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SIMÃO PEDRO SILVA SANTOS

CÁLCULO DAS VARIAÇÕES DO TIPO HERGLOTZ

CALCULUS OF VARIATIONS OF HERGLOTZ TYPE

Universidade de Aveiro Departamento de Matemática
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Tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, Programa Doutoral em Matemática 2013-2017, realizada sob a orientação científica da Professora Doutora Natália da Costa Martins, Professora Auxiliar do Departamento de Matemática da Universidade de Aveiro, e coorientação do Professor Doutor Delfim Fernando Marado Torres, Professor Catedrático do Departamento de Matemática da Universidade de Aveiro.

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palavras-chave

Cálculo das Variações, Controlo Ótimo, cálculo das variações generalizado, equações diferenciais de Euler-Lagrange, condições naturais de fronteira, problemas variacionais generalizados de ordem-superior, sistemas com delay, condição necessária de otimalidade de DuBois-Reymond, invariância, teoremas de Noether, leis de conservação de Noether.

resumo

Consideramos vários problemas com base no problema variacional generalizado de Herglotz.
Dois capítulos são dedicados à extensão do problema variacional generalizado de Herglotz para ordem superior e para problemas de primeira ordem com atraso no tempo, utilizando uma abordagem variacional.
Nos últimos quatro capítulos, reescrevemos os problemas de Herglotz na forma do controlo ótimo e usamos essa abordagem. Demonstramos equações generalizadas de Euler-Lagrange de ordem superior, inicialmente sem e depois com atraso no tempo; condições de fronteira de ordem superior; o primeiro teorema de Noether para o problema de Herglotz de primeira ordem com atraso no tempo; o primeiro teorema de Noether para problemas de ordem superior de Herglotz sem e com atraso no tempo; e a existência de leis de conservação de Noether numa versão do segundo teorema de Noether do controlo ótimo.

keywords

Calculus of Variations, Optimal Control, generalized calculus of variations, Euler-Lagrange differential equations, natural boundary conditions, higher-order generalized variational problems, retarded systems, DuBois-Reymond necessary optimality condition, invariance, Noether's theorems, Noether currents.

abstract

We consider several problems based on Herglotz’s generalized variational problem.
We dedicate two chapters to extensions on Herglotz’s generalized variational problem to higher-order and first-order problems with time delay, using a variational approach.
In the last four chapters, we rewrite Herglotz's type problems in the optimal control form and use an optimal control approach. We prove generalized higher-order Euler-Lagrange equations, first without and then with time delay; higher-order natural boundary conditions; Noether's first theorem for the first-order problem of Herglotz with time delay; Noether's first theorem for higher-order problems of Herglotz without and with time delay; and existence of Noether currents as a version of Noether's second theorem of optimal control.

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INTRODUCTION

When we sat down together for the first time in the end of 2012, with the aim of planning the following years of my Ph.D. work, we could not imagine we would walk this path. We knew we would focus in some areas of the Calculus of Variations, but we did not have yet in mind the idea of crossing the border to the Optimal Control field and develop a twofold investigation.

Our attention had already been called to the variational problem proposed by Herglotz, mostly by the work of Guenther *et al.* "The Herglotz lectures on contact transformations" [37], which led us to the original work of Herglotz [39, 40] and to the most recent work at the date on Herglotz's variational principle, by Georgieva *et al.* [29, 30, 31, 32, 33].

After that first meeting, we agreed to dedicate initially our attention and efforts in the attempt of generalizing the first-order generalized variational problem of Herglotz to the higher-order case. This investigation took us the second semester of 2013 and resulted in the publication in 2014 of our first joint work "Higher-order variational problems of Herglotz" [59], which is the basis of Chapter 4, and a public communication which constituted the evaluation of 'Seminário I', one of the first year Ph.D. disciplines.

In that first paper, we used the classical technique of introducing an admissible variation and study the necessary conditions of optimality; we also recurred to two important higher-order results: the higher-order fundamental lemma of the Calculus of Variations [51] and the higher-order integration by parts formula [53]. We were then able to prove a higher-order Euler–Lagrange equation and natural boundary conditions for the generalization of the variational problem of Herglotz to the higher-order case.

Meanwhile, we had already in mind the study of problems with time delay and we dedicated the second semester of 2014 to this task. We were aware of the classical results on delayed problems by El'sgol'c [19], Agrawal [2], Maurer [34] and Hughes [43] but we were also aware that only recently Frederico and Torres generalized the important Noether's first theorem to Optimal Control problems with time delay [24]. The investigation of Herglotz's type problems with time delay was long but led to the publication in 2015 of our second paper "Variational problems of Herglotz type with time delay: DuBois–Reymond condition and Noether's first theorem", which is the basis of Chapter 5.

We based our arguments in the classical ones, introducing again an admissible variation and making convenient changes of variables. We managed to prove two optimality conditions for the delayed first-order problem of Herglotz: generalized Euler–Lagrange equations and a DuBois–Reymond condition. Moreover, we studied invariance of the delayed Herglotz's problem and proved the existence of conservations laws resulting in the main result of the paper: a Noether's theorem for the first-order problem of Herglotz with time delay. This theorem was a major advance in the Ph.D. work in the sense that it generalized Georgieva's results, which we considered benchmarks in the generalized variational problems of Herglotz type.

Although 2015 was the more prolific year, with the publication of three papers, the second semester of 2014 played a decisive role in the development of our work; the choice of 'Controlo Ótimo' as an optional discipline of the first year of the Ph.D. course was a decisive step to take this thesis to the Optimal Control field. As a result, we started looking at Herglotz's variational problems as particular cases of Optimal Control problems in the Bolza form.

The new Optimal Control view motivated us to the publication of the paper entitled "An Optimal Control approach to Herglotz variational problems" [61], which is presented in Chapter 6. In this paper, we used existing Optimal Control results, such as Pontryagin's maximum principle and DuBois–Reymond condition [57], and Noether's theorem [67]. We made several transformations and rewrote Herglotz's first-order problem as an Optimal Control problem: we then applied previous Optimal Control results and derived a generalized Euler–Lagrange equation, a transversality condition, a DuBois–Reymond necessary optimality condition and Noether's theorem for Herglotz's fundamental problem, valid for the wider class of piecewise smooth functions and considering a more general notion of invariance.

With this new look over Herglotz's type problems, it was a quick step from the third to the forth paper: "Noether's theorem for higher-order variational problems of Herglotz type" (Chapter 7). We were acquainted with the technique of dealing with first-order Herglotz's type problems as Optimal Control problems and rapidly extended it to the higher-order case by proving a generalized Euler–Lagrange equation, transversality conditions and DuBois–Reymond necessary optimality condition for Herglotz's type higher-order variational problems; but the biggest contribution of this paper was the proof of a Noether's theorem for higher-order problems of Herglotz, something that has not yet been done for any kind of trajectories.

By the end of 2015, we were working well on both sides of Calculus of Variations and Optimal Control and perfectly convinced that we could improve the results of our work on delayed problem [60]. Namely, we were convinced we could exempt two additional hypotheses introduced with no justification, but only for technical reasons; and we were convinced we would be able to disregard them trough the Optimal Control approach. This eventually happened, and in 2016 we wrote the paper entitled "Higher-order variational problems of Herglotz with time delay" [63], which we present in Chapter 8.

The main results of this fifth paper are higher-order Euler–Lagrange and DuBois–Reymond necessary optimality conditions as well as a higher-order Noether type theorem for delayed variational problems of Herglotz type. We used again the technique of writing the addressed problem in Bolza's optimal control form, but made a major change inspired by Guinn's work [38]: we investigated and managed to write the higher-order delayed problem of Herglotz as a non--delayed optimal control problem and only then we applied the available results. With these arguments and results we were able to generalize most of the results of classical calculus of variations, but also on Herglotz' type problems.

In early 2016, we started thinking and discussion the possibility of writing a thesis and finishing the task. We thought, however, that we could go further and produce a more self-contained document if we addressed a final chapter on Noether's second theorem for higher--order variational problems of Herglotz type with time delay. We made then the clear decision of dealing with this final chapter using the optimal control approach, namely on the existence of Noether currents when the generalized variational problem is semi-invariant. This work lead to the submission of the paper entitled "Noether currents for higher-order variational problems of Herglotz type with time delay" (Chapter 9), in which we prove a type of Noether's second theorem for optimal control adapted for the higher-order delayed Herglotz's framework.

To the best of our knowledge, at the date we started thinking in our first contribution, nobody had approached Herglotz type problems since Guenther, Georgieva and their collaborators. We are flattered to notice that our investigation has motivated some of our colleagues, namely Almeida, who considered the variational problem of Herglotz in the context of scale calculus [3], Almeida and Malinowska, that considered the variational problem of Herglotz in the context of fractional calculus [4], and after them Abrunheiro, Machado and Martins, who did it in the general context of Riemannian manifolds [1].

Part I

Synthesis

CHAPTER 1

CLASSICAL CALCULUS OF VARIATIONS

The Calculus of Variations had its beginning in the end of the 17th century with the nowadays well known Brachistochrone problem proposed by Johann Bernoulli in 1696. The statement of the Brachistochrone problem is as follows: let two points A and B be given in the vertical plane. Find the curve along which a weighted particle must follow that, starting from A, it reaches B in the shortest time under its own gravity.

The problem proposed by Johann Bernoulli attracted the attention of several important mathematicians including Jakob Bernoulli (Johann's brother), Newton, Leibniz, L'Hôpital and Euler. The solution to this problem is a cycloid and is called brachistochrone or curve of fastest descent.

A decisive step in the foundations of the Calculus of Variations was achieved in the 18th century with the work of Euler and Lagrange who found a systematic way of dealing with this kind of problems by introducing what is now known as the Euler–Lagrange equation.

In the next century, Jacobi and Weierstrass made significant developments in the area, who were consolidated in the early 20th century by Hilbert, Noether, Tonelli, Lebesgue, Hadamard and Herglotz. As noted by Forsyth and cited in [74], Calculus of Variations

> "Attracted a rather fickle attention at more or less isolated intervals in its growth."

The Calculus of Variations deals with the search for extrema for some functional and, in this sense, can be considered a branch of optimization. The applications of this subject are immense and extend from physics, to economics, but mainly mechanics (see e.g. [16, 17, 28, 46, 74]).

We emphasise as examples Fermat's Principle of Minimum Time in geometrical optics and Hamilton's Principle in classic mechanics. As referred by Carathéodory in [12]:

> "I have never lost sight of the fact that the Calculus of Variations, as it is presented in Part II, should above all be a servant of mechanics."

The subject is far from dead and, as cited in [74], Stampacchia in 1974 also stated:

> "The natural development of the Calculus of Variations has produced new branches of mathematics which have assumed different aspects and appear quite different from the Calculus of Variations."

The most basic problem of the classical calculus of variations consists of finding the trajectories $x(\cdot)$ that extremize (minimize or maximize) the functional

$$\mathcal{L}[x] = \int_a^b L(t, x(t), \dot{x}(t))dt \tag{1.1}$$

with $x(\cdot) \in C^2([a,b];\mathbb{R})$, satisfying the boundary conditions $x(a) = \alpha$ and $x(b) = \beta$, for some $\alpha, \beta \in \mathbb{R}$, and L satisfying some smoothness properties.

Definition 1.1. *We say that $x(\cdot)$ is an admissible trajectory to the basic problem of the calculus of variations* (1.1) *if $x(\cdot) \in C^2([a,b];\mathbb{R})$ and satisfies $x(a) = \alpha$ and $x(b) = \beta$.*

Definition 1.2. *We say that an admissible trajectory $x^*(\cdot)$ is a (relative) extremizer to the basic problem of the calculus of variations* (1.1) *if $\mathcal{L}[x] - \mathcal{L}[x^*]$ has the same signal for all x that satisfies $\|x - x^*\|_0 < \epsilon$ for some positive real ϵ, where $\|\cdot\|_0$ denotes the $0-$norm, that is, $\|x\|_0 = \max\limits_{a \leq t \leq b} |x(t)|$.*

Euler and Lagrange proved the following necessary optimality condition for the basic problem of the calculus of variations:

$$\frac{\partial L}{\partial x}(t, x(t), \dot{x}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t)) = 0, \tag{1.2}$$

called the Euler–Lagrange equation.

Definition 1.3. *We say that an admissible trajectory $x(\cdot)$ is an extremal to the basic problem of the calculus of variations if it is solution of* (1.2).

It is well-known that the notions of symmetry and conservation laws play an important role in physics, engineering and mathematics [67, 73]. The interrelation between symmetry and conservation laws in the context of the Calculus of Variations is given by the first Noether theorem [54]. The first Noether theorem, usually known simply as Noether's theorem, guarantees that the invariance of a variational integral under a group of transformations depending smoothly on a parameter ϵ implies the existence of a conserved quantity along the Euler–Lagrange extremals. Such transformations are global transformations. Noether's theorem explains all conservation laws of mechanics, such as: conservation of energy comes from invariance of the system under time translations; conservation of linear momentum comes from invariance of the system under spatial translations; and conservation of angular momentum reflects invariance with respect to spatial rotations. The first Noether theorem is nowadays a well-known tool in modern theoretical physics, engineering and the Calculus of Variations [70]. Inexplicably, it is still not well-known that the famous paper of Emmy Noether [54] includes another important result: the second Noether theorem [69]. Noether's second theorem states that if a variational integral has an infinite-dimensional Lie algebra of infinitesimal symmetries parametrized linearly by r arbitrary functions and their derivatives up to a given order m, then there are r identities between Euler–Lagrange expressions and their derivatives up to order m. Such transformations are local transformations because can affect every part of the system differently.

Noether proved that properties of invariance lead to conservation laws, quantities that remain constant along extremals. Conservation laws have major applications, both physical and mathematical. For example, Lax and DiPerna applied conservation laws to the study of shock waves, Poincaré and Lyapunov used them to initiate stability theory and Morawetz and Strauss to scattering theory (for more details, see [29]).

In the last decades, Noether's theorems have been formulated in various other contexts: see [6, 7, 14, 22, 21, 23, 24, 35, 47, 48, 52, 68, 69, 71] and references therein.

We present next a simple version of the first Noether theorem, preceded by the respective definition of invariance under a one-parameter group of transformations. We will also present a version of Noether's second theorem, but only in Chapter 3; we will state there the Optimal Control version of Noether's second theorem.

Definition 1.4 (Classical invariance under a one-parameter group of transformations).

Let h^ϵ be a one-parameter group C^1 invertible transformations

$$\begin{aligned} h^\epsilon &: [a,b] \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}, \\ h^\epsilon(t,x(t)) &= (\mathcal{T}^\epsilon(t,x(t)), \mathcal{X}^\epsilon(t,x(t))), \\ h^0(t,x) &= (t,x), \quad \forall (t,x) \in [a,b] \times \mathbb{R}. \end{aligned}$$

The basic problem of the calculus of variations is said to be invariant under the one-parameter group of transformations h^ϵ if for all admissible $x(\cdot)$ the following condition holds:

$$\int_a^b L(t,x(t),\dot{x}(t))dt = \int_{a^\epsilon}^{b^\epsilon} L\left(\mathcal{T}^\epsilon, \mathcal{X}^\epsilon, \frac{d\mathcal{X}^\epsilon}{d\mathcal{T}^\epsilon}\right) d\mathcal{T}^\epsilon, \quad \text{with } \frac{d\mathcal{X}^\epsilon}{d\mathcal{T}^\epsilon} = \frac{\frac{d\mathcal{X}^\epsilon}{dt}}{\frac{d\mathcal{T}^\epsilon}{dt}},$$

where $a^\epsilon = \mathcal{T}^\epsilon(a,x(a))$ and $b^\epsilon = \mathcal{T}^\epsilon(b,x(b))$.

Theorem 1.5 (First Noether theorem [28, 46, 54, 74])**.** *If the basic problem of the calculus of variations is invariant under a one-parameter group of transformations in the sense of Definition 1.4, then the quantity*

$$\frac{\partial L}{\partial \dot{x}} X + \left(L - \frac{\partial L}{\partial \dot{x}} \dot{x}\right) T \tag{1.3}$$

is constant in t along every extremal of the basic problem of the calculus of variations, where

$$T = \left.\frac{\partial \mathcal{T}^\epsilon}{\partial \epsilon}\right|_{\epsilon=0} \quad \text{and} \quad X = \left.\frac{\partial \mathcal{X}^\epsilon}{\partial \epsilon}\right|_{\epsilon=0}.$$

Meanwhile, several extensions of the basic problem of the calculus of variations were made. We highlight two: the extension to higher-order problems and the extension to problems with time delay. The first one can be formulated as follows: determine the trajectories $x(\cdot) \in C^{2n}([a,b];\mathbb{R})$ such that

$$\begin{cases} \mathcal{L}[x] = \int_a^b L\left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t)\right) dt \to \text{extr}, \\ \text{subject to} \\ x(a) = \alpha_0, \quad x(b) = \beta_0 \\ \quad \vdots \\ x^{(n-1)}(a) = \alpha_{n-1}, \quad x^{(n-1)}(b) = \beta_{n-1} \end{cases} \tag{1.4}$$

where $n \in \mathbb{N}$, $a,b \in \mathbb{R}$ with $a < b$ and $\alpha_i, \beta_i \in \mathbb{R}$, $i = 0, \ldots, n-1$. We assume that the Lagrangian function L has continuous partial derivatives up to the order $n+1$ with respect to all its arguments, except with respect to t.

Theorem 1.6 (Classical higher-order Euler–Lagrange equation [28, 46, 74]). *If $x(\cdot)$ is an extremizer to the higher-order problem of the calculus of variations, then $x(\cdot)$ verifies the following higher-order Euler–Lagrange equation:*

$$\sum_{j=0}^{n}(-1)^j \frac{d^j}{dt^j}\left(\frac{\partial L}{\partial x^{(j)}}\left(t, x(t), \ldots, x^{(n)}(t)\right)\right) = 0, \quad t \in [a, b]. \tag{1.5}$$

The classical problem of the calculus of variations with time delay consists in extremizing the functional defined by

$$\mathcal{L}_\tau[x] = \int_a^b L(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau))dt, \tag{1.6}$$

subject to $x(t) = \mu(t), \quad t \in [a-\tau, a]$, where the Lagrangian $L : [a, b] \times \mathbb{R}^4 \to \mathbb{R}$ is a C^1 function for all arguments, $x(\cdot)$ is a C^2 function, τ is a real number such that $0 \leq \tau \leq b - a$ and μ is a given piecewise smooth function.

Theorem 1.7 (Classical Euler–Lagrange equations with time delay [2, 43]). *If a trajectory $x(\cdot)$ is an extremizer to the first-order delayed problem* (1.6)*, then $x(\cdot)$ satisfies the delayed Euler–Lagrange equations*

$$\frac{\partial L}{\partial x_\tau}[x]_\tau(t+\tau) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_\tau}[x]_\tau(t+\tau) + \frac{\partial L}{\partial x}[x]_\tau(t) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}[x]_\tau(t) = 0, \quad a \leq t \leq b-\tau \tag{1.7}$$

and

$$\frac{\partial L}{\partial x}[x]_\tau(t) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}[x]_\tau(t) = 0, \quad b-\tau \leq t \leq b, \tag{1.8}$$

where $[x]_\tau(t) := (t, x(t), \dot{x}(t), x_\tau(t), \dot{x}_\tau(t))$ and x_τ (resp. $\dot{x}_\tau$) refer to trajectories x (resp. $\dot{x}$) evaluated at $t - \tau$.

Remark 1.8. *The results of this chapter are trivially generalized for the case of vector functions $x : [a, b] \to \mathbb{R}^m$, $m \in \mathbb{N}$; this is the kind of trajectory that will be considered along the thesis.*

CHAPTER 2

HERGLOTZ'S VARIATIONAL PROBLEMS

It is well-known that the classical variational principle described in the previous chapter is a powerful tool in various disciplines such as physics, engineering and mathematics. However, the classical variational principle cannot describe many important physical processes.

In 1930, Gustav Herglotz [39, 40] proposed a generalized variational principle which generalizes the classical one.

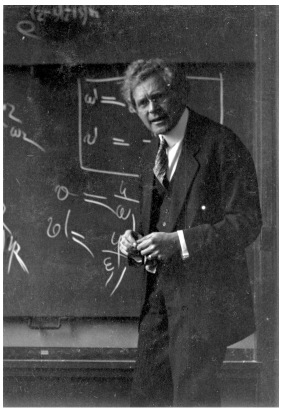

Figure 2.1: Gustav Herglotz, Göttingen, 1932

Gustav Herglotz[1] (1881–1953), see Figure 2.1, was a czech-born german mathematical physicist. Although his work is meaningful, not much of it has become widely known. He studied

[1]Author of photography: Kay Piene, Source: Ragni Piene and the archives of the Mathematisches Forschungsinstitut Oberwolfach

and taught mathematics and astronomy in Vienna, Munich and Göttingen. His branches of work included relativity theory, differential equations, number and function theory, geophysics, astronomy and applied mathematics to theoretical physics. Besides his undeniable scientific contributions, Salomon Bochner [9], who contacted personally with Herglotz, describes him as possessing great charm and perfect gentlemanliness, while Weisstein's World of Biography website [75] describes Herglotz as an enchanting lecturer, detailing that his lectures frequently attracted far more people than the university lecture halls could contain.

Herglotz was motivated to advance with his variational principle by the writings of Lie and Carathéodory and his own research on contact transformations and its connections with Hamiltonian systems and Poisson brackets. Several historical details on this matter are available in [12].

The generalized variational problem proposed by Herglotz in 1930 [39] can be formulated as follows:

$$\begin{gathered} z(b) \longrightarrow \text{extr} \\ \text{with } \dot{z}(t) = L(t, x(t), \dot{x}(t), z(t)), \quad t \in [a,b], \\ \text{subject to } z(a) = \gamma, \quad \gamma \in \mathbb{R}, \end{gathered} \tag{$\mathbf{H^1}$}$$

where by extr we mean minimize or maximize. Herglotz's variational problem consists in the determination of trajectories $x(\cdot) \equiv (x_1(\cdot), \ldots, x_m(\cdot))$ (and function $z(\cdot)$) subject to some initial condition $x(a) = \alpha$, $\alpha \in \mathbb{R}^m$, that extremize the value $z(b)$, where $L \in C^1([a,b] \times \mathbb{R}^{2m+1}; \mathbb{R})$, $x(\cdot) \in C^2([a,b]; \mathbb{R}^m)$ and $z(\cdot) \in C^1([a,b]; \mathbb{R})$.

Definition 2.1 (Admissible pair to problem $(\mathbf{H^1})$)**.** *We say that a pair $(x(\cdot), z(\cdot))$ with $x(\cdot) \in C^2([a,b]; \mathbb{R}^m)$ and $z(\cdot) \in C^1([a,b]; \mathbb{R})$ is an admissible pair to problem $(\mathbf{H^1})$ if it satisfies the equation*

$$\begin{gathered} \dot{z}(t) = L(t, x(t), \dot{x}(t), z(t)), \quad t \in [a,b], \\ \textit{subject to } z(a) = \gamma, \quad \gamma \in \mathbb{R}. \end{gathered}$$

Observe that equation $\dot{z}(t) = L(t, x(t), \dot{x}(t), z(t))$ represents a family of differential equations: for each function x a different differential equation arises. Therefore, z depends on x, a fact that can be made explicit by writing $z(t, x(t), \dot{x}(t))$ or $z[x; t]$, but for brevity and convenience of notation it is usual to write simply $z(t)$.

It is clear that Herglotz's problem $(\mathbf{H^1})$ reduces to the classical fundamental problem of the Calculus of Variations (1.1) if the Lagrangian L does not depend on the variable z. In fact, if

$\dot{z}(t) = L(t, x(t), \dot{x}(t))$, $t \in [a, b]$, then $(\mathbf{H^1})$ is equivalent to the classical variational problem

$$z(b) = \int_a^b \tilde{L}(t, x(t), \dot{x}(t))dt \longrightarrow \text{extr}, \qquad \text{subject to } z(a) = \gamma, \quad \gamma \in \mathbb{R}, \tag{2.1}$$

where

$$\tilde{L}(t, x, \dot{x}) = L(t, x, \dot{x}) + \frac{\gamma}{b-a}.$$

Herglotz proved that a necessary optimality condition for a pair $(x(\cdot), z(\cdot))$ to be a solution of the generalized variational problem $(\mathbf{H^1})$ is given by the system of equations

$$\frac{\partial L}{\partial x_i}(t, x(t), \dot{x}(t), z(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}(t, x(t), \dot{x}(t), z(t)) + \frac{\partial L}{\partial z}(t, x(t), \dot{x}(t), z(t))\frac{\partial L}{\partial \dot{x}_i}(t, x(t), \dot{x}(t), z(t)) = 0, \quad i = 1, \ldots, m, \tag{2.2}$$

$t \in [a, b]$. Equations (2.2) are known as the generalized Euler–Lagrange equations.

The system of the Euler–Lagrange equations (2.2) can be written in the condensed form

$$\frac{\partial L}{\partial x}(t, x(t), \dot{x}(t), z(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t), z(t)) + \frac{\partial L}{\partial z}(t, x(t), \dot{x}(t), z(t))\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t), z(t)) = 0, \quad t \in [a, b]. \tag{2.3}$$

Observe that for the classical problem of the Calculus of Variations one has $\frac{\partial L}{\partial z} = 0$ and equation (2.3) reduces (with $m = 1$) to the classical Euler–Lagrange equation (1.2).

Definition 2.2 (Generalized extremals—cf. [29, 31])**.** *The solutions $x(\cdot) \in C^2([a, b]; \mathbb{R}^m)$ of the generalized Euler–Lagrange equation* (2.3) *are called generalized extremals.*

As reported in [31, 32], unlike the classical variational principle, the variational principle of Herglotz gives a variational description of non-conservative processes, even when the Lagrangian is autonomous. For the importance to include nonconservativism in the Calculus of Variations, we refer the reader to the recent book [50].

According to Guenther [37], the solutions of (2.3) determine implicitly a family of contact transformations, that is, transformations that take two unions of elements with a common element and transform them into two new unions of elements, again with a common element. For the importance and applicability of these transformations in mathematics and physics we refer the reader to [12, 37, 55].

The generalized variational problem of Herglotz attracted the interest of the mathematical community only in 1996, with the publications [36, 37] by Ronald Guenther *et al.* Guenther eventually became Georgieva's Ph.D. supervisor; the main goal of Georgieva's thesis was to generalize the well known Noether's theorems to variational problems of Herglotz type [29, 30, 31, 32, 33].

Before presenting the generalization of the first Noether theorem to variational problems of Herglotz [31, 32], we introduce the definition of invariance under a one-parameter group of transformations:

Definition 2.3 (Invariance of problem $(\mathbf{H^1})$ under a one-parameter group of transformations [31]). *Let h^ϵ be a one-parameter family of C^1 invertible maps*

$$\begin{gathered}
h^\epsilon : [a,b] \times \mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^m, \\
h^\epsilon(t, x(t)) = (\mathcal{T}^\epsilon(t, x(t)), \mathcal{X}^\epsilon(t, x(t))), \\
h^0(t,x) = (t,x), \quad \forall (t,x) \in [a,b] \times \mathbb{R}^m.
\end{gathered}$$

Problem $(\mathbf{H^1})$ is said to be invariant under the one-parameter group of transformations h^ϵ if for all admissible pairs $(x(\cdot), z(\cdot))$ one has

$$\frac{d}{d\epsilon}\left[L\left(\mathcal{T}^\epsilon(t,x(t)), \mathcal{X}^\epsilon(t,x(t)), \frac{d\mathcal{X}^\epsilon}{d\mathcal{T}^\epsilon}(t,x(t)), z(\mathcal{T}^\epsilon(t,x(t)))\right)\frac{d\mathcal{T}^\epsilon}{dt}(t,x(t))\right]\bigg|_{\epsilon=0} = 0,$$

where $\frac{d\mathcal{X}^\epsilon}{d\mathcal{T}^\epsilon}(t,x(t)) = \frac{\frac{d\mathcal{X}^\epsilon}{dt}(t,x(t))}{\frac{d\mathcal{T}^\epsilon}{dt}(t,x(t))}$.

Theorem 2.4 (First Noether's theorem for the variational problem of Herglotz [31]). *If problem $(\mathbf{H^1})$ is invariant under a one-parameter group of transformations in the sense of Definition 2.3, then the quantities*

$$\lambda(t)\left[\frac{\partial L}{\partial \dot{x}_i}X + \left(L - \frac{\partial L}{\partial \dot{x}_i}\dot{x}_i(t)\right)T\right], \quad i = 1, \ldots, m, \tag{2.4}$$

are conserved along extremals of $(\mathbf{H^1})$, where $\lambda(t) = e^{-\int_a^t \frac{\partial L}{\partial z} d\theta}$. Moreover, L and its partial derivatives are evaluated at $(t, x(t), \dot{x}(t), z(t))$ and T and X are the infinitesimal generators of transformations:

$$T = \frac{\partial \mathcal{T}^\epsilon}{\partial \epsilon}(t,x(t))\bigg|_{\epsilon=0}, \quad X = \frac{\partial \mathcal{X}^\epsilon}{\partial \epsilon}(t,x(t))\bigg|_{\epsilon=0}.$$

Along the thesis we will present a more general notion of invariance than the previous one and generalize the previous result to higher-order problems of Herglotz with time delay. We will also prove the existence and deduce expressions of Noether currents for this kind of problem.

CHAPTER 3

OPTIMAL CONTROL THEORY

Typically, the Classical Calculus of Variations requires, for its applicability, the differentiability of the trajectories that solve the problem. Besides that, admissible trajectories take values on open sets. A more recent branch of mathematics, Optimal Control theory, takes dynamic optimization to another level. Optimal Control theory suffered a great development since the middle part of 20th century with the works of Lev Pontryagin and his co-workers [57], namely the maximum principle that will be presented within a few paragraphs in its weak form.

The optimal control formulation focuses upon one or more control variables that play the role of instruments of optimization. The presence of a control variable at centre stage does alter the basic orientation of the dynamic optimization problem.

The basic problem of optimal control consists in extremizing the functional

$$\mathcal{J}(x(\cdot),u(\cdot)) = \int_0^T L(t,x(t),u(t))dt$$

subject to $\dot{x}(t) = \varphi(t,x(t),u(t))$ and $x(0) = \alpha$, $\alpha \in \mathbb{R}^m$, where L, x, u and φ verify certain assumptions.

There are three major equivalent formulations for the optimal control problem: the previous one, which is Lagrange's, Mayers' and Bolza's forms. We will focus in the basic problem of optimal control written in the Bolza form:

$$\begin{gathered}\mathcal{J}(x(\cdot),u(\cdot)) = \int_a^b f(t,x(t),u(t))dt + \phi(x(b)) \longrightarrow \text{extr} \\ \text{subject to } \dot{x}(t) = \varphi(t,x(t),u(t)) \text{ and } x(a) = \alpha, \quad \alpha \in \mathbb{R}^m,\end{gathered} \tag{P}$$

where $f(\cdot) \in C^1([a,b] \times \mathbb{R}^m \times \Omega; \mathbb{R})$, $\phi(\cdot) \in C^1(\mathbb{R}^m; \mathbb{R})$, $\varphi(\cdot) \in C^1([a,b] \times \mathbb{R}^m \times \Omega; \mathbb{R}^m)$, $x(\cdot) \in PC^1([a,b]; \mathbb{R}^m)$ and $u(\cdot) \in PC([a,b]; \Omega)$, with $\Omega \subseteq \mathbb{R}^r$ an open set. In the literature of Optimal Control, x and u are called the state and control variables, respectively, while ϕ is known as the pay-off or salvage term. Note that the classical problem of the Calculus of Variations is a particular case of problem (P) with $\phi(x) \equiv 0$, $\varphi(t,x,u) = u$ and $\Omega = \mathbb{R}^m$. Note also that with the optimal control formulation we can trivially approach classical variational problems in the wider class of piecewise admissible functions.

The notation PC stands for "piecewise continuous" (for the precise meaning of piecewise continuity and piecewise differentiability see, e.g., [45, Sec. 1.1]). When dealing with PC functions we often write "for $t \in [a,b]$" meaning "for almost all $t \in [a,b]$".

One of the most important results in Optimal Control theory is Pontryagin's maximum principle proved in [57]. This result, which is a first-order necessary optimality condition, provides conditions for optimization problems with differential equations as constraints. The maximum principle is still widely used for solving control problems and other problems of dynamic optimization. Moreover, basic necessary optimality conditions from classical Calculus of Variations follow from Pontryagin's maximum principle.

Theorem 3.1 (Pontryagin's maximum principle for problem (P) [57])**.** *If a pair $(x(\cdot), u(\cdot))$ with $x(\cdot) \in PC^1([a,b]; \mathbb{R}^m)$ and $u(\cdot) \in PC([a,b]; \Omega)$ is a solution to problem (P), then there exists $\psi(\cdot) \in PC^1([a,b]; \mathbb{R}^m)$ such that the following conditions hold:*

- *the optimality condition*

$$\frac{\partial H}{\partial u}(t, x(t), u(t), \psi(t)) = 0; \tag{3.1}$$

- *the adjoint system*

$$\begin{cases} \dot{x}(t) = \frac{\partial H}{\partial \psi}(t, x(t), u(t), \psi(t)) \\ \dot{\psi}(t) = -\frac{\partial H}{\partial x}(t, x(t), u(t), \psi(t)); \end{cases} \tag{3.2}$$

- *and the transversality condition*

$$\psi(b) = grad(\phi(x))(b); \tag{3.3}$$

where the Hamiltonian H is defined by

$$H(t, x, u, \psi) = f(t, x, u) + \psi \cdot \varphi(t, x, u). \tag{3.4}$$

A forth variable arises with the maximum principle, ψ, called the co-state or adjoint variable, being a generalized "Lagrange multiplier". Like state or control variables, the co-state variable also depends on time, that is, $\psi = \psi(t)$.

Definition 3.2 (Pontryagin extremal to (P))**.** *A triplet $(x(\cdot), u(\cdot), \psi(\cdot))$ with $x(\cdot), \psi(\cdot) \in PC^1([a,b];\, \mathbb{R}^m)$ and $u(\cdot) \in PC([a,b];\Omega)$ is called a Pontryagin extremal to problem (P) if it satisfies the optimality condition* (3.1)*, the adjoint system* (3.2) *and the transversality condition* (3.3)*.*

A second important result that derives from the maximum principle is the following one. It relates the total and partial derivatives of the Hamiltonian.

Theorem 3.3 (DuBois–Reymond condition of Optimal Control [57])**.** *If $(x(\cdot), u(\cdot), \psi(\cdot))$ is a Pontryagin extremal to problem (P), then the Hamiltonian* (3.4) *satisfies the equality*

$$\frac{dH}{dt}(t, x(t), u(t), \psi(t)) = \frac{\partial H}{\partial t}(t, x(t), u(t), \psi(t)), \quad t \in [a,b].$$

The famous (first) Noether theorem [54] besides being a fundamental tool of the Calculus of Variations [71], and modern theoretical physics [25], is also a central tool in Optimal Control theory [67, 68, 72]. It states that when an optimal control problem is invariant under a one-parameter family of transformations, then there exists a corresponding conservation law: an expression that is conserved along all the Pontryagin extremals of the problem (see [67, 68, 72] and references therein).

Here we use Noether's theorem as found in [67], which is formulated for optimal control problems in Lagrange form, that is, for problem (P) with $\phi \equiv 0$. In order to apply the results of [67] to the Bolza problem (P), we rewrite it in the following equivalent Lagrange form:

$$\begin{gathered}
\mathcal{I}(x(\cdot), y(\cdot), u(\cdot)) = \int_a^b \left[f(t, x(t), u(t)) + y(t)\right] dt \longrightarrow \text{extr}, \\
\begin{cases} \dot{x}(t) = \varphi\left(t, x(t), u(t)\right), \\ \dot{y}(t) = 0, \end{cases} \\
x(a) = \alpha,\ y(a) = \frac{\phi(x(b))}{b-a}.
\end{gathered} \tag{3.5}$$

Before presenting the Noether theorem for the optimal control problem (P), we need to define the concept of invariance under a one-parameter group of transformations. Here we apply the notion of invariance found in [67] to the equivalent optimal control problem (3.5). In Definition 3.4 we use the little-o notation.

Definition 3.4 (Invariance of problem (P) under a one-parameter group of transformations cf. [67]). *Let h^ϵ be a one-parameter family of invertible C^1 maps*

$$\begin{gathered} h^\epsilon : [a,b] \times \mathbb{R}^m \times \Omega \longrightarrow \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^r, \\ h^\epsilon(t,x,u) = \left(\mathcal{T}^\epsilon(t,x,u), \mathcal{X}^\epsilon(t,x,u), \mathcal{U}^\epsilon(t,x,u)\right), \\ h^0(t,x,u) = (t,x,u) \text{ for all } (t,x,u) \in [a,b] \times \mathbb{R}^m \times \Omega. \end{gathered}$$

Problem (P) is said to be invariant under transformations h^ϵ if for all $(x(\cdot),u(\cdot))$ the following two conditions hold:

(i)

$$\begin{aligned} &\left[f \circ h^\epsilon(t,x(t),u(t)) + \frac{\phi(x(b))}{b-a} + \xi\epsilon + o(\epsilon)\right] \frac{d\mathcal{T}^\epsilon}{dt}(t,x(t),u(t)) \\ &\qquad\qquad = f(t,x(t),u(t)) + \frac{\phi(x(b))}{b-a} \end{aligned} \tag{3.6}$$

for some constant ξ;

(ii)

$$\frac{d\mathcal{X}^\epsilon}{dt}(t,x(t),u(t)) = \varphi \circ h^\epsilon(t,x(t),u(t)) \frac{d\mathcal{T}^\epsilon}{dt}(t,x(t),u(t)). \tag{3.7}$$

The next result can be easily obtained from the Noether theorem proved by Torres in [67] and Pontryagin's maximum principle (Theorem 3.1).

Theorem 3.5 (Noether's theorem for the optimal control problem(P)). *If problem (P) is invariant in the sense of Definition 3.4, then the quantity*

$$(b-t)\xi + \psi(t) \cdot X(t,x(t),u(t)) - \left[H(t,x(t),u(t),\psi(t)) + \frac{\phi(x(b))}{b-a}\right] \cdot T(t,x(t),u(t))$$

is constant in t along every Pontryagin extremal $(x(\cdot),u(\cdot),\psi(\cdot))$ of problem (P), where H is defined by (3.4) and

$$\begin{aligned} T(t,x(t),u(t)) &= \frac{\partial \mathcal{T}^\epsilon}{\partial \epsilon}(t,x(t),u(t)) \bigg|_{\epsilon=0}, \\ X(t,x(t),u(t)) &= \frac{\partial \mathcal{X}^\epsilon}{\partial \epsilon}(t,x(t),u(t)) \bigg|_{\epsilon=0}. \end{aligned}$$

Proof. The result is a simple exercise obtained by applying the Noether theorem of [67] and the Pontryagin maximum principle (Theorem 3.1) to the equivalent optimal control problem (3.5) (in particular using the adjoint equation corresponding to the multiplier associated with the state variable and the respective transversality condition). □

Before presenting Noether's second theorem for the optimal control problem (P), we need to introduced the notions of Noether current and semi-invariance under a group of symmetries. We follow the definitions presented in [69].

Definition 3.6 (Noether current [69])**.** *A function $C(t, x(t), u(t), \psi(t))$, which is constant along every Pontryagin extremal $(x(\cdot), u(\cdot), \psi(\cdot))$, is called a Noether current.*

Definition 3.7 (Semi-invariance of problem (P) under a group of symmetries [69])**.** *Let $p : [a, b] \to \mathbb{R}^d$ be an arbitrary function of class C^q. Using the notation*

$$\alpha(t) := \left(t, x(t), u(t), p(t), \dot{p}(t), \dots, p^{(q)}(t)\right),$$

we say that the optimal control problem (P) is semi-invariant if there exists a C^1 transformation group

$$\begin{aligned} g : [a, b] \times \mathbb{R}^m \times \Omega \times \mathbb{R}^{d\times(q+1)} &\to \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^r, \\ g(\alpha(t)) &= (\mathsf{T}(\alpha(t)), \mathsf{X}(\alpha(t)), \mathsf{U}(\alpha(t))), \end{aligned} \tag{3.8}$$

which for $p(t) = \dot{p}(t) = \cdots = p^{(q)}(t) = 0$ coincides with the identity transformation for all $(t, x, u) \in [a, b] \times \mathbb{R}^m \times \Omega$, satisfying the following conditions:

$$\begin{aligned} &\left(\theta_0 \cdot p(t) + \theta_1 \cdot \dot{p}(t) + \cdots + \theta_q \cdot p^{(q)}(t)\right) \frac{d}{dt} f(t, x(t), u(t)) + f(t, x(t), u(t)) \\ &\quad + \frac{\phi(x(b))}{b-a} + \frac{d}{dt} F(\alpha(t)) = \left(f(g(\alpha(t))) + \frac{\phi(\mathsf{X}(\alpha(b)))}{\mathsf{T}(\alpha(b)) - \mathsf{T}(\alpha(a))} \right) \frac{d}{dt} \mathsf{T}(\alpha(t)) \end{aligned}$$

and

$$\frac{d}{dt} \mathsf{X}(\alpha(t)) = \varphi\left(g(\alpha(t))\right) \frac{d}{dt} \mathsf{T}(\alpha(t)),$$

for some function F of class C^1 and some $\theta_0, \dots, \theta_q \in \mathbb{R}^d$.

Theorem 3.8 (Noether's second theorem for the optimal control problem (P) [69])**.** *If problem (P) is semi-invariant under a group of symmetries as in Definition 3.7, then there are $d(q+1)$ Noether currents of the form*

$$\begin{aligned} &\left.\frac{\partial F(\alpha(t))}{\partial p_J^{(I)}}\right|_0 + \theta_J^I \left(f(t, x(t), u(t)) + \frac{\phi(x(b))}{b-a} \right) \\ &\qquad + \psi(t) \cdot \left.\frac{\partial \mathsf{X}(\alpha(t))}{\partial p_J^{(I)}}\right|_0 - H(t, x(t), u(t), \psi(t)) \left.\frac{\partial \mathsf{T}(\alpha(t))}{\partial p_J^{(I)}}\right|_0 \end{aligned}$$

for $I = 0, \dots, q$, $J = 1, \dots, d$, where H is defined in (3.4) and $(*)|_0$ stands for $(*)|_{p(t)=\dot{p}(t)=\cdots=p^{(q)}(t)=0}$.

NOTATIONS AND SIMPLIFICATIONS

Throughout this thesis, several notations and simplifications are made aiming to simplify reading.

When dealing with problems with time delay, τ denotes a real number such that $0 \leq \tau < b-a$ and we use the notation $x_\tau^{(k)}(t)$, $k = 0, \ldots, n$, to denote the kth derivative of x evaluated at $t-\tau$; often we use $x_\tau(t)$ for $x_\tau^{(0)}(t) = x(t-\tau)$ and $\dot{x}_\tau(t)$ for $x_\tau^{(1)}(t) = \dot{x}(t-\tau)$.

We also introduce an operator that allows simplification of the Lagrangian arguments:

$$[x;z]_\tau^n(t) := \left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), x_\tau(t), \dot{x}_\tau(t), \ldots, x_\tau^{(n)}(t), z(t)\right).$$

Since this thesis does not focus entirely in higher-order problems with time delay, we also shorten the previous operator to several other variations, as follows:

$$\begin{aligned}
[x;z]^n(t) &:= \left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), z(t)\right);\\
[x;z](t) &:= (t, x(t), \dot{x}(t), z(t));\\
[x]_\tau^n(t) &:= \left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), x_\tau(t), \dot{x}_\tau(t), \ldots, x_\tau^{(n)}(t)\right);\\
[x;z]_\tau(t) &:= (t, x(t), \dot{x}(t), x_\tau(t), \dot{x}_\tau(t), z(t));\\
[x]_\tau(t) &:= (t, x(t), \dot{x}(t), x_\tau(t), \dot{x}_\tau(t)).
\end{aligned}$$

Along the text, we use the standard conventions $x^{(0)} = \frac{d^0 x}{dt^0} = x$ and $\sum_{k=1}^{j} \Upsilon(k) = 0$ whenever $j = 0$.

Part II

Original Work

CHAPTER 4

HIGHER-ORDER VARIATIONAL PROBLEMS OF HERGLOTZ

In this first original chapter, Herglotz's problem $(\mathbf{H^1})$ is extended to the higher-order case. A generalized Euler–Lagrange differential equation and transversality optimality conditions are obtained for higher-order Herglotz-type variational problems. In order to do so, we use the classical approach of introducing a variation in an admissible trajectory and study the necessary conditions for the trajectory to be an extremizer; the higher-order fundamental lemma of the Calculus of Variations and the higher-order integration by parts formulas on time scales proved by Martins and Torres [51, 53] are also used. Illustrative examples of the new results are also given.

The higher-order variational problem of Herglotz discussed in this chapter is defined as follows:

Problem $(\mathbf{H^n})$. *Determine the trajectories $x(\cdot) \in C^{2n}([a,b];\mathbb{R}^m)$ and $z(\cdot) \in C^1([a,b];\mathbb{R})$ such that:*

$$\begin{gathered} z(b) \longrightarrow extr, \\ with\ \dot{z}(t) = L\left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), z(t)\right), \quad t \in [a,b], \\ subject\ to\ z(a) = \gamma, \quad \gamma \in \mathbb{R}, \end{gathered} \tag{$\mathbf{H^n}$}$$

where the Lagrangian L is assumed to satisfy the following hypotheses:

i. L is a $C^1([a,b] \times \mathbb{R}^{(n+1)m+1};\mathbb{R})$ function;

ii. functions $t \mapsto \dfrac{\partial L}{\partial x^{(j)}}\left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), z(t)\right)$ *and* $t \mapsto \dfrac{\partial L}{\partial z}\left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), z(t)\right)$, $j = 0, \ldots, n$, *are differentiable up to order* n *for any admissible trajectory* x.

In line with what was said in Chapter 2 about the first-order problem of Herglotz, the generalized higher-order problem $(\mathbf{H^n})$ also generalizes the classical higher-order variational problem. In fact, if the Lagrangian L is independent of z, then

$$\dot{z}(t) = L\left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t)\right), \quad t \in [a, b],$$
$$z(a) = \gamma, \quad \gamma \in \mathbb{R},$$

which implies that the problem under consideration is the classical one:

$$z(b) = \int_a^b \tilde{L}\left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t)\right) dt \longrightarrow \mathrm{extr},$$

where

$$\tilde{L}\left(t, x, \dot{x}, \ldots, x^{(n)}\right) = L\left(t, x, \dot{x}, \ldots, x^{(n)}\right) + \frac{\gamma}{b-a}.$$

This chapter is organized as follows. In Section 4.1, we recall some results from the classical Calculus of Variations that are required to derive the main results of this chapter. In Section 4.2, we obtain the generalized Euler–Lagrange equation for problem $(\mathbf{H^n})$ in the class of functions $x(\cdot) \in C^{2n}([a, b]; \mathbb{R}^m)$ satisfying given boundary conditions

$$\begin{aligned} x(a) &= \alpha_0, \ \ldots, \ x^{(n-1)}(a) = \alpha_{n-1}, \\ x(b) &= \beta_0, \ \ldots, \ x^{(n-1)}(b) = \beta_{n-1}, \end{aligned} \tag{4.1}$$

where $\alpha_0, \ \ldots, \ \alpha_{n-1}, \beta_0, \ \ldots, \ \beta_{n-1} \in \mathbb{R}^m$. The transversality conditions (or natural boundary conditions) for problem $(\mathbf{H^n})$ are obtained in Section 4.3 and, in Section 4.4, we present some illustrative examples of application of the new results.

4.1 Preliminary results

We begin with some definitions and results that are useful in the sequel.

Definition 4.1 (Admissible pair to problem $(\mathbf{H^n})$). *We say that* $(x(\cdot), z(\cdot))$ *with* $x(\cdot) \in C^{2n}([a,b];\mathbb{R}^m)$ *and* $z(\cdot) \in C^1([a,b];\mathbb{R})$ *is an admissible pair to problem* $(\mathbf{H^n})$ *if it satisfies the equation*

$$\dot{z}(t) = L(t, x(t), \dot{x}(t), \cdots, x^{(n)}(t), z(t)), \quad t \in [a,b],$$
$$\text{with } z(a) = \gamma \in \mathbb{R}.$$

Definition 4.2. *We say that* $\eta(\cdot) \in C^{2n}([a,b];\mathbb{R}^m)$ *is an admissible variation for problem* $(\mathbf{H^n})$ *subject to boundary conditions* (4.1) *if, and only if,* $\eta(a) = \eta(b) = \cdots = \eta^{(n-1)}(a) = \eta^{(n-1)}(b) = 0$.

Lemma 4.3 (Higher-order integration by parts formulas – cf. [53]). *Let* $n \in \mathbb{N}$, $a, b \in \mathbb{R}$, $a < b$, *and* $f(\cdot), g(\cdot) \in C^n([a,b];\mathbb{R})$. *The following* n *equalities hold:*

$$\int_a^b f(t)g^{(i)}(t)dt = \left[\sum_{k=0}^{i-1}(-1)^k f^{(k)}(t)g^{(i-1-k)}(t)\right]_a^b + (-1)^i \int_a^b f^{(i)}(t)g(t)dt,$$

$i = 1, \ldots, n$.

Lemma 4.4 (Higher-order fundamental lemma of the Calculus of Variations – cf. [51]). *Let* $f_0(\cdot), \ldots, f_n(\cdot) \in C([a,b];\mathbb{R})$. *If*

$$\int_a^b \left(\sum_{i=0}^{n} f_i(t)\eta^{(i)}(t)\right) dt = 0$$

for all admissible variations η *of problem* $(\mathbf{H^n})$ *with* $m = 1$, *subject to boundary conditions* (4.1), *then*

$$\sum_{i=0}^{n}(-1)^i f_i^{(i)}(t) = 0,$$

$t \in [a,b]$.

4.2 Generalized Euler–Lagrange equations

The following result gives a necessary condition of Euler–Lagrange type for an admissible pair $(x(\cdot), z(\cdot))$ to be an extremizer of the functional $z[x;b]$, where z is defined by

$$\dot{z}(t) = L(t, x(t), \dot{x}(t), \cdots, x^{(n)}(t), z(t)), \quad t \in [a,b],$$
$$\text{and } z(a) = \gamma \in \mathbb{R},$$

where $x(\cdot) \equiv (x_1(\cdot), \ldots, x_m(\cdot))$ satisfies the boundary conditions (4.1).

In order to simplify expressions, we define $[x;z]^n(t) := \left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), z(t)\right)$.

Theorem 4.5 (Generalized higher-order Euler–Lagrange equations)**.** *If $(x(\cdot), z(\cdot))$ is a solution of problem $(\mathbf{H^n})$ subject to the boundary conditions (4.1), then the following generalized Euler–Lagrange equations hold:*

$$\sum_{j=0}^{n}(-1)^j \frac{d^j}{dt^j}\left(\lambda(t)\frac{\partial L}{\partial x_i^{(j)}}[x;z]^n(t)\right) = 0, \quad i = 1, \ldots, m, \tag{4.2}$$

$t \in [a,b]$, where $\lambda(t) := e^{-\int_a^t \frac{\partial L}{\partial z}[x;z]^n(\theta)d\theta}$.

Proof. Suppose that $x(\cdot) \equiv (x_1(\cdot), \ldots, x_m(\cdot))$ is a solution of $(\mathbf{H^n})$ subject to (4.1), and let $\eta(\cdot) \equiv (\eta_1(\cdot), \ldots, \eta_m(\cdot)) \in C^{2n}([a,b];\mathbb{R}^m)$ be an admissible variation. Let ϵ be an arbitrary real number. Define $\zeta : [a,b] \to \mathbb{R}$ by

$$\zeta(t) := \frac{d}{d\epsilon} z[x+\epsilon\eta;t]\bigg|_{\epsilon=0} = \frac{d}{d\epsilon} z\left(t, x(t)+\epsilon\eta(t), \dot{x}(t)+\epsilon\dot{\eta}(t), \ldots, x^{(n)}(t)+\epsilon\eta^{(n)}(t)\right)\bigg|_{\epsilon=0}.$$

Obviously, $\zeta(a) = 0$ and, since z is a minimizer (resp. maximizer), we have

$$z\left(b, x(b)+\epsilon\eta(b), \dot{x}(b)+\epsilon\dot{\eta}(b), \ldots, x^{(n)}(b)+\epsilon\eta^{(n)}(b)\right) \geq (\text{resp. } \leq)\, z\left(b, x(b), \dot{x}(b), \ldots, x^{(n)}(b)\right).$$

Hence, $\zeta(b) = \frac{d}{d\epsilon} z[x+\epsilon\eta;b]\big|_{\epsilon=0} = 0$ and because

$$\begin{aligned}
\dot{\zeta}(t) &= \frac{d}{dt}\frac{d}{d\epsilon} z\left(t, x(t)+\epsilon\eta(t), \dot{x}(t)+\epsilon\dot{\eta}(t), \ldots, x^{(n)}(t)+\epsilon\eta^{(n)}(t)\right)\bigg|_{\epsilon=0} \\
&= \frac{d}{d\epsilon}\frac{d}{dt} z\left(t, x(t)+\epsilon\eta(t), \dot{x}(t)+\epsilon\dot{\eta}(t), \ldots, x^{(n)}(t)+\epsilon\eta^{(n)}(t)\right)\bigg|_{\epsilon=0} \\
&= \frac{d}{d\epsilon} L[x+\epsilon\eta, z]^n(t)\bigg|_{\epsilon=0},
\end{aligned}$$

we conclude that

$$\begin{aligned}
\dot{\zeta}(t) &= \sum_{i=1}^{m}\sum_{j=0}^{n}\left(\frac{\partial L}{\partial x_i^{(j)}}[x;z]^n(t)\eta_i^{(j)}(t)\right) + \frac{\partial L}{\partial z}[x;z]^n(t)\frac{d}{d\epsilon} z[x+\epsilon\eta;t]\bigg|_{\epsilon=0} \\
&= \sum_{i=1}^{m}\sum_{j=0}^{n}\left(\frac{\partial L}{\partial x_i^{(j)}}[x;z]^n(t)\eta_i^{(j)}(t)\right) + \frac{\partial L}{\partial z}[x;z]^n(t)\zeta(t).
\end{aligned}$$

Thus, ζ satisfies a first order linear differential equation whose solution is found according to

$$\dot{y} - Py = Q \Leftrightarrow e^{-\int_a^t P(\theta)d\theta} y(t) - y(a) = \int_a^t e^{-\int_a^s P(\theta)d\theta} Q(s)ds.$$

Therefore,

$$e^{-\int_a^t \frac{\partial L}{\partial z}[x;z]^n(\theta)d\theta}\zeta(t) - \zeta(a) = \int_a^t e^{-\int_a^s \frac{\partial L}{\partial z}[x;z]^n(\theta)d\theta} \left(\sum_{i=1}^m \sum_{j=0}^n \frac{\partial L}{\partial x_i^{(j)}}[x;z]^n(s)\, \eta_i^{(j)}(s) \right) ds.$$

Denoting $\lambda(t) := e^{-\int_a^t \frac{\partial L}{\partial z}[x;z]^n(\theta)d\theta}$, we get

$$\lambda(t)\zeta(t) - \zeta(a) = \int_a^t \lambda(s) \left(\sum_{i=1}^m \sum_{j=0}^n \frac{\partial L}{\partial x_i^{(j)}}[x;z]^n(s)\eta_i^{(j)}(s) \right) ds.$$

In particular, for $t = b$, we have

$$\lambda(b)\zeta(b) - \zeta(a) = \int_a^b \lambda(s) \left(\sum_{i=1}^m \sum_{j=0}^n \frac{\partial L}{\partial x_i^{(j)}}[x;z]^n(s)\eta_i^{(j)}(s) \right) ds.$$

Since $\zeta(t) = 0$ for $t \in \{a, b\}$, the left-hand side of the previous equation vanishes and we get

$$0 = \int_a^b \sum_{i=1}^m \sum_{j=0}^n \lambda(s) \frac{\partial L}{\partial x_i^{(j)}}[x;z]^n(s)\eta_i^{(j)}(s)ds.$$

Fix $i = 1, \ldots, m$ and let $\eta_k(s) = 0$ for all $k \neq i$ and $s \in [a, b]$. Using the higher-order fundamental lemma of the Calculus of Variations (Lemma 4.4), we obtain, for each $i = 1, \ldots, m$, the generalized Euler–Lagrange equation

$$\sum_{j=0}^n (-1)^j \frac{d^j}{dt^j} \left(\lambda(t) \frac{\partial L}{\partial x_i^{(j)}}[x;z]^n(t) \right) = 0,$$

$t \in [a, b]$, proving the intended result. $\square$

In order to simplify expressions, and in agreement with Theorem 4.5, from now on we use the notation $\lambda(t) := e^{-\int_a^t \frac{\partial L}{\partial z}[x;z]^n(\theta)d\theta}$.

If $n = 1$, the differential equation of problem $(\mathbf{H^n})$ reduces to $\dot{z}(t) = L\left(t, x(t), \dot{x}(t), z(t)\right)$, which defines the function z of Herglotz's variational principle $(\mathbf{H^1})$. This principle is a particular case of our Theorem 4.5 and is given in Corollary 4.6.

Corollary 4.6 (See e.g. [29, 30, 37, 39]). *If $(x(\cdot), z(\cdot))$ is a solution of the first-order problem of Herglotz* $(\mathbf{H^1})$ *subject to* (4.1)*, then the following equations hold*

$$\frac{\partial L}{\partial x_i}(t, x(t), \dot{x}(t), z(t)) + \frac{\partial L}{\partial z}(t, x(t), \dot{x}(t), z(t)) \frac{\partial L}{\partial \dot{x}_i}(t, x(t), \dot{x}(t), z(t)) \\ - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}(t, x(t), \dot{x}(t), z(t)) = 0, \quad (4.3)$$

for all $t \in [a, b]$ and $i = 1, \ldots, m$.

Observe that when $m = 1$, (4.3) coincides with (2.3).

Our generalized higher-order Euler–Lagrange equations (4.2) are also a generalization of the classical Euler–Lagrange equations for higher-order variational problems.

Corollary 4.7 (See, e.g., [28]). *Suppose that $x(\cdot)$ is a solution of problem* $(\mathbf{H^n})$ *subject to* (4.1)*, and that the Lagrangian L is independent of z. Then $x(\cdot)$ satisfies the classical higher-order Euler–Lagrange differential equations*

$$\sum_{j=0}^{n} (-1)^j \frac{d^j}{dt^j} \left(\frac{\partial L}{\partial x_i^{(j)}} \left(t, x(t), \ldots, x^{(n)}(t)\right) \right) = 0, \quad (4.4)$$

$t \in [a, b]$ and $i = 1, \ldots, m$.

The system of generalized Euler–Lagrange equations (4.2) can be written in the condensed form

$$\sum_{j=0}^{n} (-1)^j \frac{d^j}{dt^j} \left(\lambda(t) \frac{\partial L}{\partial x^{(j)}}[x; z]^n(t) \right) = 0, \quad t \in [a, b]. \quad (4.5)$$

From now on, in order to shorten notations and to be visually friendly, we will present our results in the condensed form.

4.3 Generalized natural boundary conditions

We now consider the case when the values of $x(a)$, ..., $x^{(n-1)}(a)$, $x(b)$, ..., $x^{(n-1)}(b)$, are not necessarily specified.

Theorem 4.8 (Generalized natural boundary conditions). *Suppose that $(x(\cdot), z(\cdot))$ is a solution of problem* $(\mathbf{H^n})$*. Then $(x(\cdot), z(\cdot))$ satisfies the generalized Euler–Lagrange equation* (4.5)*. Moreover,*

1. *If $x^{(k)}(b)$ is free for some $k \in \{0, \ldots, n-1\}$, then the natural boundary condition*

$$\sum_{j=1}^{n-k}(-1)^{j-1}\frac{d^{j-1}}{dt^{j-1}}\left(\lambda(t)\frac{\partial L}{\partial x^{(k+j)}}[x;z]^n(t)\right)\Bigg|_{t=b} = 0 \tag{4.6}$$

   holds.

2. *If $x^{(k)}(a)$ is free for some $k \in \{0, \ldots, n-1\}$, then the natural boundary condition*

$$\sum_{j=1}^{n-k}(-1)^{j-1}\frac{d^{j-1}}{dt^{j-1}}\left(\lambda(t)\frac{\partial L}{\partial x^{(k+j)}}[x;z]^n(t)\right)\Bigg|_{t=a} = 0 \tag{4.7}$$

   holds.

Proof. Suppose that $(x(\cdot), z(\cdot))$ is a solution of problem $(\mathbf{H^n})$. Let $\eta(\cdot) \in C^{2n}([a,b];\mathbb{R}^m)$ and define the function ζ just like in the proof of Theorem 4.5. From the arbitrariness of η, and using similar arguments as the ones in the proof of Theorem 4.5, we conclude that $(x(\cdot), z(\cdot))$ satisfies the generalized Euler–Lagrange equation (4.5). We now prove (4.6) (the proof of (4.7) follows exactly the same arguments). Suppose that $x^{(k)}(b)$ is free for some $k \in \{0, \ldots, n-1\}$. Let $J := \{j \in \{0, \ldots, n-1\} : x^{(j)}(a) \text{ is given}\}$. For any $j \in \{0, \ldots, n-1\}$, if $j \in J$, then $\eta^{(j)}(a) = 0$; otherwise, we restrict ourselves to those functions η such that $\eta^{(j)}(a) = 0$. For convenience, we also suppose that $\eta^{(n)}(a) = 0$. Using the same arguments as the ones used in the proof of Theorem 4.5, we find that ζ satisfies the first order linear differential equation

$$\dot{\zeta}(t) = \frac{\partial L}{\partial x}[x;z]^n(t)\eta(t) + \frac{\partial L}{\partial \dot{x}}[x;z]^n(t)\dot{\eta}(t) + \cdots + \frac{\partial L}{\partial x^{(n)}}[x;z]^n(t)\eta^{(n)}(t) + \frac{\partial L}{\partial z}[x;z]^n(t)\zeta(t),$$

whose solution is found by

$$\lambda(t)\zeta(t) - \zeta(a) = \int_a^t \sum_{j=0}^{n} \lambda(s)\frac{\partial L}{\partial x^{(j)}}[x;z]^n(s)\eta^{(j)}(s)ds.$$

Again, since $\zeta(t) = 0$, for $t \in \{a, b\}$, we get

$$\int_a^b \sum_{j=0}^{n} \lambda(s)\frac{\partial L}{\partial x^{(j)}}[x;z]^n(s)\eta^{(j)}(s)ds = 0$$

and, therefore,

$$\int_a^b \lambda(s)\frac{\partial L}{\partial x}[x;z]^n(s)\eta(s)ds + \sum_{j=1}^{n}\int_a^b \lambda(s)\frac{\partial L}{\partial x^{(j)}}[x;z]^n(s)\eta^{(j)}(s)ds = 0.$$

Using the higher-order integration by parts formula (Lemma 4.3) in the second parcel we get

$$\int_a^b \lambda(s)\frac{\partial L}{\partial x}[x;z]^n(s)\eta(s)ds$$
$$+\sum_{j=1}^{n}\left(\left[\lambda(s)\frac{\partial L}{\partial x^{(j)}}[x;z]^n(s)\eta^{(j-1)}(s)+\sum_{i=1}^{j-1}(-1)^i\left(\lambda(s)\frac{\partial L}{\partial x^{(j)}}[x;z]^n(s)\right)^{(i)}\eta^{(j-1-i)}(s)\right]_a^b\right.$$
$$\left.+(-1)^j\int_a^b\left(\lambda(s)\frac{\partial L}{\partial x^{(j)}}[x;z]^n(s)\right)^{(j)}\eta(s)ds\right)=0,$$

which is equivalent to

$$\int_a^b\sum_{j=0}^{n}(-1)^j\left(\lambda(s)\frac{\partial L}{\partial x^{(j)}}[x;z]^n(s)\right)^{(j)}\eta(s)ds$$
$$+\sum_{j=1}^{n}\left[\lambda(s)\frac{\partial L}{\partial x^{(j)}}[x;z]^n(s)\eta^{(j-1)}(s)+\sum_{i=1}^{j-1}(-1)^i\left(\lambda(s)\frac{\partial L}{\partial x^{(j)}}[x;z]^n(s)\right)^{(i)}\eta^{(j-1-i)}(s)\right]_a^b=0.$$

Using the generalized Euler–Lagrange equation (4.2) into the last equation we get

$$\sum_{j=1}^{n}\left[\lambda(s)\frac{\partial L}{\partial x^{(j)}}[x;z]^n(s)\eta^{(j-1)}(s)+\sum_{i=1}^{j-1}(-1)^i\left(\lambda(s)\frac{\partial L}{\partial x^{(j)}}[x;z]^n(s)\right)^{(i)}\eta^{(j-1-i)}(s)\right]_a^b=0$$

and since $\eta(a)=\dot{\eta}(a)=\cdots=\eta^{(n-1)}(a)=0$, we conclude that

$$\sum_{j=1}^{n}\left(\lambda(s)\frac{\partial L}{\partial x^{(j)}}[x;z]^n(s)\eta^{(j-1)}(s)+\sum_{i=1}^{j-1}(-1)^i\left(\lambda(s)\frac{\partial L}{\partial x^{(j)}}[x;z]^n(s)\right)^{(i)}\eta^{(j-1-i)}(s)\right)\Bigg|_{s=b}=0.$$

This equation is equivalent to

$$\sum_{i=0}^{n-1}\left(\sum_{j=1}^{n-i}(-1)^{j-1}\left(\lambda(s)\frac{\partial L}{\partial x^{(i+j)}}[x;z]^n(s)\right)^{(j-1)}\eta^{(i)}(s)\right)\Bigg|_{s=b}=0.$$

Let $I:=\left\{i\in\{0,\ldots,n-1\}: x^{(i)}(b) \text{ is given}\right\}$. Note that $k\notin I$. For any $i\in\{0,\ldots,n-1\}$, if $i\in I$, then $\eta^{(i)}(b)=0$; otherwise, for $i\neq k$, we restrict ourselves to those functions η such that $\eta^{(i)}(b)=0$. From the arbitrariness of $\eta^{(k)}(b)$, it follows that

$$\sum_{j=1}^{n-k}(-1)^{j-1}\frac{d^{j-1}}{dt^{j-1}}\left(\lambda(s)\frac{\partial L}{\partial x^{(k+j)}}[x;z]^n(s)\right)\Bigg|_{s=b}=0.$$

This concludes the proof. $\square$

Remark 4.9. *If $(x(\cdot), z(\cdot))$ is a solution to problem $(\mathbf{H^n})$ without any of the $2n$ boundary conditions (4.1), then $(x(\cdot), z(\cdot))$ satisfies the generalized higher-order Euler–Lagrange equations (4.5), the n transversality conditions (4.6) and the n transversality conditions (4.7). In general, for each boundary condition missing in (4.1), there is a corresponding natural boundary condition, as given by Theorem 4.8.*

Next we remark that our generalized transversality conditions (4.6) and (4.7) are generalizations of the classical transversality conditions for higher-order variational problems (cf. $\psi^k = 0$, $k = 0, \ldots, n-1$, with ψ^k given as in [71, Section 5]).

Corollary 4.10. *Suppose that $x(\cdot)$ is a solution of problem $(\mathbf{H^n})$ with L independent of z. Then x satisfies the classical higher-order Euler–Lagrange equations (4.4). Moreover,*

1. *If $x^{(k)}(b)$ is free for some $k \in \{0, \ldots, n-1\}$, then the natural boundary condition*

$$\sum_{j=1}^{n-k} (-1)^{j-1} \frac{d^{j-1}}{dt^{j-1}} \left(\frac{\partial L}{\partial x^{(k+j)}} \right) \left(b, \dot{x}(b), \ldots, x^{(n)}(b)\right) = 0$$

holds.

2. *If $x^{(k)}(a)$ is free for some $k \in \{0, \ldots, n-1\}$, then the natural boundary condition*

$$\sum_{j=1}^{n-k} (-1)^{j-1} \frac{d^{j-1}}{dt^{j-1}} \left(\frac{\partial L}{\partial x^{(k+j)}} \right) \left(a, \dot{x}(a), \ldots, x^{(n)}(a)\right) = 0$$

holds.

4.4 Illustrative examples

We illustrate the usefulness of our results with some examples that are not covered by previous available results in the literature. Let us consider the particular case of Theorem 4.5 with $n = 2$ and $m = 1$.

Corollary 4.11. *Let z be a solution of $\dot{z}(t) = L\left(t, x(t), \dot{x}(t), \ddot{x}(t), z(t)\right)$, $t \in [a, b]$, subject to the boundary conditions $z(a) = \gamma$, $x(a) = \alpha_0$, $\dot{x}(a) = \alpha_1$, $x(b) = \beta_0$, and $\dot{x}(b) = \beta_1$, where γ, α_0, α_1, β_0, and β_1, are given real numbers. If $(x(\cdot), z(\cdot))$ is a solution of the* *second-order problem of Herglotz, then $(x(\cdot), z(\cdot))$ satisfies the differential equation*

$$\begin{aligned}
&\frac{\partial L}{\partial x}[x;z]^2(t) + \frac{\partial L}{\partial z}[x;z]^2(t)\frac{\partial L}{\partial \ddot{x}}[x;z]^2(t) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}[x;z]^2(t) + \left(\frac{\partial L}{\partial z}[x;z]^2(t)\right)^2 \frac{\partial L}{\partial \ddot{x}}[x;z]^2(t) \\
&- 2\frac{\partial L}{\partial z}[x;z]^2(t)\frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}[x;z]^2(t) - \left(\frac{d}{dt}\frac{\partial L}{\partial z}[x;z]^2(t)\right)\frac{\partial L}{\partial \ddot{x}}[x;z]^2(t) + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}}[x;z]^2(t) = 0,
\end{aligned} \tag{4.8}$$

$t \in [a,b]$, *where* $[x;z]^2(t) = (t, x(t), \dot{x}(t), \ddot{x}(t), z(t))$.

We now apply Corollary 4.11 to concrete situations.

Example 4.12. *Let us consider the following Herglotz's higher-order variational problem:*

$$\begin{gathered}
z(1) \longrightarrow \min, \\
\dot{z}(t) = \ddot{x}^2(t) + z^2(t), \quad t \in [0,1], \quad z(0) = \frac{1}{2}, \\
x(0) = 0, \quad \dot{x}(0) = 1, \quad x(1) = 1, \quad \dot{x}(1) = 1.
\end{gathered} \tag{4.9}$$

For this problem, the necessary optimality condition (4.8) *asserts that*

$$x^{(4)}(t) - 4z(t)x^{(3)}(t) + \left(4z^2(t) - 2\dot{z}(t)\right)x^{(2)}(t) = 0. \tag{4.10}$$

Solving the system formed by (4.10) *and* $\dot{z}(t) = \ddot{x}^2(t) + z^2(t)$, *subject to the given boundary conditions, gives the extremal*

$$x(t) = t, \quad z(t) = \frac{1}{2-t},$$

for which $z(1) = 1$.

Example 4.13. *Consider problem* (4.9) *with* $z(0) = z_0$ *free. We show that such problem is not well defined. Indeed, if a solution exists, we obtain the optimality system*

$$\begin{cases} x^{(4)}(t) - 4z(t)x^{(3)}(t) + \left(4z^2(t) - 2\dot{z}(t)\right)x^{(2)}(t) = 0 \\ \dot{z}(t) = \ddot{x}^2(t) + z^2(t) \end{cases} \tag{4.11}$$

subject to $x(0) = 0$ *and* $\dot{x}(0) = x(1) = \dot{x}(1) = 1$. *It follows that*

$$x(t) = t, \quad z(t) = \frac{z_0}{1 - z_0 t},$$

and we conclude that the problem has no solution: the infimum is $-\infty$ *obtained when* $z_0 \to 1^+$.

Example 4.14. *Consider now the following problem:*

$$\begin{gathered} z(1) \longrightarrow \min, \\ \dot{z}(t) = \ddot{x}^2(t) + z(t), \quad t \in [0,1], \quad z(0) = 1, \\ x(0) = 0, \quad \dot{x}(0) = 1, \quad x(1) = 1, \quad \dot{x}(1) = 0. \end{gathered} \tag{4.12}$$

For problem (4.12)*, the necessary optimality condition* (4.8) *asserts that*

$$x^{(4)}(t) - 2x^{(3)}(t) + x^{(2)}(t) = 0. \tag{4.13}$$

Solving the system formed by (4.13) *and* $\dot{z}(t) = \ddot{x}^2(t) + z(t)$*, subject to the given boundary conditions, gives the extremal*

$$x(t) = \frac{(1-t)e^{t+1} + (2t-1)e^t + (e-3)et - e + 1}{e^2 - 3e + 1},$$

$$z(t) = \frac{[(1+t^2)e^{t+2} - 2(2t^2+t+2)e^{t+1} + (4t^2+4t+5)e^t + e^4 - 6\,e^3 + 10\,e^2 - 2\,e - 4]\,e^t}{(e^2 - 3\,e + 1)^2},$$

for which $z(1) = \frac{(e^2-e-4)e}{e^2-3e+1} \gtrsim 7,78$.

Our last example shows the usefulness of Theorem 4.8.

Example 4.15. *We now consider problem* (4.12) *with* $\dot{x}(1)$ *free. In this case, solving*

$$\begin{cases} x^{(4)}(t) - 2x^{(3)}(t) + x^{(2)}(t) = 0 \\ \dot{z}(t) = \ddot{x}^2(t) + z(t) \end{cases}$$

subject to the boundary conditions $z(0) = 1$, $x(0) = 0$, $\dot{x}(0) = 1$, $x(1) = 1$, *and the natural boundary condition* (4.6) *for* $n = 2$ *and* $k = 1$*, that in the present situation simplifies to* $\ddot{x}(1) = 0$*, gives the extremal*

$$x(t) = t, \quad z(t) = e^t,$$

for which $\dot{x}(1) = 1$ *and* $z(1) = e \lesssim 2,72$.

4.5 Conclusions

The results of this chapter generalize both the classical higher-order problem of the Calculus of Variations [28, 46, 74] and the first-order Herglot'z problem [39]. We were able to prove generalized higher-order Euler–Lagrange equations for higher-order variational problems of Herglotz and natural boundary conditions for case of unspecified initial or final conditions.

The original results of this chapter were published in 2014 in [59]. They were also presented by the author in the EURO mini Conference on Optimization in the Natural Sciences, February 5–9, 2014, Aveiro, Portugal, in a contributed talk entitled "Higher-order variational problems of Herglotz-type".

CHAPTER 5

FIRST-ORDER VARIATIONAL PROBLEMS OF HERGLOTZ WITH TIME DELAY

In this chapter, we generalize Herglotz's problem $(\mathbf{H^1})$ by considering the generalized variational problem of Herglotz in which the trajectories also depend on past arguments.

Dynamical systems with time delay are very important in modelling real-life phenomena in several fields, such as mathematics, biology, chemistry, economics and mechanics. Indeed, several process outcomes are determined not only by variables at present time, but also by its behaviour in the past. Motivated by the importance of problems with time delay, many works generalized the classical results of the Calculus of Variations to the delayed case. The first one in this direction seems to have been published by Èl'sgol'c [19]. Since then, several authors have worked on various aspects of variational problems with time delay arguments (see [2, 34, 38, 43, 56, 58] and references therein).

Although several generalizations of variational problems have been made, only recently Frederico and Torres generalized the important Noether's first theorem to Optimal Control problems with time delay [20, 24]. For more recent works on optimal control problems with time delay see [8, 15, 34] and references therein. The importance of variational problems of Herglotz, as well as the wide applicability of problems with time delay, allied to the impossibility of applying the classical Noether theorem to these problems, constituted the main motivation to the paper [60] who is the basis of the present chapter.

The main goal of this chapter is to extend the generalized Euler–Lagrange equation, the DuBois–Reymond optimality condition and Noether's theorem to variational problems of Herglotz type with time delay.

Throughout the text, τ denotes a real number such that $0 \leq \tau < b-a$. To simplify notation, we write $z[x]_\tau(t) := z(t, x(t), \dot{x}(t), x_\tau(t), \dot{x}_\tau(t))$ and $[x; z]_\tau(t) := (t, x(t), \dot{x}(t), x_\tau(t), \dot{x}_\tau(t), z(t))$, where $x_\tau(t) = x(t-\tau)$ and $\dot{x}_\tau(t) = \dot{x}(t-\tau)$. When there is no possibility of ambiguity, we sometimes suppress arguments.

In this chapter we consider the following first-order delayed problem of Herglotz type:

Problem $(\mathbf{H}_\tau)$. *Let τ be a real number such that $0 \leq \tau < b-a$. Determine the trajectories $x(\cdot) \in C^2([a-\tau, b]; \mathbb{R}^m)$ and $z(\cdot) \in C^1([a, b]; \mathbb{R})$ such that:*

$$\begin{gathered} z(b) \longrightarrow extr, \\ \text{with } \dot{z}(t) = L\left(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau), z(t)\right), \quad t \in [a, b], \\ \text{subject to } z(a) = \gamma, \quad \gamma \in \mathbb{R}, \\ \text{and to } x(b) = \beta \quad \text{and} \quad x(t) = \mu(t), \quad t \in [a-\tau, a], \end{gathered} \tag{$\mathbf{H}_\tau$}$$

where $\beta \in \mathbb{R}^m$ and $\mu(\cdot) \in C^2([a-\tau, a]; \mathbb{R}^m)$ is a given initial function, and the Lagrangian L is assumed to satisfy the following hypotheses:

i. L is a $C^1([a, b] \times \mathbb{R}^{4m+1}; \mathbb{R})$ function;

ii. functions $t \mapsto \frac{\partial L}{\partial z}[x; z]_\tau(t)$, $t \mapsto \frac{\partial L}{\partial x^{(j)}}[x; z]_\tau(t)$ and $t \mapsto \frac{\partial L}{\partial x_\tau^{(j)}}[x; z]_\tau(t)$, $j = 0, 1$, are differentiable for any admissible trajectory x.

Observe that the previous problem reduces to the classical fundamental problem of the Calculus of Variations with time delay if the Lagrangian L does not depend on z. Also note that problem $(\mathbf{H}_\tau)$ reduces to the generalized variational problem of Herglotz $(\mathbf{H^1})$ when $\tau = 0$.

The structure of the chapter is as follows. We begin by reviewing some preliminaries about the generalized variational calculus (without time delay). In particular, we recall the notion of invariance and the first Noether theorem for variational problems of Herglotz type. Our main results are given thereafter: in Section 5.2, a generalized Euler–Lagrange necessary optimality condition (Theorem 5.6) and a DuBois–Reymond necessary optimality condition (Theorem 5.9); and in Section 5.3, a Noether's first theorem for variational problems of Herglotz type with time delay (Theorem 5.14). We end with an illustrative example of our results in Section 5.4.

5.1 Review of Noether's theorem for variational problems of Herglotz type

For the convenience of the reader, we present here the definition of invariance functional z, defined by $\dot{z} = L(t, x, \dot{x}, z)$ and $z(a) = \gamma$, under a one-parameter group of transformations and we recall Noether's first theorem for the generalized variational problem of Herglotz type.

Consider a one-parameter group of infinitesimal transformations on $\mathbb{R}^{1+m}$,

$$\bar{t} = \mathcal{T}(t, x, \epsilon), \quad \bar{x} = \mathcal{X}(t, x, \epsilon), \tag{5.1}$$

in which ϵ is the parameter and $\mathcal{T}$ and $\mathcal{X}$ are invertible C^1 functions such that $\mathcal{T}(t, x, 0) = t$ and $\mathcal{X}(t, x, 0) = x$. The infinitesimal representation of transformations (5.1) is given by

$$\begin{aligned} \bar{t} &= t + T(t, x)\epsilon + o(\epsilon), \\ \bar{x} &= x + X(t, x)\epsilon + o(\epsilon), \end{aligned}$$

where T and X denote the first degree coefficients of ϵ. Explicitly,

$$T(t, x) = \frac{\partial \mathcal{T}}{\partial \epsilon}(t, x, \epsilon)\Big|_{\epsilon=0}, \qquad X(t, x) = \frac{\partial \mathcal{X}}{\partial \epsilon}(t, x, \epsilon)\Big|_{\epsilon=0}.$$

Definition 5.1 (Invariance under a one-parameter group of transformations—cf. Proposition 3.1 of [31]). *The one-parameter group of transformations* (5.1) *leaves invariant the functional z, defined by $\dot{z} = L(t, x, \dot{x}, z)$ and $z(a) = \gamma$ for some fixed real number γ, if*

$$\frac{d}{d\epsilon}\left[L\left(\bar{t}, \bar{x}(\bar{t}), \frac{d\bar{x}}{d\bar{t}}(\bar{t}), \bar{z}(\bar{t})\right) \cdot \frac{d\bar{t}}{dt}\right]\Big|_{\epsilon=0} = 0.$$

We now prove the following useful result.

Lemma 5.2 (Necessary condition for invariance). *If the functional $z = z[x; t]$ defined by $\dot{z}(t) = L(t, x(t), \dot{x}(t), z(t))$ and $z(a) = \gamma$, for some fixed real number γ, is invariant under the one-parameter group of transformations* (5.1)*, then*

$$\frac{d\bar{z}}{d\epsilon}(t)\Big|_{\epsilon=0} = 0$$

for each $t \in [a, b]$.

Proof. Note that

$$\frac{d\bar{z}}{d\bar{t}}(\bar{t}) = L\left(\bar{t}, \bar{x}(\bar{t}), \frac{d\bar{x}}{d\bar{t}}(\bar{t}), \bar{z}(\bar{t})\right)$$

and by multiplying both sides of the equality by $\frac{d\bar{t}}{dt}$ we have, by the chain rule, that

$$\frac{d\bar{z}}{dt}(t) = \frac{d\bar{z}}{d\bar{t}}(\bar{t}) \cdot \frac{d\bar{t}}{dt}(t) = L\left(\bar{t}, \bar{x}(\bar{t}), \frac{d\bar{x}}{d\bar{t}}(\bar{t}), \bar{z}(\bar{t})\right) \cdot \frac{d\bar{t}}{dt}(t).$$

Now, differentiating with respect to ϵ and setting $\epsilon = 0$, we find, by definition of invariance, that

$$\frac{d}{dt}\left(\frac{d\bar{z}}{d\epsilon}\right)\bigg|_{\epsilon=0} = \frac{d}{d\epsilon}\left(\frac{d\bar{z}}{dt}\right)\bigg|_{\epsilon=0} = \frac{d}{d\epsilon}\left[L\left(\bar{t}, \bar{x}(\bar{t}), \frac{d\bar{x}}{d\bar{t}}(\bar{t}), \bar{z}(\bar{t})\right) \cdot \frac{d\bar{t}}{dt}\right]\bigg|_{\epsilon=0} = 0.$$

Defining $h(t) := \frac{d\bar{z}}{d\epsilon}(t)\big|_{\epsilon=0}$, we get that $\frac{dh}{dt}(t) = 0$ for all $t \in [a, b]$, and since we are supposing the initial condition $z(a)$ to be fixed ($z(a) = \gamma$), then $\bar{z}(\bar{a})$ is also fixed ($\bar{z}(\bar{a}) = \bar{\gamma}$) and hence $\frac{d}{d\epsilon}(\bar{z}(\bar{a}))\Big|_{\epsilon=0} = 0$. Observe that if $\bar{a} = a$, then $\frac{d\bar{z}}{d\epsilon}(a)\Big|_{\epsilon=0} = 0$; if $\bar{a} \neq a$, then

$$0 = \frac{d}{d\epsilon}(\bar{z}(\bar{a}))\Big|_{\epsilon=0} = \frac{d\bar{z}}{d\epsilon}(\bar{a})\Big|_{\epsilon=0} \frac{d\bar{a}}{d\epsilon}\Big|_{\epsilon=0} = \frac{d\bar{z}}{d\epsilon}(a)\Big|_{\epsilon=0} T(a, x)$$

and because $T(a, x) \neq 0$, we can write that $\frac{d\bar{z}}{d\epsilon}(a)\Big|_{\epsilon=0} = 0$. By definition of h, this means that $h(a) = 0$. Since h is constant on $[a, b]$, we conclude that

$$h(t) := \frac{d\bar{z}}{d\epsilon}(t)\bigg|_{\epsilon=0} = 0$$

for all $t \in [a, b]$. □

Theorem 5.3 (Noether's first theorem for variational problems of Herglotz type [29, 31])**.** *If functional $z = z[x; t]$ defined by $\dot{z} = L(t, x(t), \dot{x}(t), z(t))$ and $z(a) = \gamma$, for some fixed real number γ, is invariant under the one-parameter group of transformations* (5.1)*, then*

$$\lambda(t) \cdot \left(\left[L[x; z](t) - \dot{x}\frac{\partial L}{\partial \dot{x}}[x; z](t)\right] T(t, x) + \frac{\partial L}{\partial \dot{x}}[x; z](t) X(t, x)\right)$$

is conserved along the generalized extremals, where $\lambda(t) := e^{-\int_a^t \frac{\partial L}{\partial z}[x;z]_\tau(\theta) d\theta}$.

5.2 Necessary optimality conditions for Herglotz's problem with time delay

Definition 5.4 (Admissible pair to problem $(\mathbf{H}_\tau)$). *We say that $(x(\cdot), z(\cdot))$ with $x(\cdot) \in C^2([a-\tau, b]; \mathbb{R}^m)$ and $z(\cdot) \in C^1([a, b]; \mathbb{R})$ is an admissible pair to problem $(\mathbf{H}_\tau)$ if it satisfies the equation*

$$\dot{z}(t) = L\left(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau), z(t)\right), \quad t \in [a, b],$$
$$\textit{subject to } z(a) = \gamma, \quad \gamma \in \mathbb{R},$$
$$\textit{and to } x(b) = \beta \quad \textit{and} \quad x(t) = \mu(t), \quad t \in [a-\tau, a].$$

Definition 5.5 (Admissible variation). *We say that $\eta(\cdot) \in C^2\left([a-\tau, b]; \mathbb{R}^m\right)$ is an admissible variation for problem $(\mathbf{H}_\tau)$ if $\eta(t) = 0$ for $t \in [a-\tau, a]$ and $\eta(b) = 0$.*

The following result gives a necessary condition of Euler–Lagrange type for an admissible pair $(x(\cdot), z(\cdot))$ to be a solution of problem $(\mathbf{H}_\tau)$.

Theorem 5.6 (Generalized Euler–Lagrange equations for variational problems of Herglotz type with time delay). *If $(x(\cdot), z(\cdot))$ is a solution of problem $(\mathbf{H}_\tau)$, then the following generalized Euler–Lagrange equations with time delay are satisfied:*

$$\begin{aligned}\lambda(t+\tau)\left[\frac{\partial L}{\partial x_\tau}[x;z]_\tau(t+\tau) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(t+\tau) + \frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(t+\tau)\frac{\partial L}{\partial z}[x;z]_\tau(t+\tau)\right]&\\ + \lambda(t)\left[\frac{\partial L}{\partial x}[x;z]_\tau(t) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(t) + \frac{\partial L}{\partial \dot{x}}[x;z]_\tau(t)\frac{\partial L}{\partial z}[x;z]_\tau(t)\right] &= 0, \quad (5.2)\end{aligned}$$

$a \leq t \leq b-\tau$, where $\lambda(t) := e^{-\int_a^t \frac{\partial L}{\partial z}[x;z]_\tau(\theta)d\theta}$, and

$$\frac{\partial L}{\partial x}[x;z]_\tau(t) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(t) + \frac{\partial L}{\partial \dot{x}}[x;z]_\tau(t)\frac{\partial L}{\partial z}[x;z]_\tau(t) = 0, \quad (5.3)$$

$b-\tau \leq t \leq b$.

Proof. Suppose $x(\cdot) \in C^2([a-\tau, b]; \mathbb{R}^m)$ is a solution to problem $(\mathbf{H}_\tau)$ and let η be an admissible variation. Let ϵ be an arbitrary real number and define $\zeta : [a, b] \to \mathbb{R}$ by

$$\zeta(t) := \frac{d}{d\epsilon} z[x+\epsilon\eta]_\tau(t)\Bigg|_{\epsilon=0}.$$

Obviously, $\zeta(a) = 0$ and, since z is an extremizer, we conclude that $\zeta(b) = 0$. Observe that

$$\dot{\zeta}(t) = \frac{d}{dt}\frac{d}{d\epsilon} z[x+\epsilon\eta]_\tau(t)\Bigg|_{\epsilon=0} = \frac{d}{d\epsilon}\frac{d}{dt} z[x+\epsilon\eta]_\tau(t)\Bigg|_{\epsilon=0} = \frac{d}{d\epsilon} L[x+\epsilon\eta, z]_\tau(t)\Bigg|_{\epsilon=0},$$

which means that

$$\dot{\zeta}(t) = \frac{\partial L}{\partial x}[x;z]_\tau(t)\eta(t) + \frac{\partial L}{\partial \dot{x}}[x;z]_\tau(t)\dot{\eta}(t) + \frac{\partial L}{\partial x_\tau}[x;z]_\tau(t)\eta(t-\tau) \\ + \frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(t)\dot{\eta}(t-\tau) + \frac{\partial L}{\partial z}[x;z]_\tau(t)\zeta(t).$$

Consequently, ζ is solution of the first order linear differential equation

$$\dot{\zeta} = \frac{\partial L}{\partial x}\eta(t) + \frac{\partial L}{\partial \dot{x}}\dot{\eta}(t) + \frac{\partial L}{\partial x_\tau}\eta(t-\tau) + \frac{\partial L}{\partial \dot{x}_\tau}\dot{\eta}(t-\tau) + \frac{\partial L}{\partial z}\zeta.$$

and ζ satisfies the equation

$$\lambda(t)\zeta(t) - \zeta(a) = \int_a^t \lambda(s)\left[\frac{\partial L}{\partial x}[x;z]_\tau(s)\eta(s) + \frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s)\dot{\eta}(s) \right. \\ \left. + \frac{\partial L}{\partial x_\tau}[x;z]_\tau(s)\eta(s-\tau) + \frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s)\dot{\eta}(s-\tau)\right]ds,$$

where $\lambda(t) := e^{-\int_a^t \frac{\partial L}{\partial z}[x;z]_\tau(\theta)d\theta}$. The previous equation is valid for all $t \in [a,b]$, in particular for $t = b$ and because $\zeta(a) = \zeta(b) = 0$, we have

$$\int_a^b \lambda(s)\left[\frac{\partial L}{\partial x}[x;z]_\tau(s)\eta(s) + \frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s)\dot{\eta}(s)\right]ds \\ + \int_a^b \lambda(s)\left[\frac{\partial L}{\partial x_\tau}[x;z]_\tau(s)\eta(s-\tau) + \frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s)\dot{\eta}(s-\tau)\right]ds = 0.$$

Applying the change of variable $s = t+\tau$ in the second integral and recalling that η is null in $[a-\tau, a]$, we obtain that

$$\int_a^b \lambda(s)\left[\frac{\partial L}{\partial x}[x;z]_\tau(s)\eta(s) + \frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s)\dot{\eta}(s)\right]ds \\ + \int_a^{b-\tau} \lambda(s+\tau)\left[\frac{\partial L}{\partial x_\tau}[x;z]_\tau(s+\tau)\eta(s) + \frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s+\tau)\dot{\eta}(s)\right]ds = 0,$$

that is,

$$\int_a^{b-\tau}\left[\lambda(s)\frac{\partial L}{\partial x}[x;z]_\tau(s) + \lambda(s+\tau)\frac{\partial L}{\partial x_\tau}[x;z]_\tau(s+\tau)\right]\eta(s)ds \\ + \int_a^{b-\tau}\left[\lambda(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s) + \lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s+\tau)\right]\dot{\eta}(s)ds \\ + \int_{b-\tau}^b \lambda(s)\left[\frac{\partial L}{\partial x}[x;z]_\tau(s)\eta(s) + \frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s)\dot{\eta}(s)\right]ds = 0.$$

Integration by parts gives

$$\begin{aligned}
&\int_a^{b-\tau}\Bigg\{\lambda(s)\frac{\partial L}{\partial x}[x;z]_\tau(s)+\lambda(s+\tau)\frac{\partial L}{\partial x_\tau}[x;z]_\tau(s+\tau)\\
&-\frac{d}{ds}\left[\lambda(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s)+\lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s+\tau)\right]\Bigg\}\eta(s)ds\\
&+\left[\left(\lambda(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s)+\lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s+\tau)\right)\eta(s)\right]_a^{b-\tau}\\
&+\int_{b-\tau}^{b}\left[\lambda(s)\frac{\partial L}{\partial x}[x;z]_\tau(s)-\frac{d}{ds}(\lambda(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s))\right]\eta(s)ds\\
&+\left[\lambda(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s)\eta(s)\right]_{b-\tau}^{b}=0.
\end{aligned}$$

Since previous equation holds for all admissible variations, it holds also for those admissible variations η such that $\eta(t)=0$ for all $t\in[b-\tau,b]$ and, therefore, we get

$$\begin{aligned}
&\int_a^{b-\tau}\Bigg\{\lambda(s)\frac{\partial L}{\partial x}[x;z]_\tau(s)+\lambda(s+\tau)\frac{\partial L}{\partial x_\tau}[x;z]_\tau(s+\tau)\\
&-\frac{d}{ds}\left[\lambda(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s)+\lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s+\tau)\right]\Bigg\}\eta(s)ds=0.
\end{aligned}$$

From the fundamental lemma of the Calculus of Variations (see, e.g., [28]), we conclude that

$$\begin{aligned}
&\lambda(t+\tau)\frac{\partial L}{\partial x_\tau}[x;z]_\tau(t+\tau)+\lambda(t)\frac{\partial L}{\partial x}[x;z]_\tau(t)\\
&\qquad-\frac{d}{dt}\left[\lambda(t+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(t+\tau)+\lambda(t)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(t)\right]=0
\end{aligned}$$

for $a\leq t\leq b-\tau$, proving equation (5.2). Now, if we restrict ourselves to those admissible variations η such that $\eta(t)=0$ for all $t\in[a,b-\tau]$ we get

$$\int_{b-\tau}^{b}\left[\lambda(s)\frac{\partial L}{\partial x}[x;z]_\tau(s)-\frac{d}{ds}(\lambda(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s))\right]\eta(s)ds=0$$

and again, from the fundamental lemma of the Calculus of Variations we conclude that

$$\lambda(t)\frac{\partial L}{\partial x}[x;z]_\tau(t)-\frac{d}{dt}(\lambda(t)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(t))=0$$

for $b-\tau\leq t\leq b$, proving equation (5.3). $\square$

Definition 5.7 (Generalized extremals with time delay). *Admissible pairs to problem* $(\mathbf{H}_\tau)$ *that are solutions of the Euler–Lagrange equations* (5.2)–(5.3) *are called generalized extremals with time delay.*

Remark 5.8. *Note that if there is no time delay, that is, if* $\tau = 0$, *then problem* $(\mathbf{H}_\tau)$ *reduces to the classical variational problem of Herglotz* $(\mathbf{H^1})$ *and the generalized Euler–Lagrange equation* (2.3) *follows from Theorem 5.6.*

The following theorem gives a generalization of the DuBois–Reymond condition for classical variational problems [13] and generalizes the Dubois–Reymond condition for variational problems with time delay of [24].

Theorem 5.9 (DuBois–Reymond conditions for variational problems of Herglotz type with time delay). *If a pair* $(x(\cdot), z(\cdot))$ *is a generalized extremal with time delay such that*

$$\frac{\partial L}{\partial x_\tau}[x;z]_\tau(t+\tau)\cdot\dot{x}(t) + \frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(t+\tau)\cdot\ddot{x}(t) = 0 \tag{5.4}$$

for all $t \in [a-\tau, b-\tau]$, *then* $x(\cdot)$ *satisfies the following equations:*

$$\frac{d}{dt}\left\{\lambda(t)L[x;z]_\tau(t) - \dot{x}(t)\left[\lambda(t)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(t) + \lambda(t+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(t+\tau)\right]\right\} = \lambda(t)\frac{\partial L}{\partial t}[x;z]_\tau(t) \tag{5.5}$$

for $a \le t \le b-\tau$, *and*

$$\frac{d}{dt}\left\{\lambda(t)\left[L[x;z]_\tau(t) - \dot{x}(t)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(t)\right]\right\} = \lambda(t)\frac{\partial L}{\partial t}[x;z]_\tau(t) \tag{5.6}$$

for $b-\tau \le t \le b$.

Proof. In order to prove equation (5.5), let $t \in [a, b-\tau]$ be arbitrary. Note that

$$\begin{aligned}
&\int_a^t \frac{d}{ds}\left\{\lambda(s)L[x;z]_\tau(s) - \dot{x}(s)\left[\lambda(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s) + \lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s+\tau)\right]\right\}ds\\
&= \int_a^t \Big\{ - \frac{\partial L}{\partial z}[x;z]_\tau(s)\lambda(s)L[x;z]_\tau(s) + \lambda(s)\Big[\frac{\partial L}{\partial t}[x;z]_\tau(s) + \frac{\partial L}{\partial x}[x;z]_\tau(s)\dot{x}(s)\\
&+ \frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s)\ddot{x}(s) + \frac{\partial L}{\partial x_\tau}[x;z]_\tau(s)\dot{x}(s-\tau) + \frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s)\ddot{x}(s-\tau)\\
&+ \frac{\partial L}{\partial z}[x;z]_\tau(s)L[x;z]_\tau(s)\Big] - \ddot{x}(s)\Big[\lambda(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s) + \lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s+\tau)\Big]\\
&- \dot{x}(s)\frac{d}{ds}\Big[\lambda(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s) + \lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s+\tau)\Big]\Big\}ds.
\end{aligned}$$

Cancelling symmetrical terms, we get

$$\begin{aligned}
&\int_a^t \frac{d}{ds}\left\{\lambda(s)L[x;z]_\tau(s) - \dot{x}(s)\left[\lambda(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s) + \lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s+\tau)\right]\right\} ds \\
&= \int_a^t \Big(\lambda(s)\frac{\partial L}{\partial t}[x;z]_\tau(s) + \lambda(s)\frac{\partial L}{\partial x}[x;z]_\tau(s)\dot{x}(s) - \ddot{x}(s)\lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s+\tau) \\
&- \dot{x}(s)\frac{d}{ds}\left[\lambda(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s) + \lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s+\tau)\right]\Big) ds \\
&+ \int_a^t \Big(\lambda(s)\frac{\partial L}{\partial x_\tau}[x;z]_\tau(s)\dot{x}(s-\tau) + \lambda(s)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s)\ddot{x}(s-\tau)\Big) ds.
\end{aligned}$$

Observe that, by hypothesis (5.4), the last integral is null and by substitution of the Euler–Lagrange equation (5.2) one gets

$$\begin{aligned}
&\int_a^t \frac{d}{ds}\left\{\lambda(s)L[x;z]_\tau(s) - \dot{x}(s)\left[\lambda(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s) + \lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s+\tau)\right]\right\} ds \\
&= \int_a^t \Big(\lambda(s)\frac{\partial L}{\partial t}[x;z]_\tau(s) - \lambda(s+\tau)\big[\frac{\partial L}{\partial x_\tau}[x;z]_\tau(s+\tau)\dot{x}(s) + \ddot{x}(s)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s+\tau)\big]\Big) ds.
\end{aligned}$$

Using hypothesis (5.4) in the right hand side of the last equation, we conclude that

$$\begin{aligned}
&\int_a^t \frac{d}{ds}\left\{\lambda(s)L[x;z]_\tau(s) - \dot{x}(s)\left[\lambda(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s) + \lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s+\tau)\right]\right\} ds \\
&\qquad = \int_a^t \lambda(s)\frac{\partial L}{\partial t}[x;z]_\tau(s) ds.
\end{aligned}$$

Condition (5.5) follows from the arbitrariness of $t \in [a, b-\tau]$. In order to prove equation (5.6), let $t \in [b-\tau, b]$ be arbitrary. Note that

$$\begin{aligned}
&\int_t^b \frac{d}{ds}\left\{\lambda(s)L[x;z]_\tau(s) - \lambda(s)\dot{x}(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s)\right\} ds \\
&= \int_t^b \Big\{ -\frac{\partial L}{\partial z}[x;z]_\tau(s)\lambda(s)L[x;z]_\tau(s) + \lambda(s)\Big[\frac{\partial L}{\partial t}[x;z]_\tau(s) + \frac{\partial L}{\partial x}[x;z]_\tau(s)\dot{x}(s) \\
&+ \frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s)\ddot{x}(s) + \frac{\partial L}{\partial x_\tau}[x;z]_\tau(s)\dot{x}(s-\tau) + \frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s)\ddot{x}(s-\tau) \\
&+ \frac{\partial L}{\partial z}[x;z]_\tau(s)L[x;z]_\tau(s)\Big] - \ddot{x}(s)\lambda(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s) - \dot{x}(s)\frac{d}{ds}\left[\lambda(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s)\right]\Big\} ds.
\end{aligned}$$

Cancelling symmetrical terms, the previous equation becomes

$$
\begin{aligned}
&\int_t^b \frac{d}{ds}\left\{\lambda(s)L[x;z]_\tau(s)-\lambda(s)\dot{x}(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s)\right\}ds\\
&=\int_t^b \left\{\lambda(s)\big(\frac{\partial L}{\partial t}[x;z]_\tau(s)+\frac{\partial L}{\partial x}[x;z]_\tau(s)\dot{x}(s)\big)-\dot{x}(s)\frac{d}{ds}\left[\lambda(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s)\right]\right\}ds\\
&+\int_t^b \left\{\lambda(s)\big(\frac{\partial L}{\partial x_\tau}[x;z]_\tau(s)\dot{x}(s-\tau)+\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s)\ddot{x}(s-\tau)\big)\right\}ds.
\end{aligned}
$$

Substituting the Euler–Lagrange equation (5.3) and using hypothesis (5.4) in the last integral, we conclude that

$$
\int_t^b \frac{d}{ds}\left\{\lambda(s)L[x;z]_\tau(s)-\lambda(s)\dot{x}(s)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s)\right\}ds=\int_t^b \lambda(s)\frac{\partial L}{\partial t}[x;z]_\tau(s)ds.
$$

Condition (5.6) follows from the arbitrariness of $t \in [b-\tau, b]$. □

Remark 5.10. *For the classical variational problem and for the variational problem of Herglotz (without delayed arguments), hypothesis* (5.4) *is trivially satisfied.*

5.3 Noether's theorem for the problem of Herglotz with time delay

Before presenting the extension of the famous Noether's first theorem to variational problems of Herglotz type with time delay, we introduce the definition of invariance under a one-parameter group of transformations and give two useful necessary conditions for invariance.

Definition 5.11 (Invariance of problem $(\mathbf{H}_\tau)$ under a one-parameter group of transformations)**.** *The one-parameter group of invertible C^1 transformations*

$$
\begin{cases}
\bar{t}=\mathcal{T}(t,x,\epsilon)=t+T(t,x)\epsilon+o(\epsilon)\\
\bar{x}=\mathcal{X}(t,x,\epsilon)=x+X(t,x)\epsilon+o(\epsilon)
\end{cases}
\tag{5.7}
$$

leaves problem $(\mathbf{H}_\tau)$ *invariant if*

$$
\frac{d}{d\epsilon}\left[L\left(\bar{t},\bar{x}(\bar{t}),\frac{d\bar{x}}{d\bar{t}}(\bar{t}),\bar{x}(\bar{t}-\tau),\frac{d\bar{x}}{d\bar{t}}(\bar{t}-\tau),\bar{z}(\bar{t})\right)\cdot\frac{d\bar{t}}{dt}\right]\Bigg|_{\epsilon=0}=0.
$$

Lemma 5.12 (Necessary condition for invariance with time delay I)**.** *If problem* $(\mathbf{H}_\tau)$ *is invariant under the one-parameter group of transformations* (5.7)*, then*

$$\frac{d\bar{z}}{d\epsilon}(t)\Big|_{\epsilon=0} = 0$$

for each $t \in [a, b]$*.*

Proof. The proof is very similar to the one of Lemma 5.2. □

The next result is a consequence of Lemma 5.12 and is useful in the proof of Noether's first theorem for variational problems of Herglotz with time delay.

Lemma 5.13 (Necessary condition for invariance with time delay II)**.** *If problem* $(\mathbf{H}_\tau)$ *is invariant under the one-parameter group of transformations* (5.7)*, then*

$$\begin{aligned}\int_a^t \lambda(s)\Bigg[&\frac{\partial L}{\partial t}[x;z]_\tau(s)T(s) + \frac{\partial L}{\partial x}[x;z]_\tau(s)X(s) + \frac{\partial L}{\partial \dot{x}}[x;z]_\tau(s)(\dot{X}(s) - \dot{x}(s)\dot{T}(s))\\ &+ \frac{\partial L}{\partial x_\tau}[x;z]_\tau(s)X(s-\tau) + \frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(s)\left(\dot{X}(s-\tau) - \dot{x}(s-\tau)\dot{T}(s-\tau)\right)\\ &+ L[x;z]_\tau(s)\dot{T}(s)\Bigg]ds = 0\end{aligned} \tag{5.8}$$

for each $t \in [a, b]$*.*

Proof. Since

$$\frac{d\bar{z}}{dt}(t) = L\left(\bar{t}, \bar{x}(\bar{t}), \frac{d\bar{x}}{d\bar{t}}(\bar{t}), \bar{x}(\bar{t}-\tau), \frac{d\bar{x}}{d\bar{t}}(\bar{t}-\tau), \bar{z}(\bar{t})\right) \cdot \frac{d\bar{t}}{dt}(t)$$

and

$$\frac{d\bar{t}}{dt}(t)\Big|_{\epsilon=0} = 1, \quad \frac{d}{d\epsilon}\frac{d\bar{t}}{dt}(t)\Big|_{\epsilon=0} = \frac{d}{dt}T(t,x),$$

we get

$$\frac{d}{d\epsilon}\left(\frac{d\bar{z}}{dt}\right)(t)\Big|_{\epsilon=0} = \frac{dL}{d\epsilon}\Big|_{\epsilon=0} \cdot \frac{d\bar{t}}{dt}(t)\Big|_{\epsilon=0} + L \cdot \frac{d}{d\epsilon}\frac{d\bar{t}}{dt}(t)\Big|_{\epsilon=0} = \frac{dL}{d\epsilon}\Big|_{\epsilon=0} + L \cdot \frac{d}{dt}T(t,x).$$

Defining $h(t) := \frac{d\bar{z}}{d\epsilon}(t)\big|_{\epsilon=0}$,

$$\begin{aligned}\dot{h}(t) = \frac{\partial L}{\partial t}\frac{d\bar{t}}{d\epsilon}(t)\Big|_{\epsilon=0} + \frac{\partial L}{\partial x}\frac{d\bar{x}}{d\epsilon}(t)\Big|_{\epsilon=0} + \frac{\partial L}{\partial \dot{x}}\frac{d}{d\epsilon}\frac{d\bar{x}}{d\bar{t}}(t)\Big|_{\epsilon=0} + \frac{\partial L}{\partial x_\tau}\frac{d\bar{x}}{d\epsilon}(t-\tau)\Big|_{\epsilon=0}\\ + \frac{\partial L}{\partial \dot{x}_\tau}\frac{d}{d\epsilon}\frac{d\bar{x}}{d\bar{t}}(t-\tau)\Big|_{\epsilon=0} + \frac{\partial L}{\partial z}\frac{d\bar{z}}{d\epsilon}(t)\Big|_{\epsilon=0} + L\dot{T}.\end{aligned} \tag{5.9}$$

Next we prove that

$$\frac{d}{d\epsilon}\frac{d\bar{x}}{d\bar{t}}\bigg|_{\epsilon=0} = \dot{X} - \dot{x}\dot{T}.$$

Because

$$\frac{d\bar{x}}{dt} = \frac{d\bar{x}}{d\bar{t}} \cdot \frac{d\bar{t}}{dt} = \frac{d\bar{x}}{d\bar{t}} \cdot \left(\frac{\partial \bar{t}}{\partial t} + \frac{\partial \bar{t}}{\partial x}\dot{x}\right),$$

one has

$$\begin{aligned} \frac{d}{d\epsilon}\frac{d\bar{x}}{dt}\bigg|_{\epsilon=0} &= \frac{d}{d\epsilon}\left(\frac{d\bar{x}}{d\bar{t}} \cdot \left(\frac{\partial \bar{t}}{\partial t} + \frac{\partial \bar{t}}{\partial x}\dot{x}\right)\right)\bigg|_{\epsilon=0} \\ &= \frac{d}{d\epsilon}\left(\frac{d\bar{x}}{d\bar{t}}\right)\bigg|_{\epsilon=0} + \frac{d\bar{x}}{d\bar{t}}\bigg|_{\epsilon=0} \cdot \frac{d}{d\epsilon}\left(\frac{\partial \bar{t}}{\partial t} + \frac{\partial \bar{t}}{\partial x}\dot{x}\right)\bigg|_{\epsilon=0}. \end{aligned} \tag{5.10}$$

On the other hand, since

$$\frac{d}{d\epsilon}\frac{d\bar{x}}{dt}\bigg|_{\epsilon=0} = \frac{d}{d\epsilon}\left(\frac{\partial \bar{x}}{\partial t} + \frac{\partial \bar{x}}{\partial x}\dot{x}\right)\bigg|_{\epsilon=0},$$

we get from equality (5.10) that

$$\frac{\partial}{\partial t}\frac{d\bar{x}}{d\epsilon}\bigg|_{\epsilon=0} + \dot{x}\frac{\partial}{\partial x}\frac{d\bar{x}}{d\epsilon}\bigg|_{\epsilon=0} = \frac{d}{d\epsilon}\frac{d\bar{x}}{d\bar{t}}\bigg|_{\epsilon=0} + \dot{x}\left(\frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\dot{x}\right)$$

and therefore

$$\frac{\partial X}{\partial t} + \dot{x}\frac{\partial X}{\partial x} = \frac{d}{d\epsilon}\frac{d\bar{x}}{d\bar{t}}\bigg|_{\epsilon=0} + \dot{x}\dot{T},$$

which is equivalent to

$$\frac{d}{d\epsilon}\frac{d\bar{x}}{d\bar{t}}\bigg|_{\epsilon=0} = \dot{X} - \dot{x}\dot{T}. \tag{5.11}$$

Substituting (5.11) into (5.9), we get

$$\begin{aligned} \dot{h} = \frac{\partial L}{\partial t}T + \frac{\partial L}{\partial x}X + \frac{\partial L}{\partial \dot{x}}(\dot{X} - \dot{x}\dot{T}) + \frac{\partial L}{\partial x_\tau}X(t-\tau) \\ + \frac{\partial L}{\partial \dot{x}_\tau}(\dot{X}(t-\tau) - \dot{x}(t-\tau)\dot{T}(t-\tau)) + \frac{\partial L}{\partial z}h + L\dot{T}. \end{aligned}$$

Therefore, h satisfies a first order differential equation whose solution is

$$\begin{aligned} \lambda(t)h(t) - h(a) = \int_a^t \lambda(s)\left[\frac{\partial L}{\partial t}T + \frac{\partial L}{\partial x}X + \frac{\partial L}{\partial \dot{x}}(\dot{X} - \dot{x}\dot{T}) + \frac{\partial L}{\partial x_\tau}X(s-\tau)\right. \\ \left. + \frac{\partial L}{\partial \dot{x}_\tau}\left(\dot{X}(s-\tau) - \dot{x}(s-\tau)\dot{T}(s-\tau)\right) + L\dot{T}\right] ds. \end{aligned}$$

Finally, since problem $(\mathbf{H}_\tau)$ is invariant under the one-parameter group of transformations (5.7), we have by Lemma 5.12 that $h \equiv 0$ and we obtain (5.8). $\square$

The next result establishes an extension of the celebrated Noether first theorem to variational problems of Herglotz type with time delay.

Theorem 5.14 (Noether's first theorem for variational problems of Herglotz type with time delay). *If problem* $(\mathbf{H}_\tau)$ *is invariant under the one-parameter group of transformations* (5.7), *then the quantities defined by*

$$\left[\lambda(t)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(t) + \lambda(t+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(t+\tau)\right] X(t) \\ + \left[\lambda(t)L[x;z]_\tau(t) - \dot{x}(t)\left(\lambda(t)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(t) + \lambda(t+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(t+\tau)\right)\right] T(t) \quad (5.12)$$

for $a \leq t \leq b-\tau$, *and*

$$\lambda(t)\left[\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(t)X(t) + (L[x;z]_\tau(t) - \dot{x}(t)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau(t))T(t)\right] \quad (5.13)$$

for $b-\tau \leq t \leq b$, *are conserved along the generalized extremals with time delay that satisfy*

$$\frac{\partial L}{\partial x_\tau}[x;z]_\tau(t+\tau)\cdot\dot{x}(t) + \frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(t+\tau)\cdot\ddot{x}(t) = 0 \quad (5.14)$$

for all $t \in [a-\tau, b-\tau]$, *and*

$$\frac{\partial L}{\partial x_\tau}[x;z]_\tau(t+\tau)X(t) + \frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau(t+\tau)\left(\dot{X}(t) - \dot{x}(t)\dot{T}(t)\right) = 0 \quad (5.15)$$

for all $t \in [a, b-\tau]$.

Proof. Suppose that problem $(\mathbf{H}_\tau)$ is invariant under the one-parameter group of transformations (5.7) and that $x(\cdot)$ is a solution of the delayed generalized Euler–Lagrange equations (5.2)–(5.3). From the necessary condition for invariance with time delay II (Lemma 5.13), we get that

$$\int_a^t \lambda(s)\left[\frac{\partial L}{\partial t}T + \frac{\partial L}{\partial x}X + \frac{\partial L}{\partial \dot{x}}(\dot{X} - \dot{x}\dot{T}) + \frac{\partial L}{\partial x_\tau}X(s-\tau) \right. \\ \left. + \frac{\partial L}{\partial \dot{x}_\tau}\left(\dot{X}(s-\tau) - \dot{x}(s-\tau)\dot{T}(s-\tau)\right) + L\dot{T}\right] ds = 0$$

for each $t \in [a,b]$. Proceeding with a linear change of variable and noticing that we can assume X and T to be null outside $[a,b]$, the previous equation is equivalent to

$$\begin{aligned}\int_a^{t-\tau} \lambda(s)\left[\frac{\partial L}{\partial t}T+\frac{\partial L}{\partial x}X+\frac{\partial L}{\partial \dot{x}}(\dot{X}-\dot{x}\dot{T})+L\dot{T}\right]\\ +\lambda(s+\tau)\left[\frac{\partial L}{\partial x_\tau}(s+\tau)X+\frac{\partial L}{\partial \dot{x}_\tau}(s+\tau)\left(\dot{X}(s)-\dot{x}(s)\dot{T}(s)\right)\right]ds\\ +\int_{t-\tau}^{t}\lambda(s)\left[\frac{\partial L}{\partial t}T+\frac{\partial L}{\partial x}X+\frac{\partial L}{\partial \dot{x}}(\dot{X}-\dot{x}\dot{T})+L\dot{T}\right]ds=0.\end{aligned} \tag{5.16}$$

Using hypothesis (5.15), equation (5.16) implies that

$$\int_a^{t} \lambda(s)\left[\frac{\partial L}{\partial t}T+\frac{\partial L}{\partial x}X+\frac{\partial L}{\partial \dot{x}}(\dot{X}-\dot{x}\dot{T})+L\dot{T}\right]ds=0.$$

From the arbitrariness of $t\in[a,b]$ we conclude that

$$\frac{\partial L}{\partial t}T+\frac{\partial L}{\partial x}X+\frac{\partial L}{\partial \dot{x}}(\dot{X}-\dot{x}\dot{T})+L\dot{T}=0 \tag{5.17}$$

for all $t\in[a,b]$. Then, equation (5.16) becomes

$$\begin{aligned}\int_a^{t-\tau}\Bigg(\lambda(s)\frac{\partial L}{\partial t}T+\left[\lambda(s)\frac{\partial L}{\partial x}+\lambda(s+\tau)\frac{\partial L}{\partial x_\tau}(s+\tau)\right]X\\ +\left[\lambda(s)\frac{\partial L}{\partial \dot{x}}+\lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}(s+\tau)\right]\dot{X}\\ +\left[\lambda(s)L-\dot{x}\left(\lambda(s)\frac{\partial L}{\partial \dot{x}}+\lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}(s+\tau)\right)\right]\dot{T}\Bigg)ds=0\end{aligned}$$

for $t\in[a+\tau,b]$. Using integration by parts, one has

$$\begin{aligned}\int_a^{t-\tau}\Bigg(\lambda(s)\frac{\partial L}{\partial t}T+\left[\lambda(s)\frac{\partial L}{\partial x}+\lambda(s+\tau)\frac{\partial L}{\partial x_\tau}(s+\tau)\right]X\\ -\frac{d}{ds}\left[\lambda(s)\frac{\partial L}{\partial \dot{x}}+\lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}(s+\tau)\right]X\\ -\frac{d}{ds}\left[\lambda(s)L-\dot{x}\left(\lambda(s)\frac{\partial L}{\partial \dot{x}}+\lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}(s+\tau)\right)\right]T\Bigg)ds\\ +\Bigg[\left(\lambda(s)\frac{\partial L}{\partial \dot{x}}+\lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}(s+\tau)\right)X\\ +\left(\lambda(s)L-\dot{x}\left(\lambda(s)\frac{\partial L}{\partial \dot{x}}+\lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}(s+\tau)\right)\right)T\Bigg]_a^{t-\tau}=0.\end{aligned}$$

Observe that the terms in X inside the integral are null because x satisfies the delayed generalized Euler–Lagrange equation on $[a,b-\tau]$ and that, from the DuBois–Reymond equation (5.5), the sum of the remaining terms of the integral is zero. This leads to

$$\Bigg[\left(\lambda(s)\frac{\partial L}{\partial \dot{x}}+\lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}(s+\tau)\right)X \\ +\left(\lambda(s)L-\dot{x}\left(\lambda(s)\frac{\partial L}{\partial \dot{x}}+\lambda(s+\tau)\frac{\partial L}{\partial \dot{x}_\tau}(s+\tau)\right)\right)T\Bigg]_a^{t-\tau}=0$$

for every $t\in[a+\tau,b]$, which means that

$$\left[\lambda(s)\frac{\partial L}{\partial \dot{x}}+\lambda(t+\tau)\frac{\partial L}{\partial \dot{x}_\tau}(t+\tau)\Big)X+\Big(\lambda(s)L-\dot{x}\left(\lambda(s)\frac{\partial L}{\partial \dot{x}}+\lambda(t+\tau)\frac{\partial L}{\partial \dot{x}_\tau}(t+\tau)\right)\right]T$$

is constant for $t\in[a,b-\tau]$. Consider $[t_1,t_2]\subseteq[b-\tau,b]$. From equation (5.17) one has

$$\int_{t_1}^{t_2}\left(\lambda(s)\frac{\partial L}{\partial t}T+\lambda(s)\frac{\partial L}{\partial x}X+\lambda(s)\frac{\partial L}{\partial \dot{x}}\dot{X}+\lambda(s)\left(L-\dot{x}\frac{\partial L}{\partial \dot{x}}\right)\dot{T}\right)ds=0.$$

Using integration by parts, we get

$$\int_{t_1}^{t_2}\left(\lambda(s)\frac{\partial L}{\partial t}T+\lambda(s)\frac{\partial L}{\partial x}X-\frac{d}{ds}\left(\lambda(s)\frac{\partial L}{\partial \dot{x}}\right)X-\frac{d}{ds}\left[\lambda(s)\left(L-\dot{x}\frac{\partial L}{\partial \dot{x}}\right)\right]T\right)ds \\ +\left[\lambda(s)\frac{\partial L}{\partial \dot{x}}X+\lambda(s)(L-\dot{x}\frac{\partial L}{\partial \dot{x}})T\right]_{t_1}^{t_2}=0.$$

Observe that the terms in X inside the integral are null because x satisfies Euler–Lagrange equation (5.3) and that, from DuBois–Reymond equation (5.6), the sum of the remaining terms of the integral is zero. This leads to

$$\left[\lambda(s)\frac{\partial L}{\partial \dot{x}}X+\lambda(s)\left(L-\dot{x}\frac{\partial L}{\partial \dot{x}}\right)T\right]_{t_1}^{t_2}=0.$$

From the arbitrariness of $t_1,t_2\in[b-\tau,b]$, we conclude that

$$\lambda(s)\frac{\partial L}{\partial \dot{x}}X+\lambda(s)\left(L-\dot{x}\frac{\partial L}{\partial \dot{x}}\right)T$$

is constant in $[b-\tau,b]$. This ends the proof of our main result. □

Remark 5.15. *In the classical variational problem and in the variational problem of Herglotz, hypotheses* (5.14) *and* (5.15) *are trivially satisfied.*

Remark 5.16. *Our first Noether-type theorem is a generalization of Noether's first theorem for the classical variational problem of Herglotz type presented in [29, 31], that is, Theorem 5.3 is a corollary of Theorem 5.14.*

Our results also provide generalizations of the variational results with time delay presented in [24]. If the Lagrangian does not depend on z, then $\frac{\partial L}{\partial z} \equiv 0$ and $\lambda(t) \equiv 1$. In that case, problem $(\mathbf{H}_\tau)$ reduces to the classical variational problem with time delay. The Euler–Lagrange equations, the DuBois–Reymond conditions and Noether's first theorem with time delay obtained by Frederico and Torres in [24] are particular cases of Theorem 5.6, Theorem 5.9 and Theorem 5.14, respectively.

Corollary 5.17 (See [24]). *If $x(\cdot)$ is an extremizer of the functional*

$$\int_a^b L(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau))dt, \tag{5.18}$$

then $x(\cdot)$ satisfies the Euler–Lagrange equations

$$\frac{\partial L}{\partial x_\tau}[x]_\tau(t+\tau) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_\tau}[x]_\tau(t+\tau) + \frac{\partial L}{\partial x}[x]_\tau(t) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}[x]_\tau(t) = 0, \tag{5.19}$$

$a \leq t \leq b-\tau$, and

$$\frac{\partial L}{\partial x}[x]_\tau(t) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}[x]_\tau(t) = 0, \tag{5.20}$$

$b-\tau \leq t \leq b$.

Corollary 5.18 (Cf. [24]). *If $x(\cdot)$ is an extremizer of the functional* (5.18) *and*

$$\frac{\partial L}{\partial x_\tau}[x]_\tau(t+\tau) \cdot \dot{x}(t) + \frac{\partial L}{\partial \dot{x}_\tau}[x]_\tau(t+\tau) \cdot \ddot{x}(t) = 0,$$

$t \in [a-\tau, b-\tau]$, then $x(\cdot)$ satisfies the DuBois–Reymond equations

$$\frac{d}{dt}\left\{L[x]_\tau(t) - \dot{x}(t)\left[\partial_3 L[x]_\tau(t) + \frac{\partial L}{\partial \dot{x}_\tau}[x]_\tau(t+\tau)\right]\right\} = \frac{\partial L}{\partial t}[x]_\tau(t),$$

$a \leq t \leq b-\tau$, and

$$\frac{d}{dt}\left\{L[x]_\tau(t) - \dot{x}(t)\frac{\partial L}{\partial \dot{x}}[x]_\tau(t)\right\} = \frac{\partial L}{\partial t}[x]_\tau(t),$$

$b-\tau \leq t \leq b$.

Corollary 5.19 (Cf. [24]). *If functional* (5.18) *is invariant in the sense of Definition 5.1, then the quantities*

$$\left[\frac{\partial L}{\partial \dot{x}}[x]_\tau(t) + \frac{\partial L}{\partial \dot{x}_\tau}[x]_\tau(t+\tau)\right] X(t) \\ + \left[L[x]_\tau(t) - \dot{x}(t)\left(\frac{\partial L}{\partial \dot{x}}[x]_\tau(t) + \frac{\partial L}{\partial \dot{x}_\tau}[x]_\tau(t+\tau)\right)\right] T(t),$$

$a \leq t \leq b - \tau$, and

$$\frac{\partial L}{\partial \dot{x}}[x]_\tau(t)X(t) + [L[x]_\tau(t) - \dot{x}(t)\frac{\partial L}{\partial \dot{x}}[x]_\tau(t)]T(t),$$

$b - \tau \leq t \leq b$, are conserved along the solutions of the Euler–Lagrange equations (5.19)–(5.20) *that satisfy*

$$\frac{\partial L}{\partial x_\tau}[x]_\tau(t+\tau) \cdot \dot{x}(t) + \frac{\partial L}{\partial \dot{x}_\tau}[x]_\tau(t+\tau) \cdot \ddot{x}(t) = 0,$$

$t \in [a - \tau, b - \tau]$, and

$$\frac{\partial L}{\partial x_\tau}[x]_\tau(t+\tau)X(t) + \frac{\partial L}{\partial \dot{x}_\tau}[x]_\tau(t+\tau)\left(\dot{X}(t) - \dot{x}(t)\dot{T}(t)\right) = 0,$$

$t \in [a, b - \tau]$.

5.4 Illustrative example

We present an example that shows the usefulness of our results. Consider the following Herglotz's variational problem with time delay $\tau = 1$ and $m = 1$:

$$\begin{gathered} z(2) \longrightarrow \text{extr}, \\ \dot{z}(t) = L[x; z]_1(t) := (\dot{x}(t-1))^2 + z(t), \quad t \in [0, 2], \\ x(t) = -t, \quad t \in [-1, 0], \\ x(2) = 1, \quad z(0) = 0. \end{gathered} \tag{5.21}$$

For this problem, Euler–Lagrange optimality conditions (5.2)–(5.3) given by Theorem 5.6 assert that

$$\begin{cases} \dot{x}(t) - \ddot{x}(t) = 0, \quad t \in [0, 1], \\ 0 = 0, \quad t \in [1, 2]. \end{cases}$$

Solving the equation of previous system with the initial condition $x(0) = 0$, we obtain

$$x(t) = -k + ke^t, \quad t \in [0, 1],$$

for some constant $k \in \mathbb{R}$. Since in $[0, 1]$ z is defined by $\dot{z}(t) = 1 + z(t)$, with $z(0) = 0$, we obtain

$$z(t) = e^t - 1, \quad t \in [0, 1].$$

In order to illustrate our remaining results (Theorems 5.9 and 5.14), we look for trajectories x that satisfy hypothesis (5.4): $2\dot{x}(t) \cdot \ddot{x}(t) = 0$, $t \in [-1, 1]$. This condition is trivially satisfied in the interval $[-1, 0]$, but leads to $k = 0$ and, consequently, $x(t) = 0$ in $[0, 1]$. Hence, we get a family $x_{\mathcal{F}}$ of generalized extremals with time delay given by

$$x_{\mathcal{F}}(t) = \begin{cases} -t, & t \in [-1, 0], \\ 0, & t \in [0, 1], \\ \mathcal{F}(t), & t \in [1, 2], \\ 1, & t = 2, \end{cases} \tag{5.22}$$

where the continuous function $\mathcal{F}$ is chosen to guarantee that $x_{\mathcal{F}}$ is a C^2 function. With x defined by (5.22) for some $\mathcal{F}$, and z defined in $[1, 2]$ as $\dot{z}(t) = z(t)$ with $z(1) = e - 1$, it follows that $z(t) = e^{t-1}(e - 1)$ for $t \in [1, 2]$ and, consequently,

$$z(t) = \begin{cases} e^t - 1, & t \in [0, 1], \\ e^{t-1}(e - 1), & t \in [1, 2], \end{cases} \tag{5.23}$$

for which $z(2) = e^2 - e$. Next we show that DuBois–Reymond conditions (5.5)–(5.6) given by Theorem 5.9 are valid for x and z given by (5.22)–(5.23). In this case, (5.5) reduces to

$$\frac{d}{dt}\left[\lambda(t)\left(\dot{x}^2(t-1) + z(t)\right) - \dot{x}(t)\left(2\lambda(t+1)\dot{x}(t)\right)\right] = 0, \quad t \in [0, 1],$$

which is equivalent to

$$\frac{d}{dt}\left[\lambda(t)e^t\right] = 0, \quad t \in [0, 1].$$

Since $\lambda(t) = e^{-t}$, condition (5.5) holds for $t \in [0, 1]$. Similarly, it can be proved that condition (5.6) holds for $t \in [1, 2]$. Finally, we show the relevance of our main result (Theorem 5.14). First we define a one-parameter group of transformations on t and x with generators $T(t) \equiv 1$ and $X(t) \equiv 0$, respectively. Since the Lagrangian defined in (5.21) is autonomous, i.e., does not depend explicitly on t, then it is invariant in the sense of Definition 5.11. Observe that in this case hypothesis (5.15) is trivially satisfied. Theorem 5.14 asserts that (5.12) and (5.13) are constant in t, in intervals $[0, 1]$ and $[1, 2]$, respectively, along generalized extremals with time delay that satisfy hypotheses (5.14) and (5.15). Observe that (5.12) is equal to

$$\begin{aligned} \left[\lambda(t)L[x; z]_1(t) - 2\left(\dot{x}(t)\right)^2 \lambda(t+1)\right] T(t) &= e^{-t}\left[\dot{x}^2(t-1) + z(t)\right] \\ &= e^{-t}\left[1 + e^t - 1\right], \quad t \in [0, 1], \end{aligned}$$

which is equal to one. Similarly, it can be easily proved that quantity (5.13) is also constant in t and equal to $1 - e^{-1}$.

5.5 Conclusions

In this chapter, we proved some interesting results for variational problems of Herglotz with time delay: a generalized Euler–Lagrange necessary optimality condition, a DuBois–Reymond necessary optimality condition and a Noether's first theorem. Our results extend some classical results for variational problems with time delay [2, 43], but extend also Herglot'z original result [39] and Georgieva's results on first-order Herglot'z type problems [29, 31].

The original results of this chapter were published in 2015 in [60]. They were also presented by the author in the international conference Optimization 2014, July 28-30, 2014, Guimarães, Portugal, in a contributed talk entitled "Noether's first theorem for variational problems of Herglotz type with time delay".

CHAPTER 6

OPTIMAL CONTROL APPROACH TO HERGLOTZ'S VARIATIONAL PROBLEMS

Since the celebrated work [57] by Pontryagin *et al.*, the Calculus of Variations is seen as part of Optimal Control. In this chapter we approach the first-order Herglotz problem $(\mathbf{H^1})$ from an Optimal Control point of view. While in Chapter 2 and in [32, 37, 39, 59] the admissible functions are $x(\cdot) \in C^2([a,b];\mathbb{R}^m)$ and $z(\cdot) \in C^1([a,b];\mathbb{R})$, here we consider $(\mathbf{H^1})$ in a wider class of functions. We consider the following problem:

Problem $(\mathbf{H^{1*}})$. *Determine the trajectories $x(\cdot) \in PC^1([a,b];\mathbb{R}^m)$ and $z(\cdot) \in PC^1([a,b];\mathbb{R})$ such that:*

$$\begin{gathered} z(b) \longrightarrow extr \\ with\ \dot{z}(t) = L(t, x(t), \dot{x}(t), z(t)), \quad t \in [a,b], \\ subject\ to\ x(a) = \alpha, \quad z(a) = \gamma, \quad \alpha \in \mathbb{R}^m, \gamma \in \mathbb{R}, \end{gathered} \tag{$\mathbf{H^{1*}}$}$$

where the Lagrangian L is assumed to satisfy the following hypotheses:

i. $L \in C^1([a,b] \times \mathbb{R}^{2m} \times \mathbb{R};\mathbb{R})$;

ii. *functions $t \mapsto \frac{\partial L}{\partial z}[x;z](t)$, $t \mapsto \frac{\partial L}{\partial x}[x;z](t)$ and $t \mapsto \frac{\partial L}{\partial \dot{x}}[x;z](t)$ are differentiable for any admissible trajectory x.*

We make use of Pontryagin's maximum principle (Theorem 3.1) to generalize the Euler–Lagrange equation and the transversality condition for problem ($\mathbf{H^1}$) found in [59] to admissible functions $x(\cdot) \in PC^1([a,b];\mathbb{R}^m)$ and $z(\cdot) \in PC^1([a,b];\mathbb{R})$ (Theorem 6.2). We use the DuBois–Reymond condition of Optimal Control (Theorem 3.3) to obtain a DuBois–Reymond necessary optimality condition for problem ($\mathbf{H^{1*}}$) (Theorem 6.4). We also use first Noether's theorem of Optimal Control proved in [67, 68, 72] (cf. Theorem 3.5) to prove a generalization of the Noether's theorem [31] (Theorem 6.6).

6.1 Necessary optimality conditions for Herglotz' problems

We begin by introducing some basic definitions for the generalized variational problem of Herglotz ($\mathbf{H^{1*}}$).

Definition 6.1 (Admissible pair to problem ($\mathbf{H^{1*}}$))**.** *We say that $(x(\cdot), z(\cdot))$ with $x(\cdot) \in PC^1([a,b];\mathbb{R}^m)$ and $z(\cdot) \in PC^1([a,b];\mathbb{R})$ is an admissible pair to problem ($\mathbf{H^{1*}}$) if it satisfies the equation*

$$\dot{z}(t) = L(t, x(t), \dot{x}(t), z(t)), \quad t \in [a,b],$$

and the initial conditions $x(a) = \alpha$ and $z(a) = \gamma$, $\alpha \in \mathbb{R}^m, \gamma \in \mathbb{R}$.

We now present a necessary condition for a pair $(x(\cdot), z(\cdot))$ to be a solution of problem ($\mathbf{H^{1*}}$). The following result generalizes [37, 39, 59] by considering a more general class of functions.

In order to shorten notations, we use $[x;z](t) := (t, x(t), \dot{x}(t), z(t))$. When there is no possibility of ambiguity, we sometimes suppress arguments.

Theorem 6.2 (Euler–Lagrange equation and transversality condition for problem ($\mathbf{H^{1*}}$))**.** *If $(x(\cdot), z(\cdot))$ is a solution of problem ($\mathbf{H^{1*}}$), then the Euler–Lagrange equation*

$$\frac{\partial L}{\partial x}[x;z](t) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)[x;z](t) + \frac{\partial L}{\partial z}[x;z](t)\frac{\partial L}{\partial \dot{x}}[x;z](t) = 0 \tag{6.1}$$

holds for all $t \in [a,b]$. Moreover, the following transversality condition holds:

$$\frac{\partial L}{\partial \dot{x}}[x;z](b) = 0. \tag{6.2}$$

Proof. Observe that Herglotz's problem ($\mathbf{H^{1*}}$) is a particular case of problem (P), defined in the very beginning of Chapter 3, obtained by considering x and z as state variables (two components of one vectorial state variable), $\dot{x}$ as the control variable u, and by choosing $f \equiv 0$ and $\phi(x,z) = z$. Note that since $x(t) \in \mathbb{R}^m$, we have $u(t) \in \mathbb{R}^m$ (i.e., for Herglotz's problem ($\mathbf{H^{1*}}$) one has $r = m$). In this way, the problem of Herglotz, described as an optimal control problem, takes the form

$$\begin{gathered} z(b) \longrightarrow \text{extr}, \\ \text{subject to } \begin{cases} \dot{x}(t) = u(t), \\ \dot{z}(t) = L(t, x(t), u(t), z(t)), \end{cases} \\ \text{and } x(a) = \alpha,\ z(a) = \gamma, \quad \alpha \in \mathbb{R}^m, \gamma \in \mathbb{R}. \end{gathered} \tag{6.3}$$

It follows from Pontryagin's maximum principle (Theorem 3.1) that there exists $\psi_x \in PC^1([a,b];\mathbb{R}^m)$ and $\psi_z \in PC^1([a,b];\mathbb{R})$ such that the following conditions hold:

- the optimality condition

$$\frac{\partial H}{\partial u}(t, x(t), u(t), z(t), \psi_x(t), \psi_z(t)) = 0; \tag{6.4}$$

- the adjoint system

$$\begin{cases} \dot{x}(t) = \frac{\partial H}{\partial \psi_x}(t, x(t), u(t), z(t), \psi_x(t), \psi_z(t)) \\ \dot{z}(t) = \frac{\partial H}{\partial \psi_z}(t, x(t), u(t), z(t), \psi_x(t), \psi_z(t)) \\ \dot{\psi}_x(t) = -\frac{\partial H}{\partial x}(t, x(t), u(t), z(t), \psi_x(t), \psi_z(t)) \\ \dot{\psi}_z(t) = -\frac{\partial H}{\partial z}(t, x(t), u(t), z(t), \psi_x(t), \psi_z(t)); \end{cases} \tag{6.5}$$

- and the transversality conditions

$$\begin{cases} \psi_x(b) = 0, \\ \psi_z(b) = 1, \end{cases} \tag{6.6}$$

where the Hamiltonian H is defined by

$$H(t, x, u, z, \psi_x, \psi_z) = \psi_x \cdot u + \psi_z \cdot L(t, x, u, z).$$

Observe that the adjoint system (6.5) implies that

$$\begin{cases} \dot{\psi}_x = -\psi_z \frac{\partial L}{\partial x} \\ \dot{\psi}_z = -\psi_z \frac{\partial L}{\partial z}. \end{cases} \tag{6.7}$$

This means that ψ_z is solution of a first-order linear differential equation, which is solved using an integrand factor to find that $\psi_z = ke^{-\int_a^t \frac{\partial L}{\partial z} d\theta}$ with k a constant. From the second transversality condition in (6.6), we obtain that $k = e^{\int_a^b \frac{\partial L}{\partial z} d\theta}$ and, consequently,

$$\psi_z = e^{\int_t^b \frac{\partial L}{\partial z} d\theta}.$$

The optimality condition (6.4) is equivalent to $\psi_x + \psi_z \frac{\partial L}{\partial u} = 0$ and, after derivation, we obtain that

$$\dot{\psi}_x = -\frac{d}{dt}\left(\psi_z \frac{\partial L}{\partial u}\right) = -\dot{\psi}_z \frac{\partial L}{\partial u} - \psi_z \frac{d}{dt}\left(\frac{\partial L}{\partial u}\right) = \psi_z \frac{\partial L}{\partial z}\frac{\partial L}{\partial u} - \psi_z \frac{d}{dt}\left(\frac{\partial L}{\partial u}\right).$$

Now, comparing with (6.7), we have

$$-\psi_z \frac{\partial L}{\partial x} = \psi_z \frac{\partial L}{\partial z}\frac{\partial L}{\partial u} - \psi_z \frac{d}{dt}\left(\frac{\partial L}{\partial u}\right).$$

Since $\psi_z(t) \neq 0$ for all $t \in [a, b]$ and $\dot{x} = u$, we obtain the generalized Euler–Lagrange equation (6.1):

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}} = 0.$$

Note that from the optimality condition (6.4) we obtain that $\psi_x = -\psi_z \frac{\partial L}{\partial u} = -\psi_z \frac{\partial L}{\partial \dot{x}}$, which together with transversality condition (6.6) for ψ_x leads to the transversality condition (6.2):

$$\frac{\partial L}{\partial \dot{x}}(b, x(b), \dot{x}(b), z(b)) = 0.$$

This concludes the proof. $\square$

Definition 6.3 (Extremal to problem $(\mathbf{H^{1*}})$)**.** *We say that an admissible pair* $(x(\cdot), z(\cdot))$ *is an extremal to problem* $(\mathbf{H^{1*}})$ *if it satisfies the Euler–Lagrange equation* (6.1) *and the transversality condition* (6.2)*.*

Theorem 6.4 (DuBois–Reymond condition for problem$(\mathbf{H^{1*}})$)**.** *If* $(x(\cdot), z(\cdot))$ *is a solution of problem* $(\mathbf{H^{1*}})$*, then*

$$\frac{d}{dt}\left(-\psi_z(t)\frac{\partial L}{\partial \dot{x}}[x; z](t)\dot{x}(t) + \psi_z(t)L[x; z](t)\right) = \psi_z(t)\frac{\partial L}{\partial t}[x; z](t),$$

$t \in [a, b]$*, where* $\psi_z(t) = e^{\int_t^b \frac{\partial L}{\partial z}[x;z](\theta)d\theta}$.

Proof. The result follows from Theorem 3.3, rewriting problem $(\mathbf{H^{1*}})$ as the optimal control problem (6.3). $\square$

6.2 Noether's theorem for Herglotz's problem

We start this section by defining invariance for problem $(\mathbf{H^{1*}})$.

Definition 6.5 (Invariance of problem $(\mathbf{H^{1*}})$ under a one-parameter group of transformations). *Let h^ϵ be a one-parameter family of C^1 invertible maps*

$$h^\epsilon : [a,b] \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}^m \times \mathbb{R},$$
$$h^\epsilon(t, x(t), z(t)) = (\mathcal{T}^\epsilon[x;z](t), \mathcal{X}^\epsilon[x;z](t), \mathcal{Z}^\epsilon[x;z](t)),$$
$$\text{with } h^0(t,x,z) = (t,x,z), \quad \forall (t,x,z) \in [a,b] \times \mathbb{R}^m \times \mathbb{R}.$$

Problem $(\mathbf{H^{1*}})$ is said to be invariant under the one-parameter group of transformations h^ϵ if for all admissible pairs $(x(\cdot), z(\cdot))$ the following two conditions hold:

(i)

$$\left(\frac{z(b)}{b-a} + \xi\epsilon + o(\epsilon)\right)\frac{d\mathcal{T}^\epsilon}{dt}[x;z](t) = \frac{z(b)}{b-a}, \quad \text{for some constant } \xi; \tag{6.8}$$

(ii)

$$\frac{d\mathcal{Z}^\epsilon}{dt}[x;z](t) = L\left(\mathcal{T}^\epsilon[x;z](t), \mathcal{X}^\epsilon[x;z](t), \frac{d\mathcal{X}^\epsilon}{d\mathcal{T}^\epsilon}[x;z](t), \mathcal{Z}^\epsilon[x;z](t)\right)\frac{d\mathcal{T}^\epsilon}{dt}[x;z](t), \tag{6.9}$$

where

$$\frac{d\mathcal{X}^\epsilon}{d\mathcal{T}^\epsilon}[x;z](t) = \frac{\frac{d\mathcal{X}^\epsilon}{dt}[x;z](t)}{\frac{d\mathcal{T}^\epsilon}{dt}[x;z](t)}.$$

Follows the main result of the chapter.

Theorem 6.6 (Noether's theorem for problem $(\mathbf{H^{1*}})$). *If problem $(\mathbf{H^{1*}})$ is invariant in the sense of Definition 6.5, then the quantity*

$$\psi_z(t)\left[\frac{\partial L}{\partial \dot{x}}[x;z](t)X[x;z](t) - Z[x;z](t) + \left(L[x;z](t) - \frac{\partial L}{\partial \dot{x}}[x;z](t)\dot{x}(t)\right)T[x;z](t)\right] \tag{6.10}$$

is constant in t along every extremal of problem $(\mathbf{H^{1*}})$, where

$$T[x;z](t) = \frac{\partial \mathcal{T}^\epsilon}{\partial \epsilon}[x;z](t)\bigg|_{\epsilon=0},$$
$$X[x;z](t) = \frac{\partial \mathcal{X}^\epsilon}{\partial \epsilon}[x;z](t)\bigg|_{\epsilon=0},$$
$$Z[x;z](t) = \frac{\partial \mathcal{Z}^\epsilon}{\partial \epsilon}[x;z](t)\bigg|_{\epsilon=0}$$

and $\psi_z(t) = e^{\int_t^b \frac{\partial L}{\partial z}[x;z](\theta)d\theta}$.

Proof. As before, we rewrite problem ($\mathbf{H^{1*}}$) in the equivalent optimal control form (6.3), where x and z are the state variables and $u = \dot{x}$ the control. We prove that if problem ($\mathbf{H^{1*}}$) is invariant in the sense of Definition 6.5, then problem (6.3) is invariant in the sense of Definition 3.4. First, observe that if equation (6.8) holds, then (3.6) holds for problem (6.3): here $f \equiv 0$, $\phi(x, z) = z$ and (3.6) simplifies to $\left[\frac{z(b)}{b-a} + \xi\epsilon + o(\epsilon)\right] \frac{d\mathcal{T}^\epsilon}{dt}[x; z](t) = \frac{z(b)}{b-a}$. Note that the first equation of the control system of problem (6.3) ($u(t) = \dot{x}(t)$) defines naturally $\mathcal{U}^\epsilon := \frac{d\mathcal{X}^\epsilon}{d\mathcal{T}^\epsilon}$, that is,

$$\frac{d\mathcal{X}^\epsilon}{dt}[x; z](t) = \mathcal{U}^\epsilon[x; z](t)\frac{d\mathcal{T}^\epsilon}{dt}[x; z](t). \tag{6.11}$$

Hence, if equation (6.9) and (6.11) holds, then there is also invariance of the control system of (6.3) in the sense of (3.7) and consequently problem (6.3) is invariant in the sense of Definition 3.4. We are now in conditions to apply Theorem 3.5 to problem (6.3), which guarantees that the quantity

$$\begin{aligned}(b-t)\xi + \psi_x(t) \cdot X(t, x(t), u(t), z(t)) + \psi_z(t) \cdot Z(t, x(t), u(t), z(t))\\ - \left(H(t, x(t), u(t), z(t), \psi_x(t), \psi_z(t)) + \frac{z(b)}{b-a}\right) \cdot T(t, x(t), u(t), z(t))\end{aligned}$$

is constant in t along every Pontryagin extremal of problem (6.3), where

$$H(t, x, u, z, \psi_x, \psi_z) = \psi_x u + \psi_z L(t, x, u, z).$$

This means that the quantity

$$\begin{aligned}(b-t)\xi + \psi_x(t)X[x; z](t) + \psi_z(t)Z[x; z](t)\\ - \left(\psi_x(t)\dot{x}(t) + \psi_z(t)L[x; z](t) + \frac{z(b)}{b-a}\right) T[x; z](t)\end{aligned}$$

is constant in t along all extremals of problem ($\mathbf{H^{1*}}$), where

$$\psi_x(t) = -\psi_z(t)\frac{\partial L}{\partial u}[x; z](t) = -\psi_z(t)\frac{\partial L}{\partial \dot{x}}[x; z](t).$$

Equivalently,

$$\begin{aligned}(b-t)\xi - \frac{z(b)}{b-a}T[x; z](t) - \psi_z(t)\Bigg[\frac{\partial L}{\partial \dot{x}}[x; z](t)X[x; z](t) - Z[x; z](t)\\ + \left(L[x; z](t) - \frac{\partial L}{\partial \dot{x}}[x; z](t)\dot{x}(t)\right) T[x; z](t)\Bigg]\end{aligned}$$

is a constant along the extremals. To conclude the proof, we just need to prove that the quantity

$$(b-t)\xi - \frac{z(b)}{b-a}T[x;z](t) \tag{6.12}$$

is a constant. From the invariance condition (6.8) we know that

$$\left(z(b)+\xi(b-a)\epsilon+o(\epsilon)\right)\frac{d\mathcal{T}^\epsilon}{dt}[x;z](t) = z(b).$$

Integrating from a to t, we conclude that

$$\begin{aligned}&\left(z(b)+\xi(b-a)\epsilon+o(\epsilon)\right)\mathcal{T}^\epsilon[x;z](t)\\&\qquad = z(b)(t-a)+\left(z(b)+\xi(b-a)\epsilon+o(\epsilon)\right)\mathcal{T}^\epsilon[x;z](a).\end{aligned} \tag{6.13}$$

Differentiating (6.13) with respect to ϵ, and then putting $\epsilon=0$, we obtain

$$\xi(b-a)t + z(b)T[x;z](t) = \xi(b-a)a + z(b)T[x;z](a). \tag{6.14}$$

We conclude from (6.14) that expression (6.12) is the constant $(b-a)\xi - \frac{z(b)}{b-a}T[x;z](a)$. This ends the proof. □

6.3 Conclusions

We introduced a different approach to the generalized variational problem of Herglotz, by looking to Herglotz's problem as an optimal control problem. A Noether type theorem for Herglotz's problem was first proved by Georgieva and Guenther in [31]: under the condition of invariance

$$\frac{d}{d\epsilon}\left[L\left(\mathcal{T}^\epsilon[x;z](t),\mathcal{X}^\epsilon[x;z](t),\frac{d\mathcal{X}^\epsilon}{d\mathcal{T}^\epsilon}[x;z](t),z\left(\mathcal{T}^\epsilon[x;z](t)\right)\right)\frac{d\mathcal{T}^\epsilon}{dt}[x;z](t)\right]\Bigg|_{\epsilon=0}=0, \tag{6.15}$$

they obtained

$$\lambda(t)\left[\frac{\partial L}{\partial \dot{x}}[x;z](t)X[x;z](t)+\left(L[x;z](t)-\frac{\partial L}{\partial \dot{x}}[x;z](t)\dot{x}(t)\right)T[x;z](t)\right], \tag{6.16}$$

where $\lambda(t) = e^{-\int_a^t \frac{\partial L}{\partial z}[x;z](\theta)d\theta}$, as a conserved quantity along the extremals of problem $(\mathbf{H^{1*}})$. Our results improve those of [31] in three ways: (i) we consider a wider class of piecewise admissible functions; (ii) we consider a more general notion of invariance whose transformations

$\mathcal{T}^\epsilon$, $\mathcal{X}^\epsilon$ and $\mathcal{Z}^\epsilon$ may also depend on velocities, i.e., on $\dot{x}(t)$ (note that if (6.9) holds with $\mathcal{Z}^\epsilon[x;z] = z$, then (6.15) also holds); (iii) the conserved quantity (6.16), up to multiplication by a constant, is a particular case of (6.10) when there is no transformation in z ($Z = \frac{\partial \mathcal{Z}^\epsilon}{\partial \epsilon}\Big|_{\epsilon=0} = 0$).

The approach introduced in this chapter will be explored further in the following chapters in order to deal with higher-order problems and delayed problems.

The original results of this chapter were published in 2015 in [61]. They were also presented by the author in the 5th Iberian Mathematical Meeting, October 3–5, 2014, Aveiro, Portugal, in a contributed talk entitled "An Optimal Control approach to Herglotz variational problems".

CHAPTER 7

OPTIMAL CONTROL APPROACH TO HIGHER-ORDER VARIATIONAL PROBLEMS OF HERGLOTZ

In this chapter, we approach higher-order variational problems of Herglotz type from an optimal control point of view. Using Optimal Control theory, we derive a generalized Euler–Lagrange equation, transversality conditions, DuBois–Reymond necessary optimality condition and Noether's theorem for Herglotz's type higher-order variational problems, valid for piecewise smooth functions.

In [59] (presented in Chapter 4), we have introduced higher-order variational problems of Herglotz type and obtained a generalized Euler–Lagrange equation and transversality conditions for these problems. In particular, we considered the problem of determining the trajectories $x(\cdot)$ and $z(\cdot)$ such that:

$$\begin{gathered} z(b) \longrightarrow \text{extr} \\ \text{with } \dot{z}(t) = L(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), z(t)), \quad t \in [a, b], \\ \text{subject to } z(a) = \gamma, \quad \gamma \in \mathbb{R}. \end{gathered}$$

We proved (see Theorem 4.5) that if a pair $(x(\cdot), z(\cdot))$ is a solution of the previous higher-order problem, then it satisfies the higher-order generalized Euler–Lagrange equation

$$\sum_{j=0}^{n} (-1)^j \frac{d^j}{dt^j} \left(\psi_z(t) \frac{\partial L}{\partial x^{(j)}}[x; z]^n(t) \right) = 0, \quad t \in [a, b],$$

and the transversality conditions $\psi_j(b) = \psi_j(a) = 0$, for $j = 1, \ldots, n$, where

$$\begin{cases} \psi_z(t) = e^{\int_t^b \frac{\partial L}{\partial z}[x;z]^n(\theta)d\theta} \\ \psi_j(t) = \sum_{i=0}^{n-j}(-1)^{i+1}\frac{d^i}{dt^i}\left(\psi_z(t)\frac{\partial L}{\partial x^{(i+j)}}[x;z]^n(t)\right), \quad j = 1, \ldots, n. \end{cases}$$

While in [59] the admissible functions are $x(\cdot) \in C^{2n}([a,b];\mathbb{R}^m)$ and $z(\cdot) \in C^1([a,b];\mathbb{R})$, here we extend the higher-order Herglotz's problem to the wider class of functions $x(\cdot) \in PC^n([a,b];\mathbb{R}^m)$ and $z(\cdot) \in PC^1([a,b];\mathbb{R})$.

Problem $(\mathbf{H^{n*}})$. *Determine the trajectories $x(\cdot) \in PC^n([a,b];\mathbb{R}^m)$ and function $z(\cdot) \in PC^1([a,b];\mathbb{R})$ such that:*

$$\begin{gathered} z(b) \longrightarrow extr, \\ \text{with } \dot{z}(t) = L\left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), z(t)\right), \quad t \in [a,b], \\ \text{subject to } z(a) = \gamma, \quad \gamma \in \mathbb{R}, \text{ and} \\ x(a) = \alpha_0,\ \dot{x}(a) = \alpha_1, \ldots, x^{(n-1)}(a) = \alpha_{n-1}, \quad \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}^m, \end{gathered} \tag{$\mathbf{H^{n*}}$}$$

where the Lagrangian L is assumed to satisfy the following hypotheses:

i. L is a $C^1([a,b] \times \mathbb{R}^{(n+1)m+1};\mathbb{R})$ function;

ii. functions $t \mapsto \frac{\partial L}{\partial z}[x;z]^n(t)$ and $t \mapsto \frac{\partial L}{\partial x^{(j)}}[x;z]^n(t)$, $j = 0, \ldots, n$, are differentiable up to order n for any admissible pair $(x(\cdot), z(\cdot))$.

In this work we show how the results on the higher-order variational problem of Herglotz $(\mathbf{H^n})$ obtained in [59] can be generalized by using the theory of Optimal Control. Similarly to the first-order case (see [61], presented in Chapter 6) we rewrite the generalized higher--order variational problem of Herglotz $(\mathbf{H^{n*}})$ as a standard optimal control problem (P), and then we apply available results of Optimal Control theory. In detail, we extend the higher-order Euler–Lagrange equation and the transversality conditions for problem $(\mathbf{H^n})$ found in [59] to admissible functions $x(\cdot) \in PC^n([a,b];\mathbb{R}^m)$ and $z(\cdot) \in PC^1([a,b];\mathbb{R})$ (Theorem 7.2); we obtain a DuBois–Reymond necessary optimality condition (Theorem 7.4); and we generalize the first Noether theorem to higher-order variational problems of Herglotz type $(\mathbf{H^{n*}})$ (Theorem 7.6).

7.1 Necessary optimality conditions for higher-order Herglotz's problems

We begin by introducing some definitions for the higher-order variational problem of Herglotz $(\mathbf{H^{n*}})$.

Definition 7.1 (Admissible pair to problem $(\mathbf{H^{n*}})$)**.** *We say that $(x(\cdot), z(\cdot))$ with $x(\cdot) \in PC^n([a,b];\mathbb{R}^m)$ and $z(\cdot) \in PC^1([a,b];\mathbb{R})$ is an admissible pair to problem $(\mathbf{H^{n*}})$ if it satisfies the equation*

$$\dot{z}(t) = L(t, x(t), \dot{x}(t), \cdots, x^{(n)}(t), z(t)), \quad t \in [a,b],$$

and the initial conditions $z(a) = \gamma \in \mathbb{R}$ and

$$x(a) = \alpha_0,\ \dot{x}(a) = \alpha_1, \dots, x^{(n-1)}(a) = \alpha_{n-1}, \quad \alpha_0, \alpha_1, \dots, \alpha_{n-1} \in \mathbb{R}^m.$$

We now present a necessary condition for a pair $(x(\cdot), z(\cdot))$ to be a solution of problem $(\mathbf{H^{n*}})$. The following result generalizes [59] by considering a more general class of functions.

When there is no possibility of ambiguity, we sometimes suppress arguments.

Theorem 7.2 (Higher-order Euler–Lagrange equation and transversality conditions for problem $(\mathbf{H^{n*}})$)**.** *If $(x(\cdot), z(\cdot))$ is a solution of problem $(\mathbf{H^{n*}})$, then the Euler–Lagrange equation*

$$\sum_{j=0}^{n} (-1)^j \frac{d^j}{dt^j}\left(\psi_z(t) \frac{\partial L}{\partial x^{(j)}}[x;z]^n(t)\right) = 0 \tag{7.1}$$

holds, for $t \in [a,b]$. Moreover, the following transversality conditions hold:

$$\psi_j(b) = 0, \quad j = 1, \dots, n, \tag{7.2}$$

where

$$\begin{cases} \psi_z(t) = e^{\int_t^b \frac{\partial L}{\partial z}[x;z]^n(\theta) d\theta} \\ \psi_j(t) = \sum_{i=0}^{n-j} (-1)^{i+1} \frac{d^i}{dt^i}\left(\psi_z(t) \frac{\partial L}{\partial x^{(i+j)}}[x;z]^n(t)\right), \quad j = 1, \dots, n. \end{cases} \tag{7.3}$$

Proof. Observe that the higher-order problem of Herglotz $(\mathbf{H^{n*}})$ is a particular case of problem (P) when we consider a $n+1$ coordinates state variable $(x_0, x_1, \dots, x_{n-1}, z)$ with $x_0 = x$, $x_1 = \dot{x}$, $\dots$, $x_{n-1} = x^{(n-1)}$, a control $u = x^{(n)}$ and choose $f \equiv 0$ and $\phi(x_0, \dots, x_{n-1}, z) = z$.

The higher-order problem of Herglotz can now be described as an optimal control problem as follows:

$$z(b) \longrightarrow \text{extr}$$
$$\begin{cases} \dot{x}_0(t) = x_1(t), \\ \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_3(t), \\ \qquad \vdots \\ \dot{x}_{n-2}(t) = x_{n-1}(t), \\ \dot{x}_{n-1}(t) = u(t) = x_n(t), \\ \dot{z}(t) = L(t, x_0(t), \ldots, x_{n-1}(t), u(t), z(t)), \end{cases} \tag{7.4}$$
$$z(a) = \gamma, \quad \gamma \in \mathbb{R}, \text{ and}$$
$$x_0(a) = \alpha_0, \ldots, x_{n-1}(a) = \alpha_{n-1}, \alpha_0, \ldots, \alpha_{n-1} \in \mathbb{R}^m.$$

From Pontryagin's Maximum Principle for problem (P) (Theorem 3.1), there are $(\psi_1, \ldots, \psi_n, \psi_z) \in PC^1([a, b]; \mathbb{R}^{n \times m+1})$ such that the following conditions hold:

- the optimality condition

$$\frac{\partial H}{\partial u}(t, x_0(t), \ldots, x_{n-1}(t), u(t), z(t), \psi_1(t), \ldots, \psi_n(t), \psi_z(t)) = 0, \tag{7.5}$$

- the adjoint system

$$\begin{cases} \dot{x}_{j-1}(t) = \frac{\partial H}{\partial \psi_j}(t, x_0(t), \ldots, x_{n-1}(t), u(t), z(t), \psi_1(t), \ldots, \psi_n(t), \psi_z(t)), \ j = 1, \ldots, n, \\ \dot{\psi}_j(t) = -\frac{\partial H}{\partial x_{j-1}}(t, x_0(t), \ldots, x_{n-1}(t), u(t), z(t), \psi_1(t), \ldots, \psi_n(t), \psi_z(t)), \ j = 1, \ldots, n, \\ \dot{\psi}_z(t) = -\frac{\partial H}{\partial z}(t, x_0(t), \ldots, x_{n-1}(t), u(t), z(t), \psi_1(t), \ldots, \psi_n(t), \psi_z(t)), \end{cases} \tag{7.6}$$

- the transversality conditions

$$\begin{cases} \psi_j(b) = 0, \quad j = 1, \ldots, n, \\ \psi_z(b) = 1, \end{cases} \tag{7.7}$$

where the Hamiltonian H is defined by

$$\begin{aligned}
&H(t, x_0, \ldots, x_{n-1}, u, z, \psi_1, \ldots, \psi_n, \psi_z)\\
&\qquad\qquad = \psi_1 \cdot x_1 + \ldots + \psi_{n-1} \cdot x_{n-1} + \psi_n \cdot u + \psi_z \cdot L(t, x_0, \ldots, x_{n-1}, u, z).
\end{aligned}$$

Observe that the optimality condition (7.5) implies that $\psi_n = -\psi_z \frac{\partial L}{\partial u}$ and that the adjoint system (7.6) implies that

$$\begin{cases}
\dot{\psi}_1 = -\psi_z \frac{\partial L}{\partial x_0},\\
\dot{\psi}_j = -\psi_{j-1} - \psi_z \frac{\partial L}{\partial x_{j-1}}, \qquad \text{for } j = 2, \ldots, n,\\
\dot{\psi}_z = -\psi_z \frac{\partial L}{\partial z}.
\end{cases}$$

Hence, ψ_z is solution of a first-order linear differential equation, which is solved using an integrand factor to find that $\psi_z(t) = ke^{-\int_a^t \frac{\partial L}{\partial z} d\theta}$ with k a constant. From the last transversality condition in (7.7), we obtain that $k = e^{\int_a^b \frac{\partial L}{\partial z} d\theta}$ and, consequently,

$$\psi_z(t) = e^{\int_t^b \frac{\partial L}{\partial z} d\theta}.$$

Note also that for $j = n$ we obtain $\dot{\psi}_n = -\psi_{n-1} - \psi_z \frac{\partial L}{\partial x_{n-1}}$, which is equivalent to

$$\psi_{n-1} = \frac{d}{dt}\left(\psi_z \frac{\partial L}{\partial x_n}\right) - \psi_z \frac{\partial L}{\partial x_{n-1}}.$$

By differentiation of the previous expression, we obtain that

$$\dot{\psi}_{n-1} = \frac{d^2}{dt^2}\left(\psi_z \frac{\partial L}{\partial x_n}\right) - \frac{d}{dt}\left(\psi_z \frac{\partial L}{\partial x_{n-1}}\right)$$

and noting that $\dot{\psi}_{n-1} = -\psi_{n-2} - \psi_z \frac{\partial L}{\partial x_{n-2}}$, we find an expression for ψ_{n-2}:

$$\psi_{n-2} = -\frac{d^2}{dt^2}\left(\psi_z \frac{\partial L}{\partial x_n}\right) + \frac{d}{dt}\left(\psi_z \frac{\partial L}{\partial x_{n-1}}\right) - \psi_z \frac{\partial L}{\partial x_{n-2}}.$$

Similarly, we obtain that

$$\psi_{n-3} = \frac{d^3}{dt^3}\left(\psi_z \frac{\partial L}{\partial x_n}\right) - \frac{d^2}{dt^2}\left(\psi_z \frac{\partial L}{\partial x_{n-1}}\right) + \frac{d}{dt}\left(\psi_z \frac{\partial L}{\partial x_{n-2}}\right) - \psi_z \frac{\partial L}{\partial x_{n-3}}.$$

Applying the same argument to the next multipliers and noting that $\psi_1 = -\dot{\psi}_2 - \psi_z \frac{\partial L}{\partial x_1}$, we have

$$\begin{aligned}
\dot{\psi}_1 &= -\psi_z \frac{\partial L}{\partial x_0}\\
&= (-1)^n \frac{d^n}{dt^n}\left(\psi_z \frac{\partial L}{\partial x_n}\right) + (-1)^{n-1} \frac{d^{n-1}}{dt^{n-1}}\left(\psi_z \frac{\partial L}{\partial x_{n-1}}\right) + \cdots - \frac{d}{dt}\left(\psi_z \frac{\partial L}{\partial x_1}\right)
\end{aligned}$$

or, equivalently,

$$(-1)^n \frac{d^n}{dt^n}\left(\psi_z \frac{\partial L}{\partial x_n}\right) + (-1)^{n-1}\frac{d^{n-1}}{dt^{n-1}}\left(\psi_z \frac{\partial L}{\partial x_{n-1}}\right) + \cdots - \frac{d}{dt}\left(\psi_z \frac{\partial L}{\partial x_1}\right) + \psi_z \frac{\partial L}{\partial x_0} = 0.$$

Rewriting previous equation in terms of problem $(\mathbf{H^{n*}})$ and in the form of a summation, one gets

$$\sum_{j=0}^{n}(-1)^j \frac{d^j}{dt^j}\left(\psi_z \frac{\partial L}{\partial x^{(j)}}\right) = 0$$

as intended. Observe also that we were also able to derive expressions for the multipliers:

$$\psi_j = \sum_{i=0}^{n-j}(-1)^i \frac{d^i}{dt^i}\left(-\psi_z \frac{\partial L}{\partial x^{(i+j)}}\right), \quad j = 1, \ldots, n,$$

which together with (7.7) lead to the transversality conditions

$$\sum_{i=0}^{n-j}(-1)^i \frac{d^i}{dt^i}\left(\psi_z \frac{\partial L}{\partial x^{(i+j)}}\right)\Bigg|_{t=b} = 0, \quad j = 1, \ldots, n.$$

This concludes the proof. □

Definition 7.3 (Extremal to problem $(\mathbf{H^{n*}})$)**.** *We say that an admissible pair* $(x(\cdot), z(\cdot))$ *is an extremal of problem* $(\mathbf{H^{n*}})$ *if it satisfies the Euler–Lagrange equation* (7.1) *and the transversality conditions* (7.2)*.*

Next we present the DuBois–Reymond condition for the higher-order variational problem of Herglotz $(\mathbf{H^{n*}})$.

Theorem 7.4 (DuBois–Reymond condition for problem$(\mathbf{H^{n*}})$)**.** *If* $(x(\cdot), z(\cdot))$ *is a solution of problem* $(\mathbf{H^{n*}})$*, then*

$$\frac{d}{dt}\left(\sum_{i=1}^{n}\psi_i(t)x^{(i)}(t) + \psi_z(t)L[x;z]^n(t)\right) = \psi_z(t)\frac{\partial L}{\partial t}[x;z]^n(t),$$

where $\psi_z(t)$ *and* $\psi_i(t)$ *are defined in* (7.3)*.*

Proof. Rewrite $(\mathbf{H^{n*}})$ as the optimal control problem (7.4) and apply Theorem 3.3. □

7.2 Higher-order Noether's symmetry theorem

We begin by introducing the definition of invariance under a one-parameter group of transformations.

Definition 7.5 (Invariance of problem ($\mathbf{H^{n*}}$) under a one-parameter group of transformations)**.** *Let h^ϵ be a one-parameter family of invertible C^1 maps $h^\epsilon : [a,b] \times \mathbb{R}^m \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}$,*

$$h^\epsilon(t, x(t), z(t)) = (\mathcal{T}^\epsilon[x;z]^n(t), \mathcal{X}^\epsilon[x;z]^n(t), \mathcal{Z}^\epsilon[x;z]^n(t)),$$
$$\text{with } h^0(t,x,z) = (t,x,z), \quad \forall (t,x,z) \in [a,b] \times \mathbb{R}^m \times \mathbb{R}.$$

Problem ($\mathbf{H^{n*}}$) is said to be invariant under the one-parameter group of transformations h^ϵ if for all admissible pairs $(x(\cdot), z(\cdot))$ the following two conditions hold:

(i)

$$\left(\frac{z(b)}{b-a} + \xi\epsilon + o(\epsilon)\right)\frac{d\mathcal{T}^\epsilon}{dt}[x;z]^n(t) = \frac{z(b)}{b-a}, \text{ for some constant } \xi; \tag{7.8}$$

(ii)

$$\frac{d\mathcal{Z}^\epsilon}{dt}[x;z]^n(t) = L\Bigg(\mathcal{T}^\epsilon[x;z]^n(t), \mathcal{X}^\epsilon[x;z]^n(t), \frac{d\mathcal{X}^\epsilon}{d\mathcal{T}^\epsilon}[x;z]^n(t), \ldots$$
$$\ldots, \frac{d^n\mathcal{X}^\epsilon}{d(\mathcal{T}^\epsilon)^n}[x;z]^n(t), \mathcal{Z}^\epsilon[x;z]^n(t)\Bigg)\frac{d\mathcal{T}^\epsilon}{dt}[x;z]^n(t), \tag{7.9}$$

where

$$\frac{d\mathcal{X}^\epsilon}{d\mathcal{T}^\epsilon}[x;z]^n(t) = \frac{\frac{d\mathcal{X}^\epsilon}{dt}[x;z]^n(t)}{\frac{d\mathcal{T}^\epsilon}{dt}[x;z]^n(t)} \text{ and } \frac{d^i\mathcal{X}^\epsilon}{d(\mathcal{T}^\epsilon)^i}[x;z]^n(t) = \frac{\frac{d}{dt}\left(\frac{d^{i-1}\mathcal{X}^\epsilon}{d(\mathcal{T}^\epsilon)^{i-1}}[x;z]^n(t)\right)}{\frac{d\mathcal{T}^\epsilon}{dt}[x;z]^n(t)} \tag{7.10}$$

for $i = 2, \ldots, n$.

Next we present the main result of this chapter.

Theorem 7.6 (Noether's theorem for problem($\mathbf{H^{n*}}$))**.** *If problem ($\mathbf{H^{n*}}$) is invariant in the sense of Definition 7.5, then the quantity*

$$\sum_{i=1}^{n}\psi_i(t)X_{i-1}[x;z]^n(t) + \psi_z(t)Z[x;z]^n(t)$$
$$- \left(\sum_{i=1}^{n}\psi_i(t)x^{(i)}(t) + \psi_z(t)L[x;z]^n(t)\right)T[x;z]^n(t)$$

is constant in t along every extremal to problem $(\mathbf{H^{n*}})$, *where*

$$T = \left.\frac{\partial \mathcal{T}^\epsilon}{\partial \epsilon}\right|_{\epsilon=0}, \quad X_0 = \left.\frac{\partial \mathcal{X}^\epsilon}{\partial \epsilon}\right|_{\epsilon=0}, \quad Z = \left.\frac{\partial \mathcal{Z}^\epsilon}{\partial \epsilon}\right|_{\epsilon=0},$$
$$X_i = \frac{d}{dt}X_{i-1} - x^{(i)}\frac{d}{dt}\left(\left.\frac{\partial \mathcal{T}^\epsilon}{\partial \epsilon}\right|_{\epsilon=0}\right) \quad \textit{for } i = 1, \ldots, n-1,$$

ψ_i *is defined by* (7.3) *and* $\psi_z(t) = e^{\int_t^b \frac{\partial L}{\partial z} d\theta}$.

Proof. As before, we deal with problem $(\mathbf{H^{n*}})$ in its equivalent optimal control form (7.4). We now prove that if problem $(\mathbf{H^{n*}})$ is invariant in the sense of Definition 7.5, then (7.4) is invariant in the sense of Definition 3.4. First, observe that if (7.8) holds, then (3.6) holds for (7.4) with $f \equiv 0$ and $\phi(x_0, \ldots, x_{n-1}, z) = z$. Second, note that the control system of (7.4) defines naturally $\mathcal{U}^\epsilon := \frac{d\mathcal{X}^\epsilon_{n-1}}{d\mathcal{T}^\epsilon}$ and $\mathcal{X}^\epsilon_i := \frac{d\mathcal{X}^\epsilon_{i-1}}{d\mathcal{T}^\epsilon}$, that is,

$$\begin{cases} \frac{d\mathcal{X}^\epsilon_{i-1}}{dt}[x;z]^n(t) = \mathcal{X}^\epsilon_i[x;z]^n(t)\frac{d\mathcal{T}^\epsilon}{dt}[x;z]^n(t), \quad i = 1, \ldots, n-1, \\ \frac{d\mathcal{X}^\epsilon_{n-1}}{dt}[x;z]^n(t) = \mathcal{U}^\epsilon[x;z]^n(t)\frac{d\mathcal{T}^\epsilon}{dt}[x;z]^n(t). \end{cases} \tag{7.11}$$

This means that if (7.9) and (7.11) hold, then there is also invariance in the sense of (3.7) and problem (7.4) is invariant in the sense of Definition 3.4. This invariance gives conditions to apply Theorem 3.5 to problem (7.4), which assures that the quantity

$$(b-t)\xi + \sum_{i=1}^{n} \psi_i(t) X_{i-1}[x;z]^n(t) + \psi_z(t) Z[x;z]^n(t) - \left[\sum_{i=1}^{n} \psi_i(t) x_i(t) + \psi_z(t) L[x;z]^n(t) + \frac{\phi(x(b))}{b-a}\right] T[x;z]^n(t),$$

where $X_i = \frac{\partial}{\partial \epsilon}\left.\frac{d^i \mathcal{X}^\epsilon}{d(\mathcal{T}^\epsilon)^i}\right|_{\epsilon=0}$, is constant in t along every Pontryagin extremal of problem (7.4). This means that the quantity

$$(b-t)\xi - \frac{\phi(x(b))}{b-a} T[x;z]^n(t) + \sum_{i=1}^{n} \psi_i(t) X_{i-1}[x;z]^n(t) + \psi_z(t) Z[x;z]^n(t) - \left[\sum_{i=1}^{n} \psi_i(t) x^{(i)}(t) + \psi_z(t) L[x;z]^n(t)\right] T[x;z]^n(t)$$

is constant in t along every extremal of problem $(\mathbf{H^{n*}})$. Observe that $X_0 = \left.\frac{\partial \mathcal{X}^\epsilon}{\partial \epsilon}\right|_{\epsilon=0}$, which together with (7.10) lead to

$$\begin{aligned}
X_i = \frac{\partial}{\partial \epsilon} \frac{d^i \mathcal{X}^\epsilon}{d(\mathcal{T}^\epsilon)^i}\bigg|_{\epsilon=0} &= \frac{\partial}{\partial \epsilon} \left(\frac{\frac{d}{dt}\left(\frac{d^{i-1}\mathcal{X}^\epsilon}{d(\mathcal{T}^\epsilon)^{i-1}}\right)}{\frac{d\mathcal{T}^\epsilon}{dt}} \right)\Bigg|_{\epsilon=0} \\
&= \frac{d}{dt}\left(\frac{\partial}{\partial \epsilon} \frac{d^{i-1}\mathcal{X}^\epsilon}{d(\mathcal{T}^\epsilon)^{i-1}}\bigg|_{\epsilon=0} \right) - x^{(i)} \frac{d}{dt}\left(\frac{\partial \mathcal{T}^\epsilon}{\partial \epsilon}\bigg|_{\epsilon=0} \right) \\
&= \frac{d}{dt} X_{i-1} - x^{(i)} \frac{d}{dt}\left(\frac{\partial \mathcal{T}^\epsilon}{\partial \epsilon}\bigg|_{\epsilon=0} \right).
\end{aligned}$$

To end the proof we only need to prove that the quantity

$$(b-t)\xi - \frac{z(b)}{b-a} T[x;z]^n(t) \tag{7.12}$$

is a constant. But this has already been done in the previous chapter in the proof of Theorem 6.6, from where we concluded that expression (7.12) is the constant

$$(b-a)\xi - \frac{z(b)T[x;z]^n(a)}{b-a}.$$

The proof is then complete. □

7.3 Conclusions

In this chapter we investigated the higher-order variational problem of Herglotz from an optimal control point of view. The higher-order generalized Euler–Lagrange equation and the transversality conditions proved in [59] were obtained in the wider class of piecewise admissible functions. Moreover, we proved two important new results: a DuBois–Reymond necessary condition and Noether's theorem for higher-order variational problems of Herglotz type.

The original results of this chapter were published in 2015 in [62]. They were also presented by the author in the AMS-EMS-SPM International Meeting 2015, June 10–13, 2015, Porto, Portugal, in a contributed talk entitled "Noether's theorem for higher-order variational problems of Herglotz type", and in the Portuguese Meeting on Optimal Control EPCO 2015, September 15, 2015, Guimarães, Portugal, in a contributed talk entitled "An optimal control approach to higher-order variational problems of Herglotz type".

CHAPTER 8

OPTIMAL CONTROL APPROACH TO HIGHER-ORDER DELAYED VARIATIONAL PROBLEMS OF HERGLOTZ

In this chapter, we focus again on time delayed problems. As already mentioned in Chapter 5, variational problems with time delay play an important role in the modelling of phenomena in several fields.

The main goal of this chapter is to generalize the results of [59, 60, 61, 62] (presented in the previous four chapters) by considering higher-order variational problems of Herglotz type with time delay, proving the corresponding Euler–Lagrange equations, transversality conditions, the DuBois–Reymond necessary optimality condition and Noether's first theorem. In particular, in relation to our previous work with time delay [60] (see Chapter 5), we improved its results by considering a wider class of admissible functions. Moreover, we extend the results of [60] to the higher-order case. Precisely, we generalize several Herglotz's based problems: $(\mathbf{H^1})$, $(\mathbf{H}_\tau)$, $(\mathbf{H^{1*}})$ and $(\mathbf{H^{n*}})$, by considering the following higher-order variational problem with time delay:

Problem $(\mathbf{H}_\tau^{\mathbf{n}*})$. *Let τ be a real number such that $0 \leq \tau < b - a$. Determine the piecewise* *trajectories* $x(\cdot) \in PC^n([a-\tau,b];\mathbb{R}^m)$ *and the function* $z(\cdot) \in PC^1([a,b];\mathbb{R})$ *such that:*

$$
\begin{gathered}
z(b) \longrightarrow extr, \\
\text{with the pair } (x(\cdot), z(\cdot)) \text{ satisfying for all } t \in [a,b] : \\
\dot{z}(t) = L\left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), x(t-\tau), \dot{x}(t-\tau), \ldots, x^{(n)}(t-\tau), z(t)\right), \\
\text{subject to } z(a) = \gamma \in \mathbb{R} \\
\text{and } x^{(k)}(t) = \mu^{(k)}(t), \text{for all } t \in [a-\tau, a], \quad k = 0, \ldots, n-1,
\end{gathered}
\tag{$\mathbf{H}_\tau^{\mathbf{n}*}$}
$$

where $\mu(\cdot) \in PC^n([a-\tau,a];\mathbb{R}^m)$ *is a given initial function and the Lagrangian* L *is assumed to satisfy the following hypotheses:*

i. $L \in C^1([a,b] \times \mathbb{R}^{2m(n+1)+1};\mathbb{R})$;

ii. functions $t \mapsto \frac{\partial L}{\partial z}[x;z]^n_\tau(t)$, $t \mapsto \frac{\partial L}{\partial x^{(j)}}[x;z]^n_\tau(t)$ *and* $t \mapsto \frac{\partial L}{\partial x^{(j)}_\tau}[x;z]^n_\tau(t)$, $k = 0, \ldots, n$, *are differentiable up to order* n *for any admissible trajectory* x.

In the rest of the chapter we use the notation

$$[x;z]^n_\tau(t) := \left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), x_\tau(t), \dot{x}_\tau(t), \ldots, x^{(n)}_\tau(t), z(t)\right).$$

The structure of the chapter is as follows. In Section 8.1, we use a very interesting technique that allows to deal with a delayed problem as a non-delayed one. In Section 8.2, we formulate and prove higher-order Euler–Lagrange equations and transversality conditions for generalized variational problems with time delay (Theorem 8.3) and the DuBois–Reymond optimality condition (Theorem 8.7). Finally, in Section 8.3, we prove a Noether's theorem for higher-order variational problems of Herglotz type with time delay (Theorem 8.10).

8.1 Reduction to a non-delayed problem

We generalize the technique of reduction of a delayed first-order optimal control problem to a non-delayed problem proposed by Guinn in [38] to our higher-order delayed problem. In order to reduce the higher-order problem of Herglotz with time delay to a non-delayed first-order problem, we assume, without loss of generality, the initial time to be zero ($a = 0$) and the final time to be an integer multiple of τ, that is, $b = N\tau$ for $N \in \mathbb{N}$ (see Remark 8.1). We divide the interval $[a,b]$ into N equal parts and fix $t \in [0,\tau]$. We also introduce the variables $x^{k;i}$ and z_j with $k = 0, \ldots, n$, $i = 0, \ldots, N$, and $j = 1, \ldots, N+1$. The variable k is related to the order of the derivative of x, i is related to the ith subinterval of $[-\tau, N\tau]$, and j is related to the jth subinterval of $[0, (N+1)\tau]$ as follows:

$$\begin{aligned} &x^{k;i}(t) = x^{(k)}(t + (i-1)\tau), \quad z_j(t) = z(t + (j-1)\tau), \\ &\dot{z}_j(t) = L_j(t), \quad x^{k;N+1}(t) = 0, \quad \dot{z}_{N+1}(t) = L_{N+1} = 0 \end{aligned} \tag{8.1}$$

with

$$L_j(t) := L\left(t + (j-1)\tau, x^{0;j}(t), \ldots, x^{n;j}(t), x^{0;j-1}(t), \ldots, x^{n;j-1}(t), z_j(t)\right).$$

Finally, the higher-order problem of Herglotz with time delay $(\mathbf{H}^{\mathbf{n}}_{\tau})$ can be written as an optimal control problem without time delay as follows:

$$z_N(\tau) \longrightarrow \mathrm{extr}, \quad \text{subject to}$$

$$\begin{cases} \dot{x}^{k;i}(t) = x^{k+1;i}(t), \\ x^{k;N+1}(t) = 0, \\ \dot{z}_j(t) = L_j(t), \\ \dot{z}_{N+1}(t) = L_{N+1}(t) = 0 \end{cases} \tag{8.2}$$

$$\text{for all } t \in [0, \tau], \quad k = 0, \ldots, n-1, \ i = 0, \ldots, N, \ j = 1, \ldots, N,$$

and with the initial conditions

$$x^{k;0}(0) = \mu^{(k)}(-\tau), \quad x^{k;i}(0) = x^{k;i-1}(\tau),$$

$$z_1(0) = \gamma, \quad \gamma \in \mathbb{R}, \quad z_j(0) = z_{j-1}(\tau), \text{ for } i \geq 1 \text{ and } j \geq 2.$$

In this form we look to $x^{k;i}$ and z_j as state variables and to $u_i := x^{n;i}$ as control variables.

Remark 8.1. *We considered the case of b being an integer multiple of τ. If b is not an integer multiple of τ, then there is an integer N such that $(N-1)\tau < b < N\tau$. In that case, the only modification required in the change of variables given in* (8.1) *is to consider the variables $x^{k;N}$, $k = 0, \ldots, n$, and $\dot{z}_N$ as defined in* (8.1) *for $t \in [0, b-(N-1)\tau]$ and zero for $t \in]b-(N-1)\tau, \tau]$. With this slight change, the function to be extremized remains the same and we can consider, without loss of generality, b to be an integer multiple of τ.*

8.2 Necessary optimality conditions for higher-order Herglotz's problems with time delay

Before the proof of the first result of this chapter we introduce some definitions concerning the variational problem of Herglotz with time delay $(\mathbf{H}^{\mathbf{n}*}_{\tau})$.

Definition 8.2 (Admissible pair to problem $(\mathbf{H}_\tau^{\mathbf{n}*})$)**.** *We say that* $(x(\cdot), z(\cdot))$ *with* $x(\cdot) \in PC^n([a-\tau, b]; \mathbb{R}^m)$ *and* $z(\cdot) \in PC^1([a, b]; \mathbb{R})$ *is an admissible pair to problem* $(\mathbf{H}_\tau^{\mathbf{n}*})$ *if it satisfies the equation*

$$\dot{z}(t) = L[x; z]_\tau^n(t), \quad t \in [a, b],$$

subject to

$$z(a) = \gamma, \; x^{(k)}(t) = \mu^{(k)}(t)$$

for all $k = 0, 1, \ldots, n-1$*,* $t \in [a-\tau, a]$ *and* $\gamma \in \mathbb{R}$*.*

We now prove a necessary condition for a pair $(x(\cdot), z(\cdot))$ to be an extremizer to problem $(\mathbf{H}_\tau^{\mathbf{n}*})$. Along the proofs we sometimes suppress arguments for expressions whose arguments have been clearly stated before.

Theorem 8.3 (Higher-order delayed Euler–Lagrange and transversality conditions)**.** *If* $(x(\cdot), z(\cdot))$ *is a solution of problem* $(\mathbf{H}_\tau^{\mathbf{n}*})$*, then the two Euler–Lagrange equations*

$$\sum_{l=0}^{n} (-1)^l \frac{d^l}{dt^l} \left(\psi_z(t) \frac{\partial L}{\partial x^{(l)}}[x; z]_\tau^n(t) + \psi_z(t+\tau) \frac{\partial L}{\partial x_\tau^{(l)}}[x; z]_\tau^n(t+\tau) \right) = 0, \tag{8.3}$$

for $t \in [a, b-\tau]$*, and*

$$\sum_{l=0}^{n} (-1)^l \frac{d^l}{dt^l} \left(\psi_z(t) \frac{\partial L}{\partial x^{(l)}}[x; z]_\tau^n(t) \right) = 0, \tag{8.4}$$

for $t \in [b-\tau, b]$ *and* ψ_z *defined by*

$$\psi_z(t) = e^{\int_t^b \frac{\partial L}{\partial z}[x; z]_\tau^n(\theta) d\theta}, \quad t \in [a, b],$$

hold. Furthermore, the following transversality conditions hold:

$$\sum_{l=0}^{n-k} (-1)^l \frac{d^l}{dt^l} \left(\psi_z(t) \frac{\partial L}{\partial x^{(l+k)}}[x; z]_\tau^n(t) \right) \Bigg|_{t=b} = 0, \tag{8.5}$$

$k = 1, \ldots, n.$

Proof. In order to prove both Euler–Lagrange equations, consider problem $(\mathbf{H}_\tau^{\mathbf{n}*})$ in the non-delayed optimal control form (8.2). Applying Pontryagin's maximum principle for problem (P) to problem $(\mathbf{H}_\tau^{\mathbf{n}*})$ in the form (8.2), we conclude that there are multipliers

$\phi_{k;i}$ and ψ_j for $k = 1, \ldots, n$, $i = 0, \ldots, N$ and $j = 1, \ldots, N+1$, such that, with the Hamiltonian defined by

$$H = \sum_{l=1}^{n} \left(\sum_{i=0}^{N} \phi_{l;i}(t) \cdot x^{l;i}(t) \right) + \sum_{j=1}^{N+1} \psi_j(t) L_j(t), \tag{8.6}$$

the following conditions hold:

- the optimality conditions
$$\frac{\partial H}{\partial u_i} = 0,$$
- the adjoint system
$$\begin{cases} \dot{x}^{k-1;i} = \frac{\partial H}{\partial \phi_{k;i}}, \\ \dot{z}_j = \frac{\partial H}{\partial \psi_j}, \\ \dot{\phi}_{k;i} = -\frac{\partial H}{\partial x^{k-1;i}}, \\ \dot{\psi}_j = -\frac{\partial H}{\partial z_j}, \end{cases}$$
- the transversality conditions
$$\begin{cases} \phi_{k;i}(\tau) = 0, \\ \psi_j(\tau) = 1. \end{cases}$$

Observe that the forth equation in the adjoint system is equivalent to the differential equation $\dot{\psi}_j = -\psi_j \frac{\partial L_j}{\partial z_j}$. Together with the transversality condition, we obtain that the multipliers ψ_j, $j = 1, \ldots, N+1$, are given by

$$\psi_j(t) = e^{\int_t^\tau \frac{\partial L_j}{\partial z_j} d\theta}. \tag{8.7}$$

From the third equation in the adjoint system, we obtain that

$$\dot{\phi}_{k;i} = -\phi_{k-1;i} - \psi_i \frac{\partial L_i}{\partial x^{k-1;i}} - \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{k-1;i}}, \tag{8.8}$$

$k, i = 1, \ldots, n$, which for the particular case of $k = n$ reduces to

$$\dot{\phi}_{n;i} = -\phi_{n-1;i} - \psi_i \frac{\partial L_i}{\partial x^{n-1;i}} - \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{n-1;i}}.$$

This equality, together with the differentiation of the optimality condition

$$\begin{aligned} \dot{\phi}_{n;i} &= -\frac{d}{dt}\left(\psi_i \frac{\partial L_i}{\partial u_i}\right) - \frac{d}{dt}\left(\psi_{i+1} \frac{\partial L_{i+1}}{\partial u_i}\right) \\ &= -\frac{d}{dt}\left(\psi_i \frac{\partial L_i}{\partial x^{n;i}} + \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{n;i}}\right), \end{aligned}$$

leads to

$$\phi_{n-1;i} = -\psi_i \frac{\partial L_i}{\partial x^{n-1;i}} - \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{n-1;i}} + \frac{d}{dt}\left(\psi_i \frac{\partial L_i}{\partial x^{n;i}} + \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{n;i}}\right).$$

By differentiation of the previous expression and comparison with (8.8) for $k = n-1$, we find the expression for $\phi_{n-2;i}$:

$$\begin{aligned}\phi_{n-2;i} = &-\psi_i \frac{\partial L_i}{\partial x^{n-2;i}} - \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{n-2;i}} \\ &+ \frac{d}{dt}\left(\psi_i \frac{\partial L_i}{\partial x^{n-1;i}} + \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{n-1;i}}\right) - \frac{d^2}{dt^2}\left(\psi_i \frac{\partial L_i}{\partial x^{n;i}} + \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{n;i}}\right).\end{aligned}$$

Using recursively the technique of derivation of $\phi_{k;i}$ and comparison with (8.8), we find the expression for $\phi_{k;i}$ $(k = 1, \ldots, n)$:

$$\phi_{k;i} = \sum_{l=0}^{n-k} (-1)^{l+1} \frac{d^l}{dt^l}\left(\psi_i \frac{\partial L_i}{\partial x^{l+k;i}} + \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{l+k;i}}\right), \quad i = 1, \ldots, N. \tag{8.9}$$

Considering $\phi_{1;i}$ given by the previous equation and comparing it with

$$\phi_{1;i} = -\dot{\phi}_{2;i} - \psi_i \frac{\partial L_i}{\partial x^{1;i}} - \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{1;i}},$$

given by (8.8) for $k = 2$, we obtain that

$$\sum_{l=0}^{n} (-1)^l \frac{d^l}{dt^l}\left(\psi_i \frac{\partial L_i}{\partial x^{l;i}} + \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{l;i}}\right) = 0, \quad i = 1, \ldots, N. \tag{8.10}$$

Since $L_{N+1} = 0$, the previous equation for $i = N$ reduces to

$$\sum_{l=0}^{n} (-1)^l \frac{d^l}{dt^l}\left(\psi_N \frac{\partial L_N}{\partial x^{l;N}}\right) = 0. \tag{8.11}$$

The final step is to rewrite the results obtained inverting the changes of variables (8.1). For this purpose, define $\psi_z(t)$, $t \in [0, b+\tau]$, by

$$\psi_z(t) = \psi_i(t-(i-1)\tau), \quad (i-1)\tau \le t \le i\tau, \quad i = 1, \ldots, N+1,$$

and $\phi_k(t)$, $k = 1, \ldots, n$, $t \in [-\tau, b]$, by

$$\phi_k(t) = \phi_{k;i}(t-(i-1)\tau), \quad (i-1)\tau \le t \le i\tau, \quad i = 1, \ldots, N.$$

This allows to write

$$\psi_z(t) = e^{\int_t^b \frac{\partial L}{\partial z}[x;z]_\tau^n(\theta)d\theta}, \quad t \in [a, b], \tag{8.12}$$

and

$$\phi_k(t) = \sum_{l=0}^{n-k}(-1)^{l+1}\frac{d^l}{dt^l}\left(\psi_z(t+\tau)\frac{\partial L}{\partial x_\tau^{(l+k)}}[x;z]_\tau^n(t+\tau)\right), \quad t \in [a-\tau, a],$$

$$\begin{aligned}\phi_k(t) = \sum_{l=0}^{n-k}(-1)^{l+1}\frac{d^l}{dt^l}\Bigg(&\psi_z(t)\frac{\partial L}{\partial x^{(l+k)}}[x;z]_\tau^n(t) \\ &+\psi_z(t+\tau)\frac{\partial L}{\partial x_\tau^{(l+k)}}[x;z]_\tau^n(t+\tau)\Bigg), \quad t \in [a,b],\end{aligned} \tag{8.13}$$

$k = 1, \ldots, n$. Note that if $t \in [b-\tau, b]$, then $L[x;z]_\tau^n(t+\tau)$ is, by definition, null. Finally, equations (8.10)–(8.11) lead to the Euler–Lagrange equations for the higher-order problem of Herglotz with time delay $(\mathbf{H_\tau^{n*}})$:

$$\sum_{l=0}^{n}(-1)^l\frac{d^l}{dt^l}\left(\psi_z(t)\frac{\partial L}{\partial x^{(l)}}[x;z]_\tau^n(t) + \psi_z(t+\tau)\frac{\partial L}{\partial x_\tau^{(l)}}[x,z]_\tau^n(t+\tau)\right) = 0$$

for $t \in [a, b-\tau]$ and

$$\sum_{l=0}^{n}(-1)^l\frac{d^l}{dt^l}\left(\psi_z(t)\frac{\partial L}{\partial x^{(l)}}[x;z]_\tau^n(t)\right) = 0$$

for $t \in [b-\tau, b]$. From (8.9) and the transversality conditions for $\phi_{k;i}$, we obtain the transversality conditions $\phi_k(b) = 0$, that is,

$$\sum_{l=0}^{n-k}(-1)^l\frac{d^l}{dt^l}\left(\psi_z(t)\frac{\partial L}{\partial x^{(l+k)}}[x;z]_\tau^n(t)\right)\Bigg|_{t=b} = 0,$$

$k = 1, \ldots, n$. □

Definition 8.4 (Extremal to problem $(\mathbf{H_\tau^{n*}})$). *We say that an admissible pair* $(x(\cdot), z(\cdot))$ *is an extremal to problem* $(\mathbf{H_\tau^{n*}})$ *if it satisfies the Euler–Lagrange equations* (8.3)–(8.4) *and the transversality conditions* (8.5).

Theorem 8.3 gives a generalization of the Euler–Lagrange equation and transversality conditions for the higher-order problem of Herglotz presented by the authors in [59] (see Chapter 4). It is also a generalization of the results in [61, 62] (see Chapters 6-7).

Corollary 8.5 (cf. [59, 62]). *If* $(x(\cdot), z(\cdot))$ *is a solution of the higher-order problem of Herglotz*

$$\begin{gathered} z(b) \longrightarrow extr, \\ \dot{z}(t) = L\left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), z(t)\right), \quad t \in [a,b], \\ z(a) = \gamma \in \mathbb{R}, \quad x^{(k)}(a) = \alpha_k, \quad \alpha_k \in \mathbb{R}^m, \quad k = 0, \ldots, n-1, \end{gathered} \tag{8.14}$$

then the Euler–Lagrange equation

$$\sum_{l=0}^{n}(-1)^l \frac{d^l}{dt^l}\left(\psi_z(t)\frac{\partial L}{\partial x^{(l)}}[x;z]_0^n(t)\right) = 0$$

holds for $t \in [a,b]$, *where* ψ_z *is defined in* (8.12). *Furthermore, the following transversality conditions hold:*

$$\sum_{l=0}^{n-k}(-1)^l \frac{d^l}{dt^l}\left(\psi_z(t)\frac{\partial L}{\partial x^{(l+k)}}[x;z]_0^n(t)\right)\Big|_{t=b} = 0,$$

$k = 1, \ldots, n.$

Proof. Consider Theorem 8.3 with no delay, that is, with $\tau = 0$. Recall that $[x;z]_\tau^n(t) := \left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), x_\tau(t), \dot{x}_\tau(t), \ldots, x_\tau^{(n)}(t), z(t)\right)$. □

Theorem 8.3 is also a generalization of the Euler–Lagrange equations for the first-order problem of Herglotz with time delay obtained in [60] (see Chapter 5).

Corollary 8.6 (cf. [60]). *If* $(x(\cdot), z(\cdot))$ *is a solution of the first-order problem of Herglotz with time delay*

$$\begin{gathered} z(b) \longrightarrow extr, \\ \dot{z}(t) = L\left(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau), z(t)\right), \quad t \in [a,b], \\ z(a) = \gamma \in \mathbb{R}, \quad x(t) = \mu(t), \quad t \in [a-\tau, a], \end{gathered} \tag{8.15}$$

for a given piecewise initial function μ, *then the Euler–Lagrange equations*

$$\begin{aligned} &\psi_z(t)\frac{\partial L}{\partial x}[x;z]_\tau^1(t) + \psi_z(t+\tau)\frac{\partial L}{\partial x_\tau}[x,z]_\tau^1(t+\tau) \\ &\qquad - \frac{d}{dt}\left(\psi_z(t)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau^1(t) + \psi_z(t+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x,z]_\tau^1(t+\tau)\right) = 0, \end{aligned}$$

for $t \in [a, b-\tau]$, *and*

$$\psi_z(t)\frac{\partial L}{\partial x}[x;z]_\tau^1(t) - \frac{d}{dt}\left(\psi_z(t)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau^1(t)\right) = 0,$$

for $t \in [b-\tau, b]$, *hold.*

Proof. Consider Theorem 8.3 with $n = 1$. □

Theorem 8.7 (Higher-order delayed DuBois–Reymond condition). *If the pair* $(x(\cdot), z(\cdot))$ *is a solution of problem* $(\mathbf{H}_\tau^{\mathbf{n}*})$, *then*

$$\frac{d}{dt}\left(\sum_{k=1}^{n}\phi_k(t)\cdot x^{(k)}(t)+\psi_z(t)L[x;z]_\tau^n(t)\right)=\psi_z(t)\frac{\partial L}{\partial t}[x;z]_\tau^n(t), \tag{8.16}$$

where ψ_z *and* ϕ_k *are defined by* (8.12) *and* (8.13), *respectively.*

Proof. Consider problem $(\mathbf{H}_\tau^{\mathbf{n}*})$ in the formulation given by (8.2). Theorem 3.3 asserts that $\frac{dH}{dt}=\frac{\partial H}{\partial t}$ for H given by (8.6). We obtain (8.16) by writing H in the variables ϕ_k and ψ_z. □

Theorem 8.7 is also a generalization of the DuBois–Reymond condition presented in [60] for the first-order problem of Herglotz with time delay. In that paper, for technical reasons, we added an additional hypothesis that we are able to avoid here.

Corollary 8.8 (cf. [60]). *If* $(x(\cdot), z(\cdot))$ *is a solution of first-order problem of Herglotz with time delay* (8.15), *then*

$$\begin{aligned}\psi_z(t)\frac{\partial L}{\partial t}[x;z]_\tau^1(t)=\frac{d}{dt}\Bigg(&\psi_z(t)L[x;z]_\tau^1(t)\\&-\left(\psi_z(t)\frac{\partial L}{\partial \dot{x}}[x;z]_\tau^1(t)+\psi_z(t+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]_\tau^1(t+\tau)\right)\dot{x}(t)\Bigg),\end{aligned}$$

where ψ_z *is defined by* (8.12).

Proof. Consider Theorem 8.7 with $n=1$. □

8.3 Higher-order Noether's symmetry theorem with time delay

Before presenting a Noether theorem to problem $(\mathbf{H}_\tau^{\mathbf{n}*})$, we introduce the notion of invariance under a one-parameter group of transformations.

Definition 8.9 (Invariance of problem $(\mathbf{H}_\tau^{\mathbf{n}*})$ under a one-parameter group of transformations). *Let* h^ϵ *be a one-parameter family of invertible* C^1 *maps* $h^\epsilon : [a-\tau, b]\times\mathbb{R}^m\times\mathbb{R}\longrightarrow$

$\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}$,

$$h^\epsilon(t, x(t), z(t)) = (\mathcal{T}^\epsilon[x; z]^n_\tau(t), \mathcal{X}^\epsilon[x; z]^n_\tau(t), \mathcal{Z}^\epsilon[x; z]^n_\tau(t)),$$
$$h^0(t, x, z) = (t, x, z), \quad \forall (t, x, z) \in [a - \tau, b] \times \mathbb{R}^m \times \mathbb{R}.$$

Problem $(\mathbf{H}^{\mathbf{n}*}_\tau)$ *is said to be invariant under the transformations* h^ϵ*, if for all admissible pairs* $(x(\cdot), z(\cdot))$ *the following two conditions hold:*

$$\left(\frac{z(b)}{b-a} + \xi\epsilon + o(\epsilon)\right) \frac{d\mathcal{T}^\epsilon}{dt}[x; z]^n_\tau(t) = \frac{z(b)}{b-a} \tag{8.17}$$

for some constant ξ *and*

$$\begin{aligned} \frac{d\mathcal{Z}^\epsilon}{dt}[x; z]^n_\tau(t) = \frac{d\mathcal{T}^\epsilon}{dt}[x; z]^n_\tau(t)\, L\Bigg(& \mathcal{T}^\epsilon[x; z]^n_\tau(t), \mathcal{X}^\epsilon[x; z]^n_\tau(t), \\ & \frac{d\mathcal{X}^\epsilon}{d\mathcal{T}^\epsilon}[x; z]^n_\tau(t), \dots, \frac{d^n\mathcal{X}^\epsilon}{d(\mathcal{T}^\epsilon)^n}[x; z]^n_\tau(t), \mathcal{X}^\epsilon[x, z]^n_\tau(t-\tau), \\ & \frac{d\mathcal{X}^\epsilon}{d\mathcal{T}^\epsilon}[x, z]^n_\tau(t-\tau), \dots, \frac{d^n\mathcal{X}^\epsilon}{d(\mathcal{T}^\epsilon)^n}[x, z]^n_\tau(t-\tau), \mathcal{Z}^\epsilon[x; z]^n_\tau(t)\Bigg), \end{aligned} \tag{8.18}$$

where

$$\frac{d\mathcal{X}^\epsilon}{d\mathcal{T}^\epsilon}[x; z]^n_\tau(t) = \frac{\frac{d\mathcal{X}^\epsilon}{dt}[x; z]^n_\tau(t)}{\frac{d\mathcal{T}^\epsilon}{dt}[x; z]^n_\tau(t)},$$
$$\frac{d^k\mathcal{X}^\epsilon}{d(\mathcal{T}^\epsilon)^k}[x; z]^n_\tau(t) = \frac{\frac{d}{dt}\left(\frac{d^{k-1}\mathcal{X}^\epsilon}{d(\mathcal{T}^\epsilon)^{k-1}}[x; z]^n_\tau(t)\right)}{\frac{d\mathcal{T}^\epsilon}{dt}[x; z]^n_\tau(t)},$$

$k = 2, \dots, n.$

Now we generalize the higher-order Noether's theorem of [62] (see Chapter 7) to the more general case of variational problems of Herglotz type with time delay.

Theorem 8.10 (Higher-order delayed Noether's theorem)**.** *If problem* $(\mathbf{H}^{\mathbf{n}*}_\tau)$ *is invariant in the sense of Definition 8.9, then the quantity*

$$\sum_{k=1}^{n} \phi_k(t) \cdot X_{k-1}[x; z]^n_\tau(t) + \psi_z(t) Z[x; z]^n_\tau(t)$$
$$- \left[\sum_{k=1}^{n} \phi_k(t) \cdot x^{(k)}(t) + \psi_z(t) L[x; z]^n_\tau(t)\right] T[x; z]^n_\tau(t)$$

is constant in t along all extremals of problem $(\mathbf{H}_\tau^{\mathbf{n}*})$, where the generators of the one- -parameter family of maps are given by

$$T = \frac{\partial \mathcal{T}^\epsilon}{\partial \epsilon}\bigg|_{\epsilon=0}, \quad X_0 = \frac{\partial \mathcal{X}^\epsilon}{\partial \epsilon}\bigg|_{\epsilon=0}, \quad Z = \frac{\partial \mathcal{Z}^\epsilon}{\partial \epsilon}\bigg|_{\epsilon=0},$$
$$X_k = \frac{d}{dt}X_{k-1} - x^{(k)}\frac{d}{dt}\left(\frac{\partial \mathcal{T}^\epsilon}{\partial \epsilon}\bigg|_{\epsilon=0}\right), \quad k = 1, \ldots, n-1,$$

and ψ_z, ϕ_k are defined by (8.12)–(8.13).

Proof. We start by considering problem $(\mathbf{H}_\tau^{\mathbf{n}*})$ in its non-delayed optimal control form (8.2). The first step is to prove that if problem $(\mathbf{H}_\tau^{\mathbf{n}*})$ is invariant in the sense of Definition 8.9, then (8.2) is invariant in the sense of Definition 3.4. In order to do that, observe that (8.17) is equivalent to

$$\left(\frac{z_N(\tau)}{N\tau} + \xi\epsilon + o(\epsilon)\right)\frac{d\mathcal{T}^\epsilon}{dt}[x;z]_\tau^n(t) = \frac{z_N(\tau)}{N\tau}$$

and defining $\xi_\tau := \xi N$ we have

$$\left(\frac{z_N(\tau)}{\tau} + \xi_\tau\epsilon + o(\epsilon)\right)\frac{d\mathcal{T}^\epsilon}{dt}[x;z]_\tau^n(t) = \frac{z_N(\tau)}{\tau}, \quad \text{for some } \xi_\tau. \tag{8.19}$$

Observe also that the control system of (8.2) defines $\mathcal{X}_k^\epsilon := \frac{d\mathcal{X}_{k-1}^\epsilon}{d\mathcal{T}^\epsilon}$, that is,

$$\frac{d\mathcal{X}_{k-1}^\epsilon}{dt}[x;z]_\tau^n(t) = \mathcal{X}_k^\epsilon[x;z]_\tau^n(t)\frac{d\mathcal{T}^\epsilon}{dt}[x;z]_\tau^n(t), \quad k = 1, \ldots, n.$$

Let

$$\mathcal{X}_{k;i}[x;z]_\tau^n(t) := \mathcal{X}_k^\epsilon[x;z]_\tau^n(t + (i-1)\tau),$$
$$\mathcal{T}_i[x;z]_\tau^n(t) := \mathcal{T}^\epsilon[x;z]_\tau^n(t + (i-1)\tau),$$
$$\mathcal{Z}_j[x;z]_\tau^n(t) := \mathcal{Z}^\epsilon[x;z]_\tau^n(t + (j-1)\tau).$$

One has

$$\frac{d\mathcal{X}_{k;i}}{dt}[x;z]_\tau^n(t) = \mathcal{X}_{k+1;i}[x;z]_\tau^n(t)\frac{d\mathcal{T}_i}{dt}[x;z]_\tau^n(t) \tag{8.20}$$

and

$$\frac{d\mathcal{Z}_j}{dt}[x;z]_\tau^n(t) = L_j\left(\mathcal{T}_j^\epsilon[x;z]_\tau^n(t), \mathcal{X}^\epsilon[x;z]_\tau^n(t); \mathcal{Z}^\epsilon[x;z]_\tau^n(t)\right)\frac{d\mathcal{T}_j}{dt}[x;z]_\tau^n(t), \tag{8.21}$$

$k = 0, \ldots, n-1$, $i = 0, \ldots N$, $j = 1, \ldots, N$. Equalities (8.19)–(8.21) prove that problem (8.2) is invariant in the sense of Definition 3.4. This allow us to advance to the second step: to apply Theorem 3.5 to the non-delayed optimal control problem (8.2). This theorem guarantees that the quantity

$$(\tau-t)\xi_\tau+\sum_{k=1}^{n}\sum_{i=0}^{N}\phi_{k;i}(t)\cdot X_{k-1;i}[x;z]^n_\tau(t)+\sum_{j=1}^{N}\psi_j(t)Z_j[x;z]^n_\tau(t)$$
$$-\left[\sum_{k=1}^{n}\sum_{i=0}^{N}\phi_{k;i}(t)\cdot x^{k;i}(t)+\sum_{j=1}^{N}\psi_j(t)L_j[x;z]^n_\tau(t)+\frac{z_N(\tau)}{\tau}\right]T[x;z]^n_\tau(t)$$

is constant in t along the extremals of (8.2), where $X_{k;i}=\frac{\partial}{\partial\epsilon}\frac{d^k\mathcal{X}^\epsilon_{k;i}}{d(\mathcal{T}^\epsilon)^k}\Big|_{\epsilon=0}$ and $Z_i=\frac{\partial}{\partial\epsilon}\frac{d\mathcal{Z}^\epsilon_i}{d(\mathcal{T}^\epsilon)}\Big|_{\epsilon=0}$. Rewriting in the original variables, we obtain

$$(\tau-t)\xi_\tau+\sum_{k=1}^{n}\phi_k(t)\cdot X_{k-1}[x;z]^n_\tau(t)+\psi_z(t)Z[x;z]^n_\tau(t)$$
$$-\left[\sum_{k=1}^{n}\phi_k(t)\cdot x^{(k)}(t)+\psi_z(t)L[x;z]^n_\tau(t)+\frac{z_N(\tau)}{\tau}\right]T[x;z]^n_\tau(t)$$

constant in t along the extremals of (8.2). The third step is to prove that

$$(\tau-t)\xi_\tau-\frac{z_N(\tau)}{\tau}T[x;z]^n_\tau(t) \tag{8.22}$$

is constant in t. That will be done in a very similar way to the proof of Theorem 6.6. From the invariance condition (8.19), we know that

$$\left(\frac{z_N(\tau)}{\tau}+\xi_\tau\epsilon+o(\epsilon)\right)\frac{d\mathcal{T}^\epsilon}{dt}[x;z]^n_\tau(t)=\frac{z_N(\tau)}{\tau}.$$

Integrating from 0 to t we conclude that

$$\left(\frac{z_N(\tau)}{\tau}+\xi_\tau\epsilon+o(\epsilon)\right)\mathcal{T}^\epsilon[x;z]^n_\tau(t)=\frac{z_N(\tau)}{\tau}t+\left(\frac{z_N(\tau)}{\tau}+\xi_\tau\epsilon+o(\epsilon)\right)\mathcal{T}^\epsilon[x;z]^n_\tau(0).$$

Differentiating this equality with respect to ϵ, and then putting $\epsilon=0$, we get

$$\xi_\tau t+\frac{z_N(\tau)}{\tau}T[x;z]^n_\tau(t)=\frac{z_N(\tau)}{\tau}T[x;z]^n_\tau(0). \tag{8.23}$$

We conclude from (8.23) that expression (8.22) is the constant

$$\tau\xi_\tau-\frac{z_N(\tau)}{\tau}T[x;z]^n_\tau(0).$$

Hence,

$$\sum_{k=1}^{n} \phi_k(t) \cdot X_{k-1}[x;z]_\tau^n(t) + \psi_z(t) Z[x;z]_\tau^n(t) \\ - \left[\sum_{k=1}^{n} \phi_k(t) \cdot x^{(k)}(t) + \psi_z(t) L[x;z]_\tau^n(t) \right] T[x;z]_\tau^n(t)$$

is constant in t along the extremals of problem (8.2). Finally, observe that $X_0 = \frac{\partial \mathcal{X}^\epsilon}{\partial \epsilon}\big|_{\epsilon=0}$ and

$$\begin{aligned} X_k = \frac{\partial}{\partial \epsilon} \frac{d^k \mathcal{X}^\epsilon}{d(\mathcal{T}^\epsilon)^k}\bigg|_{\epsilon=0} &= \frac{\partial}{\partial \epsilon} \left(\frac{\frac{d}{dt}\left(\frac{d^{k-1}\mathcal{X}^\epsilon}{d(\mathcal{T}^\epsilon)^{k-1}}\right)}{\frac{d\mathcal{T}^\epsilon}{dt}} \right)\Bigg|_{\epsilon=0} \\ &= \frac{d}{dt}\left(\frac{\partial}{\partial \epsilon} \frac{d^{k-1}\mathcal{X}^\epsilon}{d(\mathcal{T}^\epsilon)^{k-1}}\bigg|_{\epsilon=0} \right) - x^{(k)} \frac{d}{dt}\left(\frac{\partial \mathcal{T}^\epsilon}{\partial \epsilon}\bigg|_{\epsilon=0} \right) \\ &= \frac{d}{dt} X_{k-1} - x^{(k)} \frac{d}{dt}\left(\frac{\partial \mathcal{T}^\epsilon}{\partial \epsilon}\bigg|_{\epsilon=0} \right), \end{aligned}$$

$k = 1, \ldots, n-1$. This concludes the proof. □

Corollary 8.11 (cf. [62]). *If the higher-order problem of Herglotz* (8.14) *is invariant in the sense of Definition 8.9, then the quantity*

$$\sum_{k=1}^{n} \tilde{\phi}_k(t) \cdot X_{k-1}[x;z]_0^n(t) + \psi_z(t) Z[x;z]_0^n(t) \\ - \left[\sum_{k=1}^{n} \tilde{\phi}_k(t) \cdot x^{(k)}(t) + \psi_z(t) L[x;z]_0^n(t) \right] T[x;z]_0^n(t)$$

is constant in t along any extremal of the problem, where

$$\tilde{\phi}_k(t) = \sum_{l=0}^{n-k} (-1)^{l+1} \frac{d^l}{dt^l} \left(\psi_z(t) \frac{\partial L}{\partial x^{(l+k)}}[x;z]_0^n(t) \right),$$

$k = 1, \ldots, n$, *and* ψ_z *is given by* (8.12).

Proof. Consider Theorem 8.10 with $\tau = 0$. □

Theorem 8.10 is a generalization of Noether's theorem [60] for the first-order problem of Herglotz with time delay. Besides the improvement of dealing with piecewise differentiable functions instead of differentiable, the theorem presents a similar conserved quantity but without the imposition of two additional hypotheses (5.4)-(5.15) required in [60] (see Chapter 5). Moreover, the current definition of invariance is more general than the one considered in [60].

Corollary 8.12 (cf. [60]). *If the first-order problem of Herglotz with time delay* (8.15) *is invariant in the sense of Definition 8.9, then the quantity*

$$\left(\psi_z(t)\frac{\partial L}{\partial \dot{x}}[x;z]^1_\tau(t) + \psi_z(t+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]^1_\tau(t+\tau)\right) X_0[x;z]^1_\tau(t)$$
$$+ \psi_z(t)Z[x;z]^1_\tau(t) + \left[-\left(\psi_z(t)\frac{\partial L}{\partial \dot{x}}[x;z]^1_\tau(t)\right.\right.$$
$$\left.\left.+\psi_z(t+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]^1_\tau(t+\tau)\right)\dot{x}(t) + \psi_z(t)L[x;z]^1_\tau(t)\right] T[x;z]^1_\tau(t)$$

is constant in $t \in [a,b]$ along any extremal of the problem.

Proof. Consider Theorem 8.10 with $n = 1$. □

Remark 8.13. *If $t \in [b-\tau, b]$, then $L[x;z]^n_\tau(t+\tau)$ is, by definition, null (see* (8.1)*) and the constant of Corollary 8.12 reduces to*

$$\left(\psi_z(t)\frac{\partial L}{\partial \dot{x}}[x;z]^1_\tau(t)\right) X_0[x;z]^1_\tau(t) + \psi_z(t)Z[x;z]^1_\tau(t)$$
$$+ \left[-\left(\psi_z(t)\frac{\partial L}{\partial \dot{x}}[x;z]^1_\tau(t)\right)\dot{x}(t) + \psi_z(t)L[x;z]^1_\tau(t)\right] T[x;z]^1_\tau(t)$$

for $t \in [b-\tau, b]$, which is the second constant quantity of [60].

8.4 Conclusions

Optimal Control is a convenient tool to deal with delayed and non-delayed Herglotz type variational problems. In this chapter we have shown how some of the central results from the classical Calculus of Variations can be proved for higher-order Herglotz variational problems with time delay from analogous and well-known Optimal Control results. The techniques here developed can now be used to obtain other results. For example, our Optimal Control approach can be employed together with [69] to derive an extension of the second Noether theorem of Optimal Control to the delayed or non-delayed Herglotz's framework (see Chapter 9).

The original results of this chapter were published in 2016 in [63]. They were also presented by the author in 2016 in a meeting of the Center for Research Development in Mathematics and Applications (CIDMA), January 21-22, 2016 Aveiro, Portugal, in a talk entitled "Higher-order variational problems of Herglotz type with time delay".

CHAPTER 9

NOETHER CURRENTS FOR HIGHER-ORDER VARIATIONAL PROBLEMS OF HERGLOTZ WITH TIME DELAY

This final chapter is concerned with higher-order delayed variational problems of Herglotz type, which are invariant under a certain symmetry group of transformations. Such problems were first studied in 1918 by Emmy Noether for the particular case of first-order variational problems without time delay [54]. In her famous paper [54], Noether proved two remarkable theorems that relate the invariance of a variational integral with properties of its Euler–Lagrange equations. Since most physical systems can be described by using Lagrangians and their associated actions, the importance of Noether's two theorems is obvious [5].

As already seen in previous chapters, the first Noether's theorem, usually simply called Noether's theorem, ensures the existence of r conserved quantities along the Euler–Lagrange extremals when the variational integral is invariant with respect to a continuous symmetry transformation that depend on r parameters [71]. Noether's theorem explains all conservation laws of mechanics, for instance, invariance under translation in time implies conservation of energy; conservation of linear momentum comes from invariance of the system under spacial translations; invariance under rotations in the base space yields conservation of angular momentum.

The second Noether's theorem, less known than the first one, applies to variational problems that are invariant under a certain group of transformations that depends on arbitrary functions and their derivatives up to some order [69]. In contrast to Noether's theorem, where the transformations are global, in Noether's second theorem the transformations are local: they can affect every part of the system differently. Noether's second theorem has applications in several fields, such as, general relativity, hydromechanics, electrodynamics, and quantum chromodynamics [27, 44, 65]. Extensions of both Noether's theorems to optimal control problems were first obtained in [67, 68, 69, 72]. For systems with time delay see [24, 47, 48].

Motivated by the important applications of Noether's second theorem [48] and the applicability of higher-order dynamic systems with time delay in modelling real-life phenomena [8, 26, 66], as well as the importance of variational problems of Herglotz [37, 39], our goal in this chapter is to study generalized variational problems that are invariant under a certain group of transformations that depends on arbitrary functions and their derivatives up to some order, and deduce expressions for Noether currents, that is, expressions that are constant in time along the extremals.

Our work is related with the second Noether theorem for Optimal Control in the sense of [69], and is particularly useful because provides necessary conditions for the search of extremals. There are other different results on the Calculus of Variations, also related with the notion of invariance under a certain group of transformations that depends on arbitrary functions and their derivatives [32, 49], but they are concerned with Noether identities and not with Noether currents as we do here.

The chapter is organized in two sections. In Section 9.1, we prove our main results: the second Noether theorem for higher–order problems of Herglotz with time delay (Theorem 9.3) and two important corollaries (Corollary 9.4 and 9.5). We finish the chapter with an illustrative example (Section 9.2).

In this chapter we consider the generalized variational problem $(\mathbf{H}_{\tau}^{\mathbf{n}*})$ of Chapter 8.

9.1 Noether's second theorem for higher-order variational problems of Herglotz with time delay

The central idea of the proof of this chapter's main result, Noether's second theorem for the higher-order variational problem of Herglotz type with time delay, is to rewrite problem $(\mathbf{H}_{\tau}^{\mathbf{n}*})$ as a non-delayed optimal control problem. This procedure is done inspired by the ideas presented in [38] and [63] in a way as done in Chapter 8 (see Section 8.1).

Before presenting a second Noether's theorem to problem $(\mathbf{H}_\tau^{\mathbf{n}*})$, we define semi-invariance of problem $(\mathbf{H}_\tau^{\mathbf{n}*})$ under a group of symmetries.

Definition 9.1 (Semi-invariance of problem $(\mathbf{H}_\tau^{\mathbf{n}*})$ under a group of symmetries). *Let $p : [a, b] \to \mathbb{R}^d$ be a C^q arbitrary function of the independent variable. We say that problem $(\mathbf{H}_\tau^{\mathbf{n}*})$ is semi-invariant under a symmetry group g if there exists a C^1 transformation group*

$$g : [a, b] \times \mathbb{R}^{2m(n+1)+1} \times \mathbb{R}^{d\times(q+1)} \to \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}, \tag{9.1}$$

$$g(\alpha(t)) = (\mathsf{T}(\alpha(t)), \mathsf{X}(\alpha(t)), \mathsf{Z}(\alpha(t))),$$

with $\alpha(t)$ standing for

$$\left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), x(t-\tau), \dot{x}(t-\tau), \ldots, x^{(n)}(t-\tau), z(t), p(t), \dot{p}(t), \ldots, p^{(q)}(t)\right),$$

which for $p(t) = \dot{p}(t) = \cdots = p^{(q)}(t) = 0$ coincides with the identity transformation for all $(t, x, z) \in [a - \tau, b] \times \mathbb{R}^m \times \mathbb{R}$, and satisfies the two equations

$$\frac{z(b)}{b-a} + \frac{d}{dt}F(\alpha(t)) = \frac{\mathsf{Z}(\alpha(b))}{\mathsf{T}(\alpha(b)) - \mathsf{T}(\alpha(a))}\frac{d}{dt}\mathsf{T}(\alpha(t)) \tag{9.2}$$

and

$$\frac{d}{dt}\mathsf{Z}(\alpha(t)) = L(g(\alpha(t)))\frac{d}{dt}\mathsf{T}(\alpha(t)), \tag{9.3}$$

for some function F of class C^1, where

$$\frac{d}{d\mathsf{T}}\mathsf{X}(\alpha(t)) = \frac{\frac{d}{dt}\mathsf{X}(\alpha(t))}{\frac{d}{dt}\mathsf{T}(\alpha(t))} \text{ and } \frac{d^k}{d\mathsf{T}^k}\mathsf{X}(\alpha(t)) = \frac{\frac{d}{dt}\left(\frac{d^{k-1}}{d\mathsf{T}^{k-1}}\mathsf{X}(\alpha(t))\right)}{\frac{d}{dt}\mathsf{T}(\alpha(t))},$$

$k = 2, \ldots, n$.

Remark 9.2. *The group of transformations g (9.1) is usually called a gauge symmetry of the optimal control problem, in order to emphasize the fact that the transformations depend on arbitrary functions and, therefore, have local nature.*

We are now in a position to formulate and prove the main result of this chapter.

Theorem 9.3 (Noether's second theorem for problem $(\mathbf{H}_\tau^{\mathbf{n}*})$). *If problem $(\mathbf{H}_\tau^{\mathbf{n}*})$ is semi--invariant under a group of symmetries as in Definition 9.1, then there are $d(q+1)$ Noether* *currents of the form*

$$
\begin{aligned}
&\left.\frac{\partial F(\alpha(t))}{\partial p_J^{(I)}}\right|_0 + \theta_J^I \frac{z(b)}{b-a} \\
&\qquad + \sum_{k=1}^{n} \phi_k(t) \cdot \left.\frac{\partial}{\partial p_J^{(I)}} \left(\frac{d^{k-1}}{d\mathsf{T}^{k-1}} \mathsf{X}(\alpha(t)) \right)\right|_0 + \psi_z(t) \cdot \left.\frac{\partial \mathsf{Z}(\alpha(t))}{\partial p_J^{(I)}}\right|_0 \\
&\qquad\qquad - H(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), z(t), \phi_1(t), \ldots, \phi_n(t), \psi_z(t)) \left.\frac{\partial \mathsf{T}(\alpha(t))}{\partial p_J^{(I)}}\right|_0,
\end{aligned}
$$

$t \in [a,b]$, for $I = 0, \ldots, q$, $J = 1, \ldots, d$, and $\theta_J^I \in \mathbb{R}^d$, where H, ψ_z and ϕ_k and are defined, respectively, by (8.6)–(8.12)–(8.13) *and $(*)|_0$ stands for $(*)|_{p(t)=\dot{p}(t)=\cdots=p^{(q)}(t)=0}$.*

Proof. In order to prove the result, we start by considering problem $(\mathbf{H}_\tau^{\mathbf{n}*})$ in its optimal control and non-delayed form (8.2). First, we prove that if $(\mathbf{H}_\tau^{\mathbf{n}*})$ is semi-invariant under a group of symmetries as in Definition 9.1, then the non-delayed optimal control problem (8.2) is invariant in the sense of Definition 3.7. Observe that if (9.2) holds, then there is $\tilde{F}$ of class C^1 such that

$$
\frac{z_N(\tau)}{\tau} + \frac{d}{dt}\tilde{F}(\alpha(t)) = \frac{\mathsf{Z}_N(\alpha(\tau))}{\mathsf{T}(\alpha(\tau))} \frac{d}{dt}\mathsf{T}(\alpha(t)). \tag{9.4}
$$

Now, defining

$$
\begin{aligned}
\mathsf{X}_{k;i}(\alpha(t)) &:= \frac{d^k}{d\mathsf{T}^k}\mathsf{X}(\alpha(t+(i-1)\tau)), \\
\mathsf{T}_i(\alpha(t)) &:= \mathsf{T}(\alpha(t+(i-1)\tau)), \\
\mathsf{Z}_j(\alpha(t)) &:= \mathsf{Z}(\alpha(t+(j-1)\tau))
\end{aligned}
$$

for fixed $t \in [0,\tau]$, we have

$$
\frac{d}{dt}\mathsf{X}_{k;i}(\alpha(t)) = \mathsf{X}_{k+1;i}(\alpha(t)) \frac{d}{dt}\mathsf{T}_i(\alpha(t)) \tag{9.5}
$$

and

$$
\frac{d}{dt}\mathsf{Z}_j(\alpha(t)) = L_j\left(g(\alpha(t))\right) \frac{d}{dt}\mathsf{T}_j(\alpha(t)), \tag{9.6}
$$

for $k = 0, \ldots, n-1$, $i = 0, \ldots N$, and $j = 1, \ldots, N$. From (9.4)–(9.6), we conclude that the non-delayed optimal control problem (8.2) is semi-invariant in the sense of Definition 3.7. This kind of semi-invariance is the required condition for application of the second Noether theorem for Optimal Control (Theorem 3.8), which asserts the existence of $d(q+1)$ Noether currents of the form

$$\left.\frac{\partial F(\alpha(t))}{\partial p_J^{(I)}}\right|_0 + \theta_J^I \frac{z_N(\tau)}{\tau} + \sum_{k=1}^{n}\sum_{i=0}^{N} \phi_{k;i}(t)\cdot \left.\frac{\partial \mathsf{X}_{k-1;i}(\alpha(t))}{\partial p_J^{(I)}}\right|_0 + \sum_{j=1}^{N}\psi_j(t)\cdot \left.\frac{\partial \mathsf{Z}_j(\alpha(t))}{\partial p_J^{(I)}}\right|_0 - \left[\sum_{k=1}^{n}\sum_{i=0}^{N}\phi_{k;i}(t)\cdot x^{k;i}(t) + \sum_{j=1}^{N}\psi_j(t)L_j(t)\right] \left.\frac{\partial \mathsf{T}(\alpha(t))}{\partial p_J^{(I)}}\right|_0,$$

$t\in[0,\tau]$, for $I=0,\dots,q$, $J=1,\dots,d$, where ψ_j and $\phi_{k;i}$ are defined in (8.7)–(8.9):

$$\phi_{k;i}(t) = \phi_k(t+(i-1)\tau) \text{ and } \psi_j(t) = \psi_z(t+(i-1)\tau),$$

for $i=0,\dots,N$ and $j=1,\dots,N$. Finally, we rewrite the result in the original variables, obtaining that there are $d(q+1)$ Noether currents of the form

$$\left.\frac{\partial F(\alpha(t))}{\partial p_J^{(I)}}\right|_0 + \theta_J^I \frac{z(b)}{b-a} + \sum_{k=1}^{n}\phi_k(t)\cdot \left.\frac{\partial \mathsf{X}_k(\alpha(t))}{\partial p_J^{(I)}}\right|_0 + \psi_z(t)\cdot \left.\frac{\partial \mathsf{Z}(\alpha(t))}{\partial p_J^{(I)}}\right|_0 - H(t,x(t),\dot{x}(t),\dots,x^{(n)}(t),z(t),\phi_1(t),\dots,\phi_n(t),\psi_z(t)) \left.\frac{\partial \mathsf{T}(\alpha(t))}{\partial p_J^{(I)}}\right|_0.$$

This conclude the proof. □

Our result is new even for first-order generalized variational problems.

Corollary 9.4. *If the first-order problem of Herglotz with time delay*

$$z(b) \longrightarrow \text{extr},$$
$$\dot{z}(t) = L\left(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau), z(t)\right), \quad t\in[a,b],$$
$$z(a)=\gamma\in\mathbb{R}, \quad x(t)=\mu(t), \quad t\in[a-\tau,a],$$

where μ is a given piecewise differentiable initial function, is semi-invariant in the sense of Definition 9.1, then there exist $d(q+1)$ Noether currents of the form

$$\left.\frac{\partial F(\alpha(t))}{\partial p_J^{(I)}}\right|_0 + \theta_J^I \frac{z(b)}{b-a} + \phi_1(t)\cdot \left.\frac{\partial \mathsf{X}(\alpha(t))}{\partial p_J^{(I)}}\right|_0 + \psi_z(t)\cdot \left.\frac{\partial \mathsf{Z}(\alpha(t))}{\partial p_J^{(I)}}\right|_0 - \left[\phi_1(t)\dot{x}(t) + \psi_z(t)L[x;z]_\tau^1(t)\right] \left.\frac{\partial \mathsf{T}(\alpha(t))}{\partial p_J^{(I)}}\right|_0,$$

$t\in[a,b]$, for $I=0,\dots,q$, $J=1,\dots,d$, where ϕ_1 is given by (8.13) *and ψ_z by* (8.12).

Proof. Consider Theorem 9.3 with $n = 1$. □

As a corollary of Corollary 9.4, we obtain a new result for delayed classical problems of the Calculus of Variations.

Corollary 9.5. *If the first-order variational problem with time delay*

$$\int_a^b L(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau))dt \longrightarrow \text{extr},$$

with $x(t) = \mu(t)$, $t \in [a-\tau, a]$, for a given piecewise differentiable initial function μ, is semi-invariant in the sense of Definition 9.1, then there exists $d(q+1)$ Noether currents of the form

$$\left.\frac{\partial F(\alpha(t))}{\partial p_J^{(I)}}\right|_0 + \phi_1(t) \cdot \left.\frac{\partial \mathsf{X}(\alpha(t))}{\partial p_J^{(I)}}\right|_0 + \theta_J^I \frac{z(b)}{b-a}$$
$$- \left[\phi_1(t)\dot{x}(t) + L\left(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau)\right)\right] \left.\frac{\partial \mathsf{T}(\alpha(t))}{\partial p_J^{(I)}}\right|_0,$$

$t \in [a, b]$, for $I = 0, \ldots, q$, $J = 1, \ldots, d,$, where ϕ_1 is given by (8.13).

Proof. Consider Corollary 9.4 with L not depending on z. □

9.2 Illustrative example

In order to illustrate our results, we present a simple example that cannot be covered using available results in the literature. Consider an arbitrary interval $[a, b]$ and let $\tau \in \mathbb{R}$ be a nonnegative real number such that $\tau < b - a$. We address the following problem with $m = d = q = 1$:

$$\begin{gathered} z(b) \to \text{extr}, \\ \dot{z}(t) = x(t-\tau)z(t), \quad t \in [a, b], \\ \text{subject to } z(a) = \gamma, \quad x(t) = \mu(t), \quad t \in [a-\tau, a], \end{gathered} \tag{9.7}$$

where $\mu(\cdot) \in PC^1([a-\tau, a]; \mathbb{R})$ is a given initial function. Let p be a $C^1([a, b]; \mathbb{R})$ function and consider the C^1 group of symmetries

$$g(\alpha(t)) = \left(t + p(t), \frac{x(t-\tau)}{1+\dot{p}(t)}, z(t)\right),$$

that is,

$$\begin{aligned}\mathsf{T}(\alpha(t)) &= T\left(t, p(t)\right) = t + p(t),\\ \mathsf{X}(\alpha(t)) &= X\left(x(t-\tau), \dot{p}(t)\right) = \frac{x(t-\tau)}{1+\dot{p}(t)},\\ \mathsf{Z}(\alpha(t)) &= Z\left(z(t)\right) = z(t),\end{aligned}$$

which for $p(t) = \dot{p}(t) = 0$, $t \in [a,b]$, reduce to the identity transformations. Observe that the problem under study is semi-invariant. Indeed, (9.2) is verified with

$$F(t) = \frac{z(b)}{b-a+p(b)-p(a)}\left(t+p(t)\right) - \frac{z(b)}{b-a}t$$

and (9.3) is also valid because

$$\frac{d}{dt}\mathsf{Z}(\alpha(t)) = \dot{z}(t) = \frac{x(t-\tau)}{1+\dot{p}(t)}z(t)(1+\dot{p}(t)) = L(g(\alpha(t)))\frac{d}{dt}\mathsf{T}(\alpha(t)).$$

From Theorem 9.3, we have that there are two Noether currents of the form

$$\begin{aligned}&\left.\frac{\partial F(\alpha(t))}{\partial p^{(I)}}\right|_0 + \theta^I \frac{z(b)}{b-a} + \phi_1(t)\cdot\left.\frac{\partial \mathsf{X}(\alpha(t))}{\partial p^{(I)}}\right|_0 + \psi_z(t)\cdot\left.\frac{\partial \mathsf{Z}(\alpha(t))}{\partial p^{(I)}}\right|_0\\ &\qquad - \left[\phi_1(t)\dot{x}(t) + \psi_z(t)L[x;z]_\tau^1(t)\right]\left.\frac{\partial \mathsf{T}(\alpha(t))}{\partial p^{(I)}}\right|_0, \quad I = 0, 1.\end{aligned}$$

Noting that $\phi_1(t) = 0$ and $\psi_z(t) = e^{\int_t^b x(s-\tau)ds}$, $t \in [a,b]$, the second Noether current reduces to a constant while the first gives a nontrivial conclusion: it asserts that

$$x(t-\tau)z(t)e^{\int_t^b x(s-\tau)ds}$$

is constant along the extremals of problem (9.7).

9.3 Conclusions

We have deduced new necessary conditions for higher-order generalized variational problems with time delay that are semi-invariant under a group of transformations that depends on arbitrary functions. The conditions are potentially useful, because for many variational problems, the Euler–Lagrange equations and transversality conditions are not enough to obtain an explicit solution. The main result of this chapter is new even for classical delayed variational problems.

The original results of this chapter were in 2017 accepted for publication [64].

CONCLUSIONS AND FUTURE WORK

In this thesis, we generalized the variational problem of Herglotz in several directions. We used classical variational techniques to prove higher-order results such as generalized Euler–Lagrange equations and natural boundary conditions, but also to prove first-order results on the delayed problem of Herglotz: Euler–Lagrange equations, DuBois–Reymond condition and Noether's first theorem.

We made a major change of approach when we started looking at Herglotz's based problems on their optimal control form. We were then able to generalize Herglotz's [39] and Georgieva's [29] first-order results to the wider class of piecewise differentiable functions. We continued on this path and proved several other important and new results valid for piecewise differentiable functions: Euler–Lagrange equations, Dubois–Reymond optimality condition and Noether's first theorem for the higher-order generalized problem of Herglotz.

We managed to improve and generalize our first-order delayed results to the higher-order case. After rewriting the main problem in the optimal control form, we also described it as a non-delayed problem, proving then Euler–Lagrange equations, transversality conditions, Dubois–Reymond condition and Noether's first and second theorems for the higher-order problem of Herglotz with time delay.

This thesis introduced new results and a new approach to generalized variational problems of Herglotz type. However, there are still many open questions related with this kind of problems. Some possible directions for future work are:

- to consider the isoperimetric problem, that is, when the admissible trajectories satisfy the boundaries conditions $x(a) = \alpha$, $x(b) = \beta$, $z(a) = \gamma$ and are such that the functional

$\int_a^b G(t, x(t), \dot{x}(t), z(t))dt$, for some fixed Lagrangian G, takes a fixed real value l;

- to consider the free terminal point problem, that is, we intend to find the value of $T \in [a, b]$, such that the value of $z(T)$ is maximum (or minimum), where $x(a) = \alpha$, $z(a) = \gamma$ and no constraint is imposed on $x(T)$;

- to prove sufficient conditions for variational problems of Herglotz type.

We would also like to generalize the variational problem of Herglotz to the context of time scales calculus. The theory of time scales had its beginning in 1988 with the Ph.D. thesis of Hilger [41], providing a powerful theory that unify discrete and continuous mathematics in one theory [10, 11]. With a short time this unification aspect has been supplemented by the extension and generalization features. The time scale calculus allows to consider more complex time domains, such as q-scales, periodics numbers or hybrid domains, that are important for applications. For this reason, we believe that it is relevant to consider variational problems of Herglotz in such a general context.

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