SOME LOG-MAJORIZATIONS AND AN EXTENSION OF A DETERMINANTAL INEQUALITY

RUTE LEMOS AND GRAÇA SOARES

ABSTRACT. An eigenvalue inequality involving a matrix connection and its dual is established, and some log-majorization type results are obtained. In particular, some eigenvalues inequalities considered by F. Hiai and M. Lin [9], an associated conjecture, and a singular values inequality by L. Zou [20] are revisited. A reformulation of the inequality $\det(A + U^*B) \leq \det(A + B)$, for positive semidefinite matrices A, B , with U a unitary matrix that appears in the polar decomposition of BA , is also extended, using some known norm inequalities, associated to Furuta inequality and Araki-Cordes inequality.

1. INTRODUCTION

Let $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ be vectors with the components sorted in non-increasing order, that is, $x_1 \geq \dots \geq x_n$ and $y_1 \geq \dots \geq y_n$. We say that $\mathbf{y}$ *weakly majorizes* $\mathbf{x}$ and write $\mathbf{x} \prec_w \mathbf{y}$ if

$$(1) \quad \sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i, \quad k = 1, \dots, n.$$

If $\mathbf{x} \prec_w \mathbf{y}$ and equality holds in (1) for $k = n$, we say that $\mathbf{y}$ *majorizes* $\mathbf{x}$, denoted by $\mathbf{x} \prec \mathbf{y}$. For $\mathbf{x}, \mathbf{y}$ with nonnegative components, we write $\mathbf{x} \prec_{\log} \mathbf{y}$ if $\mathbf{y}$ *log-majorizes* $\mathbf{x}$, that is,

$$(2) \quad \prod_{i=1}^k x_i \leq \prod_{i=1}^k y_i, \quad k = 1, \dots, n,$$

with equality occurring in (2) when $k = n$.

For any real valued function f defined on an interval, containing all the components of the real vector $\mathbf{x}$, we adopt the notation $f(\mathbf{x}) = (f(x_1), \dots, f(x_n))$. If all the components of $\mathbf{x}, \mathbf{y}$ are positive, then $\mathbf{x} \prec_{\log} \mathbf{y}$ if and only if $\log \mathbf{x} \prec \log \mathbf{y}$, this justifying the log-majorization terminology. If f is convex, then $\mathbf{x} \prec \mathbf{y}$ implies $f(\mathbf{x}) \prec_w f(\mathbf{y})$. In particular, the log-majorization implies the weak majorization. Additionally, if f is an increasing and convex function, then $\mathbf{x} \prec_w \mathbf{y}$ implies $f(\mathbf{x}) \prec_w f(\mathbf{y})$. For instance, $f(t) = \ln(1 + e^t)$ is a strictly increasing and convex function on $(0, +\infty)$. Two important resources on the topic of majorization are [2, 15].

Let M_n be the algebra of $n \times n$ complex matrices and I be the identity matrix of order n . For $A \in M_n$ with real eigenvalues, we denote by $\lambda(A)$ the n -tuple of eigenvalues of A arranged as follows $\lambda_1(A) \geq \dots \geq \lambda_n(A)$. If $A, B \in M_n$, then AB and BA have the same eigenvalues, including multiplicities [11, Theorem 1.3.20], hence $\lambda(AB) = \lambda(BA)$.

For simplicity of notation, if $A, B \in M_n$ have real eigenvalues, then we write $A \prec_w B$ whenever $\lambda(A) \prec_w \lambda(B)$; moreover, if $A, B \in M_n$ have nonnegative eigenvalues, we write $A \prec_{\log} B$ when $\lambda(A) \prec_{\log} \lambda(B)$. Majorization is a powerful tool for establishing determinantal and matrix norm inequalities. In particular, if $A \prec_{\log} B$, then $\det(I + A) \leq \det(I + B)$. On the other hand, some classical determinantal inequalities can find their majorization counterparts.

Key words and phrases. Log-majorization; matrix connections and means; eigenvalues; singular values; trace and determinantal inequalities; Löwner-Heinz inequality; Furuta inequality; Araki-Cordes inequality

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For $A \in M_n$, the unique positive semidefinite square root of A^*A is denoted by $|A|$. For $A, B \in M_n$, Ky Fan Dominance Theorem [15] asserts that $|A| \prec_w |B|$ if and only if $\|A\| \leq \|B\|$ holds for any unitarily invariant norm $\|\cdot\|$ in M_n . We recall that a norm $\|\cdot\|$ is said to be *unitarily invariant* in M_n if $\|UAV\| = \|A\|$ for all $A \in M_n$ and all unitary matrices $U, V \in M_n$. Considering the singular values of $A \in M_n$, that is, the eigenvalues of $|A|$, ordered as follows $s_1(A) \geq \cdots \geq s_n(A)$, the Ky Fan k -norms of A defined by

$$\|A\|_{(k)} = \sum_{i=1}^k s_i(A), \quad 1 \leq k \leq n,$$

including the spectral (or operator) norm $\|A\|$, when $k = 1$, are examples of unitarily invariant norms in M_n .

As usual, $A \geq B$ means that $A, B \in M_n$ are Hermitian and $A - B$ is positive semidefinite; $A > 0$ means that $A \in M_n$ is Hermitian and positive definite. Let $0 \leq \alpha \leq 1$. The famous *Löwner-Heinz inequality* [14] states that $A \geq B \geq 0$ implies $A^\alpha \geq B^\alpha$. Kubo and Ando [12] introduced the α -power mean of positive semidefinite matrices A, B as

$$A \sharp_\alpha B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^\alpha A^{\frac{1}{2}},$$

when A is invertible, and extended to any non-invertible A by continuity as follows:

$$A \sharp_\alpha B = \lim_{\epsilon \rightarrow 0^+} (A + \epsilon I) \sharp_\alpha B.$$

If $AB = BA$, then $A \sharp_\alpha B = A^{1-\alpha} B^\alpha$. In general, $A \sharp_{1-\alpha} B = B \sharp_\alpha A$ and $(A, B) \mapsto A \sharp_\alpha B$ is jointly monotone, as a consequence of Löwner-Heinz inequality. In particular, $\sharp = \sharp_{1/2}$ denotes the *geometric mean*. We recall that $A \sharp B$ is the unique positive solution of the Riccati equation $XA^{-1}X = B$, also characterized by Pusz and Woronowicz [18] as

$$A \sharp B = \max \left\{ X \geq 0 : \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0 \right\}.$$

This note is organized as follows. In Section 2, recalling the Kubo-Ando axiomatic theory of matrix connections [12], an eigenvalue inequality is established, involving a matrix connection and its dual. In Section 3, the antisymmetric tensor power technique is used to prove some log-majorizations. As a consequence, previous known results are revisited. In particular, an eigenvalues inequality, involving the α -power mean, considered by F. Hiai and M. Lin [9, Theorem 2.5], as well as a singular value inequality for the geometric mean due to L. Zou [20, Theorem 2.10] are reobtained. A conjecture, considering eigenvalues replaced by singular values, in the same spirit of the one presented in [9, Conjecture 2.6] is also raised. In Section 4, a reformulation of the determinantal inequality

$$\det(A + U^*B) \leq \det(A + B),$$

for positive semidefinite matrices A, B , where U is a unitary matrix that appears at the polar decomposition of BA , formulated by K. M. R. Audenaert [1], when comparing geodesics induced by different metrics, and further complemented by M. Lin [13], is extended. Such an extension is obtained, using the interplay between majorization relations and determinantal inequalities. The main tools are a norm inequality [17, Theorem 1], which is a simultaneous extension of Araki-Cordes inequality [5] and Bebiano-Lemos-Providência inequality [4], and its reverse [17, Theorem 2], as well as another inequality for unitarily invariant norms obtained via log-majorization [7] from Furuta inequality [6].

2. EIGENVALUE INEQUALITY FOR MATRIX CONNECTIONS

The axiomatic theory of connections and means for pairs of positive operators was developed by F. Kubo and T. Ando [12]. A binary operation σ on the cone of $n \times n$ positive semidefinite matrices, satisfying for all $A, B, C, D \geq 0$ the following conditions:

C1. (joint monotonicity) $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$;

C2. (transformer inequality) $X^*(A\sigma B)X \leq (X^*AX)\sigma(X^*BX)$ for $X \in M_n$;

C3. (joint continuity from above) for $A_n, B_n \geq 0$, if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n\sigma B_n \downarrow A\sigma B$

is called a (matrix) *connection*. A (matrix) *mean* is a connection σ , satisfying $I\sigma I = I$.

For each connection σ , there exists a unique operator monotone function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, such that $f(t)I = I\sigma(tI)$, $t \in \mathbb{R}^+$. Such function f is called the *representing function* of σ . The formula

$$A\sigma B = A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$$

holds for $A > 0, B \geq 0$, where the right hand side is defined via the analytic functional calculus, and it can be extended to $A \geq 0$ by continuity as follows:

$$A\sigma B = \lim_{\epsilon \rightarrow 0^+} (A + \epsilon I)\sigma B.$$

The *dual* of a connection σ is the connection $\sigma^\perp$ defined for $A, B > 0$ by

$$A\sigma^\perp B = (B^{-1}\sigma A^{-1})^{-1}$$

and extended by continuity to $A, B \geq 0$ as usual. Since $tf(t)^{-1}$ is the representing function of $\sigma^\perp$, when the representing function of σ is $f(t)$, it is clear that

$$(3) \quad (A\sigma B)X(A\sigma^\perp B)X \quad \text{and} \quad AXBX$$

have the same determinant for any $X \in M_n$. If σ is the right trivial mean, then its dual is the left trivial mean and the matrices (3) are trivially equal. An easy consequence of the properties (C1)-(C3) of the definition of σ is the next inequality between the maximum eigenvalue of the matrices (3) for any $A, B, X \geq 0$.

Theorem 2.1. *Let $A, B, X \geq 0$ and σ be a connection. Then*

$$(4) \quad \lambda_1((A\sigma B)X(A\sigma^\perp B)X) \leq \lambda_1(AXBX).$$

Proof. Firstly, let $A, B > 0$. If $X = I$, then we only need to show that $\lambda_1(AB) \leq 1$ implies

$$(5) \quad \lambda_1((A\sigma B)(A\sigma^\perp B)) \leq 1,$$

because both sides of (4) have the same order of homogeneity for A, B , so that we can multiply A, B by a positive scalar. Since A, B are invertible, from $\lambda_1(AB) \leq 1$, we have $A \leq B^{-1}$, as well as $B \leq A^{-1}$. By the joint monotonicity of σ , we get

$$A\sigma B \leq B^{-1}\sigma A^{-1} = (A\sigma^\perp B)^{-1},$$

consequently, (5) holds. Now, let $X > 0$. The transformer inequality (C2) becomes an equality, when X is invertible. Thus, denoting $X^{\frac{1}{2}}AX^{\frac{1}{2}}$ by A_X , we have

$$\lambda_1((A\sigma B)X(A\sigma^\perp B)X) = \lambda_1((A_X\sigma B_X)(A_X\sigma^\perp B_X)) \leq \lambda_1(A_X B_X) = \lambda_1(AXBX).$$

When A, B, X are not invertible, we may replace A, B, X in (4) by $A + \epsilon I, B + \epsilon I, X + \epsilon I > 0$, for $\epsilon > 0$, respectively, and then we use a continuity argument, letting $\epsilon \downarrow 0$. $\blacksquare$

Corollary 2.2. *Let $A, B \geq 0$ such that $A + B$ is invertible. Then*

$$\lambda_1((A + B)A(A + B)^{-1}B) \leq \lambda_1(AB),$$

occurring equality when $AB = BA$.

Proof. If $A, B, A + B$ are invertible and $A\sigma B = A + B$, then

$$A\sigma^\perp B = (A^{-1} + B^{-1})^{-1} = A(A + B)^{-1}B$$

and the required eigenvalue inequality readily follows from Theorem 2.1 for $X = I$. If $A, B \geq 0$ and either A or B is not invertible, then $A_\epsilon = A + \epsilon I$, $B_\epsilon = B + \epsilon I$, $\epsilon > 0$, are positive definite and the result is obtained by a continuity argument. If $AB = BA$, then $(A + B)A = A(A + B)$ or, equivalently,

$$(A + B)A(A + B)^{-1}B = AB,$$

that is, the eigenvalue inequality occurs as equality. ■

In general, the following eigenvalues inequalities do not hold

$$\lambda_i((A + B)A(A + B)^{-1}B) \leq \lambda_i(AB), \quad i = 2, \dots, n,$$

as the following counterexample shows.

Example Consider the positive definite matrices

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalues of $(A + B)A(A + B)^{-1}B$ are

$$7.9161, \quad 2.7485, \quad 0.7354,$$

while the eigenvalues of AB are

$$8.6185, \quad 2.6918, \quad 0.6897.$$

3. SOME LOG-MAJORIZATION RESULTS

For $k = 1, \dots, n$ and $n_k = \binom{n}{k}$, we denote the k th compound or k th antisymmetric tensor power of $A \in M_n$ by $A^{\wedge k}$, that is, the matrix in M_{n_k} with entries given by the minors $\det A(\mathbf{i}, \mathbf{j})$, where the index sets $\mathbf{i}, \mathbf{j} \subset \{1, \dots, n\}$ have cardinality k and are lexicographically ordered. As usual, $A(\mathbf{i}, \mathbf{j})$ denotes the submatrix of A that lies in rows and columns indexed, respectively, by $\mathbf{i}, \mathbf{j}$. We list some essential properties of these matrices [2] for $A, B \in M_n$:

P1. $(AB)^{\wedge k} = A^{\wedge k} B^{\wedge k}$ (Binet-Cauchy formula);

P2. $(A^{\wedge k})^r = (A^r)^{\wedge k}$, $r > 0$, and if A is invertible, then $(A^{\wedge k})^{-1} = (A^{-1})^{\wedge k}$.

Hence, any expression involving products and fractional matrix powers “commutes” with the k th antisymmetric tensor power. Moreover, $\lambda_i(A^{\wedge k}) = \prod_{j=1}^k \lambda_{i_j}(A)$, $1 \leq i_1 < \dots < i_k \leq n$ holds, so that

$$\mathbf{P3.} \quad \|A^{\wedge k}\| = s_1(A^{\wedge k}) = \prod_{i=1}^k s_i(A), \quad k = 1, \dots, n.$$

If $A, B \in M_n$ have nonnegative eigenvalues, it follows that

P4. $A \prec_{\log} B$ if and only if $\det A = \det B$ and $\lambda_1(A^{\wedge k}) \leq \lambda_1(B^{\wedge k})$, $k = 1, \dots, n$.

In this section, we illustrate the potential of using the antisymmetric tensor power technique to derive some log-majorizations and as a consequence some known results are revisited.

The next result is a corollary of Theorem 2.1.

Corollary 3.1. *If $A, B, X \geq 0$ and $\alpha \in [0, 1]$, then*

$$(6) \quad (A \sharp_{\alpha} B)X(A \sharp_{1-\alpha} B)X \prec_{\log} AXBX.$$

Proof. Theorem 2.1 holds with A, B, X replaced by their k th compounds $A^{\wedge k}, B^{\wedge k}, X^{\wedge k} \geq 0$ for any connection σ and each $k = 1, \dots, n$. In particular, if $\sigma = \sharp_{\alpha}$, then $\sigma^{\perp} = \sharp_{1-\alpha}$. By properties P1 and P2, we have

$$(A^{\wedge k} \sharp_{\alpha} B^{\wedge k})X^{\wedge k}(A^{\wedge k} \sharp_{1-\alpha} B^{\wedge k})X^{\wedge k} = ((A \sharp_{\alpha} B)X(A \sharp_{1-\alpha} B)X)^{\wedge k},$$

$$A^{\wedge k} X^{\wedge k} B^{\wedge k} X^{\wedge k} = (AXBX)^{\wedge k},$$

$k = 1, \dots, n$. Then the required log-majorization holds, recalling the equality between the determinants of the matrices in (6) and using P4. $\blacksquare$

The log-majorization (6) may be equivalently formulated, for $A, B, X \geq 0$ and $\alpha \in [0, 1]$, by

$$(7) \quad \prod_{i=k}^n \lambda_i((A \sharp_{\alpha} B)X(A \sharp_{1-\alpha} B)X) \geq \prod_{i=k}^n \lambda_i(AXBX), \quad k = 1, \dots, n,$$

with equality for $k = n$. When $X = I$, these inequalities are included in [9, Theorem 2.5] considered by F. Hiai and M. Lin.

For $s, t \in \mathbb{R}$ and $A, B > 0$, for simplicity of notation, we will consider

$$A \natural_{s,t} B = A^{\frac{s}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{s}{2}},$$

extended to $A, B \geq 0$ by continuity as usual. For $0 \leq \alpha \leq 1$, we have $A \natural_{1,\alpha} B = A \sharp_{\alpha} B$.

If $A, B \geq 0$, then Corollary 3.1 with $X = A^{\frac{s+r}{2}-1}$, $r, s \in \mathbb{R}$ and $t \in [0, 1]$ yields

$$(A \natural_{r,t} B)(A \natural_{s,1-t} B) \prec_{\log} A^{r+s-1} B.$$

It is natural to ask if eigenvalues may be replaced by singular values in the inequalities of the previous log-majorization as Hiai and Lin [9, Conjecture 2.6] did for (7) when $X = I$, proving it in case $t \in [\frac{1}{4}, \frac{3}{4}]$. In a similar way, the following conjecture can be formulated.

Conjecture. *If $A, B \geq 0$, $r, s \in \mathbb{R}$ and $0 \leq t \leq 1$, then*

$$|(A \natural_{r,t} B)(A \natural_{s,1-t} B)| \prec_{\log} |A^{r+s-1} B|.$$

We will prove the conjecture in a particular case, using the next proposition.

Proposition 3.2. *Let $A, B > 0$, $r, s \geq 0$ and $\frac{r}{r+s} \leq 2t \leq \frac{2r+s}{r+s}$. If $B \leq A^{1-r-s}$, then*

$$(8) \quad (A \natural_{s,1-t} B)^2 \leq (A \natural_{r,t} B)^{-2}.$$

Proof. Let $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Then

$$(A \natural_{r,t} B)(A \natural_{s,1-t} B) = A^{\frac{r}{2}} C^t A^{\frac{r+s}{2}} C^{1-t} A^{\frac{s}{2}}.$$

From $B \leq A^{1-r-s}$, we have $C \leq A^{-(r+s)}$ and $A^{r+s} \leq C^{-1}$. Using Löwner-Heinz inequality, under the hypothesis that $r, s \geq 0$ and $\frac{r}{r+s} \leq 2t \leq \frac{2r+s}{r+s}$, this two last inequalities imply

$$\begin{aligned}
(A \natural_{r,t} B)(A \natural_{s,1-t} B)^2(A \natural_{r,t} B) &= A^{\frac{r}{2}} C^t A^{\frac{r+s}{2}} C^{1-t} A^s C^{1-t} A^{\frac{r+s}{2}} C^t A^{\frac{r}{2}} \\
&\leq A^{\frac{r}{2}} C^t A^{\frac{r+s}{2}} C^{2(1-t)-\frac{s}{r+s}} A^{\frac{r+s}{2}} C^t A^{\frac{r}{2}}, & 0 \leq \frac{s}{r+s} \leq 1 \\
&\leq A^{\frac{r}{2}} C^t A^{2(r+s)(t-1)+s} C^t A^{\frac{r}{2}}, & 0 \leq 2(1-t) - \frac{s}{r+s} \leq 1 \\
&\leq A^{\frac{r}{2}} C^{\frac{r}{r+s}} A^{\frac{r}{2}}, & 0 \leq 2t - 1 + \frac{s}{r+s} \leq 1 \\
&\leq A^{\frac{r}{2}} A^{-r} A^{\frac{r}{2}}, & 0 \leq \frac{r}{r+s} \leq 1 \\
&= I.
\end{aligned}$$

Hence, (8) occurs. ■

Remark. It follows from the previous proposition that

$$(9) \quad \|(A \natural_{r,t} B)(A \natural_{s,1-t} B)\| \leq \|A^{r+s-1} B\|$$

holds for $r, s \geq 0$ and $\frac{r}{r+s} \leq 2t \leq 1 - \frac{s}{r+s}$. In fact, if $\|A^{r+s-1} B\| \leq 1$, then $B^2 \leq A^{2(1-r-s)}$. By Löwner-Heinz inequality, we have $B \leq A^{1-r-s}$, this implying

$$\|(A \natural_{r,t} B)(A \natural_{s,1-t} B)\| \leq 1,$$

by Proposition 3.2. Replacing the matrices in (9) by their k th compounds, by properties P1, P2 and P3, we may confirm the conjecture in that special case of r, s, t . In particular, if $r + s = 2$, then $\frac{r}{4} \leq t \leq 1 - \frac{s}{4}$. The case confirmed by Hiai and Lin occurs when $r = s = 1$.

Theorem 3.3. *If $A, B \geq 0$, then*

$$A(A \sharp B) B(A \sharp B) \prec_{\log} A^2 B^2.$$

Proof. Let $A, B \geq 0$. It is clear that $A(A \sharp B) B(A \sharp B)$ and $A^2 B^2$ have the same determinant. Assuming A, B invertible, let us prove that

$$(10) \quad \lambda_1(A(A \sharp B) B(A \sharp B)) \leq \lambda_1(A^2 B^2).$$

If $\lambda_1(A^2 B^2) \leq 1$, then $B^2 \leq A^{-2}$. By Löwner-Heinz inequality, we have $B \leq A^{-1}$. Therefore

$$(A \sharp B) B(A \sharp B) \leq (A \sharp B) A^{-1}(A \sharp B) = B \leq A^{-1},$$

because $(A \sharp B) A^{-1}(A \sharp B) = B$. We conclude that

$$\lambda_1(A(A \sharp B) B(A \sharp B)) \leq 1.$$

If A, B are not invertible, we may use a continuity argument, replacing A by $A + \epsilon I$ and B by $B + \epsilon I$ to obtain (10). By properties P1 and P2, we have $(A^{\wedge k})^2 (B^{\wedge k})^2 = (A^2 B^2)^{\wedge k}$ and

$$(A(A \sharp B) B(A \sharp B))^{\wedge k} = A^{\wedge k} (A^{\wedge k} \sharp B^{\wedge k}) B^{\wedge k} (A^{\wedge k} \sharp B^{\wedge k}).$$

Finally, the result follows from inequality (10) applied to the matrices $A^{\wedge k}, B^{\wedge k}$, $k = 1, \dots, n$, using property P4. ■

We remark that the singular values of $A^{\frac{1}{2}}(A \sharp B) B^{\frac{1}{2}}$ and AB coincide with the square roots of the eigenvalues of $A(A \sharp B) B(A \sharp B)$ and $A^2 B^2$, respectively. Hence, we have the following easy consequence of Theorem 3.3.

Corollary 3.4. *If $A, B \geq 0$, then $|A^{\frac{1}{2}}(A \sharp B) B^{\frac{1}{2}}| \prec_{\log} |AB|$.*

Proof. The log-majorization in Theorem 3.3 can be equivalently formulated as

$$B^{\frac{1}{2}}(A\sharp B)A(A\sharp B)B^{\frac{1}{2}} \prec_{\log} |AB|^2$$

and the matrix in the LHS of the previous log-majorization is $|A^{\frac{1}{2}}(A\sharp B)B^{\frac{1}{2}}|^2$. Now the result follows, because a log-majorization between two positive semidefinite matrices implies the corresponding log-majorization between the square roots of such matrices. ■

Corollary 3.4 contains the singular values inequalities

$$\prod_{i=1}^k s_i(A^{\frac{1}{2}}(A\sharp B)B^{\frac{1}{2}}) \leq \prod_{i=1}^k s_i(AB), \quad k = 1, \dots, n,$$

proved by L. Zou [20, Theorem 2.10], using a different approach.

Inspired by the previous result, it is natural to ask if the following log-majorization holds:

$$|A^\alpha(A\sharp_\alpha B)B^{1-\alpha}| \prec_{\log} |AB|,$$

for $A, B \geq 0$ and $\alpha \in [0, 1]$, being the case $\alpha = \frac{1}{2}$ presented in Corollary 3.4.

4. AN EXTENSION OF A DETERMINANTAL INEQUALITY

A. Matsumoto, R. Nakamoto and M. Fujii [17, Theorem 1] proved for $A, B \geq 0$ that

$$(11) \quad \|A^{\frac{s+t}{2}} B^t A^{\frac{s+t}{2}}\| \leq \|A^{\frac{s}{2}} (A^{\frac{r}{2}} B^r A^{\frac{r}{2}})^{\frac{t}{r}} A^{\frac{s}{2}}\|, \quad 0 \leq t \leq r, \quad s \geq 0,$$

which reduces to Araki-Cordes inequality [5] if $s = 0$ and to Bebiano-Lemos-Providência inequality [4, Theorem 2.1] if $s = 1$. When $0 \leq s \leq r \leq t$ and $r > 0$, they also proved that (11) holds with the reverse inequality sign [17, Theorem 2].

In this last section, we observe that it is easy to extend the determinantal inequality

$$(12) \quad \det(A^2 + |BA|) \leq \det(A^2 + AB)$$

formulated by K. M. R. Audenaert [1] for $A, B \geq 0$. This is a reformulation of the inequality

$$\det(A + U^* B) \leq \det(A + B),$$

where U is a unitary matrix that appears in the polar decomposition of BA for $A, B \geq 0$. According to Audenaert [1], this determinantal inequality has arisen in the study of interpolation methods for image processing in diffusion tensor imaging, when comparing geodesics induced by different metrics.

M. Lin [13] obtained a slightly more general inequality for $A, B \geq 0$, namely,

$$(13) \quad \det(A^2 + |BA|^t) \leq \det(A^2 + A^t B^t), \quad 0 \leq t \leq 2.$$

In order to further extend it, we recall the norm inequality

$$(14) \quad \|A^{\frac{c}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha A^{\frac{c}{2}}\| \leq \|A^{\frac{c-\alpha}{2}} B^\alpha A^{\frac{c-\alpha}{2}}\|, \quad 0 \leq \alpha \leq 1, \quad c \geq \alpha,$$

for $A > 0, B \geq 0$, obtained in [7, Corollary 3.1 (iii)], using Furuta inequality [6]. Replacing A, B by A^{-r}, B^r , respectively, considering $\alpha = \frac{t}{r}$ and $c = -\frac{s}{r}$ in (14) yields the reverse of (11) for $0 \leq t \leq r$ and $-s \geq t$. As a consequence, we have the following extension of the determinantal inequality (13).

Proposition 4.1. *Let $A > 0$ and $B \geq 0$. Then*

$$\det(A^{-s} + (A^{\frac{r}{2}} B^r A^{\frac{r}{2}})^{\frac{t}{r}}) \leq \det(A^{-s} + A^t B^t)$$

holds if either (i) $0 \leq t \leq r$ and $-s \geq t$ or (ii) $0 \leq s \leq r \leq t$ with $r > 0$; the reverse inequality holds if (iii) $0 \leq t \leq r$ and $s \geq 0$; occurring equality if $s \in \mathbb{R}$ and either $t = 0$ or $t = r$.

Proof. The norm inequality (11) implies the determinantal inequality

$$\det(I + A^{s+t}B^t) \leq \det(I + A^s(A^{\frac{r}{2}}B^rA^{\frac{r}{2}})^{\frac{t}{r}}), \quad 0 \leq t \leq r, \quad s \geq 0,$$

occurring equality if $s \in \mathbb{R}$ and either $t = 0$ or $t = r$. On the other hand, the norm inequality (11) holds with the reverse sign, which implies the reverse of the previous determinantal inequality, in the cases (i) $0 \leq s \leq r \leq t$ with $r > 0$; (ii) $0 \leq t \leq r$ and $-s \geq t$. As observed previously, we find the case (ii) as a consequence of (14). Hence, the result follows, multiplying both hand sides of the previous inequalities by $\det(A^{-s}) > 0$. ■

The particular case $r = 2$ in Proposition 4.1 includes, for $A > 0$ and $B \geq 0$, the inequality

$$\det(A^{-s} + |BA|^t) \leq \det(A^{-s} + A^tB^t)$$

if either $0 \leq t \leq 2$ and $-s \geq t$ or $0 \leq s \leq 2 \leq t$, as well as the reverse inequality if $0 \leq t \leq 2$ and $s \geq 0$. The case $s = -2$ yields (13) due to M. Lin.

Remark. The norm inequality (14) due to Furuta can be restated as follows. If $A > 0$, $B \geq 0$, $0 < \alpha < 1$ and $c \geq \alpha$, then

$$(15) \quad A \natural_{c,\alpha} B \prec_{\log} A^{c-\alpha}B^\alpha,$$

which holds trivially if $c \in \mathbb{R}$ and either $\alpha = 0$ or $\alpha = 1$. This log-majorization was obtained by J. S. Matharu and J. S. Aujla [16, Theorem 2.10] if $c = 1$ and by D. T. Hoa [10, Proposition 2.1] if $c = 2$. If $\alpha = \frac{1}{2}$ and $c = 2$, the particular case

$$A^{\frac{1}{2}}(A \sharp B)A^{\frac{1}{2}} \prec_{\log} A^{\frac{3}{4}}B^{\frac{1}{2}}A^{\frac{3}{4}}$$

was considered by Bhatia, Lim and Yamazaki [3, Theorem 2]. Further, we observe that the case $r = 1$, $s = -c$, $t = \alpha$, replacing A by A^{-1} , in the norm inequality (11) and in its reverse by Matsumoto, Nakamoto and Fujii, yields (15) with the reverse log-majorization sign if $0 \leq \alpha \leq 1$ and $c \leq 0$; and again (15), whenever $\alpha \geq 1$ and $-1 \leq c \leq 0$.

If A, B are density matrices, that is, positive semidefinite matrices of trace one, then

$$S(A, B) = \text{Tr}(A(\log A - \log B))$$

is the Umegaki relative entropy [19] of A, B . Fujii and Kamei introduced the variant

$$\hat{S}(A|B) = A^{\frac{1}{2}} \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}},$$

that is, the connection associated to the logarithmic function. A logarithmic trace inequality is now presented, inspired by the corresponding case $s = t = 1$ by F. Hiai and D. Petz [8].

Proposition 4.2. *Let $A, B \geq 0$. If $t, s \geq 0$, then*

$$(16) \quad \text{Tr}(A^s(\log A^t + \log B^t)) \leq \text{Tr}(A^s \log(A^{\frac{r}{2}}B^rA^{\frac{r}{2}})^{\frac{t}{r}}), \quad r > 0,$$

and the LHS converges to the RHS as $r \downarrow 0$.

Proof. The norm inequality (11) implies the trace inequality

$$\text{Tr}(A^sA^tB^t) \leq \text{Tr}(A^s(A^{\frac{r}{2}}B^rA^{\frac{r}{2}})^{\frac{t}{r}}), \quad 0 \leq t \leq r, \quad s \geq 0,$$

occurring trace equality when $t = 0$. Taking the derivatives of the RHS and LHS of the previous inequality at $t = 0$, observing that

$$(17) \quad \frac{d}{dt}(A^tB^t)|_{t=0} = \log A + \log B,$$

$$(18) \quad \frac{d}{dt}((A^{\frac{r}{2}}B^rA^{\frac{r}{2}})^{\frac{t}{r}})|_{t=0} = \log(A^{\frac{r}{2}}B^rA^{\frac{r}{2}})^{\frac{1}{r}}, \quad r > 0,$$

yields a trace inequality. Multiplying both hand sides of the obtained trace inequality by $t \geq 0$, provides (16). By the parametric Lie-Trotter formula (see, for instance, [2, Exercise IX.1.5]), we may see that (18) converges to (17) as $r \downarrow 0$, that is, the convergence of the LHS of (16) to its RHS holds. ■

Using relative entropy terminology, the case $t = s$ of Proposition 4.2, replacing B by B^{-1} , may be written in the condensed form

$$S(A^s, B^s) \leq -\frac{s}{r} \operatorname{Tr}(\hat{S}(A^r|B^r)A^{s-r}), \quad s \geq 0, \quad r > 0,$$

this providing an upper bound for the relative entropy $S(A, B)$, when $s = 1$.

If $r = s = 2$ in Proposition 4.2, then we have the trace inequality

$$\operatorname{Tr}(A^2(\log A + \log B)) \leq \operatorname{Tr}(A^2 \log |BA|),$$

in a parallel line to Audenaert's determinantal inequality (12), which motivated the considerations of this last section.

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(Rute Lemos) CIDMA, MATHEMATICS DEPARTMENT, UNIVERSITY OF AVEIRO, 3810-193 AVEIRO, PORTUGAL

E-mail address, Rute Lemos: **rute@ua.pt**

(Graça Soares) MATHEMATICS CENTER CMAT, POLE CMAT-UTAD, CEMAT-IST-UL, UNIVERSIDADE DE TRÁS-OS-MONTES E ALTO DOURO, UTAD, ESCOLA DAS CIÊNCIAS E TECNOLOGIA, QUINTA DOS PRADOS, 5000-801 VILA REAL, PORTUGAL, WWW.UTAD.PT

E-mail address, Graça Soares: **gsoares@utad.pt**