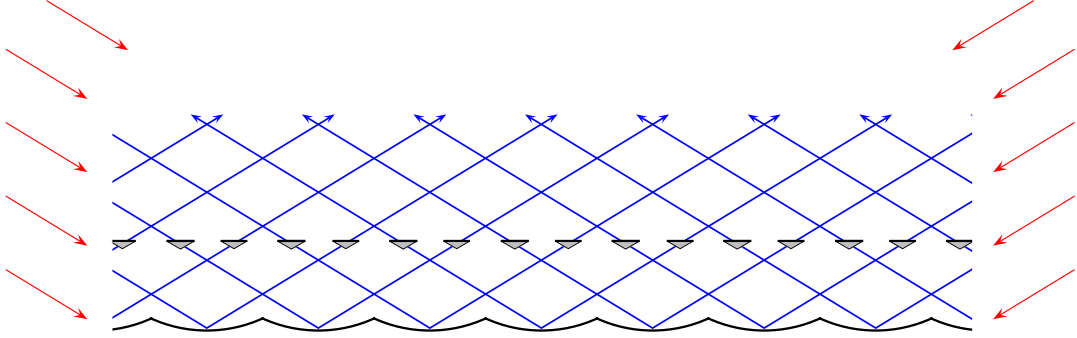


Figure 12: A periodic sequence of hollows; two incident flows; the axes of the reflected flows; and two disjoint periodic sequences of s-triangles. The points of intersection of the axes correspond to the values $l_m(\varepsilon)$.

Take c_0 as in Lemma 3 and set $c = c_0 + 2/(r \cos \alpha \sin \alpha)$; then one can choose a piecewise continuous function $l(\varepsilon) \in [c_0, c]$ so that the values $|l(\varepsilon) - l_m(\varepsilon)|$, $m \in \mathbb{Z}$ are separated from zero by a constant, namely

$$\min_{m \in \mathbb{Z}} |l(\varepsilon) - l_m(\varepsilon)| \geq \frac{1}{r \cos \alpha \sin \alpha}.$$

This choice guarantees that the s-triangles in the two periodic sequences

$$m_1(2\varepsilon, 0) + \Delta_{\varepsilon, r, \alpha, l(\varepsilon)}, \quad m_1 \in \mathbb{Z} \quad \text{and} \quad m_2(2\varepsilon, 0) + \Delta_{\varepsilon, r, \pi-\alpha, l(\varepsilon)}, \quad m_2 \in \mathbb{Z} \quad (10)$$

are disjoint for ε sufficiently small (see Fig. 12).

Now enumerate all s-triangles in both sequences according to $m = 1, 2, \dots$. sequence of horizontal vectors $V = \{v_0^1, v_0^2, \dots\}$ with $v_0^m = (\pm 1, 0)$, then choose sequence of disjoint intervals $\mathcal{U} = \{U^m = (\tau_1^m, \tau_2^m), m = 1, 2, \dots\}$ in $[0, 1]$, m th triangle ($m = 1, 2, \dots$) assign an s-set so as the m th reflected thin flow interval U^m (more precisely, is contained between the corresponding points B of the m th triangle) and has the direction v_0^m ; see Fig 13. The non-overlapping guarantees that each particle makes reflections in only one s-triangle.

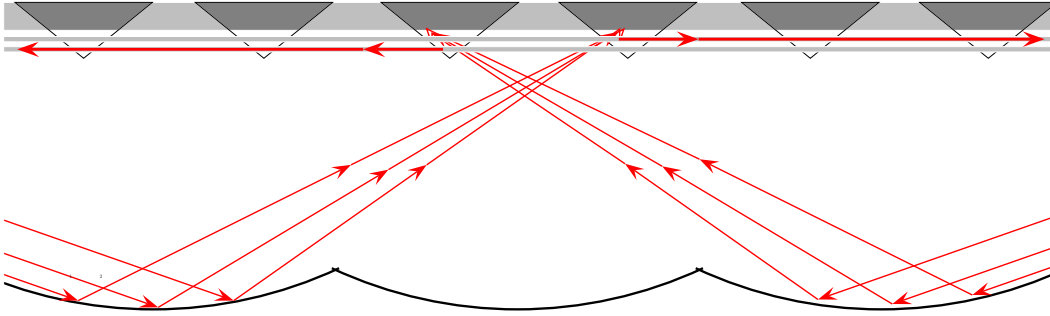


Figure 13: A collection of hollows and s-triangles: a schematic representation. Only two portions of incident flows at angles α and $\pi - \alpha$ are shown. They are reflected first by the corresponding hollows and then by the s-sets 1 and 2. The resulting thin flows are directed to the right and to the left.

Definition. The collection of these s-sets and the associated s-triangles (10) will be called the $(\varepsilon, r, \alpha, V, \mathcal{U})$ -system of the first kind (or just an α -system, if appropriate). In the case $\alpha = \pi/2$, a $(\pi/2)$ -system is a collection of s-triangles in the periodic sequence

$$m(2\varepsilon, 0) + \Delta_{\varepsilon, r, \pi/2, c_0}, \quad m \in \mathbb{Z} \quad (11)$$

and the corresponding s-sets. Thus, an α -system is defined for all $0 < \alpha \leq \pi/2$.

Summarizing, for $0 < \alpha < \pi/2$ we have two sequences of s-sets associated with two $(2\varepsilon, 0)$ -periodic sequences of disjoint s-triangles (10) with the size $O(\varepsilon^{3/2})$, $\varepsilon \rightarrow 0$ and at the same height $R_\alpha(1) + O(\varepsilon)$. (Notice that $0 < R_\alpha(1) \leq r/2$). If $\alpha = \pi/2$, we have only one sequence of s-sets associated with the sequence of s-triangles (11) at the height $r/2$. For an incident particle, with the angle of incidence α or $\pi - \alpha$, 3 possibilities can be realized.

(a) It is reflected once by a hollow and three times by an s-set, and then moves horizontally to the right or to the left. All reflected light rays are organized in (infinitely many) horizontal thin flows.

(b) The particle hits the base of an s-triangle and goes away. The length of each base is $\varepsilon^{3/2}$ and the distance between nearest bases is $O(\varepsilon)$; therefore only a portion $O(\varepsilon^{1/2})$ of the incident flow is reflected this way.

(c) The particle hits a vertex at the base of an s-triangle or an endpoint of a hollow. There are countably many such particles, and they will be neglected in what follows.

5 Constructing a system of multilevel reflecting sets

Consider a line segment on the boundary of a half-plane and n parallel flows in this half-plane incident on the segment at the angles $\alpha_j = \frac{2j-1}{2n}\pi$, $j = 1, \dots, n$. Take an isosceles trapezoid in the half-plane with the height r and with the angle $\pi/(2n)$ at the larger base, such that the smaller base coincides with the segment. Substitute this segment with a finite sequence of hollows (circular arcs outside the half-plane). Take the coordinate system xOy such that the hollows are described by formula (9) and the half-plane takes the form $y \geq -\sqrt{r^2 - \varepsilon^2}$, and assume that the length of the segment is a multiple of 2ε .

Take several sequences of horizontal vectors V_j and sequences of disjoint intervals \mathcal{U}_j , and consider the $(\varepsilon, r, \alpha_j, V_j, \mathcal{U}_j)$ -systems ($j = 1, \dots, \lfloor \frac{n+1}{2} \rfloor$) defined in the previous subsection (here $\lfloor \dots \rfloor$ means the integer part). We take only those s-sets and s-triangles of the systems that correspond to the given finite sequence of hollows covering the segment (for a schematic representation of these s-triangles see Fig. 14). Note that the s-triangles lie on heights $\leq r/2$ and in the $O(\varepsilon^{3/2})$ -neighborhood of the axes of reflected flows; on the other hand, the axes form angles $\geq \pi/(2n)$ with the segment, and therefore, their $O(\varepsilon)$ -neighborhoods (at the heights $< r$) lie in the trapezoid. This implies that the s-triangles under the consideration also lie in the trapezoid.

According to formula (8), the s-triangles of an α_j -system are situated on the height of $\frac{r}{2} \sin^2 \alpha_j + O(\varepsilon)$; therefore the triangles of different systems lie on different heights and do not overlap for ε sufficiently small.

Consider the finite collection of s-triangles and s-sets in all $(\varepsilon, r, \alpha_j, V_j, \mathcal{U}_j)$ -systems, $j = 1, \dots, \lfloor \frac{n+1}{2} \rfloor$ corresponding to the given finite sequence of hollows. It is still not quite satisfactory for our purposes; let us see why.

For an incident particle at an angle α_j , there are three possibilities. (Here and in the sequel we exclude from consideration finitely many particles that make the first reflection at a vertex of the base of an s-triangle or at an endpoint of a hollow.)

(a) It is reflected once by a hollow and three times by the s-set corresponding to this

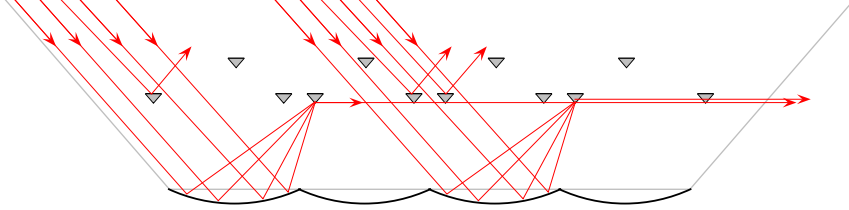


Figure 14: A schematic representation of a collection of s-triangles contained in a trapezoid. Only two portions of a flow incident on the hollows are shown. There are two periodic sequences of s-triangles corresponding to inclined flows and a sequence corresponding to the vertical flow ($\alpha = \pi/2$), with 4 s-triangles in each sequence.

hollow and to the given incident flow, and then moves horizontally (to the left or to the right).

(b) It hits the base of an s-triangle. (We do not care about what happens afterwards.)

(c) It is reflected by a hollow and then hits an irrelevant s-set (corresponding to a different flow).

The case (c) is undesirable, since the trajectory after hitting an irrelevant set cannot be controlled. To exclude this case, we need to add some more sequences of triangles (let them call "false s-triangles") shielding the triangles of our collection from irrelevant reflected flows (see Fig. 15). All the triangles of the original collection (that is, those that generate s-sets) will be called *true s-triangles*, in order to distinguish them from the false ones.

The false s-triangles are also isosceles, with the horizontal base and the angle $\pi/(2n)$ at the base, and with the apex turned downward. They should be placed higher than the true ones, that is, on a height in the interval $(r/2, r)$. We shall take the height of their bases to be $3r/4$. With all these restrictions, we take a false s-triangle to be the smallest triangle that shields a fixed (true) s-triangle from a fixed piece of flow. This choice uniquely defines all false triangles.

Note that the portion of the reflected flow corresponding to the angle of incidence α_j and to a certain hollow can hit at most two triangles of an irrelevant α_i -system, provided that $\frac{n+1}{2} - i > |\frac{n+1}{2} - j|$, and does not hit triangles of this system otherwise. Indeed, this inequality ensures that the triangles of the α_i -system are at a lower height than the triangles of the α_j -system (and in particular $\alpha_i < \pi/2$). Since the α_i -system is composed of two periodic sequences with the period 2ε and the width of the reflected flow is smaller

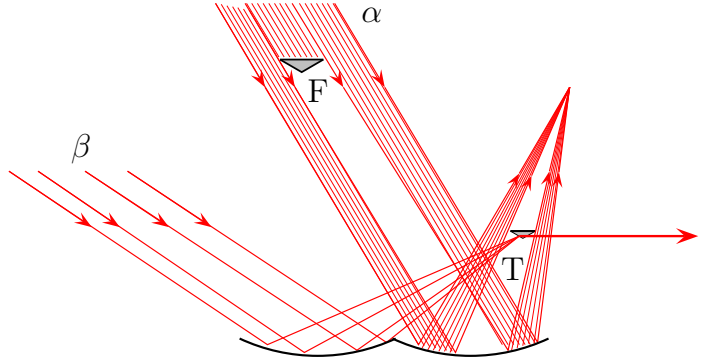


Figure 15: Two incident flows and two s-triangles are shown. The true s-triangle T is related to the β -flow, and is shielded by the false s-triangle F from the α -flow.

than 2ε , only one triangle of each sequence can be hit.

Thus, the number of false s-triangles on the way of a portion of the j th flow does not exceed $n - 1 - |n + 1 - 2j|$ (and therefore is zero for the flows incident at the smallest angles $\pi/(2n)$ and $(2n-1)\pi/(2n)$). Any such false triangle is a representative of a periodic sequence of false triangles; the number of sequences does not exceed $\lfloor \frac{n+1}{2} \rfloor (\lfloor \frac{n+1}{2} \rfloor - 1)$. Thus, the total number of false triangles is $O(1/\varepsilon)$. Their size is $O(\varepsilon^{3/2})$, and they may intersect each other (but do not intersect true s-triangles).

Definition. The collection of sets including the finite sequence of hollows resting on the given line segment, the s-sets of all $(\varepsilon, r, \alpha_j, V_j, \mathcal{U}_j)$ -systems ($j = 1, \dots, \lfloor \frac{n+1}{2} \rfloor$) related to this sequence of hollows, as well as the associated true s-triangles, and the false s-triangles, is called the $[\varepsilon, r, b, V, \mathcal{U}]$ -system of the second kind induced by the trapezoid. Here b is the length of the segment, and the collection $V(\mathcal{U})$ is composed of all sequences of vectors V_j (sequences of intervals \mathcal{U}_j).

Recall that the segment coincides with the smaller base of the trapezoid and its length b is supposed to be a multiple of 2ε .

It follows from the construction that a system of the second kind satisfies the following properties.

- The collection of (true and false) s-triangles in the system does not depend on the choice of V and \mathcal{U} .
- The true s-triangles corresponding to the flow incident at a fixed angle α_j form a finite $(2\varepsilon, 0)$ -periodic sequence, and the sequences corresponding to α_j and $\pi - \alpha_j$ are mutually symmetric with respect to the perpendicular bisector of the segment (and in

particular, the sequence of triangles corresponding to $\pi/2$ is itself symmetric with respect to this bisector). The set of false triangles is also symmetric with respect to the bisector.

- For the incident particles only the cases (a) and (b) can be realized. Only a part $O(\varepsilon^{1/2})$ of the incident flows satisfies (b).

6 Constructing a polygonal system of reflecting sets

In this and the next section we consider $2n$ parallel flows with the directions being the external bisectors of the angles of P_1 .

Let us give some more definitions. Let O be the center of the polygon P_1 . The image of ∂P_1 under a dilation centered at O is called a *polygonal contour*. A domain bounded by two polygonal contours is called a *polygonal ring*, or a *p-ring*. These contours are called the *inner* and *outer boundaries* of the p-ring.

Consider the rhombus ($ABCD$ in Fig. 16 (a)) bounded by two adjacent sides of the outer polygon of the p-ring and by the extensions of the corresponding sides of the inner polygon. Divide it into two triangles by the larger diagonal, and apply the dilation with the ratio 3 to the outer one, the center of the dilation being the center of the rhombus. The resulting triangle (shaded in Fig. 16 (a)) is called a *v-triangle* ("v" from vertex) associated with the p-ring.

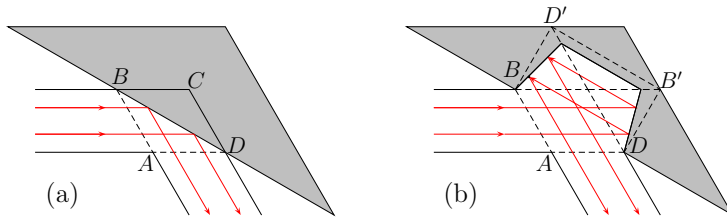


Figure 16: A v-triangle and a v-set are shown in figures (a) and (b). Reflection of the flow from a v-set preserves the order of light rays, as shown in figure (b).

In Figure 16 (b), the segments BB' and DD' are extensions of sides of the outer polygon, and the segments BD' and DB' are obviously orthogonal to BD . Draw a line through B until the intersection with DD' , and a line through D until the intersection with BB' , both lines making the angle $\pi/(4n)$ with BD' , clockwise and counterclockwise. Joining the points of intersection, as a result we obtain a closed hexagonal broken line. The domain bounded by this line is called a *v-set associated with this polygonal ring* (shaded in Fig. 16 (b)).

Now consider the union of all v-sets associated with a given p-ring, and consider a particle traveling in the p-ring with velocity parallel to one of its sides. When hitting a v-set, the particle is reflected according to the billiard law (see Fig. 16 (b)). One easily sees that the trajectory of the particle is closed and is symmetric with respect to all lines of symmetry of P_1 .

Take a polygonal contour and a finite collection of disjoint closed line segments $\{S_k\}$ on the contour symmetric with respect to all lines of symmetry of P_1 . Fix ε and r , and for each k take the regular trapezoid (outside the contour) whose smaller base coincides with the k th segment, S_k , and with the height r and the angle $\pi/(2n)$ at the larger base (see Fig. 17). We choose r small enough, so that the trapezoids are disjoint. For each k take the $[\varepsilon, r, b_k, V_k, \mathcal{U}_k]$ -system associated with the k th trapezoid, where b_k is the length of the segment S_k , and V_k and \mathcal{U}_k will be specified below. We assume that all b_k are multiples of 2ε . The collection of s-triangles in all these systems is symmetric with respect to all lines of symmetry of P_1 . Substitute each segment S_k with the corresponding finite sequence of hollows.

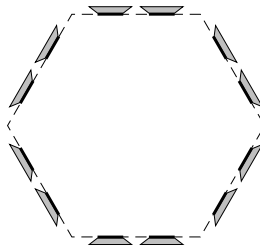


Figure 17: A finite collection of segments (shown in bold) and the corresponding trapezoids.

Consider the collection of true s-triangles in these systems. There is a natural pairing of triangles (and correspondingly of s-sets). Namely, each true s-triangle corresponds to a certain parallel flow and a certain hollow. Consider the symmetry with respect to a main diagonal of P_1 . The triangles corresponding to symmetric hollows and to opposite flows orthogonal to this diagonal form a pair (see Fig. 18). Note that the triangles in a pair are responsible for flows with complementary angles of incidence, say α and $\pi - \alpha$, and therefore are situated at the same height above the contour.

Recall that the collections of vectors $\{V_k\}$ indicate the directions of the thin flows for all s-sets (the vectors $(1, 0)$ and $(-1, 0)$ correspond to the directions "to the right" and "to the left", respectively). The collections of intervals $\{\mathcal{U}_k\}$ are responsible for the heights and widths of the thin flows. Also recall that each s-set is the disjoint union of two domains, the larger one and the smaller one.

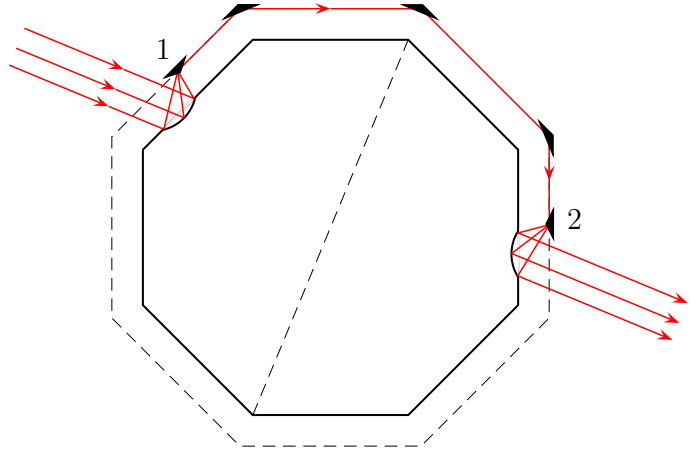


Figure 18: The true s-triangles in the figure (black triangles labeled by "1" and "2") form a pair. The small p-ring associated with this pair is shown by a (partly dashed and partly solid) broken line. A part of this ring is traversed by the compressed flow generated by the triangles 1 and 2. Some v-triangles at the vertices of the ring are also shown. The main diagonal of the octagon orthogonal to the incident flow is shown dashed; the triangles in the pair are mutually symmetric with respect to this diagonal.

Each s-set naturally induces two (*large* and *small*) p-rings; namely, the minimal p-rings containing the larger and the smaller domain, respectively. In the same way, the false triangles induce a p-ring, the same for all triangles. The large p-rings induced by two s-sets coincide, if these s-sets are responsible for incident flows with the same or complementary (α and $\pi - \alpha$) angles of incidence (and therefore are at the same level), and do not intersect otherwise. Also, large p-rings do not intersect small p-rings, and both do not intersect the p-ring induced by the false triangles.

The small p-rings induced by two s-sets coincide, if these s-sets are at the same level and the two intervals corresponding to the s-sets coincide. The small p-rings are always disjoint, if the corresponding intervals are disjoint.

Figure 13 may be viewed as a small part of a larger picture, where a part of a sequence of s-triangles at the same level above a certain polygonal contour is shown. The wider strip shown in light gray is a part of the large p-ring induced by these triangles, and the two thin strips are parts of the small p-rings induced by the s-sets labeled by "1" and "2".

Choose the collection of vectors $\{V_k\}$ in such a way that the vectors corresponding to the s-triangles in a pair are opposite (that is, one of them is $(1, 0)$ and the other $(-1, 0)$). In Fig. 18, the vectors corresponding to the triangles 1 and 2 are $(-1, 0)$ and $(1, 0)$, respectively. Choose the collections of intervals $\{U_k\}$ so that the intervals corresponding

to paired s-triangles coincide, and are disjoint otherwise. This implies, in particular, that the small p-rings induced by a pair of s-sets coincide.

For each small p-ring induced by a pair of s-sets consider the associated v-sets. (In Fig. 18 the induced octagonal ring is indicated by a broken line, and 3 of 8 associated v-triangles are shown.) The minimal p-ring containing the small p-ring and these v-sets will be called the *extended small p-ring induced by the pair of s-sets*; note that its width is twice as much as that of the original small p-ring.

Require, additionally, that the extended small p-rings induced by different pairs do not intersect. It suffices to require that the distance between nearest intervals in the collection $\{\mathcal{U}_k\}$ is larger than the sum of their lengths.

The above conditions imposed on $\{V_k\}$ and $\{\mathcal{U}_k\}$ guarantee that a portion of an incident flow, after four reflections from a hollow and an s-set, turns into a thin flow, then goes along the induced p-ring making reflections from the associated v-sets, then makes again four reflections from the paired s-set and the corresponding hollow, and finally is transformed into another portion of a flow, which is a continuation of the original one (see Fig. 18). Moreover, each trajectory in this portion is symmetric with respect to the diagonal of P_1 orthogonal to the incident flow, and therefore is invisible.

Definition. The collection of sets including all the sets (the hollows, true and false s-triangles, and s-sets) in the $[\varepsilon, r, b_k, V_k, \mathcal{U}_k]$ -systems, as well as all v-sets associated with the small p-rings, is called an (ε, r) -*polygonal system resting on the segments S_k* , or for brevity a *p-system*. The bases of the (true and false) s-triangles, as well as the lateral sides of the v-triangles, are called the *segments generated by the p-system*. The corresponding trapezoids (resting on the segments S_k) are called the *trapezoids of the p-system*. The union of these trapezoids is called the *envelope of this p-system*. The polygonal contour containing the segments S_k is called the *polygonal contour of the p-system*.

Slightly abusing the language, we also say that the p-system is resting on the corresponding collection of hollows.

All segments generated by a p-system lie on polygonal contours, and their union is symmetric with respect to all lines of symmetry of P_1 .

A fragment of a p-system (including a trapezoids resting on several hollows, two s-triangles and two v-triangles) is schematically represented in Fig. 19. The segments generated by the p-system are shown in bold.

The minimal p-ring containing the trapezoids is called the *full p-ring of this p-system*. The large p-rings and the extended small p-rings induced by the s-sets, as well as the p-ring induced by the false s-triangles are called the *occupied p-rings of the p-system*. They are disjoint and lie in the full p-ring. The complement of the union of occupied p-rings in the full p-ring is again the union of finitely many rings, and is called the *free p-rings of*

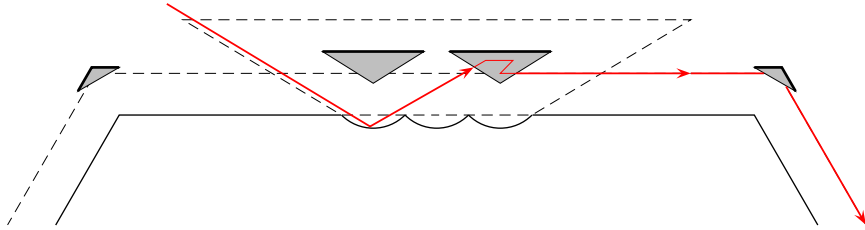


Figure 19: A fragment of a p-system including two s-triangles and two v-triangles. The trapezoid resting on three hollows is shown dashed, and an initial part of a billiard trajectory is also shown.

the p-system. Thus, the free and occupied p-rings are disjoint, and their union is the full ring of the system. (Moreover, the smallest and the largest p-rings in this union are free p-rings.) The free p-rings do not intersect the sets forming the p-system, and all segments generated by the p-system are contained in the inner boundaries of free p-rings.

Let a line segment lie on a polygonal contour. The isosceles triangle outside the contour with the angle $\pi/(2n)$ at the base and with the base coinciding with the segment is called the *semi-shadow of the segment* (see Fig. 20 (a)). Taking if necessary ε sufficiently small, one can ensure that the semi-shadows of all segments generated by the p-system lie in the free p-rings of this system.

Take again a polygonal contour and consider the $2n$ -gon formed by the external bisectors of its vertices. The part of the $2n$ -gon outside the contour is called the *the semi-shadow of the polygonal contour* (shown gray in Fig. 20 (b)). (Thus, the semi-shadow of a polygonal contour is the union of the semi-shadows of its sides.) Now consecutively enumerate the vertices and consider the union of the two n -gons formed by the external bisectors through the even and through the odd vertices. It is called the *weak semi-shadow of the contour* (the hexagonal star in Fig. 20 (b)).

The semi-shadow of a segment generated by a p-system lies either in the envelope of the system (if it corresponds to an s-triangle), or in the complement of the semi-shadow of the corresponding polygonal contour (if it corresponds to a v-triangle).

The union of semi-shadows of the segments generated by a p-system is called the *semi-shadow generated by this p-system*. We conclude that it is the union of two subsets, with the former one lying in the envelope of the system, and the latter one in the complement of the semi-shadow of the polygonal contour.

Consider the billiard outside the union of all hollows, s-sets, v-sets, and false s-triangles

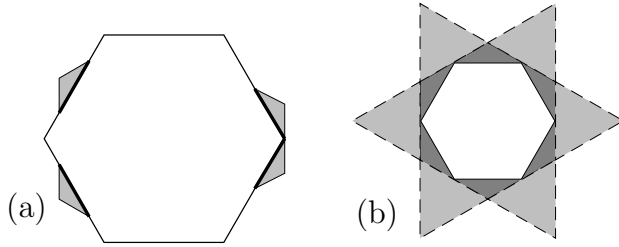


Figure 20: (a) Some segments on a polygonal contour and their semi-shadows (shown light gray). (b) The semi-shadow (gray) and the weak semi-shadow (the hexagonal star) of a polygonal contour.

of a p-system. For a particle of one of the $2n$ incident flows (except for finitely many particles) there are two possibilities.

(a) It makes the first reflection from the interior part of a hollow, then makes 3 reflections from an s-set, then goes along the corresponding polygonal ring making reflections from v-sets, then makes again 3 reflections from the paired s-set and a reflection from the corresponding hollow, and goes away along the same straight line as initially. The trajectory of this particle is invisible.

(b) It makes the first reflection from something else. We do not care about what happens after that.

In the case (a) the part of the trajectory between the first and fourth reflections (as well as its symmetric part, between the first and fourth reflections from the end) lies in the envelope of the p-system outside the semi-shadow generated by the system (see Fig. 21). The intermediate part of the trajectory, between the fourth reflection and the fourth reflection from the end, lies in an occupied p-ring of the p-system.

7 Constructing a hierarchy of polygonal systems

Take polygons P and P' homothetic to P_1 with the center at O such that P' lies in the weak semi-shadow of P and $P_1 \subset P \subset P' \subset P_2$ (and moreover, both $P \setminus P_1$ and $P' \setminus P$ have nonempty interior).

Now determine an iterative procedure leading to the construction of an invisible set. At each step $i = 0, 1, 2, \dots$ of the procedure we inductively define a set B_i and a marked part of its boundary J_i . The marked part of boundary is responsible for "visibility" of the set B_i , and its length $|J_i|$ goes to zero as $i \rightarrow \infty$. We also require that J_i is the finite union of line segments lying on polygonal contours, and that these contours are inner boundaries of p-rings disjoint with B_i . Both B_i and J_i are symmetric with respect to all

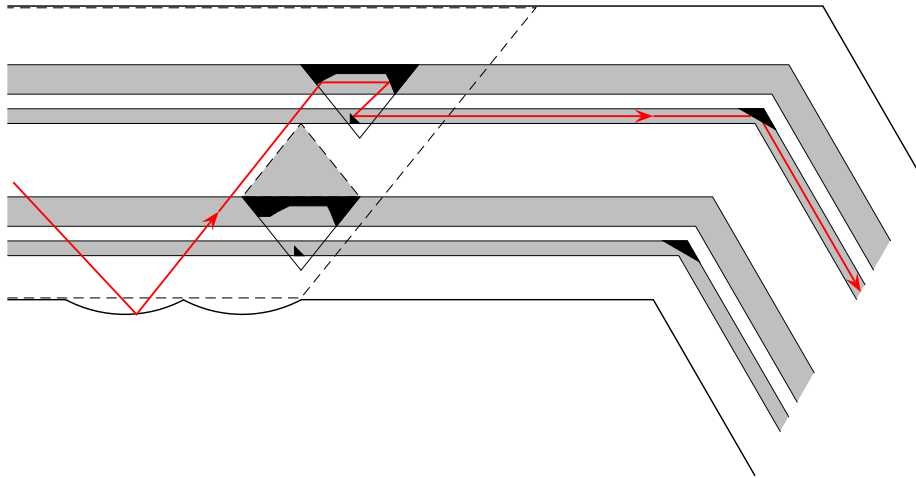


Figure 21: Fragments of the following objects are shown: a trapezoid of a p-system (a part of its boundary is shown by dashed line); two v-triangles of this system (shown black); two s-sets (shown black) and the corresponding s-triangles; the semi-shadow of an s-triangle (shown light-gray and bounded by dashed line); the full p-ring of the system composed of 5 free and 4 occupied (two large and two extended small) p-rings. A fragment of a billiard trajectory is also shown.

lines of symmetry of P , and additionally, $P_1 \subset B_i \subset P'$ and $\partial B_i \cap \partial P_1 = \emptyset = \partial B_i \cap \partial P'$.

Initially we have $B_0 = P$ and $J_0 = \partial P$; that is, the original set coincides with the polygon and all its boundary is marked.

At the i th step of the procedure we do the following.

1) Take a sub-interval of each interval in J_i , and let \tilde{J}_i be the union of these sub-intervals. We require that \tilde{J}_i is symmetric with respect to all lines of symmetry of P and the total length of the remaining part of J_i is smaller than β_i , $|J_i \setminus \tilde{J}_i| < \beta_i$, where $\lim_{i \rightarrow \infty} \beta_i = 0$.

2) By the inductive hypothesis, there are finitely many p-rings disjoint with B_i with the inner boundaries containing all the sub-intervals. Without loss of generality we assume that all these p-rings are contained in P' . For each of these p-rings, take an (ε, r) -polygonal system (p-system) resting on the corresponding sub-intervals, with ε and r being the same for all p-systems. Take the parameter r so small that each trapezoid of these p-systems lies in the corresponding p-ring, and also lies in the semi-shadow of the corresponding interval of J_i . Take ε so small that the total length of the segments generated by all the p-systems is smaller than β_i , and the hollows of the p-systems lie outside P_1 .

3) Let \tilde{B}_i be the modification of B_i obtained by substituting the chosen sub-intervals

by the added circular hollows. We have $\tilde{B}_i \subset B_i$. The new set B_{i+1} is the union of the modified set \tilde{B}_i and the sets (s-sets, v-sets, and false s-triangles) forming the added p-systems. The new marked part of boundary, J_{i+1} , is the union of $J_i \setminus \tilde{J}_i$ and the segments generated by these p-systems.

A part of J_{i+1} lies in the part of ∂B_i which is not substituted with hollows, and therefore lies in ∂B_{i+1} . The remaining part of J_{i+1} lies on the boundary of the added sets. It follows that $J_{i+1} \subset \partial B_{i+1}$.

Due to the construction of a p-system, the union of segments generated by the added p-systems is the finite union of disjoint segments and is symmetric with respect to all lines of symmetry of P . Hence J_{i+1} also satisfies these conditions.

It follows from the construction that the full p-rings of the added p-systems are disjoint. Each segment in $J_i \setminus \tilde{J}_i$ lies in the inner boundary of the full p-ring of an added p-system, and therefore also lies in the inner boundary of a free ring of this system. On the other hand, each segment generated by a p-system also lies on the inner boundary of a free ring of the system. These free rings do not contain points of B_i , and also do not contain points of the added sets. Thus, all inductive assumptions for the step $i + 1$ are satisfied.

That is, we start with a unique domain — the polygon P . At each step of the procedure we reduce the existing domains (by making hollows on their boundary) and add new domains (the sets of the added p-systems) that are mutually disjoint and are also disjoint with the existing ones. The resulting set B is the union of all reduced domains obtained in this procedure. In exact terms it can be defined as $B = \bigcup_{k=1}^{\infty} \bigcap_{i \geq k} B_i$, or (equivalently) as $B = \bigcap_{k=1}^{\infty} \bigcup_{i \geq k} B_i$.

The p-systems added in the course of the procedure form a tree (see Fig. 22). There is a unique p-system at the first step; it generates several p-systems of the second step; each of them in turn gives rise to several p-systems of the third step, etc.

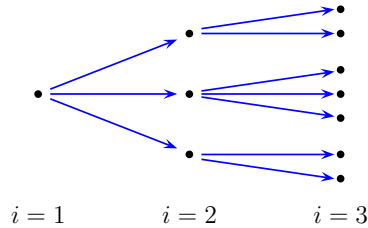


Figure 22: The hierarchy of p-systems of the procedure.

In a pair "parent – child", the full p-ring of the p-system child is contained in a free p-ring of the p-system parent. It follows that in a pair "ancestor – successor", the full p-ring of the p-system successor is contained in a free p-ring of the p-system ancestor. The

polygonal contour of the p-system child lies outside the polygonal contour of the p-system parent. It follows that the polygonal contour of the p-system successor lies outside the polygonal contour of the p-system ancestor. Further, the envelope of the p-system child is contained either in the envelope of the p-system parent (and moreover, in the semi-shadow generated by this system), or outside the semi-shadow of its polygonal contour. It follows that the same is true for a pair "ancestor – successor", and in particular, the envelope of the p-system successor is contained either in the semi-shadow generated by the p-system ancestor, or outside the envelope of this system.

Further, if two p-systems do not form a pair "ancestor – successor", then they are successors of a common p-system ancestor, and therefore their full p-rings lie in disjoint free p-rings of this system.

Now consider a p-system from the tree and a billiard trajectory outside the union of the sets (hollows, s- and v-sets, and false s-triangles) forming this system. Assume that the first reflection of this trajectory is from the interior part of a hollow (and therefore it satisfies condition (a) at the end of the previous section).

Lemma 4. *The part of the trajectory between the first and the last reflections does not intersect the sets of the other p-systems from the tree.*

Proof. We need to prove that this part of the trajectory does not intersect sets of (a) p-systems ancestors; (b) p-systems that are neither ancestors nor successors; (c) p-systems successors.

(a) This part of the trajectory lies in the full ring of the system, and therefore also lies in a free p-ring of each p-system ancestor; hence it does not intersect the sets of p-systems ancestors.

(b) Further, if a system does not form a pair "ancestor – successor" with the given p-system, then the full p-rings of these systems are disjoint, and therefore the sets of that system do not intersect the given part of the trajectory.

(c) Finally, the part of the trajectory between the first and the fourth reflections, as well as the symmetric part between the first and the fourth reflections from the end, lie in the envelope of the system outside the semi-shadows generated by it, and therefore does not intersect the sets of each p-system successor. On the other hand, the part of the trajectory between the fourth reflection and the fourth reflection from the end lies in an occupied p-ring of the p-system, and therefore does not intersect the sets of each p-system successor. \square

The following lemma finishes the proof of Theorem 2.

Lemma 5. *The set B is invisible for the $2n$ incident flows.*

Proof. First consider the s- and v-triangles included in the p-systems of the tree. We are going to show that the lateral sides of the s-triangles and the bases of the v-triangles are not accessible for the first reflection of the $2n$ incident flows.

Indeed, each v-triangle lies in the weak semi-shadow of P , and therefore is protected by P from the first reflection of $n - 1$ incident flows. Further, its base is protected from the flows with the opposite $n - 1$ directions by the v-triangle itself. Finally, it is not accessible for the two remaining opposite directions parallel to the base. Thus, the base is shielded from the first reflection.

Further, the lateral sides of an s-triangle are shielded by the s-triangle itself from the first reflection of n incident flows. They are also protected from the flows with the opposite n directions, since the s-triangle is situated in the semi-shadow generated by the p-system parent. Thus, the lateral sides of the s-triangle are not accessible for the first reflection.

B is the disjoint union of infinitely many domains of three kinds. The (unique) domain of the first kind is the reduced polygon P , with its boundary substituted by the union of infinitely many circular hollows. Domains of the second kind are reduced s-sets, with the base of each s-set substituted by the union of infinitely many hollows. Domains of the third kind are reduced v-sets, where the lateral sides of each v-set are substituted by the union of infinitely many hollows. The argument in the beginning of the proof shows that only hollows are accessible for the first reflection.

Consider a particle of a flow in B^c . If it does not make reflections from B , there is nothing to do. If it does, then the first reflection is from a hollow. Take the p-system resting on this hollow, and consider the (auxiliary) trajectory of the particle in the complement of this p-system with the same initial data (and therefore also with the same point of the first reflection) as the original particle. Lemma 4 guarantees that the part of the trajectory between the first and the last reflections does not intersect sets of the other p-systems, and therefore of course does not intersect the corresponding reduced sets. Since the part of its trajectory before the first reflection (a half-line) coincides with the original trajectory, it does not intersect the reduced sets of other p-systems, and the same is true for the symmetric part of the trajectory (after the last reflection).

It follows that the auxiliary trajectory coincides with the original one (in B^c), and since the former trajectory is invisible, so is the latter one. Hence B is invisible for the $2n$ incident flows.

Thus, Lemma 5, and therefore also Theorem 2, are proved. □

Remark. Slightly modifying the above construction, one can prove the following statement:

Given a finite number of directions, an arbitrary $\varepsilon > 0$, and two convex bounded bodies

$C_1 \subset C_2 \subset \mathbb{R}^2$ with $\partial C_1 \cap \partial C_2 = \emptyset$, there exists a set B invisible for all incident trajectories in the given directions, except for a portion ε of trajectories, such that $C_1 \subset B \subset C_2$. The set B is the disjoint union of finitely many domains with piecewise smooth boundary.

Note that no restrictions on the angles between the directions of invisibility are imposed; on the other hand, the invisibility is not perfect here. The underlying construction involves only the first step of the above infinite procedure. This statement will be proved in a forthcoming paper.

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