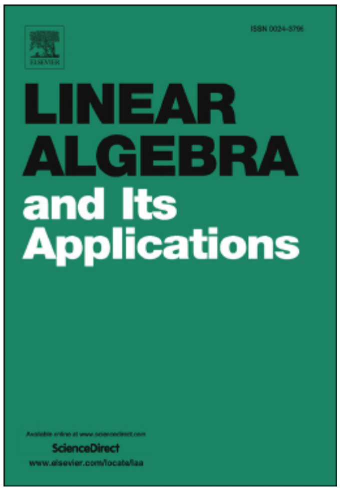

Block matrices and Guo's Index for block circulant matrices with circulant blocks

Enide Andrade, Cristina Manzaneda, Hans Nina, María Robbiano

PII: S0024-3795(18)30341-0
DOI: https://doi.org/10.1016/j.laa.2018.07.015
Reference: LAA 14657

To appear in: *Linear Algebra and its Applications*

Received date: 13 March 2018
Accepted date: 12 July 2018

Block matrices and Guo's Index for block circulant matrices with circulant blocks

Enide Andrade*

CIDMA-Center for Research and Development in Mathematics and Applications Departamento de Matemática, Universidade de Aveiro, 3810-193, Aveiro, Portugal.

Cristina Manzaneda

Departamento de Matemáticas, Facultad de Ciencias. Universidad Católica del Norte. Av. Angamos 0610 Antofagasta, Chile.

Hans Nina

Departamento de Matemáticas, Facultad de Ciencias Básicas. Universidad de Antofagasta. Av. Angamos 601 Antofagasta, Chile.

María Robbiano

Departamento de Matemáticas, Facultad de Ciencias. Universidad Católica del Norte. Av. Angamos 0610 Antofagasta, Chile.

Abstract

In this paper we deal with circulant and partitioned into n-by-n circulant blocks matrices and introduce spectral results concerning this class of matrices. The problem of finding lists of complex numbers corresponding to a set of eigenvalues of a nonnegative block matrix with circulant blocks is treated. Along the paper we call realizable list if its elements are the eigenvalues of a nonnegative matrix. The Guo's index λ_0 of a realizable list is the minimum spectral radius such that the list (up to the initial spectral radius) together with λ_0 is realizable. The Guo's index of block circulant matrices with circulant blocks is obtained, and in consequence, necessary and sufficient

*Corresponding author

Email addresses: enide@ua.pt (Enide Andrade), cmanzaneda@ucn.cl (Cristina Manzaneda), hans.nina@uantof.cl (Hans Nina), mrobbiano@ucn.cl (María Robbiano)

conditions concerning the NIEP, Nonnegative Inverse Eigenvalue Problem, for the realizability of some spectra are given.

Inverse eigenvalue problem; Structured inverse eigenvalue problem; Nonnegative matrix; Circulant matrix; Block circulant matrix; Guo index

15A18, 15A29, 15B99.

---

1. A brief review and some tools

In this section we present a brief resume related to nonnegative inverse eigenvalue problem (NIEP). Recall that a square matrix $A = (a\ )$ is non-negative ($A \quad 0$) if and only if $a \quad 0$, for each $i, j = 1, \ldots, n$. For more background material on nonnegative matrices see for example [3]. The NIEP is the problem of determining necessary and sufficient conditions for a list of complex numbers to be the spectrum of an n-by-n nonnegative matrix A. If a list σ is the spectrum of a nonnegative matrix A, then σ is and the matrix A σ (or, that is a realizing matrix for the list). This is a hard problem and it is considered by many authors since more than 50 years ago. This problem was firstly considered by Suleĭmanova [37] in 1949. Many partial results were found but the problem is still unsolved for $n \quad 5$. For $n = 3$ it was solved in [22] and for matrices of order $n = 4$ the problem was solved in [26] and [25]. In its general form it has been studied in e.g. [5, 11, 14, 19, 20, 22, 35, 36]. There are some variants of this problem namely for instance, the one called symmetric nonnegative inverse eigenvalue problem, SNIEP, (when the nonnegative realizing matrix is required to be symmetric). This is also an open problem and some work on this can be seen in [9, 15, 23, 34]. Another variants of the original problem is the question for which lists of n real numbers can occur as eigenvalues of an n-by-n nonnegative matrix and it is called real nonnegative inverse eigenvalue problem (RNIEP). Some results can be seen in e.g. [4, 8, 28, 30, 33]. The structured NIEP is an analogous problem to NIEP where the realizing matrix must be structured, for instance, the matrix can be symmetric, Toeplitz, Hankel, circulant, normal, etc. see in [9, 24, 27] and the reference therein.

To not be so extensive on the description of this problem the reader must refer to some surveys on NIEP, for instance in [13] and in the references therein.

Throughout the text, and $\sigma(A)$ denote the set of complex k-tuples and of the eigenvalues of a square matrix A, respectively. Also $\rho(A)$ denotes the spectral radius of A. Here, the identity matrix of order n is denoted by I and if the order of the identity matrix can be easily deduced then it is just denoted by I.

Since a nonnegative matrix is real, its characteristic polynomial must have real coefficients and then $\lambda^0, \ldots, \lambda_{-1} = \sigma = \overline{\sigma} = \{\overline{\lambda^0}, \ldots, \overline{\lambda_{-1}}$, where $\overline{\lambda}$ stands for the complex conjugate of λ .

Therefore consider the following definition:

Definition 1. n $(\lambda_0, \ldots, \lambda_{-1})$

$$\lambda^0, \ldots, \lambda_{-1} = \{\overline{\lambda^0}, \ldots, \overline{\lambda_{-1}},$$

The following fundamental theorem was proven in [11] and in its statement it is introduced formally the notion of Guo's index.

Theorem 2. $(\lambda_1, \ldots, \lambda_{-1})$ $(n \quad 1)$ $\lambda \quad 0$

$$\max_{1 \leq \leq -1} \lambda \qquad \lambda\ 0.$$

$(\lambda, \lambda_1, \ldots, \lambda_{-1})$ n n A λ $\lambda\ 0$ $\lambda\ 0$ $2n \max_{1 \leq \leq -1} \lambda$

The following definition generalizes the concept of circulant matrix.

Definition 3. n $n \quad 2$

This concept was introduced in [27]. The spectra of a class of permutative matrices was studied in [24]. In particular, spectral results for matrices partitioned into 2-by-2 symmetric blocks were presented and, using these results sufficient conditions on a given list to be the list of eigenvalues of a nonnegative permutative matrix were obtained and the corresponding permutative matrices were constructed. Here, in [24], it was introduced the concept of permutatively equivalent matrix.

Definition 4. Let $\tau = (\tau_1, \ldots, \tau\ )$ be an n-tuple whose elements are permutations in the symmetric group S , with $\tau_1 = id$. Let $\mathbf{a} = (a_1, \ldots, a\ )$. Define the row-vector,

$$\tau\ (\mathbf{a}) = \quad a\ _{(1)}, \ldots, a\ _{(\ )}$$

and consider the matrix

$$\tau\,(\mathbf{a}) = \begin{matrix} \tau_1\,(\mathbf{a}) \\ \tau_2\,(\mathbf{a}) \\ \vdots \\ \tau\ _{-1}\,(\mathbf{a}) \\ \tau\ \,(\mathbf{a}) \end{matrix} \quad . \tag{1}$$

An n-by-n matrix A is called τ if $A = \tau\,(\mathbf{a})$ for some n-tuple $\mathbf{a}$.

Definition 5. A B τ $\tau =$ $(\tau_1, \ldots, \tau\ )$

The paper is organized as follows: In Section 2 we introduce some concepts and results related with circulant matrices and block circulant matrices. In particular, we give a new necessary and sufficient condition for the list $\sigma = (\lambda_1, \ a + bi, \ a \ \ bi, \ldots, \ a + bi, \ a \ \ bi)$ with $a > 0, b > 0$ to be the spectrum of a nonnegative circulant matrix. This result improves the one proved in [31, Proposition 4]. We also refer the importance of circulant matrices and block circulant matrices in some applied areas. In Section 3 we present spectral results for matrices partitioned into blocks where each block is a square circulant matrix of order n then, the spectral results are applied to structured NIEP and SNIEP. In Section 4 some properties of a matrix partitioned into blocks with a certain structure are found using some already known structure on matrices of smaller size. Finally at Section 5 it is established the Guo's index for block circulant matrices with circulant blocks. Throughout the paper some illustrative examples are presented.

2. Circulant matrices and block circulant matrices

The class of circulant matrices and their properties are introduced in [7]. In [18] it was presented a spectral decomposition of four types of real circulant matrices. Among others, right circulants (whose elements topple from right to left) as well as skew right circulants (whose elements change their sign when toppling) were analyzed. The inherent periodicity of circulant matrices means that they are closely related to Fourier analysis and group theory.

Let $a = (a_0, a_1, \ldots, a_{\ -1})$ be given.

Definition 6. real right circulant matrix circulant matrix

$$A(a) = \begin{matrix} a_0 & a_1 & \ldots & \ldots & a_{\ -1} \\ a_{\ -1} & a_0 & a_1 & \ldots & a_{\ -2} \\ a_{\ -2} & & & & \\ & & & a_0 & a_1 \\ a_1 & \ldots & a_{\ -2} & a_{\ -1} & a_0 \end{matrix}$$

The matrix $A(a)$ is clearly determined by its first row. Therefore, the above circulant matrix is also sometimes denoted by $circ(a_0, a_1, \ldots, a_{\ -1})$ or, in a more simple way by $(a_0, a_1, \ldots, a_{\ -1})$. The next concepts can be seen in [18]. The entries of the unitary discrete Fourier transform (DFT) matrix $F = (f\ \ )$ are given by

$$f\ \ := \frac{1}{\sqrt{n}}\omega\ \ , \ p = 0, 1, \ldots, n \quad 1, \ q = 0, 1, \ldots, m \quad 1, \tag{2}$$

where

$$\omega = \cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m}. \tag{3}$$

The following results characterize the circulant spectra.

Theorem 7. $a = (a_{\ 0}, \ldots, a_{\ -1})$ $A(a) = circ(a_0, \ldots, a_{\ -1})$.

$$A(a) = F\Lambda(a)F^*,$$

$$\Lambda(a) = diag(\lambda_0(a), \lambda_1(a), \ldots, \lambda_{\ -1}(a))$$

$$\lambda\ \ (a) = \ \ _{=0}^{\ -1} a\ \omega\ \ \quad k = 0, 1, \ldots, m \quad 1. \tag{4}$$

Corollary 8. a

$$v := v(a) = (\lambda_0(a), \lambda_1(a), \ldots, \lambda_{\ -1}(a)) \ .$$

$$a \ = \frac{1}{m} \ {}^{-1}_{=0} \ \lambda\ \omega^{-} \qquad k = 0, 1, \ldots, m \quad 1. \tag{5}$$

Remark 9. $a = (a_{\ 0}, \ldots, a_{\ -1})$

$$a = \frac{1}{\sqrt{m}} F^* v(a) \qquad v(a) = \sqrt{m} F a.$$

The next proposition obtains the Guo's index of some spectra and it is a generalization of the result obtained by O. Rojo and R. Soto in [31, Proposition 4].

Proposition 10. n

$$\sigma = (\lambda_1, \ a + bi, \ \bar{a} \ \ bi, \ldots, \ a + bi, \ a \ \ bi)$$

$a > 0, b > 0$ λ_1 $\sqrt{a^2 + b^2}$ σ n n A

$$\lambda_1 \ \ (n \ \ 1)a + n \max \ \ 0, \frac{b}{\sqrt{n}} \ \ a \ \ . \tag{6}$$

Proof. Let s 0 and consider $\sigma' = (n \ \ 1)(a + s), \ (a + s) + bi, \ (a + s) \ \ bi, \ldots, \ (a + s) + bi, \ (a + s) \ \ bi$. We claim that there exists an s 0 such that a companion matrix

$$B = \begin{matrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ b & b_{\ -1} & b_{\ -2} & \ldots & b_2 & 0 \end{matrix}$$

realizes σ'. In fact, for the characteristic polynomial of B we have

$$p(x) = x \qquad b_{\ 2}(\sigma) x^{\ -2} \qquad b \ \ (\sigma),$$

and from the Newton identities, [13],

$$b_2(\sigma) = \frac{1}{2}[(n \quad 1)^{2}(a+s)^2 + (n \quad 1)(a+s)^{2} \quad (n \quad 1)b^{2}].$$

As B must be nonnegative, from [21, Lemma 5] it is required that $b^{2}(\sigma) \quad 0$ or, in a equivalent way, that $a + s \quad \frac{b}{\sqrt{n}}$. Thus

$$s \quad \max \quad 0, \frac{b}{\sqrt{n}} \quad a \quad . \tag{7}$$

Therefore, σ is realizable by the matrix $A = B + sI \quad 0$ if and only if $\lambda_1 \quad (n \quad 1)a + ns$, which shows the result. ∎

Remark 11.

C

$$C = \begin{matrix} c_0 & c_1 & \dots & \dots & c \\ c & c_0 & c_1 & \dots & c_{\ -1} \\ c_{\ -1} & c & c_0 & \dots & \\ & & & & \\ c_1 & c_2 & c_3 & \dots & c_0 \end{matrix} ,$$

$$\begin{matrix} c_0 \\ c_1 \\ \\ c \end{matrix} = \frac{1}{n} \begin{matrix} 1 & 1 & \dots & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{\ -1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(\ -1)} \\ & & & & \\ 1 & \omega^{\ -1} & \omega^{2(\ -1)} & \dots & \omega^{(\ -1)} \end{matrix} \quad \begin{matrix} \lambda_1 \\ a+ib \\ a \quad ib \\ \\ a \quad ib \end{matrix} .$$

Definition 12.

$$\begin{matrix} A_0 & A_1 & \dots & A_{\ -1} \\ A_{\ -1} & A_0 & \dots & A_{\ -2} \\ & & & \\ A_1 & A_2 & \dots & A_0 \end{matrix} , \tag{8}$$

$A \quad n \quad n$

The partitioned into blocks matrices have particular importance in many areas. We refer here for instance Engineering, see [16] where the authors study forced vibration of symmetric structures. They present a method to calculate the eigenvectors of these matrices and then from the discrete structure they establish relationships for continuum structures. After discussing the dynamics aspects of the structures they consider subjects from earthquake engineering and spectral analysis from such structures. Moreover, block circulant matrices are used in coding theory. For instance, in [17] the author used the canonical form based in circulant matrices to found many good codes: quadratic residue codes and high quality group codes. Some LDPC codes can also be defined by a matrix partitioned into blocks where each block are circulant matrices [1]. See more applications in coding theory in [10], and the references therein. More on circulant block matrices with the circulant or factor circulant structure was considered for instance in [2, 6, 32, 38].

3. Eigenpairs for square matrices partitioned into circulant blocks

In this section we present spectral results for matrices partitioned into blocks where each block is a circulant matrix of order n. The next theorem is valid in an algebraically closed field K of characteristic 0. For instance, when $K =$. We recall that it is a particular case of [39, Theorem 1] when $A = circ(0, 1, 0, \dots, 0)$ however, its statement establishes the notation of this work. Therefore, we include it here.

Theorem 13. K 0
$A = (A(i,j))$ mn mn n 2
m m 1 i, j $n,$

$$A = (A(i,j)), \qquad A(i,j) = circ(a(i,j)), \tag{9}$$

$$a(i,j) = (a_0(i,j), a_1(i,j), \dots, a_{-1}(i,j)),$$

a (i,j) $K,$ 1 i,j n $k = 0, 1, \dots, m$ $1.$ $k =$ $0, 1, \dots, m$ $1,$

$$\mathbf{e} = 1, \omega, \omega^2, \dots, \omega^{(-1)}, \tag{10}$$

ω

$$\sigma(A) = \sum_{=0}^{-1} \sigma(S\ ), \tag{11}$$

$$S\ = (s\ (i,j))_{1\leq\ \leq}\ , \qquad s\ (i,j) = \mathbf{e}\quad a(i,j)\ , \tag{12}$$

$k = 0,1,\ldots,m \quad 1 \qquad 1 \quad i,j \quad n.$

The next result is a direct consequence of (5).

Corollary 14. $\ell = 0,1,\ldots,m \quad 1$ S $a(u,v) := (a_0(u,v),\ldots,a_{-1}(u,v))$ $A(u,v) = circ(a(u,v))$

$$\begin{matrix} a\ (1,1) & a\ (1,2) & \ldots & a\ (1,n) \\ a\ (n,1) & a\ (n,2) & \ldots & a\ (n,n) \end{matrix} = \frac{1}{m}\sum_{=0}^{-1} S\,\omega^{-}\ ,$$

$k = 0,1,\ldots,m \quad 1.$

Remark 15. A

$k = 0,1,\ldots,m \quad 1,$

$$L\ := \frac{1}{m}\sum_{=0}^{-1} S\,\omega^{-} \tag{13}$$

$S\quad 0$

$$\begin{aligned} \sum_{=0}^{-1} L &= \frac{1}{m}\sum_{=0}^{-1}\sum_{=0}^{-1} S\,\omega^{-} \\ &= \frac{1}{m}\sum_{=0}^{-1} S \sum_{=0}^{-1} \omega^{-} \\ &= \frac{1}{m}S_0 + \frac{1}{m}\sum_{=1}^{-1}\sum_{=0}^{-1} S\,\omega^{-}\ . \end{aligned}$$

$$S_0 \quad \sum_{=0}^{-1} L \quad 0$$

$$\sum_{=1}^{-1} \sum_{=0}^{-1} \omega^{-} = \sum_{=1}^{-1} \frac{1 \quad \omega^{-}}{1 \quad \omega^{-}} = 0.$$

Remark 16. $q_1, q_2, \ldots, q$ $k = 0, 1 \ldots m \quad 1,\ 1 \quad u, v \quad n$ ρ $(u,v) = q \ S \ q$

$$a \ (u,v) = \sum_{=0}^{-1} \frac{1}{m} \rho \ (u,v) \omega^{-} \ . \tag{14}$$

The next result will be important in order to present a constructive criterion.

Theorem 17. $k = 0, 1, \ldots, m \quad 1 \quad L$ $A(u,v)$ (u,v) th A $1 \quad u, v \quad n.$

$$\begin{aligned} A(u,v) &= circ(L_0, L_{\ -1}, L_{\ -2}, \ldots, L_1) \quad q \quad q \\ &= circ(q \ L_0 \ q, q \ L_1 \ q, \ldots, q \ L_{\ -1} \ q) \end{aligned}$$

Proof. From equation (14) in Remark 16, it is obtained the equality:

$$a \ (u,v) = q \ L \ q \ .$$

Then,

$$\begin{aligned} A(u,v) &= circ(a_0(u,v), a_1(u,v), \ldots, a_{\ -1}(u,v)) \\ &= circ(\ L_0 \ q \quad , \ L_1 \ q \quad , \ldots, \ L_{\ -1} \ q \quad) \quad q \\ &= circ(L_0 \ q \ , L_1 \ q \ , \ldots, L_{\ -1} \ q \ ) \quad q \\ &= circ(L_0, L_{\ -1}, \ldots, L_1) \quad q \quad q \ . \end{aligned}$$

Therefore the statement is verified. □

From Theorem 13 and Theorem 17 the proof of the following result is clear.

Corollary 18. S $^{-1}_{=0}$ $m\,n$ n
A

$$\sigma\ A\ =\ \overset{-1}{\underset{=0}{}}\ \sigma\ S\ , \tag{15}$$

Proof. For $k = 0, 1, \ldots, m$ 1 let us consider the matrix L defined as in (13) by using S instead of S. Define the block matrix A as in (9) with the circulant blocks defined as in Theorem 17 by using L instead of L. From Theorem 13 the spectrum of A is given by the union in (11). Moreover, from Theorem 17 the (u, v)-th circulant block $A\ (u, v)$ of A is obtained. Since, this last circulant block and the original circulant block in the (u, v)-th position coincide, the union referred in (11) and in (15) coincide. Thus the matrix A has spectrum equal to the set in (15). □

The following example shows that for a given list, there exists a better splitting of this list such that there exists a matrix A that realizes it.

Example 19. $1, \omega_1, \omega_2$
$\omega_1 = \omega$ nd $\omega_2 = \omega^2$, w ω $m = 3.$
$4,\ 3,\ \frac{1}{2} + i, \frac{1}{2}\ \ i,\ \frac{1}{2} + i, \frac{1}{2}\ \ i$

$$4,\ 3\ \ \frac{1}{2} + i, \frac{1}{2}\ \ i\ \ \ \frac{1}{2} + i, \frac{1}{2}\ \ i\ \ .$$

$$S_0 = \begin{matrix} \frac{1}{2} & \frac{7}{2} \\ \frac{7}{2} & \frac{1}{2} \end{matrix}\ , \qquad 4,\ 3,$$

$$S_1 = \begin{matrix} \frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{matrix}\ , \qquad \frac{1}{2} + i, \frac{1}{2}\ \ i,$$

$$S_2 = \begin{matrix} \frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{matrix}\ , \qquad \frac{1}{2} + i, \frac{1}{2}\ \ i.$$

$\omega_1 = \frac{-1+i\sqrt{3}}{2}$ $\omega_2 = \frac{-1-i\sqrt{3}}{2}$,

$$\begin{aligned} 3L_0 &= S_0 + S_1 + S_2 = \begin{matrix} \frac{3}{2} & \frac{3}{2} \\ \frac{11}{2} & \frac{3}{2} \end{matrix} , \\ 3L_1 &= S_0 + \omega_2 S_1 + \omega_1 S_2 = \begin{matrix} 0 & \frac{9}{2} \\ \frac{5}{2} & 0 \end{matrix} , \\ 3L_2 &= S_0 + \omega_1 S_1 + \omega_2 S_2 = \begin{matrix} 0 & \frac{9}{2} \\ \frac{5}{2} & 0 \end{matrix} . \end{aligned}$$

$$A(1,1) = circ \quad \frac{1}{2}, 0, 0 \quad , \; A(1,2) = circ \quad \frac{1}{2}, \frac{3}{2}, \frac{3}{2} \quad ,$$
$$A(2,2) = circ \quad \frac{1}{2}, 0, 0 \quad , \; A(2,1) = circ \quad \frac{11}{6}, \frac{5}{6}, \frac{5}{6} \quad .$$

$$A = \begin{matrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} \\ \frac{11}{6} & \frac{5}{6} & \frac{5}{6} & \frac{1}{2} & 0 & 0 \\ \frac{5}{6} & \frac{11}{6} & \frac{5}{6} & 0 & \frac{1}{2} & 0 \\ \frac{5}{6} & \frac{5}{6} & \frac{11}{6} & 0 & 0 & \frac{1}{2} \end{matrix}$$

$$4, \quad 3, \quad \frac{1}{2} + i, \frac{1}{2} \quad i, \quad \frac{1}{2} + i, \frac{1}{2} \quad i \quad .$$

$$4, \frac{1}{2} + i, \frac{1}{2} \quad i \qquad\qquad 3, \; \frac{1}{2} + i, \frac{1}{2} \quad i \quad ,$$

re w ,

$$S_1 = \frac{1}{3} \quad \begin{matrix} 2 & \sqrt{3} - \frac{7}{2} & -\sqrt{3} - \frac{7}{2} \\ -\sqrt{3} - \frac{7}{2} & 2 & \sqrt{3} \quad \frac{7}{2} \\ \sqrt{3} \quad \frac{7}{2} & \sqrt{3} \quad \frac{7}{2} & 2 \end{matrix} \quad = \quad \begin{matrix} -\frac{2}{3} & \frac{\sqrt{3}}{3} \quad \frac{7}{6} & -\frac{\sqrt{3}}{3} \quad \frac{7}{6} \\ -\frac{\sqrt{3}}{3} \quad \frac{7}{6} & -\frac{2}{3} & \frac{\sqrt{3}}{3} \quad \frac{7}{6} \\ \frac{\sqrt{3}}{3} \quad \frac{7}{6} & \frac{\sqrt{3}}{3} \quad \frac{7}{6} & \frac{2}{3} \end{matrix}$$

$$\frac{1}{2} + i, \frac{1}{2} \quad i, \quad 3,$$

$$S_0 = \frac{1}{3} \quad \begin{matrix} 5 & \sqrt{3} + \frac{7}{2} & -\sqrt{3} + \frac{7}{2} \\ -\sqrt{3} + \frac{7}{2} & 5 & \sqrt{3} + \frac{7}{2} \\ \sqrt{3} + \frac{7}{2} & \sqrt{3} + \frac{7}{2} & 5 \end{matrix} \quad = \quad \begin{matrix} \frac{5}{3} & \frac{\sqrt{3}}{3} + \frac{7}{6} & \frac{7}{6} - \frac{\sqrt{3}}{3} \\ \frac{7}{6} - \frac{\sqrt{3}}{3} & \frac{5}{3} & \frac{\sqrt{3}}{3} + \frac{7}{6} \\ \frac{\sqrt{3}}{3} + \frac{7}{6} & \frac{7}{6} \quad \frac{\sqrt{3}}{3} & \frac{5}{3} \end{matrix}$$

$\frac{1}{2} + i, \frac{1}{2}$ $i, 4.$

$$S_0 + S_1 = \begin{matrix} 1 & \frac{2\sqrt{3}}{3} & -\frac{2\sqrt{3}}{3} \\ -\frac{2\sqrt{3}}{3} & 1 & \frac{2\sqrt{3}}{3} \\ \frac{2\sqrt{3}}{3} & \frac{2\sqrt{3}}{3} & 1 \end{matrix}$$

4. On structured matrices partitioned into circulant blocks matrices

In this section we search conditions to obtain a given structure on a partitioned into blocks matrix A with circulant blocks as in (9) using the structure of the matrices S in (12). Here, a matrix partitioned into blocks is called when all its row blocks (up to the first one) are permutations of precisely its first row block.

Theorem 20. A

S $k = 1, 2, \ldots, m$ $1.$

1. S $k = 0, 1, \ldots, m$ 1 A

2. A

   S $k = 0, 1, \ldots, m$ $1.$

3. A

   S $k = 0, 1, \ldots, m$

   $1.$

4. A

   S $k =$

   $0, 1, \ldots, m$ $1.$

Proof. Suppose that for all $\ell = 0, 1, \ldots, m$ 1 the matrices S are diagonal. We will prove that $A = (A(u, v))$ is a diagonal block matrix with circulant blocks. It is clear that for u $= v$, q S q $= 0$ therefore, for all $k = 0, 1, \ldots . m$ 1

$$q \quad L \quad q \quad = \frac{1}{} \sum_{=0}^{-1} \omega^{-} \quad q \quad S \quad q \quad = 0.$$

Then, for $u \;\; = v$, by Theorem 17, $A(u,v) = 0$. Thus A is a diagonal block matrix with circulant blocks.

Suppose that A is a block circulant matrix with circulant blocks, since the entries of S follow the distribution of the blocks of A the matrices S are circulant. Conversely, assume that for all $k = 0, 1, \ldots, m \;\; 1$, S is circulant, then for $k = 1, 2, \ldots, m$ the matrix L defined in (13) is also circulant. Let us suppose that

$$L \;\; = circ \;\; \varrho_0 \;\; , \varrho_1 \;\; , \ldots , \varrho_{(\;\; -1)} \;\; ,$$

then

$$q \;\; L \;\; q \;\; = \begin{matrix} \varrho_{(\;\; - \;\;)} & 1 \;\; u \;\; v \;\; n, \\ \varrho_{(\;\; - \;\; + \;\;)} & 1 \;\; v < u \;\; n; \end{matrix}$$

since

$$\begin{aligned} A(u,v) &= circ(q \;\; L_0 q \;\; , q \;\; L_1 q \;\; , \ldots , q \;\; L_{\;\; -1} q \;\;) \\ &= \begin{matrix} circ \;\; \varrho_{(\;\; - \;\;)0}, \varrho_{(\;\; - \;\;)1}, \ldots , \varrho_{(\;\; - \;\;)(\;\; -1)} & 1 \;\; u \;\; v \;\; n, \\ circ \;\; \varrho_{(\;\; - \;\; + \;\;)0}, \varrho_{(\;\; - \;\; + \;\;)1}, \ldots , \varrho_{(\;\; - \;\; + \;\;)(\;\; -1)} & 1 \;\; v < u \;\; n. \end{matrix} \end{aligned}$$

Thus the matrix $A = (A(u,v))$ partitioned into blocks is block circulant.

Let us suppose now that A is a block permutative matrix. Then there exists a permutation vector

$$\nu = (\nu_0, \nu_1, \ldots, \nu_{\;\; -1}) \tag{16}$$

with $\nu_0 = id$, such that

$$A = \begin{matrix} A(1,1) & A(1,2) & \ldots & A(1,n) \\ A(1,\nu_1(1)) & A(1,\nu_1(2)) & \ldots & A(1,\nu_1(n)) \\ \vdots & \vdots & \ddots & \vdots \\ A(1,\nu_{\;\; -1}(1)) & \ldots & \ldots & A(1,\nu_{\;\; -1}(n)) \end{matrix} \;\; .$$

Thus

$$S \;\; = (s \;\; (i,j)) = \begin{matrix} s \;\; (1,1) & s \;\; (1,2) & \ldots & s \;\; (1,n) \\ s \;\; (1,\nu_1(1)) & s \;\; (1,\nu_1(2)) & \ldots & s \;\; (1,\nu_1(n)) \\ \vdots & \vdots & \ddots & \vdots \\ s \;\; (1,\nu_{\;\; -1}(1)) & \ldots & \ldots & s \;\; (1,\nu_{\;\; -1}(n)) \end{matrix}$$

this means that the set of matrices S are permutatively equivalent. Conversely, if S are permutatively equivalent matrices for all $k = 0, 1, \ldots, m \;\; 1$, then the matrix L defined in (13) is also permutative. Let us suppose that there exist a vector $b = (b_1, b_2, \ldots, b)$ and a permutation vector as in (16) such that

$$L = \begin{pmatrix} b_1 & b_2 & \ldots & b \\ b_{(1),} & b_{(2),} & \ldots & b_{(\,),} \\ \vdots & \vdots & \ddots & \vdots \\ b_{(1),} & b_{(2),} & \ldots & b_{(\,),} \end{pmatrix}.$$

Then

$$q\ L\ q = b_{(\,),\ ,}$$

since

$$\begin{aligned} A(u,v) &= circ(q\ L_0 q\ , q\ L_1 q\ , \ldots, q\ L_{-1} q\ ) \\ &= circ\ b_{(\,),0}, b_{(\,),1}, \ldots, b_{(\,),\ -1} \ . \end{aligned}$$

From now on, in order to simplify the notation and unless we say the contrary, it is written $(a_0, a_1, \ldots, a_{-1})$ instead of $circ(a_0, a_1, \ldots, a_{-1})$. Thus, A can be constructed as follows:

Therefore, A is a block permutative matrix with circulant blocks.

In order to prove Item 4., suppose that A is symmetric partitioned into circulant block matrices. Since the entries of S follow the distribution of the blocks of A then S is symmetric, for all $k = 0, 1, \ldots, m\ \ 1$. Conversely, assume that for all $k = 0, 1, \ldots, m\ \ 1$, S is a symmetric real matrix. Since for all $k = 0, 1, \ldots, m\ \ 1$, $s\ (u,v)$ is an eigenvalue of $A(u,v)$, then the circulant matrices $A(u,v)$ have only real eigenvalues, so they should be symmetric (see in [29]). Moreover, for all $k = 0, 1, \ldots, m\ \ 1$, the L is the matrix defined in (13) and is symmetric. Suppose that

$$L = \begin{pmatrix} \varrho\ (1,1) & \varrho\ (1,2) & \ldots & \varrho\ (1,n) \\ \varrho\ (1,2) & \varrho\ (2,2) & \ldots & \varrho\ (2,n) \\ \vdots & \vdots & \ddots & \vdots \\ \varrho\ (1,n) & \ldots & \ldots & \varrho\ (n,n) \end{pmatrix}$$

then

$$
\begin{aligned}
A(v,u) &= circ(q \;\; L_0 q \;\; , q \;\; L_1 q \;\; , \ldots , q \;\; L \;\; _{-1} q \;\;) \\
&= circ\left(\varrho_0 (v,u), \varrho_1 (v,u), \ldots, \varrho \;\; _{-1} (v,u)\right) \\
&= circ\left(\varrho_0 (u,v), \varrho_1 (u,v), \ldots, \varrho \;\; _{-1} (u,v)\right) \\
&= A(u,v).
\end{aligned}
$$

Therefore,

$$A \quad = (A(u,v)) \quad = \quad A(v,u) \qquad = (A(v,u)) = A.$$

Thus, the matrix A is symmetric. □

5. An inverse problem related to block circulant matrices with circulant blocks

In this section we study the Guo index for some structured matrices. Namely, we dedicate our attention to block circulant matrices with circulant blocks, which are a type of permutative matrices. To this purpose we rise to the following inverse problem.

Problem 21. $q_1, q_2, \ldots, q$ $E = (\varepsilon \quad)$ n
m E E
ε_{11}
E
E re $S \quad _0$
$\ell = 1, \ldots, \frac{m}{2}$

$$Eq \quad _{+1} = \overline{Eq \quad _{- \quad +1}} \tag{17}$$

$(m \quad \ell + 1)$ E res
$(\ell + 1)$ E $\ell = 1, \ldots, \frac{m}{2}$.
$m = 2h$ $Eq \quad _{+1}$ E

A

E

In order to give a response to this question one needs to introduce the following DFT matrix

$$G = \frac{1}{\sqrt{n}} \begin{matrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & \tau & \tau^2 & \dots & \tau^{\ -2} & \tau^{\ -1} \\ 1 & \tau^2 & \tau^4 & \dots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \tau^{\ -1} & \tau^{2(\ -1)} & \dots & \dots & \tau^{(\ -1)} \end{matrix} .$$

where

$$\tau = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n} . \tag{18}$$

Moreover, it is necessary to define .

Definition 22. $\Lambda = (\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{\ -1})$ Λ

1. $\lambda_0 = \rho = \max \ \lambda \ : j = 1, 2, \dots, n$
2. $\lambda_{\ -\ } = \overline{\lambda}$ $k = 1, 2, \dots, n \ 1.$

The following result gives the Guo's Index for circulant matrices.

Theorem 23. $\Lambda = (\lambda_0, \lambda_1, \dots, \lambda_{\ -1})$

$$= \ \alpha \quad : \alpha(0) = 0 \ \mathit{and} \ \alpha(n \quad k) = n \quad \alpha(k), k \quad 1, 2, \dots, n \quad 1 \quad .$$

Λ

$$\lambda_0 \quad \min_{\in \mathcal{P}} \max_{0 \leq \ \leq 2} \quad \begin{matrix} 2 \ {}_{=1} \ \mathrm{Re}\lambda_{\ (j)} \cos \frac{2kj\pi}{2m+1} \\ 2 \ {}_{=1} \ \mathrm{Im}\lambda_{\ (j)} \sin \frac{2kj\pi}{2m+1} \end{matrix} \tag{19}$$

$n = 2m \quad 1$

$$\lambda_0 \quad \min_{\in \mathcal{P}} \max_{0 \leq \ \leq 2 \ +1} \quad \begin{matrix} 2 \ {}_{=1}^{-1} \ \mathrm{Re}\lambda_{\ (j)} \cos \frac{2kj\pi}{m+1} \quad (\ 1)^{\ k} \lambda_m \\ 2 \ {}_{=1}^{-1} \ \mathrm{Im}\lambda_{\ (j)} \sin \frac{2kj\pi}{m+1} \end{matrix} \tag{20}$$

$n = 2m \quad 2$

When the matrix is block circulant with circulant blocks we have the following result.

Theorem 24. $E = (\varepsilon \quad)$ n m

$S \quad _0$.

ε_{11} E E *lut*

E $\ell = 1, \ldots, \frac{m}{2}$

$$Eq \quad _{+1} = \overline{Eq \quad _{- \quad +1}}. \tag{21}$$

$$\varepsilon_{11} \quad \Phi \tag{22}$$

$$\Phi = \max_{\in\{0 \quad \}} \sum_{=1}^{-1} \varepsilon_{(\ +1)1} \tau^{-} + \sum_{=1}^{-1} \omega^{-} \varepsilon_{1(\ +1)} + \sum_{=1}^{-1} \sum_{=1}^{-1} \varepsilon_{(\ +1)(\ +1)} \omega^{-} \tau^{-}$$

$k = 0, 1, \ldots, m \quad 1,$ E A

Proof. By the conditions of the statement there exists a nonnegative circulant matrix

$$S_0 := circ \ s_{00}, s_{10}, \ldots, s_{(\ -1)0}$$

whose spectrum is Eq^1 (the set of the entries in Eq^1). The condition in (21) implies that for $\ell = 1, \ldots, \frac{m}{2}$ the circulant matrices $S \ _{+1}$ and $S \ _{-}$ whose spectrum are $Eq_{(\ +1)}$ and $Eq_{(\ - \ +1)}$, respectively, are related by $S^*_{\ +1} = S \ _{- \ +1}$. For $\ell = 1, \ldots, m \quad 1$, suppose that

$$S \ = circ\,(s\,(\ell))\,, \text{ with } s\,(\ell) = \ s_{0} , s_{1} , \ldots, s_{(\ -1)} \quad ,$$

where

$$s\,(\ell) \quad = \frac{1}{\sqrt{n}} G^* Eq \ . \tag{23}$$

The entries of the n-by-n nonnegative circulant matrices can be obtained, using equation (5) and the entries of the sums

$$L \ = \frac{1}{m} S_0 + \frac{1}{m} \sum_{=1}^{-1} S\,\omega^{-} \quad , \tag{24}$$

$k = 0, 1, \ldots, m \quad 1$. Recalling that the linear combination of circulant matrices are circulant, [24], then the matrices L in (24) are circulant. Suppose that

$$L \quad = circ\,(a_0(k), \ldots, a_{\ -1}(k))\,.$$

From (24), for $j = 0, 1, \ldots, n \quad 1$ the following holds:

$$a\ (k) \quad = \quad \frac{1}{m} \quad s_{\ 0} + \sum_{=1}^{-1} \omega^{-} \quad s \quad .$$

Using (23), we have

$$s \quad = \quad \frac{1}{n}[\varepsilon_{1(\ +1)} + \sum_{=1}^{-1} \varepsilon_{(\ +1)(\ +1)}\tau^{-} \quad],$$

for $j = 0, 1, \ldots, n \quad 1$.

On the other hand, from the expression of $a\ (k)$ and its nonnegativity condition we can write:

$$0 \quad (\ ) \quad = \quad \frac{1}{} \quad {}_0 + \sum_{=1}^{-1} \quad {}^{-}$$

$$= \quad \frac{1}{}\frac{1}{} \quad [\ {}_{11} + \sum_{=1}^{-1} \quad {}_{(\ +1)1} \quad {}^{-} \quad] + \sum_{=1}^{-1} \quad {}^{-} \quad [\ {}_{1(\ +1)} + \sum_{=1}^{-1} \quad {}_{(\ +1)(\ +1)} \quad {}^{-} \quad]$$

$$= \quad \frac{1}{} \quad {}_{11} + \sum_{=1}^{-1} \quad {}_{(\ +1)1} \quad {}^{-} \quad + \sum_{=1}^{-1} \quad {}^{-} \quad {}_{1(\ +1)} + \sum_{=1}^{-1}\sum_{=1}^{-1} \quad {}_{(\ +1)(\ +1)} \quad {}^{-} \quad {}^{-}$$

for all $\quad = 0 \ 1 \qquad 1$ Therefore, the last condition implies the inequality in (22). □

Now, one can formulate the following question. Under which conditions the multiset E formed with the entries of an n-by-m matrix $E = (\varepsilon \quad)$ as in Problem 21 is the spectrum of a nonnegative block matrix with circulant blocks. Let us consider the set

$$= \quad f : \ E \qquad E \ \ : f \ is\ bijective$$

Definition 25. $\quad E \qquad f \quad E\,(f) = (f\,(\varepsilon \quad)) \qquad E$

E $f(\varepsilon_{11})$ $f(\varepsilon\ ),$ $E(f)$ $(m\ \ \ell+1)$ $E(f)$ $(\ell+1)$ $E(f)$

$$E(f)\,q_{(\ \ -\ \ +1)} = \overline{E(f)}q_{(\ \ +1)},$$

$\ell = 1, \dots, \frac{m}{2}$.

For instance:

1. The identity function of E into E is clearly E-NNSS.
2. $f:\ E\ \ \ E$ defined by

$$f(\varepsilon\ ) = \begin{cases} \varepsilon & j\ \ = 2 \text{ and } j\ \ = m, \\ \varepsilon & j = 2, \\ \varepsilon_{\ 2} & j = m; \end{cases}$$

   is E-NNSS.
3. $f:\ E\ \ \ E$ defined by

$$f(\varepsilon\ ) = \overline{\varepsilon} \quad \text{for all}\, i, j$$

   is E-NNSS.
4. $f:\ E\ \ \ E$ defined by

$$f(\varepsilon\ ) = \begin{cases} \overline{\varepsilon} & \text{if } j = 1, \\ \varepsilon & \text{if } j\ \ = 1; \end{cases}$$

   is E-NNSS.

We denote by * the subset of formed by all E-NNSS bijections of E. Note that if f P^* then $f(\varepsilon_{11}) = \varepsilon_{11}$.

Theorem 26. $E = (\varepsilon\ )$ n m $S\ _0,$ E E ε_{11} ut E $\ell = 1, \dots, \frac{m}{2}$

$$Eq_{\ +1} = \overline{Eq_{\ -\ +1}}. \tag{25}$$

E A

$$\varepsilon_{11} \quad \min_{\in} \max_{\in\{0 \quad\}} \Theta, \tag{26}$$

$$\Theta = \sum_{=1}^{-1} f(\varepsilon_{(\ +1)1})\tau^{-} + \sum_{=1}^{-1} \omega^{-} f(\varepsilon_{1(\ +1)}) + \sum_{=1}^{-1}\sum_{=1}^{-1} f(\varepsilon_{(\ +1)(\ +1)})\omega^{-} \tau^{-}$$

Proof. Following the same steps of the above proof this time replacing ε by $f(\varepsilon\ )$ we arrive at the inequality in (22). After taking the minimum when the function f vary into *, the inequality (26) is obtained. □

Example 27.

$$E_4 = \begin{matrix} 4 & 1+i & 1 & i \\ 1 & i & & i \end{matrix}$$

$$S_0 = \begin{matrix} 1.5 & 2.5 \\ 2.5 & 1.5 \end{matrix}$$

$$S_1 = \begin{matrix} 0.5+i & 0.5 \\ 0.5 & 0.5+i \end{matrix}$$

$$S_2 = \begin{matrix} 0.5 \quad i & 0.5 \\ 0.5 & 0.5 \quad i \end{matrix}$$

$$L_0 = \begin{matrix} 0.1667 & 0.5 \\ 0.5 & 0.1667 \end{matrix}$$

$$L_1 = \begin{matrix} 1.2440 & 1 \\ 1 & 1.2440 \end{matrix}$$

$$L_2 = \begin{matrix} 0.0893 & 1 \\ 1 & 0.0893 \end{matrix}$$

$$E_3 = \begin{matrix} 3 & 1+i & 1 \quad i \\ 1 & i & i \end{matrix}$$

$$S_0 = \begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix}$$

$$S_1 = \begin{matrix} 0.5 & 0.5+i \\ 0.5+i & 0.5 \end{matrix}$$

$$S_2 = \begin{matrix} 0.5 & 0.5 \quad i \\ 0.5 \quad i & 0.5 \end{matrix}$$

$$L_0 = \begin{matrix} 0 & 0.3333 \\ 0.3333 & 0 \end{matrix}$$

$$L_1 = \begin{matrix} 0.5 & 1.4107 \\ 1.4107 & 0.5 \end{matrix}$$

$$L_2 = \begin{matrix} 0.5 & 0.2560 \\ 0.2560 & 0.5 \end{matrix}$$

$E \quad {}_3, 3$

Acknowledgments. The authors would like to thank the referee for his/her constructive suggestions that improved the final version of this paper. Enide Andrade was supported in part by the Portuguese Foundation for Science and Technology (FCT-Fundação para a Ciência e a Tecnologia), through CIDMA - Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2013. Moreover, Enide Andrade also thanks the project from University of Antofagasta UA INI-17-02. Hans Nina is supported by the projects UA INI-17-02 and FONDECYT 11170389. M. Robbiano was partially supported by project VRIDT UCN 170403003. M. Robbiano also thanks the support of CIDMA - Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2013 and the hospitality of the Mathematics Department of the University of Aveiro, Portugal, where this research was initiated.

[1] K. Andrews, S. Dolinar, J. Thorpe, IEEE Xplore, Conference: Information Theory, 2005. ISIT 2005. Proc. International Symposium on Information Theory and Its applications, 2005, DOI 10.1109/ISIT.2005.1523758.

[2] J. Baker, F. Hiergeist, G. E. Trapp, The structure of multi-blocks circulant, Kyungpook Math. J. 25 (1985): 71-75.

[3] A. Berman, R. J. Plemmons, Nonnegative matrices in the Mathematical Sciences, SIAM Publications, Philadelphia, 1994.

[4] A. Borobia. On nonnegative eigenvalue problem, Lin. Algebra Appl. 223/224 (1995): 131-140, Special Issue honoring Miroslav Fiedler and Vlastimil Pták.

[5] M. Boyle, D. Handelman, The spectra of nonnegative matrices via symbolic dynamics, Ann. of Math. 133, 2 (1991): 249-316.

[6] C. - Y. Chao, A remark on symmetric circulant matrices, Linear Algebra Appl. 103 (1988): 133-148.

[7] P.J. Davis, Circulant matrices, John Wiley & Sons, New York, Chichester, Brisbane, Toronto (1979).

[8] S. Friedland, On an inverse problem for nonnegative and eventually nonnegative matrices, Israel T. Math. 1, 29 (1978): 43-60.

[9] M. Fiedler, Eigenvalues of nonnegative symmetric matrices, Lin. Algebra Appl. 9 (1974): 119-142.

[10] S. D. Georgious, E. Lappas, Self-dual codes from circulant matrices. Des. Codes Cryptogr. 64 (2012): 129-141.

[11] W. Guo, Eigenvalues of nonnegative matrix, Linear Algebra and its Applications 266 (1997): 261–270.

[12] H. Horn, C. R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1991.

[13] C. R. Johnson, C. Marijuán, P. Paparella, M. Pisonero, The NIEP, https://arXiv:1703.10992v2 [math.SP], 2017.

[14] C. R. Johnson, Row stochastic matrices similar to doubly stochastic matrices, Lin. and Multilin. Algebra 2 (1981): 113-130.

[15] C. Johnson, T. Laffey, R. Loewy, The real and symmetric nonnegative inverse eigenvalue problems are different, Proc. Amer. Math Soc.,12, 124 (1996): 3647-3651.

[16] R. D. Kangwai, S. D. Guest, S. Pellegrino, An introduction to the analysis of symmetric structures, Computers and Structures 71 (1999): 671–688.

[17] M. Karlin, New binary coding results by circulant, IEEE Trans. Inform. Theory, 15, (1969): 81-92.

[18] H. Karner, J. Schneid, C. W. Ueberhuber, Spectral decomposition of real circulant matrices. Lin. Algebra Appl. 367 (2003): 301-311.

[19] T. Laffey, Extreme nonnegative matrices, Lin. Algebra Appl. 275/276 (1998): 349-357. Proceedings of the sixth conference of the international Linear Algebra Society (Chemnitz, 1996).

[20] T. Laffey, Realizing matrices in the nonnegative inverse eigenvalue problem, Matrices and group representations (Coimbra, 1998), Textos Mat. Sér. B, 19, Univ. Coimbra, Coimbra, (1999): 21-31.

[21] T. Laffey, H. Šmigoc, Nonnegative realization of spectra having negative real parts, Linear Algebra Appl., 384 (2004): 199–206.

[22] R. Loewy, D. London, A note on an inverse problem for nonnegative matrices, Lin. and Multilin. Algebra 6, 1 (1978/79): 83-90.

[23] R. Loewy, J. J. Mc Donald, The symmetric nonnegative inverse eigenvalue problem for 5 5 matrices, Linear Algebra Appl. 393 (2004): 275-298.

[24] C. Manzaneda, E. Andrade, M. Robbiano, Realizable lists via the spectra of structured matrices, Lin. Algebra Appl. 534 (2017): 51-72.

[25] J. Mayo Torre, M. R. Abril, E. Alarcia Estévez, C. Marijuán, M. Pisonero, The nonnegative inverse problem from the coeficientes of the characteristic polynomial EBL digraphs, Linear Algebra Appl. 426 (2007): 729-773.

[26] M. E. Meehan, Some results on matrix spectra, Phd thesis, National University of Ireland, Dublin, 1998.

[27] P. Paparella, Realizing Suleĭmanova-type spectra via permutative matrices, Electron. J. Linear Algebra, 31 (2016): 306-312.

[28] H. Perfect, On positive stochastic matrices with real characteristic roots, Proc. Cambridge Philos. Soc. 48 (1952): 271-276.

[29] O. Rojo, R. L. Soto, Guo perturbations for symmetric nonnegative circulant matrices, Linear Algebra Appl. 431 (2009): 594-607.

[30] O. Rojo, R. L. Soto, Applications of a Brauer Theorem in the nonnegative inverse eigenvalue problem, Linear Algebra Appl. 416 (2007): 1-18.

[31] O. Rojo, R. L. Soto, Existence and construction of nonnegative matrices with complex spectrum, Linear Algebra Appl. 368 (2003): 53–69.

[32] R. L. Smith, Moore- Penrose Inverses of block circulant and block k-circulant matrices, Linear Algebra Appl. 16 (1977): 237-245.

[33] R. Soto, O. Rojo, C. Manzaneda, On the nonnegative realization of partitioned spectra, Electron. J. Linear Algebra, 22 (2011): 557-572.

[34] G. W. Soules, Constructing symmetric nonnegative matrices, Linear and Multilin. Algebra 13, 3 (1983): 241-251.

[35] H. Šmigoc, The inverse eigenvalue problem for nonnegative matrices, Linear Algebra Appl. 393 (2004): 365-374.

[36] H. Šmigoc, Construction of nonnegative matrices and the inverse eigenvalue problem, Lin. and Multilin. Algebra 53, 2 (2005): 85-96.

[37] H. R. Suleĭmanova, Stochastic matrices with real characteristic numbers, Doklady, Akad. Nuk SSSR (N. S.) 66 (1949): 343-345.

[38] G. E. Trapp, Inverses of circulant matrices and block circulant matrices, Kyungpook Math. J. 13 (1973) 11-20.

[39] J. Williamson, The latent roots a matrix of special type, Bull. Am. Math. Soc. 37, (1931), 585-590.

[40] F. Zhang, Matrix Theory, Basic Results and Techniques. Second Edition. Universitext. Springer. http://www.springer.com/series/223.