Exploring symmetry in professional development courses for mathematics teachers

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Abstract

In a way, mathematics can be generally described as a science that studies structure, relations, order or patterns through its own language, in order to help us better understand the world we live in. Structure, relations, order and patterns are also fundamental in art and therefore it is not surprising that mathematics and art share a long historical relationship. Mathematics has inspired several artists and art works are sometimes a good motivation for mathematical reasoning or learning.

Combining artistic creativity and mathematical reasoning turns out to be a very appealing way to explore several mathematical topics, in particular symmetry. Learning the mathematical principles of symmetry by first looking at art works and exploring one's own creativity has proven to be a very powerful and attractive strategy. During the last years I have explored symmetry with mathematics teachers of all grades through several professional development courses. In these courses teachers had the opportunity to create their own art works using different materials. The topics explored in these courses covered all types of symmetry in the plane: rosettes, friezes and wallpaper patterns.

In this paper we shall present some results of three of these courses, involving about 50 primary and secondary school teachers (grades 1 to 12). In all courses we studied symmetry and the groups of symmetry of plane figures. We produced applications using two specific crafts techniques: patchwork/quilting and ceramics.

Introduction

Symmetry is a very important feature in visual perception of images. Because of this, symmetry has been a recurring feature in art, architecture and other artifacts of human construction for centuries. Symmetry is indeed a fundamental principle of visual organization and has been shown to be important for multiple aspects of visual perception, such as perceptual grouping and pattern recognition (Stanford University, 2016). Symmetry in itself embeds the notion of repetition, regularity or congruence. As Wade (2006) sais, symmetry is a universal principal and "it is as much interest to mathematicians as it is to artists, and is as relevant to physics as it is to architecture".

Elliot Eisner, a pioneer in arts education, suggested that an artistic approach to education could improve its quality and lead to a new vision for teaching and learning (Eisner, 2002). Some topics of the mathematics school curriculum make a perfect setting for a deeper contribution of art to mathematical education. One such topic is the study of symmetry and isometry present along the school mathematics curriculum from elementary to secondary levels. We believe that the learning/teaching of symmetry and isometry can be greatly facilitated by taking the role of an artist and creating works of art, eventually inspired by well-known artists.

It is very engaging to use applications and examples of symmetry in the real world to address the topic of symmetry in the classroom. The visual arts are a nourishing field for that purpose. Symmetry can be found in paintings, ceramic pieces, pavements, textile works, iron works, sculptures, architecture, etc. Most mathematics teachers have a like for the arts but failed to have had training in this area. Increasing teacher's skills in arts provides them with additional tools for addressing symmetry (and other mathematical topics) in their classrooms.

In today's fast changing world, professional development courses for school teachers are an important means to keep up to date with new strategies and research results on education and science in general. These courses allow teachers to increase their knowledge and to reinforce their previous knowledge. In addition, they provide a good opportunity for teachers to stimulate their creativity and enroll in multidisciplinary projects, developing other skills beyond mathematics. Simultaneously, professional development courses are an important means to share teaching experiences and create working networks which may help teachers in their future work.

Being aware of the importance of professional development in teacher's careers, several continuing education courses for mathematics teachers have been proposed and took place at the University of Aveiro, Portugal, over the last years. In some of these courses a strong link between mathematics and other areas of knowledge was established. In this paper we shall present the outcome of three professional development courses for mathematics teachers involving about 50 primary and secondary school teachers (grades 1 to 12). In one of the course, Rosceas a torto e a direito (Rosettes in every way) we explored rosette groups of symmetry and the teachers produced several examples of figures, through patchwork and quilting, and through real glazed and colored ceramic pieces. Rosettes can exhibit reflection and /or rotational symmetry. The works produced illustrate a wide variety ofl possibilities, from the least symmetrical one to figures with multiple reflection and rotational symmetries. In another course, procura das simetrias do patchwork (Looking for the symmetries of patchwork), we explored friezes and wallpaper patterns through patchwork and quilting, obtaining examples of all the possible groups of symmetry of friezes wallpapers. In the third course, Vamos fazer azulejos com matemtica (Let's do ceramic tiles with mathematics), we explored tessellations using the same type of techniques used by the Dutch artist M.C. Escher in some of his tessellation works, producing figurative shapes, starting from some simple polygons such as squares, triangles and rectangles. These tessellations were then used to produce ceramic panels.

Symmetry

During the last century, Physics and Mathematics together gave an important contribution to the development of the study of symmetry. The need to understand the properties of the microscopic structure of materials, especially crystals, lead to a full description of symmetry in the plane and in space. Symmetry plays an important role not just in the structure of matter, but also in many physical phenomena. As Richard Feynman (Feynman, Leighton & Sands, 2013) said, "the marvelous thing about it all is that for such a wide range of important, strong phenomena - nuclear forces, electrical phenomena, and even weak ones like gravitation - over a tremendous range of physics, all the laws for these seem to be symmetrical".

Usually we think of symmetry of visual objects (in the plane or in space). This type of symmetry is defined over metric spaces. But symmetry can be expanded to other contexts, such as topological spaces. In this paper we shall consider only symmetry defined on metric spaces, in particular symmetry in the plane. In this sense, symmetry is strongly related to isometry. Since there are four types of isometries in the plane, there are also four types of symmetry in the plane: reflection, rotation, translation and glide reflection.

The set of all symmetries of a figure in the plane together with geometric composition forms a group. These groups can be classified according to the types of symmetries in them. If there are no infinitely small translations or rotations in the group, it is said to be discrete. Discrete groups are classified into 3 large categories: rosette, frieze (periodic pattern along one direction) and wallpaper (periodic pattern along two directions) groups. The definition of each of these categories can be made in several equivalent ways, one of which relates to the number of fixed points of the symmetries involved.

The most basic isometry is the identity which fixes all the points of the plane. A reflection fixes the reflection axis, point by point. A non-trivial rotation fixes only one point, the center of rotation. Finally a non-trivial translation or a glide-reflection has no fixed points.

Rosette groups are discrete groups of symmetry in the plane which have a fixed point. These groups have a finite number of symmetries. Rosettes only have rotational and/or reflection symmetry. Their groups of symmetry can only be of two different types: cyclic, C_n (with only rotational symmetry, $C_n, n \geq 2$, or no symmetry besides the identity, C_1), or dihedral, D_n, (with at least one reflection symmetry). Cyclic groups, C_n, have exactly n elements, all of them being rotations with a common center and rotation angles $360k/n$, $k = 1, \ldots, n$. Dihedral groups, D_n, have exactly $2n$ elements, n rotations like in the cyclic groups and n reflections along n straight lines meeting at the center of rotation. Figure 1 gives some examples of rosettes.

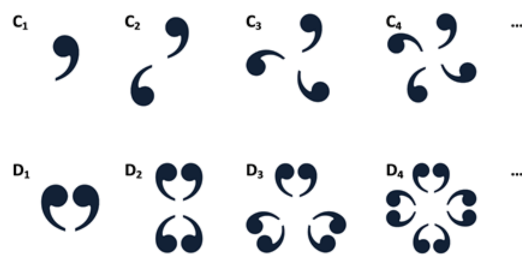

Figure 1: examples of rosettes

Frieze and wallpaper groups are discrete groups of symmetry in the plane which have no fixed points. These groups have an infinite number of symmetries.

Friezes exhibit only 7 types of symmetry groups. All the groups have at least translation symmetries. There are several designation systems and we shall use one from crystallography: p111, p1a1, pm11, pma2, p112, p1m1, pmm2 (see for instance Frieze Patterns (2009)). The first letter, p, refers to the periodicity of the frieze; the second digit may be an m if there is a vertical reflection symmetry or 1 if not; the third digit may be an m if there is a horizontal reflection symmetry, an a if there is a glide reflection or 1 otherwise; the last digit may be 2 if there is a rotation symmetry of order 2 or 1 if not. Figure 2 contains an example of all types of friezes.

Figure 2: examples of friezes

Wallpaper patterns exhibit 17 symmetry groups. There are several designation systems and we shall use one from crystallography: p1, p2, pm, pg, cm, pmm, pmg, pgg, cmm, p4, p4m, p4g, p3, p3m1, p31m, p6, p6m. The numbers relate to the highest order of the rotational symmetries. Letters m and g indicate mirror (reflection) and glide reflection symmetry. Letters p and c have to do with the type of cell (primitive or centered). A detailed description of all the groups can be found in many geometry text books or in several internet sites (see for instance Martin (1982) or the site Wikipedia (2016)).

Washburn and Crowe (1988) provide a simple flowchart to classify any wallpaper pattern. Figure 3 contains an example of the 17 wallpaper groups of symmetry using the same basic motive. The figure was adapted from an illustration by Wade (1993) retrieved from University of Stanford (2016).

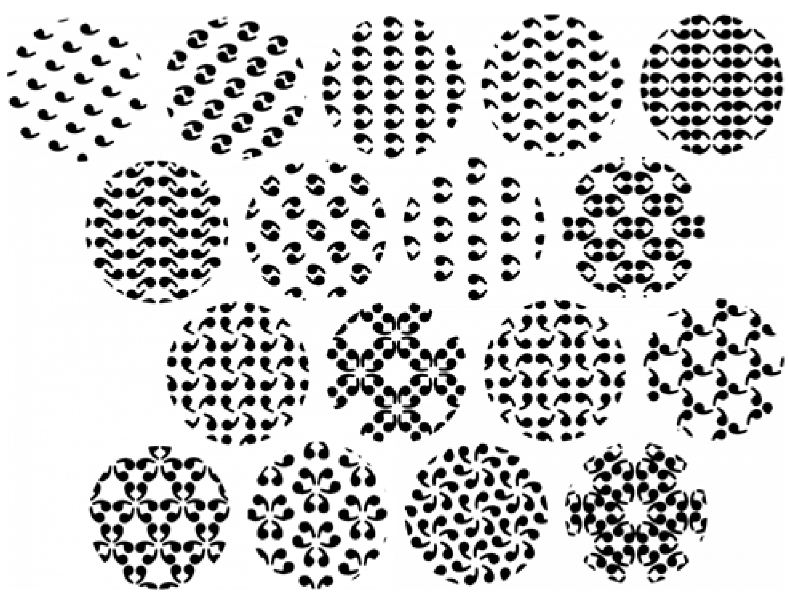

Figure 3: the 17 wallpaper groups of symmetry. Left to right, top to bottom: p1, p2, pm, pg, pmm, pmg, pgg, cm, cmm, p4, p4m, p4g, p3, p3m1, p31m, p6, p6m

Rosettes in every way

Rosettes in every way was a professional development course for mathematics teachers which took place at the University of Aveiro, Portugal, in 2015, from April 15th to June 27th. Like all professional development courses for Portuguese teachers, the course was acknowledged by the National Scientific and Pedagogical Committee for Teacher's Professional Development (Conselho Cientfico-Pedaggico da Formao Contnua), and was registered with the number CCPFC/ACC- 79097/14. This course consisted of 15 hours of contact between all participants and 15 hours of individual work (including work in the teacher's classrooms). It had the collaboration of a ceramist, Maria da Purificao Barros, and a quilter, Carla Santos, who coached part of the course. The course had 17 participants who taught mathematics from grades 1 to 12. Given the wide range of grades taught and the specificity of each level of teaching, different activities were proposed for teachers of different levels (elementary and secondary).

The topics covered in this course were symmetry and isometry, with the goal of exploring rosette groups of symmetry. The general structure of the course consisted of three parts: in the first part some concepts were provided and some activities were carried out, aiming at the deepening of the understanding and knowledge about the topic; in the second part of the course teachers were asked to explore activities in their classroom, and a few weeks later the experiences were shared and discussed with all the participants; in the third part of the course teachers were asked to develop individual projects applying the concepts of the course into practical works. Each participant was challenged to create his own pieces of art. The materials used in the individual projects were fabrics (patchwork) and ceramics. Teachers were asked to produce examples of rosettes belonging to different groups of symmetry, from both types, C_n and D_n, with n ranging from 1 to 8.

Colouring is an important aspect of any art/craft work. Colouring may or may not affect the symmetry properties of a figure. In patchwork, fabrics most often contain patterns which taken rigorously destroy all the symmetries in the overall figure. In what follows, each fabric will be considered as one-coloured. In addition, the classification of the symmetry group of each piece will take into account its colours. Figure 4 contains photos of some of the works produced.

Figure 4: examples of patchwork and ceramic rosettes

The board made by Ana Paula Moreira (A) and the lamp shed by Dulce Mesquita (B) are two examples of cyclic rosettes with symmetry group C_4. The decorative ceramic dish, made by Amelia Sales (C) has symmetry group C_6. The remaining three examples are dihedral rosettes: a cushion by Ana Cristina Martins (D) with symmetry group D_3, a ceramic dish by Dora Alfaiate (E) with symmetry group D_5; and another ceramic dish by Paula Santiago (F) with symmetry group D_7.

Looking for the symmetries of patchwork

Looking for the symmetries of patchwork was a 25 hours professional development course for teachers that took place at the University of Aveiro, Portugal, in 2012, from March 24th to May 15th. It was registered with the number CCPFC/ACC- 66489/11 and had the collaboration of Paula Coelho, a specialized quilter, who coached part of the course. There were 24 participants, who were all secondary school mathematics teachers. The course had three stages. First we explored isometry and symmetry through examples and some theory presented. Then we looked for patchwork patterns of all symmetry types (in books and internet sites) and found out that some groups are more typical than others (fabrics are easier to use when using perpendicular lines therefore squares and rectangles are more common than equilateral triangles, for instance). We couldn't find any examples for groups p3, p31m and p6. Next we designed and explored different patterns from all the groups of symmetry of friezes and wallpapers in order to choose the patterns to use in the applied projects. Patchwork is based mostly on polygons because it is much easier to sew along straight lines than along curves. Therefore all the patterns considered for the applied works were based on polygons. Finally each participant produced two pieces of patchwork: one with a frieze and another with a wallpaper pattern. We obtained examples of all the possible groups of symmetry of friezes and wallpapers. The frieze works consisted of folder covers (A4 size) in patchwork. The wallpaper quilts were all panels for hanging on walls and they were mostly square shaped with sides measuring between 80 and 100cm.

The friezes produced were all based on a basic module quite common in patchwork, a square divided in two triangles (of different colours) across its diagonal. The friezes used were the following:

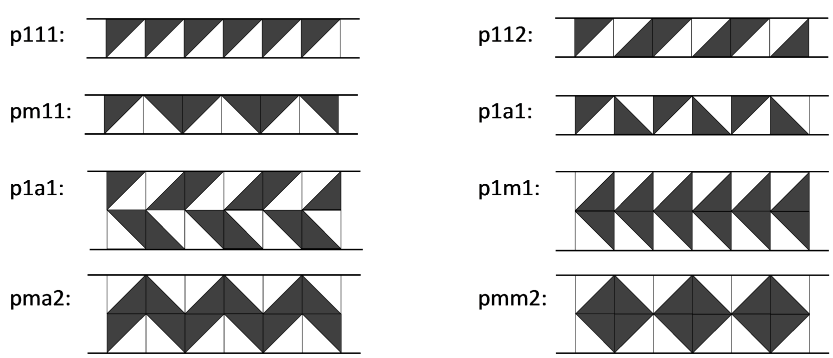

Figure 5: set of modular friezes containing all 7 types of symmetries (group p1a1 has 2 versions)

The next figures give some examples of the works produced. Figure 6 contains friezes from the following groups of symmetry (left to right, top to bottom): p1a1 (by Margarida Rodrigues), pmm2 (by Odete Silva), p111 (by Elsa Oliveira), p1a1 (by Virginia Vaz), p112 (by Catarina Silva), p1m1 (by Margarida Pereira), p1m1 (by Cristina Rodrigues), p1a1 (by Andreia Hall).

Figures 7 and 8 contain a selection of the wallpaper quilts produced. Among the 17 possible groups

Figure 6: examples of patchwork folder covers with friezes

of symmetry 5 have an underlying isometric grid (based on equilateral triangles) which is not very easy to work with through patchwork. Within these 5 groups, p3, p3m1, p31m, p6 and p6m, only the p6m is more or less common in patchwork. It is not surprising that we couldn't find any examples of groups p3, p31m and p6. Therefore, we present the works produced within these groups and another of group p3m1 which is also uncommon.

Figure 7: patchwork panels from Maria Jlia Cunha (left) with symmetry group p3m1 and Annabel Oliveira (right) with symmetry group p31m

Figure 8: patchwork panels from Jos Antnio Branco (left) with symmetry group p3 and Odete Silva (right) with symmetry group p6.

Let's do ceramic tiles with mathematics

Let's do ceramic tiles with mathematics was a 25 hours professional development course for teachers which took place at the University of Aveiro, Portugal, in 2013, from January 20th to April 4th. It was registered with the number CCPFC/ACC - 67053/11 and had the collaboration of a ceramist, Maria da Purificao Barros. The course had 14 teachers who taught mathematics from grades 1 to 12. In this course, we explored tessellations using the same type of techniques used by M.C. Escher in some of his works, producing tessellations with animal and figurative shapes. Like the previous courses this one was given in three parts. In the first part some concepts were provided and some activities were carried out, aiming at the deepening of the understanding and knowledge about the topic. In the second part of the course teachers were asked to explore activities in their classroom, taking into account the specificity of their classroom context. In the third part of the course teachers were asked to develop an individual project applying the concepts of the course into a ceramic piece of work.

Escher produced 137 wallpaper designs gathered in his notebooks and analyzed in his book entitled The regular division of the Plane. He started from some simple polygons such as squares, triangles and rectangles and by deforming the sides in specific ways he obtained different tessellations belonging to different symmetry groups. In a similar way, teachers were asked to make their own tessellations starting with simples polygons. The resulting tessellations were then used to produce real glazed and colored ceramic pieces.

The basic procedure is as follows: we start from simple polygons and perform deformations on some of the sides. The deformations are then transferred to the other sides using plane isometries. We shall use the following notation for these transformations:

The polygons we used were squares, rectangles, rhombuses, parallelograms, kites, triangles and hexagons. Many different transformations can be performed on these and other polygons to produce tessellations. Altogether we explored 32 different models. Fathauer (2008) provides many suggestions with worked examples and gives many hints on practical issues related to this type of tessellations.

Next we present photos of a selection of the ceramic pieces produced. For each work we present the underlying basic polygon and the transformations applied to produce the proto-tile. We also provide the wallpaper symmetry group of the tessellation.

The first example (Figure 9) is based on a parallelogram. The transformation procedure is based only on translation of two sides.

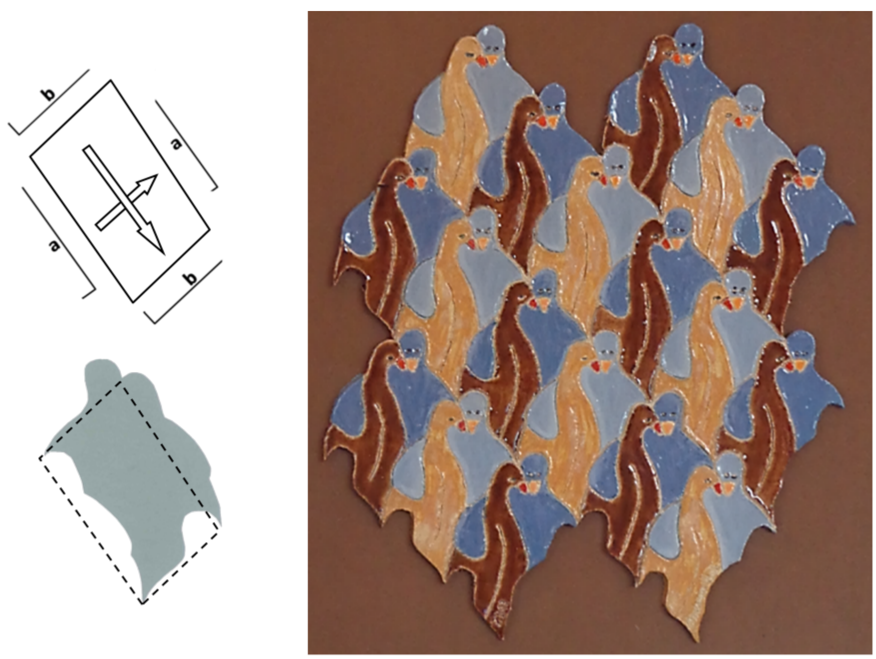

Figure 9: Ceramic tessellation by Andreia Hall with symmetry group p1

The second example (Figure 10) is based on rotations over a right triangle.

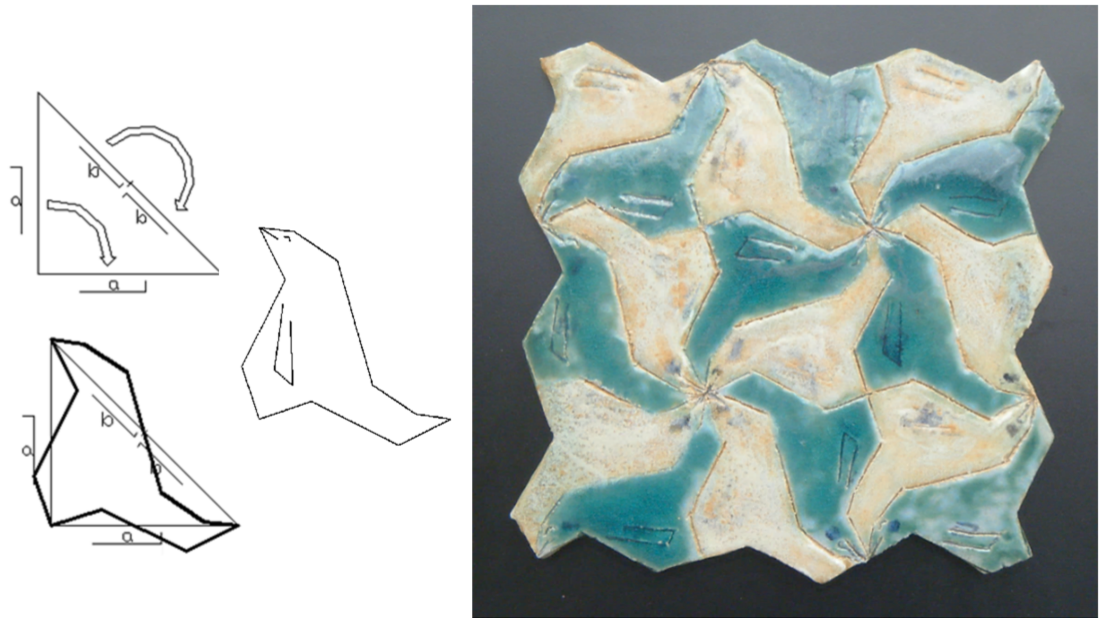

Figure 10: Ceramic tessellation by Lusa Pinheiro with symmetry group p4

The third example (Figure 11) is based on rotation and glide reflection over an equilateral triangle. For this tessellation two symmetric proto-tiles are needed.

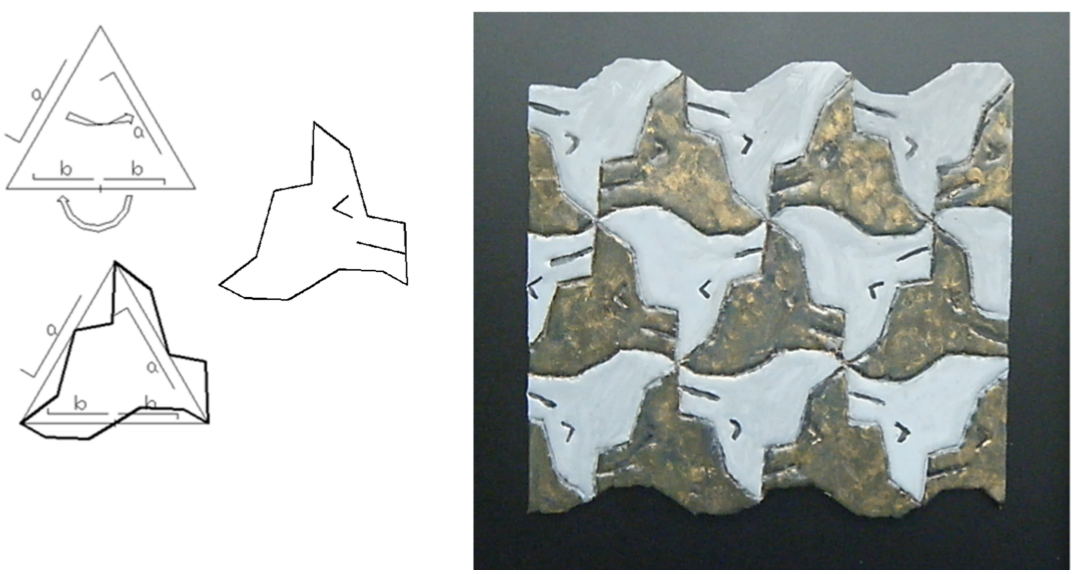

Figure 11: Ceramic tessellation by Maria Manuela Pinheiro with symmetry group pgg (ignoring colours)

The last example (Figure 12) is based on reflections over a square and for this tessellation two different proto-tiles are needed.

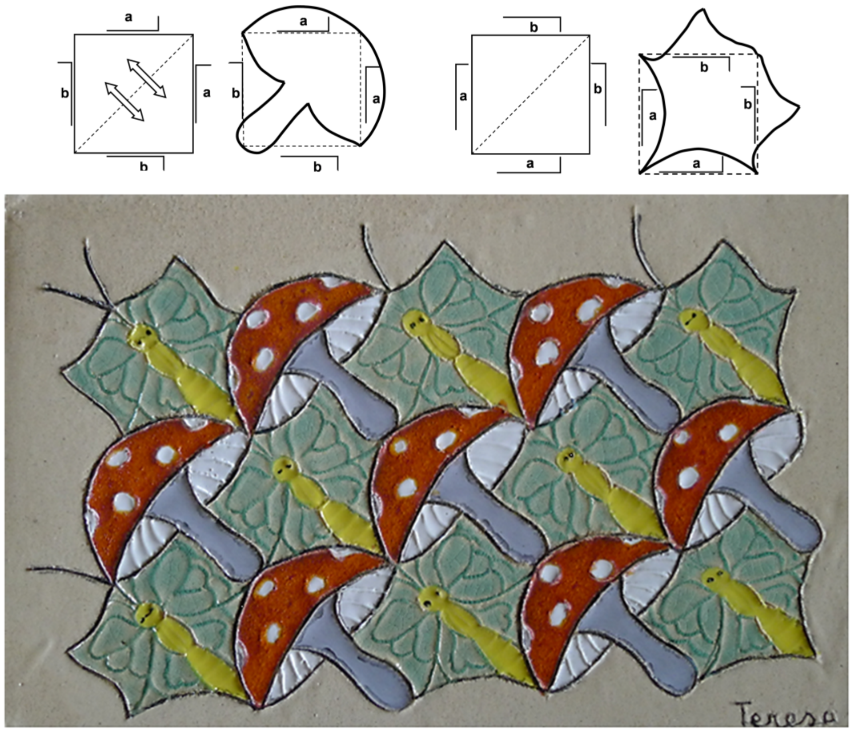

Figure 12: Ceramic tessellation by Teresa Mena with symmetry group pm

During the second part of this course some teachers developed work with their students following some of the procedures explored. The students produced their own tessellation using paper and coloring pens. Figure 13 contains some of the examples produced.

Figure 13: Tessellations by 3rd grade students of Maria Manuela Pereira

Conclusion

Working with mathematics teachers in professional development courses is undoubtedly enriching and motivating. Engagement is a fundamental task for an effective and fulfilling learning experience. Eisner (2002) believed that in the arts engagement tends to be secured by the aesthetic satisfactions obtained from the work of art itself. The work being created presents natural challenges which are related to part of these satisfactions:

> Materials resist the maker; they have to be crafted and this requires an intense focus on the modulation of forms as they emerge in a material being processes. This focus is so intense that all sense of time is lost. The work and the worker become one. (p.14)

Indeed, during the several professional learning courses that I have taught I have had the pleasure to witness the satisfaction felt by all participants when engaging with their works. The sense of time was lost for several times and the unity between work and worker could be felt repeatedly. These experiences have proven to be highly pleasurable and rewarding both for the participants and myself. In the end of each course teachers were asked to fill in an evaluation form. The overall evaluation was very positive and allows us to conclude that the goals of the courses were completely accomplished. It was very stimulating to see how the teachers could improve their mathematical skills on symmetry and at the same time learn some crafts techniques which allowed them to apply these skills and above all have fun while working.

Acknowledgments

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