Spec. Matrices 2018; 6:68–76 DE GRUYTER

Research Article Open Access

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A sharp lower bound on the signless Laplacian index of graphs with (κ, τ)-regular sets

https://doi.org/10.1515/spma-2018-0007
Received October 25, 2017; accepted January 22, 2018

Abstract: A new lower bound on the largest eigenvalue of the signless Laplacian spectra for graphs with at least one (κ, τ)-regular set is introduced and applied to the recognition of non-Hamiltonian graphs or graphs without a perfect matching. Furthermore, computational experiments revealed that the introduced lower bound is better than the known ones. The paper also gives sufficient condition for a graph to be non Hamiltonian (or without a perfect matching).

Keywords: Graph spectra, signless Laplacian index, Hamiltonian graphs, perfect matching

MSC: 15A18, 05C45, 05C50, 05C70

1 Introduction

This paper presents a new lower bound on the signless Laplacian index of graphs with at least one (κ, τ)-regular set. Such graphs appear often, since they are closely related to the Hamiltonian graphs and also to graphs with a perfect matching. On the other hand, the signless Laplacian index has gained special attention in spectral graph theory over the last decade. Several upper bounds on the signless Laplacian index as well as its relations with the classical combinatorial invariant parameters of graphs have been developed (see [7, 8]). However, regarding lower bounds, only a few results are known. The next lemma states two mostly exploited in the literature.

Lemma 1.1. *[3, 4] Let G be a graph with maximum vertex degree Δ. Then*

$$q_1(G) \geq \Delta + 1, \tag{1.1}$$

$$q_1(G) \geq \min_{i \sim j}(d_i + d_j). \tag{1.2}$$

The lower bound on the signless Laplacian index of a graph with a (κ, τ)-regular set, introduced in this paper, is considered as a necessary condition for the existence of a (κ, τ)-regular set. It is applied to the recognition of graphs without such type of combinatorial structure as it is the case for the line graphs of non-Hamiltonian graphs or the line graphs of graphs without a perfect matching. According to the performed computational tests, the introduced lower bound has successful application in the recognition of many non-Hamiltonian graphs.

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Throughout the text we consider simple graphs $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$. The order and the size of G are denoted by $n(= |V(G)|)$ and $m(= |E(G)|)$, respectively. We write $u \sim v$ whenever two vertices u and v are adjacent, and A_G stands for the (0, 1)-adjacency matrix of G. The neighborhood $N_G(i)$ of a vertex $i \in V(G)$ is the set of vertices adjacent to i; the degree of i is $d_i = |N_G(i)|$; the average vertex degree $\bar{d} = \frac{2m}{n}$ and D_G stands for the diagonal matrix of vertex degrees. The signless Laplacian matrix Q_G is equal to $A_G + D_G$; it is symmetric, hence all the zeros of its characteristic polynomial are real and then they can be considered in non-increasing order

$$q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G).$$

The largest eigenvalue of Q_G, is called the *signless Laplacian index* of G.

The subgraph of G induced by the vertex subset $T \subset V(G)$ is denoted by $G[T]$. A (κ, τ)-regular set S is a subset of $V(G)$, inducing a κ-regular subgraph such that every vertex not in the subset has τ neighbors in it. By definition, the vertex set of a κ-regular graph is considered to be $(\kappa, 0)$-regular.

The paper is organized in the following way. In Section 2 a lower bound for the signless Laplacian index of graphs with at least one (κ, τ)-regular set is deduced under different suitable hypotheses. In Section 3 a few numerical examples to illustrate the quality of the introduced bound and independence of some known results are given. Afterwards, this bound is compared with the bounds (1.1) and (1.2). In section 4 the obtained results are applied to the recognition of non-Hamiltonian graphs or graphs without a perfect matching.

2 A lower bound on the signless Laplacian index of a graph with a (κ, τ)-regular set

Through several computational experiments we arrived at the conclusion that the signless Laplacian index of a connected graph with a (κ, τ)-regular set is usually greater than or equal to $\kappa + \tau$. However, there exist some graphs not satisfying this inequality as shown in the next example.

Example 2.1. The broken wheel graph $W(7, 6)$, depicted in Figure 2.1, consists of cycle C_7 and one outer vertex connected to 6 vertices of the cycle. Its signless Laplacian index is $q_1 \approx 7.92186 < \kappa + \tau = 8$.

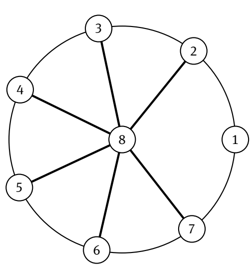

Figure 2.1: The graph $W(7, 6)$ with a $(2, 6)$-regular set $S = \{1, 2, 3, 4, 5, 6, 7\}$.

Note that for a κ-regular graph G, $q_1(G) = 2\kappa$ and the vertex set $V(G)$ is $(\kappa, 0)$-regular. Therefore, $\kappa + \tau$ is a trivial lower bound on the index of the signless Laplacian matrix of a regular graph.

We first state a sufficient condition on a connected graph G with a (κ, τ)-regular set S under which the inequality $q_1(G) \geq \kappa + \tau$ holds provided $\tau > 0$. Let $\bar{S} = V(G) \setminus S$,

$$a = \frac{|\bar{S}|}{|S|} \quad \text{and} \quad b = \frac{2|E(G[\bar{S}])|}{|\bar{S}|}, \tag{2.1}$$

i.e., b is the average vertex degree of the induced subgraph $G[\bar{S}]$. Regarding the vertex partition $V(G) = S \cup \bar{S}$ which defines the corresponding block subdivision of $Q_G = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$, the quotient matrix B_G has the following form

$$B_G = \begin{pmatrix} 2\kappa + \tau a & \tau a \\ \tau & 2\frac{2|E(G[\bar{S}])|}{|\bar{S}|} + \tau \end{pmatrix}. \tag{2.2}$$

The entries of B_G are the average row sums of the corresponding blocks. The eigenvalues of B_G are equal to

$$\gamma_{1/2} = \frac{2\kappa + 2b + \tau(1 + a) \pm \sqrt{(2\kappa + 2b + \tau(1 + a))^2 - 8(\kappa\tau + 2\kappa b + \tau ab)}}{2}.$$

These eigenvalues interlace the eigenvalues of Q_G (see [6, Corollary 2.3]), and hence $q_1(G) \geq \gamma_1$. Next we check when $\gamma_1 < \kappa + \tau$, i.e., when

$$\gamma_1 - (\kappa + \tau) = b + \frac{\tau(a - 1) + \sqrt{(2\kappa + 2b + \tau(1 + a))^2 - 8(\kappa\tau + 2\kappa b + \tau ab)}}{2} < 0.$$

The previous inequality reduces to

$$\sqrt{(2\kappa + 2b + \tau(1 + a))^2 - 8(\kappa\tau + 2\kappa b + \tau ab)} < \tau(1 - a) - 2b. \tag{2.3}$$

We assume $\tau(1 - a) - 2b > 0$, i.e,

$$b < \frac{\tau(1 - a)}{2}, \tag{2.4}$$

since otherwise (2.3) does not hold. Under the assumption (2.4), the inequality (2.3) is equivalent to

$$\kappa^2 - \kappa(2b - \tau(a - 1)) + a\tau^2 - 2b\tau(a - 1) < 0. \tag{2.5}$$

The inequality (2.5) holds (considering the left hand side as a quadratic function in κ) iff its discriminant is positive, i.e., when

$$(2b + \tau(a - 1) - 2\sqrt{a}\tau)(2b + \tau(a - 1) + 2\sqrt{a}\tau) > 0 \tag{2.6}$$

and $\kappa \in (\kappa_1, \kappa_2)$, where κ_1, κ_2 are the roots of the corresponding equation.

From (2.4), it follows $2b + \tau(a - 1) - 2\sqrt{a}\tau < 0$ and therefore (2.6) holds iff

$$b < \tau\frac{1 - a - 2\sqrt{a}}{2}.$$

It is easy to verify that $\frac{1 - a - 2\sqrt{a}}{2} > 0$ iff $a \in (0, 3 - 2\sqrt{2})$. Therefore, for $a \in (0, 3 - 2\sqrt{2})$, $b < \tau\frac{1 - a - 2\sqrt{a}}{2}$ and $\kappa \in (\kappa_1, \kappa_2)$ where $\kappa_{1/2}$ are the roots of the corresponding quadratic polynomial in (2.5), $\gamma_1 < \kappa + \tau$ holds. The previous observations are summarized in the following theorem.

Theorem 2.1. *Let G be a connected graph with a (κ, τ)-regular set $S \subset V(G)$, $\bar{S} = V(G) \setminus S$, $a = \frac{|\bar{S}|}{|S|}$, $b = \frac{2|E(G[\bar{S}])|}{|\bar{S}|}$ and*

$$g(a, b) = \frac{2b + \tau(a - 1) + \sqrt{(2\kappa + 2b + \tau(1 + a))^2 - 8(\kappa\tau + 2\kappa b + \tau ab)}}{2}.$$

If each of the following conditions

(a) $0 < a < 3 - 2\sqrt{2}$,

(b) $b < \tau\frac{1 - a - 2\sqrt{a}}{2}$,

(c) $\kappa \in (\frac{2b + \tau(1 - a) - \sqrt{D}}{2}, \frac{2b + \tau(1 - a) + \sqrt{D}}{2})$, *where* $D = 4b^2 + 4b\tau(a - 1) + \tau^2(a^2 - 6a + 1)$,

is satisfied, then $q_1(G) > \kappa + \tau - |g(a,b)|$. *Otherwise,*

$$q_1(G) \geq \kappa + \tau + |g(a,b)| . \tag{2.7}$$

For at least two graph classes (2.7) holds as equality. Those are bidegreed graphs having at least one (κ, τ)-regular set S such that all vertices in S have the same vertex degree Δ_S, while all vertices in $\overline{S}$ have the same vertex degree $\delta_{\overline{S}}$, $\Delta_S \neq \delta_{\overline{S}}$ i.e. the partition $V(G) = S \cup \overline{S}$ induces an equitable partition of G (for more details about equitable partitions see [5, p. 83]). The second class consists of regular graphs having at least one (κ, τ)-regular set S.

Corollary 2.2. *Let G be a graph with (κ_i, τ_i)-regular sets S_i, $\tau_i > 0$, $1 \leq i \leq t$. Then* $q_1(G) \geq \max\{\kappa_i + \tau_i + g_i(a_i, b_i) : 1 \leq i \leq t\}$.

Theorem 2.1 reads that if $a \geq 3 - 2\sqrt{2}$ or $b \geq \tau \dfrac{1 - a - 2\sqrt{a}}{2}$ or $\kappa \notin (\kappa_1, \kappa_2)$, then $q_1(G) \geq \kappa + \tau + |g(a,b)|$. However it turns out that a different type of constraints, mainly on a, imply the same inequality. The next lemma consider a weaker one, but more convenient for checking .

Lemma 2.3. *Let G be a connected graph with a (κ, τ)-regular set $S \subset V(G)$, $\overline{S} = V(G) \setminus S$ and $a = \dfrac{|\bar{S}|}{|S|}$. If $a \geq 1 - \dfrac{\kappa}{\tau}$, then the inequality (2.7) holds.*

Proof. Let γ_1 be the largest eigenvalue of the quotient matrix B_G (see (2.2)). We check whether $g(a,b) = \gamma_1 - (\kappa + \tau) \geq 0$ if $a \geq 1 - \dfrac{\kappa}{\tau}$. The proof is divided into two cases.

1. If $a \geq 1$ (notice $1 \geq 1 - \dfrac{\kappa}{\tau}$), then it is immediate that

$$g(a,b) = b + \frac{1}{2}\left(\tau(a-1) + \sqrt{(a\tau + 2b + 2\kappa + \tau)^2 - 8(ab\tau + 2b\kappa + \kappa\tau)}\right) \geq 0.$$

2. If $1 > a \geq 1 - \dfrac{\kappa}{\tau}$, then setting $s = |S|$ we may write $|\overline{S}| = s - \epsilon$, for some integer $\epsilon \in \{1, \ldots, s-1\}$. Let $e = |E(G[\overline{S}])|$, then by replacing a and b (see (2.1)) in the expression for $g(a,b)$, we obtain

$$g(a,b) = \frac{4es - s\tau\epsilon + \tau\epsilon^2 + \sqrt{d}}{2s(s-\epsilon)}, \tag{2.8}$$

where $d = (4es + (s-\epsilon)(2s(\kappa+\tau) - \tau\epsilon))^2 + 8s(s-\epsilon)(-4e\kappa s - \tau(s-\epsilon)(2e + \kappa s))$. Clearly, $g(a,b)$ is non-negative if $4es - s\tau\epsilon + \tau\epsilon^2 \geq 0$. Therefore, in what follows, we assume $4es - s\tau\epsilon + \tau\epsilon^2 < 0$, i.e.,

$$e = |E(G[\overline{S}])| < \frac{\epsilon\tau(s-\epsilon)}{4s}. \tag{2.9}$$

In this case $g(a,b) \geq 0$ if and only if $d \geq \left(4es - s\tau\epsilon + \tau\epsilon^2\right)^2$, which gives

$$4s(s-\epsilon)\left(e(4\tau\epsilon - 4\kappa s) + (s-\epsilon)\left(s\left(\kappa^2 + \tau^2\right) - \tau\epsilon(\kappa+\tau)\right)\right) \geq 0.$$

Since s and $s - \epsilon$ are both positive, the above inequality holds iff $y \geq 0$, where

$$y = 4(\tau\epsilon - \kappa s)e + (s-\epsilon)\left(s\left(\kappa^2 + \tau^2\right) - \tau\epsilon(\kappa+\tau)\right). \tag{2.10}$$

The inequality $a \geq 1 - \dfrac{\kappa}{\tau}$ is equivalent to $\tau\epsilon - \kappa s \leq 0$, i.e., $\epsilon \leq \dfrac{\kappa}{\tau}s$. Bearing in mind (2.9), we obtain

$$y \geq (s-\epsilon)\frac{(\tau\epsilon - s\kappa)^2 + s\tau^2(s-\epsilon)}{s} \geq 0,$$

since $s - \epsilon > 0$.

□

The inequality $q_1(G) \geq \kappa + \tau + |g(a,b)|$ straightforwardly holds for any connected graph with (κ, τ)-regular set such that $\kappa \geq \tau$, since $a \geq 1 - \frac{\kappa}{\tau}$. In the next theorem we focus on graphs with a (κ, τ)-regular set, where $\kappa < \tau$, taking into account that $\kappa < \tau$ is equivalent to $1 - \frac{\kappa^2+\tau^2}{\tau(\kappa+\tau)} < 1 - \frac{\kappa}{\tau}$.

Theorem 2.4. *Let G be a connected graph with a (κ, τ)-regular set S such that $\kappa < \tau$. If $a \geq 1 - \frac{\kappa^2+\tau^2}{\tau(\kappa+\tau)}$, then the inequality* (2.7) *holds.*

Proof. If $a \geq 1 - \frac{\kappa}{\tau}$ then (2.7) holds by Lemma 2.3. Otherwise, if $a < 1 - \frac{\kappa}{\tau}$, then $\tau\epsilon - \kappa s > 0$, i.e., $\epsilon > \frac{\kappa}{\tau}s$, where ϵ and s are defined in the same way as in the proof of Lemma 2.3. Having in mind $e \geq 0$ from (2.10) it follows that $y \geq (s-\epsilon)\left(s\left(\kappa^2+\tau^2\right) - \tau\epsilon(\kappa+\tau)\right)$. Since $s > \epsilon$, we have $y \geq 0$ if $\left(\kappa^2+\tau^2\right)s - \tau(\kappa+\tau)\epsilon \geq 0$. The last inequality is equivalent to

$$\epsilon \leq \frac{\kappa^2+\tau^2}{\tau(\kappa+\tau)}s\,.$$

Since $\epsilon = 2s - |V(G)|$, from the above inequality, we conclude that $y \geq 0$ if

$$a \geq 1 - \frac{\kappa^2+\tau^2}{\tau(\kappa+\tau)}\,.$$

□

Remark 2.1. For $\kappa = 0$ the inequality $q_1 \geq \kappa + \tau + |g(a,b)|$ holds independently of the values of a and b. This conclusion follows directly from Theorem 2.4, taking into account that under such conditions, $a \geq 1 - \frac{\kappa^2+\tau^2}{\tau(\kappa+\tau)}$ is equivalent to $a \geq 0$.

Example 2.2. A star S_n, a tree with $n-1$ vertices of degree 1 and one vertex of degree $n-1$, has a $(0, n-1)$-regular set S. By direct application of Theorem 2.4 we conclude that $q_1(S_n) \geq n$. In this case the inequality holds as an equality since stars are bidegreed graphs and $V(G) = S \cup \bar{S}$ induces an equitable partition.

3 Numerical examples

In what follows we give a few numerical examples to illustrate the quality of the lower bound (2.7) and we compare the values obtained by (2.7) and by the lower bounds (1.1) and (1.2).

Example 3.1. The graph H of order 20, depicted in Figure 3.1, has a $(2,3)$-regular set S with cardinality 18. Since $a = \frac{1}{9}$, $b = 1 > \frac{1-a-2\sqrt{a}}{2} = \frac{1}{9}$, the condition (b) in Theorem 2.1 is not satisfied and thus, applying this theorem,

$$q_1(H) \geq \kappa + \tau + |g(a,b)| = 5.7208.$$

In fact the signless Laplacian index is $q_1(H) \approx 6.28853$.

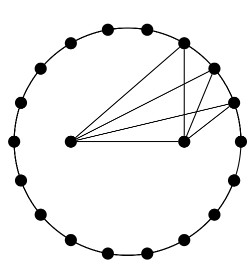

Figure 3.1: The graph H with a $(2,3)$-regular set.

Example 3.2. The broken wheel graph $W(6, 5)$ depicted in Figure 3.2 has a $(2, 5)$-regular set with cardinality 6. Since $b = 0$ and $a = \frac{1}{6}$, all conditions (a)-(c) in Theorem 2.1 are satisfied and hence

$$q_1(W(6,5)) \geq \kappa + \tau - |g(a,b)| = 6.95961.$$

In fact the signless Laplacian index is $q_1(W(6,5)) \approx 7.12783$.

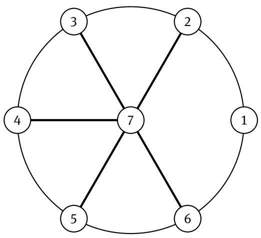

Figure 3.2: The graph $W(6, 5)$ with a $(2, 5)$-regular set $S = \{1, 2, 3, 4, 5, 6\}$.

We point out that, in Examples 3.1 and 3.2, neither the hypothesis of Theorem 2.4 nor the hypothesis of Lemma 2.3 are satisfied.

Example 3.3. Consider a graph F of order 20 having a $(8, 10)$-regular set S with cardinality 18 (see Figure 3.3). Since $a = \frac{1}{9} \geq 1 - \frac{\kappa^2 + \tau^2}{\tau(\kappa + \tau)} = 0.088889$, we may apply Theorem 2.4 and thus

$$q_1(F) \geq \kappa + \tau + |g(a,b)| = 18.7558.$$

In fact its signless Laplacian index is $q_1(F) \approx 18.9646$. Since $b = 1$ the condition (c) in Theorem 1.1 fails to hold. Despite Theorems 2.3 and 2.4 provide the same bound, it is clear that the assumption in Theorem 2.4 is easier to check.

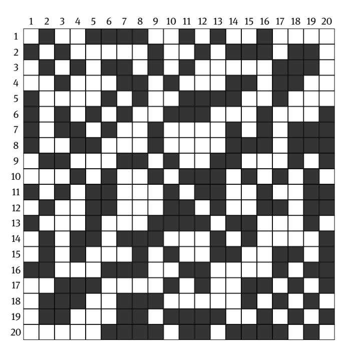

Figure 3.3: The adjacency matrix plot of graph F with a $(8, 10)$-regular set of Example 3.3.

Table 3.1 presents the lower bounds on the signless Laplacian index obtained in the previous examples by (2.7) as well as those obtained by applying (1.1) and (1.2). As we can see, in all the cases (2.7) gives the best approximation to $q_1(G)$.

We finish this section by comparing the bound (2.7) with the bounds (1.1) and (1.2) for a large number of randomly generated graphs with (κ, τ)-regular sets and orders up to 20.

For each graph G, obtained by Algorithm 1, we have computed its signless Laplacian index, q_1, and the relative errors $\frac{q_1 - \bar{q}_1}{q_1}$, where $\bar{q}_1$ is one of the lower bounds given by Theorem 2.1, (1.1) and (1.2).

Table 3.1: Comparison of the bounds.

G	$q_1(G)$	$\kappa + \tau \pm \lvert g(a,b)\rvert$	$\Delta + 1$	$\min_{i\sim j}(d_i + d_j)$
$W(7,6)$	7.92186	7.76724	7	5
H	6.28853	5.72076	5	4
$W(6,5)$	7.12783	6.95961	6	5
F	18.9646	18.7558	12	16

Algorithm 1 (random generation of a graph with a (κ, τ)-regular set).

1: generate a random κ-regular graph G_S of order $1 \le |S| \le 10$;
2: generate a random graph $G_{\overline{S}}$ with $1 \le |\overline{S}| \le 10$ vertices and $0 \le e \le \binom{|\overline{S}|}{2}$ edges;
3: randomly connect each vertex of $G_{\overline{S}}$ with τ vertices of G_S, $1 \le \tau \le |S|$;
4: check if the resulting graph is connected.

Table 3.2 presents the average relative errors of the three bounds when compared with the exact values of the signless Laplacian index q_1.

Table 3.2: Average relative errors of the lower bounds on the signless Laplacian index of the randomly generated graphs by the Algorithm 1.

Num. graphs	ERR(Th. 2.1)	ERR(1.1)	ERR(1.2)
45369	0.03	0.31	0.32

4 Computational experiments and relations with some well known combinatorial structures in graphs

Several well known combinatorial structures in graphs can be characterized by (κ, τ)-regular sets, as is the case of Hamiltonian cycles and perfect matchings whose edge sets correspond in the line graph to $(2, 4)$-regular sets and $(0, 2)$-regular sets, respectively.

In this section we deduce lower bounds on the signless Laplacian index of the line graph of a graph with a Hamiltonian cycle or a graph with a perfect matching. These bounds are then applied to the recognition of non-Hamiltonian graphs and graphs without a perfect matching.

4.1 Applications to the recognition of non-Hamiltonian graphs

A line graph $L(G)$ of a graph G, is a graph whose vertex set corresponds to the set of edges of G and two vertices of $L(G)$ are adjacent if the corresponding edges in G are adjacent, i.e., share a common vertex.

The line graph $L(G)$ of every Hamiltonian graph G has a $(2, 4)$-regular set S consisting of vertices that correspond to the edges of a Hamiltonian cycle in G (see [1]). We apply Theorem 2.4 to $L(G)$ in order to obtain a necessary condition for a graph to be Hamiltonian. Then

- $2\overline{m} = 2\,|E(L(G))| = \sum_{ij \in E(G)}(d_i + d_j - 2)$;
- $|S| = n$;
- $a = \dfrac{m-n}{n} = \dfrac{\bar{d}}{2} - 1$, where $\bar{d}$ denotes the average vertex degree of vertices in G;

– $b = 2\frac{\overline{m} - n}{m - n} - 8$.

Proposition 4.1. *Let G be a connected graph of order n with the edge set E(G) such that* $|E(G)| \neq n$ *and assume that its average degree* $\bar{d} \geq \frac{7}{3}$. *If G is Hamiltonian, then*

$$q_1(L(G)) \geq 2\frac{\overline{m} - n}{m - n} + \bar{d} - 6 + \sqrt{(2\frac{\overline{m} - n}{m - n} - \bar{d} - 10)^2 + 8(2\frac{\overline{m} - n}{m - n} - 10)}. \tag{4.1}$$

Proof. If G is a Hamiltonian graph, then $L(G)$ has a $(2, 4)$-regular set S of cardinality n. Since $1 - \frac{\kappa^2 + \tau^2}{\tau(\kappa + \tau)} = 1 - \frac{2^2 + 4^2}{4(2 + 4)} = \frac{1}{6}$, it follows that the assumption of Theorem 2.4 is satisfied, that is, $\kappa < \tau$ and

$$a \geq 1 - \frac{\kappa^2 + \tau^2}{\tau(\kappa + \tau)} \Leftrightarrow \frac{\bar{d}}{2} - 1 \geq \frac{1}{6} \Leftrightarrow \bar{d} \geq \frac{7}{3}.$$

Therefore, taking into account the values of the parameters a and b determined above, the inequality (4.1) follows as direct application of Theorem 2.4. □

The next example highlights the application of Proposition 4.1.

Example 4.1. Considering the non-Hamiltonian complete bipartite graph $K_{2,3}$ depicted in Figure 4.1, it is immediate that the inequality (4.1) does not hold. In fact $q_1(L(K_{2,3})) = 8$, while the right hand side of (4.1) gives 9.3333.

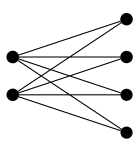

Figure 4.1: Non-Hamiltonian graph $K_{2,3}$ for which the inequality (4.1) does not hold.

Remark 4.1. Proposition 4.1 can be used for the recognition of non-Hamiltonian graphs.

We tested the inequality (4.1) for all non-Hamiltonian graphs of orders n ranging from $n = 4$ up to $n = 40$ produced by the GraphData library of Mathematica [9] with average vertex degree not less than $7/3$. Over a total of 1167 graphs, 34% have violated the inequality (4.1) and thus the Proposition 4.1 confirms that they are non-Hamiltonian.

4.2 Applications to the recognition of graphs without a perfect matching

A stable or independent set is a subset of pairwise non-adjacent vertices. A matching M in G is a set of pairwise non-adjacent edges, that is, no two edges share a common vertex. A perfect matching is a matching which matches all vertices of the graph.

In [2] it was proven that a graph G has a perfect matching M iff the line graph $L(G)$ has a vertex subset $S = L(M)$ which is $(0, 2)$-regular. Assuming that $|V(G)| = n$, $|E(G)| = m$ and $2m \neq n$, it follows that

– $|S| = \frac{n}{2}$;
– $a = \frac{m - \frac{n}{2}}{\frac{n}{2}} = \bar{d} - 1$;
– $b = 2\frac{\overline{m} - 2(m - \frac{n}{2})}{m - \frac{n}{2}} = 2(\frac{2\overline{m}}{2m - n} - 2)$.

Proposition 4.2. *Let G be a graph of order n and size m with a perfect matching. If $m \neq 2n$, then*

$$q_1(L(G)) \geq \frac{4\overline{m}}{2m-n} + \bar{d} - 4 + \sqrt{(\frac{4\overline{m}}{2m-n} + \bar{d} - 4)^2 - 8(\frac{2\overline{m}}{2m-n} - 2)(\bar{d} - 1)}. \tag{4.2}$$

Proof. Since $L(G)$ has a $(0, 2)$-regular set, the assumption of Theorem 2.4 is satisfied. Therefore, taking into account the computed values for a and b, the inequality (4.2) is obtained by direct application of Theorem 2.4. □

It is worth mentioning that from Remark 2.1, in this case the lower bound is always greater than 2.

Example 4.2. For the complete bipartite graph $K_{2,3}$ depicted in Figure 4.1, $q_1(L(K_{2,3})) = 8$ while the right hand side of (4.2) gives ≈ 8.17651.

The inequality (4.2) was also tested for all graphs of even order n with no perfect matching ranging from $n = 4$ up to $n = 60$ produced by the GraphData library of Mathematica [9]. Over a total of 165 graphs, 55% violated the inequality (4.2). We point out that odd order graphs have no perfect matching and each even order Hamiltonian graph has a perfect matching. Therefore, every even order graph with no perfect matching is non-Hamiltonian and thus the Proposition 4.2 confirms that at least 55% of the 165 tested graphs are also non-Hamiltonian.

We are aware that for large graphs calculation of $q_1(G)$ can be a computationally difficult task. However, one possibility to relax it, would be to consider some upper bound on $q_1(L(G))$ instead, bearing in mind that, in most of the cases, upper bounds on $q_1(G)$ are expressed using vertex degrees (see [8]). Since, vertex degrees in a line graph of G are related to vertex degrees in G (if $e = ij$ then $d_e = d_i + d_j - 2$), we can end up with inequalities including only parameters of G and thus the computational effort can be significantly reduced. The price to be paid is a smaller number of non Hamiltonian graphs (or graphs without perfect matching) that would be recognized.

Acknowledgement: The research of D.M. Cardoso and António Pereira was partially supported by the Portuguese Foundation for Science and Technology (“FCT-Fundação para a Ciência e a Tecnologia”), through the CIDMA - Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2013.

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