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STRONGLY NONLINEAR SECOND ORDER MULTIVALUED DIRICHLET SYSTEMS

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Abstract. We consider second order nonlinear Dirichlet systems driven by a nonlinear nonhomogeneous differential operator. The reaction term consists of a maximal monotone map $A(\cdot)$ plus a multivalued perturbation F depending also on the derivative. Using tools from multivalued analysis and from the theory of nonlinear operators of monotone type, we prove existence theorems both for the "convex" (F convex-valued) and the "nonconvex" (F nonconvex-valued) problems. We also present an example of a system with unilateral constraints.

Keywords. Maximal monotone operator; p-Laplace-like operator; Yosida approximation; Vector Sobolev space.

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1. Introduction

In this paper, we study the following second order nonlinear differential inclusion

$$\begin{cases} (a(u'(t)))' \in A(u(t)) + F(t, u(t), u'(t)) \text{ for a.a. } t \in T := [0, b], \\ u(0) = u(b) = 0. \end{cases} \tag{1.1}$$

In this problem, $a : \mathbb{R}^N \to \mathbb{R}^N$ is a monotone homeomorphism, which includes as a special case the vector p-Laplacian, $A : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map and $F : T \times \mathbb{R}^N \to 2^{\mathbb{R}^N} \setminus \{\varnothing\}$ is a set-valued nonlinearity.

Analogous second order systems were studied by Aizicovici-Papageorgiou-Staicu [1], Erbe-Krawcewicz [5], Frigon [6], Frigon-Montoki [7], Halidias-Papageorgiou [10], Kyritsi-Matzakos-Papageorgiou [12], Manasevich-Mawhin [13], Pruszko [16]. In all the above works either $A \equiv 0$ or the multifunction F is independent of u' or the conditions on F are more restrictive. The presence of the map $A(\cdot)$ in our problem and the fact that in general $D(A) \neq \mathbb{R}^N$, enables us to incorporate in our framework differential variational inequalities.

Our approach uses tools and results from multivalued analysis and from the theory of nonlinear operators of monotone type. In the next section, for the convenience of the reader, we recall the basic definitions and facts from these theories which we will use in the sequel.

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2. Mathematical Background-Hypotheses.

We start with multivalued analysis. Further details can be found in Hu-Papageorgiou [11]. So, let (Ω,Σ) be a measurable space and $(X,\|.\|)$ be a separable Banach space. We will use the following notation:

$$\mathscr{P}_{f(c)}(X)=\{A\subseteq X : A \text{ is nonempty, closed, (and convex)}\},$$

$$\mathscr{P}_{(w)k(c)}(X)=\{A\subseteq X : A \text{ is nonempty, (weakly-) compact, (and convex)}\}.$$

We will use the symbol $\xrightarrow{w}$ to designate weak convergence.

For a multifunction (set-valued map) $F:\Omega\to 2^X\setminus\{\varnothing\}$, the *graph* of F is the set

$$Gr\ F=\{(\omega,x)\in\Omega\times X : x\in F(\omega)\}.$$

We say that $F(.)$ is *graph measurable* if $Gr\ F\in\Sigma\times\mathscr{B}(X)$ where $\mathscr{B}(X)$ is the Borel $\sigma-$field of X.

If $F(.)$ is graph measurable and if μ is a $\sigma-$finite measure on Σ, then from the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [11], pp.158-159), we know that we can find a sequence of $\Sigma-$measurable selections $f_n:\Omega\to X$ ($n\in\mathbb{N}$) of $F(\cdot)$ such that

$$F(\omega)\subseteq\overline{\{f_n(\omega)\}}_{n\geq 1} \text{ for } \mu-\text{a.a. } \omega\in\Omega.$$

For a $P_f(X)-$valued multifunction $F(\cdot)$, we say that $F(\cdot)$ is *measurable*, if for every $u\in X$, the function

$$\omega\to d(u,F(\omega)):=\inf\{\|u-x\| : x\in F(\omega)\}$$

is $\Sigma-$measurable. For $\mathscr{P}_f(X)-$valued multifunctions, measurability implies graph measurability and the converse is true if there is a complete $\sigma-$finite measure $\mu(\cdot)$ on Σ.

Now suppose that (Ω,Σ,μ) is a $\sigma-$finite measure space, $1\leq p\leq\infty$ and $F:\Omega\to 2^X\setminus\{\varnothing\}$ is a multifunction. We define

$$S_F^p=\{f\in L^p(\Omega,X) : f(\omega)\in F(\omega)\ \mu-\text{a.e.}\}.$$

If $F(\cdot)$ is graph measurable, then using the Yankov-von Neumann-Aumann selection theorem, we see that $S_F^p\neq\varnothing$, if and only if $\omega\to\inf\{\|x\| : x\in F(\omega)\}$ belongs to $L^p(\Omega)$.

The set S_F^p is *decomposable,* in the sense that if $(A,f_1,f_2)\in\Sigma\times S_F^p\times S_F^p$, then

$$f_1\chi_A+f_2\chi_{\Omega\setminus A}\in S_F^p.$$

Here, for any $B\in\Sigma$, χ_B denotes the characteristic function of B.

Let Y, V be Hausdorff topological spaces and $G:Y\to 2^V\setminus\{\varnothing\}$ a multifunction. We say that $G(\cdot)$ is "upper semicontinuous" ("usc" for short) if for every open set $U\subset V$, $G^+(U):=\{y\in Y : G(y)\subset U\}$ is open. $G(\cdot)$ is said to be "lower semicontinuous" ("lsc" for short) if for every open set $U\subset V$, $G^-(U):=\{y\in Y : G(y)\cap U\neq\varnothing\}$ is open.

If V is regular space, then an usc multifunction has closed graph. The converse is true if G has closed values and is locally compact (that is, for every $y\in Y$ we can find U, a neighborhood of y such that $\overline{G(U)}$ is compact in V). Also, if V is a metric space, then a $\mathscr{P}_f(V)-$valued multifunction $G:Y\to 2^V\setminus\{\varnothing\}$ is lsc if and only if for all $v\in V$, the function $y\to d(v,G(y)):=\inf\{\|v-x\| : x\in G(y)\}$ is upper semicontinuous.

Now we present some basic definitions and facts for nonlinear operators of monotone type. For further details we refer to Gasinski-Papageorgiou [8]. So, let X be a reflexive Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*,X). A multivalued map $A:X\to 2^{X^*}$ is said to be *monotone* if for all (x,x^*), $(u,u^*)\in GrA$, we have

$$\langle u^*-x^*,u-x\rangle\geq 0.$$

If $\langle u^*-x^*,u-x\rangle=0\Longrightarrow x=u$, then we say that A is *strictly monotone*. The set

$$D(A):=\left\{x\in\mathbb{R}^N:A(x)\neq\varnothing\right\}$$

is the *domain* of A. We say that a monotone map $A(\cdot)$ is *maximal monotone* if

$$[\langle u^*-x^*,u-x\rangle\geq 0 \text{ for all } (x,x^*)\in GrA]\Longrightarrow(u,u^*)\in GrA,$$

that is, $Gr\,A$ is not properly contained in the graph of a monotone map.

Let X_w (resp. X^*_w) denote the space X (resp. X^*) endowed with the weak topology. We can easily see that for a maximal monotone map $A:X\to 2^{X^*}$, $Gr\,A$ is closed in $X_w\times X^*$ and in $X\times X^*_{w.}$.

Let $X=H$ be a Hilbert space and identify H with its dual by the Riesz-Frechet theorem (that is $H=H^*$). Let $A:H\to 2^H$ be a maximal monotone map.

For $\lambda>0$, we define the following single valued maps

$$J_\lambda:=(I+\lambda A)^{-1} \text{ (the \textit{resolvent} of } A\text{)},$$
$$A_\lambda:=\frac{1}{\lambda}(I-J_\lambda) \text{ (the \textit{Yosida approximation} of } A\text{)}.$$

The next proposition summarizes the properties of these two operators.

Proposition 2.1. *If $A:H\to 2^H$ is a maximal monotone map and $\lambda>0$, then:*

- (a) *$J_\lambda:H\to H$ is nonexpansive (that is, Lipschitz continuous with Lipschitz constant 1);*
- (b) *$A_\lambda(x)\in A(J_\lambda(x))$ for all $x\in H$;*
- (c) *$A_\lambda(.)$ is monotone and Lipschtz continuous with Lipschitz constant $\frac{1}{\lambda}$ (therefore $A_\lambda(.)$ is maximal monotone too);*
- (d) *$\|A_\lambda(x)\|\leq\left\|A^0(x)\right\|=\min\{\|x^*\|:x^*\in A(x)\}$ and $A_\lambda(x)\to A^0(x)$ as $\lambda\to 0^+$ for all $x\in D(A)$;*
- (e) *$\overline{D(A)}$ is convex and $J_\lambda(x)\to p_{\overline{D(A)}}(x)$ as $\lambda\to 0^+$ for all $x\in H$.*

Remark 2.1. The maximal monotonicity of $A(\cdot)$ implies that for all $x\in D(A)$, $A(x)\in P_{fc}(H)$ and so $A^0(x)$ is well defined as the element of minimal norm in $A(x)$ (see Proposition 2.1 (d)). Also $p_{\overline{D(A)}}(\cdot)$ denotes the metric projection on the closed convex set $\overline{D(A)}$ (see Proposition 2.1 (e)). According to part (e), $J_\lambda(.)$ can be viewed as an approximation of the identity map. Note that if $D(A)=H$, then $J_\lambda(x)\to x$ for all $x\in H$.

Let Y, V be two Banach spaces and $K:Y\to V$. We say that:

- (a) K is *completely continuous,* if

$$y_n\xrightarrow{w}y \text{ in } Y\Longrightarrow K(y_n)\to K(y) \text{ in } V.$$

- (b) K is *compact,* if it is continuous and maps bounded sets into relatively compact sets.

In general, these two notions are distinct. However, if Y is reflexive, then "complete continuity" implies "compactness". If Y is reflexive and $K \in \mathscr{L}(Y,V)$, then "complete continuity" and "compactness" of K are equivalent.

A multifunction $G : Y \to 2^V \setminus \{\varnothing\}$ is said to be *compact* if it is usc and maps bounded subsets of Y into relatively compact subsets of V.

Suppose $G : Y \to \mathscr{P}_{wkc}(V)$ is upper semicontinuous from Y into V_w, $K : V \to Y$ is completely continuous and $Q = K \circ G$. The following generalization of the Leray-Schauder Alternative Theorem (see [9], Theorem 4.93, p. 642), is due to Bader [2].

Proposition 2.2. *If Y, V, Q are as above and Q is compact, then one and only one of the following properties holds:*

(a) $S := \{y \in Y : y \in tQ(y) \text{ for some } 0 < t < 1\}$ is unbounded;

(b) $Q(\cdot)$ has a fixed point.

In what follows, by $|\cdot|$ we denote the norm of $\mathbb{R}^N$ and by $(\cdot,\cdot)_{\mathbb{R}^N}$ we denote the usual inner product on $\mathbb{R}^N$. By $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}\left((0,b);\mathbb{R}^N\right)$ and by $\|\cdot\|_p$ we denote the norm of the space $L^p\left((0,b);\mathbb{R}^N\right)$. From the Poincare inequality, we have

$$\|u\| = \left\|u'\right\|_p \text{ for all } u \in W_0^{1,p}\left((0,b);\mathbb{R}^N\right),\ 1 < p < \infty.$$

Recall that

$$\left(W_0^{1,p}\left((0,b);\mathbb{R}^N\right)\right)^* = W^{-1,p'}\left((0,b);\mathbb{R}^N\right), \text{ where } \frac{1}{p} + \frac{1}{p'} = 1.$$

By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for this dual pair.

In the sequel, for notational simplicity, we write $W_0^{1,p}$ for $W_0^{1,p}\left((0,b);\mathbb{R}^N\right)$, $W^{-1,p'}$ for

$$W^{-1,p'}\left((0,b);\mathbb{R}^N\right) = \left(W_0^{1,p}\left((0,b);\mathbb{R}^N\right)\right)^*$$

(with $\frac{1}{p} + \frac{1}{p'} = 1$), and L^q for $L^q\left(T,\mathbb{R}^N\right)$, where $T = [0,b]$ and $1 \le q \le \infty$.

We will also need some information about the spectrum of the vector Dirichlet $p-$Laplacian. So, we consider the following nonlinear vector eigenvalue problem

$$\begin{cases} \left(-\left|u'(t)\right|^{p-2} u'(t)\right)' = \widehat{\lambda}\, |u(t)|^{p-2} u(t) \text{ for a.a. } t \in T := [0,b], \\ u(0) = u(b) = 0, \end{cases} \tag{2.1}$$

where $1 < p < \infty$. We know (see Gasinski-Papageorgiou [8], Theorem 6.3.10, p.768) that (2.1) has a sequence of eigenvalues $\left\{\widehat{\lambda}_n\right\}_{n\ge1} \subset (0,+\infty)$ such that

$$\widehat{\lambda}_n = \left(\frac{n}{b}\right)^p (p-1)\left[2\int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}}\right]^p \text{ for all } n \ge 1.$$

So, $\widehat{\lambda}_n \to +\infty$ as $n \to +\infty$. The corresponding eigenfunctions are

$$\widehat{u}_n(t) = \xi u_n(t) \text{ for all } t \in T, \text{ all } n \ge 1,$$

where $\xi \in \mathbb{R}^N$ and $u_n(\cdot)$ is the corresponding scalar eigenfunction. Also recall the following variational characterization of $\widehat{\lambda}_1$ (see [8], p. 761)

$$\widehat{\lambda}_1 = \inf\left\{ \frac{\|u'\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(0,b), u \neq 0 \right\}. \tag{2.2}$$

These facts lead to the following useful result (see Motreanu-Motreanu-Papageorgiou [15], Lemma 11.3, p.305):

Proposition 2.3. *If $\eta \in L^\infty(T)$, $\eta(t) \leq \widehat{\lambda}_1$ for a.a. $t \in T$ and $\eta \neq \widehat{\lambda}_1$, then there exists $C_1 > 0$ such that*

$$\left\|u'\right\|_p^p - \int_0^b \eta(t)\,|u(t)|^p\,dt \geq C_1 \|u\|^p \text{ for all } u \in W_0^{1,p}.$$

Now we introduce our hypotheses on the data of (1.1).

$\mathbf{H}(a)$: $a : \mathbb{R}^N \to \mathbb{R}^N$ is of the form $a(x) = k(x)x$ or $a(x) = (k_m(x)x_m)_{m=1}^N$ with $k : \mathbb{R}^N \to \mathbb{R}^+$ or $k_m : \mathbb{R} \to \mathbb{R}^+$ continuous and such that

(i) $a(\cdot)$ is strictly monotone;

(ii) $(a(x), x)_{\mathbb{R}^N} \geq C_0 |x|^p$ for all $x \in \mathbb{R}^N$ some $C_0 > 0$ and with $p \geq 2$.

Remark 2.2. These conditions on $a(\cdot)$ are a little more restrictive than those employed by Manasevich-Mawhin [13]. However, they are general enough to incorporate in our analysis many differential operators of interest. For example, let

$$a(x) = |x|^{p-2}x \text{ or } a(x) = \left(|x_m|^{p-2}x_m\right)_{m=1}^N \text{ for all } x = (x_m)_{m=1}^N \in \mathbb{R}^N,\ p \geq 2.$$

Then these maps satisfy hypotheses $\mathbf{H}(a)$ and correspond to different versions of the $p-$Laplacian (see Zhang [17]). We stress that we do not impose any growth conditions on $a(\cdot)$. So, if $\beta(x) = \frac{1}{p}\left(Ce^{|x|^p} - 1\right)$ with $C > 1$, $2 \leq p < \infty$, then

$$a(x) = \nabla\beta(x) = \left(Ce^{|x|^p} - 1\right)|x|^{p-2}x,\ x \in \mathbb{R}^N$$

satisfies hypotheses $\mathbf{H}(a)$. Another possibility for $a(\cdot)$ is

$$a(x) = \xi(|x|)\,|x|^{p-2}x,\ p \geq 2,$$

with $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ continuous, $0 < C_0 \leq \xi(r)$ for all $r \geq 0$ and $r \to \xi(r)r^{p-1}$ strictly increasing on $(0,+\infty)$. For example, let $\xi(r) = \frac{r+1}{r+2}$.

Note that $a(\cdot)$ is a homeomorphism and $\left|a^{-1}(x)\right| \to +\infty$ as $|x| \to +\infty$.

$\mathbf{H}(A)$: $A : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map such that $0 \in A(0)$.

Remark 2.3. We stress that we do not assume that $D(A) = \mathbb{R}^N$. This way we incorporate in our analysis differential variational inequalities (that is, systems with unilateral constraints).

$\mathbf{H}(F)$: $F : T \times \mathbb{R}^N \times \mathbb{R}^N \to \mathscr{P}_{kc}\left(\mathbb{R}^N\right)$ is a multifunction such that

(i) for a. a. $x, y \in \mathbb{R}^N$, $t \to F(t,x,y)$ is graph measurable;

(ii) for a. a. $t \in T$, $(x,y) \to F(t,x,y)$ has closed graph;

(iii) for a.a. $t \in T$, all $x, y \in \mathbb{R}^N$ and all $v \in F(t,x,y)$, we have

$$|v| \leq \gamma_1(t,|x|) + \gamma_2(t,|x|)\,|y|^{p-1},$$

where $\gamma_i : T \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfy

$$\sup\{\gamma_1(t,r) : 0 \le r \le k\} \le \eta_{1,k}(t),\ \eta_{1,k} \in L^{p'}(T)_+$$

and

$$\sup\{\gamma_2(t,r) : 0 \le r \le k\} \le \eta_{2,k}(t),\ \eta_{2,k} \in L^{\infty}(T)_+;$$

(iv) if $m(t,x,y) := \inf\{(v,x)_{\mathbb{R}^N} : v \in F(t,x,y)\}$, then

$$\liminf_{|x|\to+\infty} \frac{\inf\{m(t,x,y) : y \in \mathbb{R}^N\}}{|x|^p} \ge -\theta(t) \text{ uniformly for a.a. } t \in T,$$

where $\theta \in L^{\infty}(T)_+$, $\theta(t) \le C_0\widehat{\lambda}_1$ for a.a. $t \in T$, $\theta \ne C_0\widehat{\lambda}_1$ with $C_0 > 0$ as in hypothesis $\mathbf{H}(a)(ii)$.

Remark 2.4. Note that $t \to F(t,x,y)$ is measurable and for a.a. $t \in T$, $F(t,\cdot,\cdot)$ is usc. Hypothesis $\mathbf{H}(F)(iv)$ is a kind of nonresonance condition. Indeed, if $a(x) = |x|^{p-2}x$ (the vector p-Laplacian operator, hence $C_0 = 1$), $A \equiv 0$ and F is single valued and independent on $y \in \mathbb{R}^N$, then hypothesis $\mathbf{H}(F)(iv)$ is a nonuniform nonresonance condition employed quite often in problems of variational character (see Zhang [17]).

3. Auxiliary results

Throughout the remainder of the paper, $p \in [2,\infty)$ is the same as in assumption $\mathbf{H}(a)(ii)$.

We start by considering the following auxiliary Dirichlet system

$$\begin{cases} -(a(u'(t)))' + |u(t)|^{p-2}u(t) = g(t) \text{ for a.a. } t \in T := [0,b], \\ u(0) = u(b) = 0. \end{cases} \tag{3.1}$$

with $g \in L^{p'}(T,\mathbb{R}^N)$ (recall that $\frac{1}{p} + \frac{1}{p'} = 1$).

Proposition 3.1. *If hypotheses* $\mathbf{H}(a)$ *hold, then problem* (3.1) *has a unique solution* $u \in C^1(T,\mathbb{R}^N)$.

Proof. For every $h \in L^{p'}(T,\mathbb{R}^N)$, the Dirichlet system

$$\begin{cases} -(a(u'(t)))' = h(t) \text{ for a.a. } t \in T := [0,b], \\ u(0) = u(b) = 0 \end{cases} \tag{3.2}$$

has a unique solution. To see this, note that from (3.2) after integration over $[0,t]$, we obtain

$$a(u'(t)) = \alpha - H(h)(t) \text{ for a.a. } t \in T,$$

where $\alpha \in \mathbb{R}^N$ and $H : L^{p'}(T,\mathbb{R}^N) \to C(T,\mathbb{R}^N)$ is defined by

$$H(h)(t) = \int_0^t h(s)\,ds.$$

So, we have

$$u'(t) = a^{-1}(\alpha - H(h)(t)) \text{ for a.a. } t \in T,\ u(0) = u(b) = 0.$$

Integrating over $[0,t]$, we obtain

$$u(t) = \int_0^t a^{-1}(\alpha - H(h)(s))\,ds.$$

Note that $u(b) = \int_0^b a^{-1}(\alpha - H(h)(s))\,ds = 0$. From Proposition 2.2 (i) of Manasevich-Mawhin [13], we know that the equation $\int_0^b a^{-1}(\alpha - H(h)(s))\,ds = 0$ in $\alpha \in \mathbb{R}^N$, has a unique solution $\alpha = \sigma(H(h))$. Therefore problem (3.2) has a unique solution

$$u(t) = \int_0^t a^{-1}(\sigma(H(h)) - H(h)(s))\,ds. \tag{3.3}$$

Now let $K : L^{p'} \to W_0^{1,p}$ be the map which to each $h \in L^{p'}(T, \mathbb{R}^N)$ assigns the unique solution (3.3) for problem (3.1).

Claim 1: $K : L^{p'} \to W_0^{1,p}$ is completely continuous.

Let $h_n \xrightarrow{w} h$ in $L^{p'}$ and set $u_n = K(h_n) \in W_0^{1,p}$ for any $n \in \mathbb{N}$. We have

$$-\left(a\left(u_n'(t)\right)\right)' = h_n(t) \text{ for a.a. } t \in T, \text{ all } n \in \mathbb{N}. \tag{3.4}$$

Taking the inner product with $u_n(t)$, integrating over $T = [0,b]$ and performing an integration by parts, we obtain

$$C_0 \left\| u_n' \right\|_p^p \leq \left\| h_n \right\|_p \left\| u_n \right\|_p$$

(see hypothesis $\mathbf{H}(a)(ii)$). Hence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}$ is bounded (by Poincare's inequality). Also, directly from (3.4) we have that

$$\left\{\left(a\left(u_n'(t)\right)\right)'\right\}_{n\geq 1} \subseteq L^{p'} \text{ is bounded.} \tag{3.5}$$

From (3.3), we know that

$$u_n'(t) = a^{-1}(\alpha - H(h)(t)) \text{ for a.a. } t \in T, \text{ all } n \in \mathbb{N}. \tag{3.6}$$

From Proposition 2.2 (ii) of Manasevich-Mawhin [13] we know that the solution map $\sigma : C(T, \mathbb{R}^N) \to \mathbb{R}^N$ is continuous and bounded (that is, maps bounded sets to bounded sets). Also, $H : L^{p'} \to C(T, \mathbb{R}^N)$ is linear continuous. Moreover, if $N_1 : C(T, \mathbb{R}^N) \to C(T, \mathbb{R}^N)$ is defined by

$$N_1(y)(\cdot) = a^{-1}(y(\cdot)),$$

then N_1 is continuous and maps bounded sets to bounded sets. So, from (3.6) it follows that we can find $C_2 > 0$ such that

$$\left|u_n'(t)\right| \leq C_2 \text{ for all } t \in T, \text{ all } n \in \mathbb{N}.$$

hence

$$\left|a\left(u_n'(t)\right)\right| \leq C_3 \text{ for some } C_3 > 0, \text{ all } t \in T, \text{ all } n \in \mathbb{N},$$

therefore

$$\left\{a\left(u_n'(\cdot)\right)\right\}_{n\geq 1} \subseteq W^{1,p'} \text{ is bounded,}$$

and we conclude that

$$\left\{a\left(u_n'(\cdot)\right)\right\}_{n\geq 1} \subseteq C(T, \mathbb{R}^N) \text{ is relatively compact,}$$

(recall that $W^{1,p'} \hookrightarrow C\left(T,\mathbb{R}^N\right)$ compactly). The continuity of $N_1(\cdot)$ implies that $\{u_n'\}_{n\geq 1} \subseteq C\left(T,\mathbb{R}^N\right)$ is relatively compact. Therefore, we can say that

$$\{u_n\}_{n\geq 1} \subseteq C^1\left(T,\mathbb{R}^N\right) \text{ is relatively compact.}$$

So, we may assume that

$$u_n \to u \text{ in } C^1\left(T,\mathbb{R}^N\right).$$

Passing to the limit as $n\to\infty$ in (3.4) we have

$$-\left(a\left(u'(t)\right)\right)' = h(t) \text{ for a.a. } t\in T,\ u(0)=u(b)=0,$$

hence $u\in K(h)$, therefore K is completely continuous. This proves Claim 1.

Next, let $N_2 : W_0^{1,p} \to L^{p'}$ be the map defined by

$$N_2(u)(\cdot) = -|u(\cdot)|^{p-2}u(\cdot)+g(\cdot).$$

Evidently, $N_2(\cdot)$ is continuous and bounded. A solution of (3.1) is a fixed point of the composition $K\circ N_2$. We shall produce such a fixed point using the classical Leray-Schauder alternative theorem.

Claim 2: The set $S=\left\{u\in W_0^{1,p} : u=\beta\,(K\circ N_2)(u) \text{ for some } \beta\in(0,1)\right\}$ is bounded.

Let $u\in S$. We have

$$-\left(a\left(\frac{1}{\beta}u'(t)\right)\right)' + |u(t)|^{p-2}u(t) = g(t) \text{ for a.a. } t\in T,\ u(0)=u(b)=0.$$

Taking the inner product with $u(t)$, integrating over $T=[0,b]$ and using as before integration by parts, we obtain

$$\frac{C_0}{\beta^{p-1}}\left\|u'\right\|_p^p + \frac{1}{\beta^{p-1}}\left\|u\right\|_p^p \leq \|g\|_{p'}\|u\|_p$$

(see hypothesis $\mathbf{H}(a)(ii)$), hence

$$\|u\|^{p-1} \leq C_4\|g\|_{p'} \text{ for some } C_4>0,$$

therefore $S\subseteq W_0^{1,p}$ is bounded. This proves Claim 2.

We apply the Leray-Schauder alternative theorem and obtain $\widehat{u}\in C^1\left(T,\mathbb{R}^N\right)$ such that $\widehat{u}=(K\circ N_2)(\widehat{u})$. Then $\widehat{u}\in C^1\left(T,\mathbb{R}^N\right)$ is a solution of (3.1). In fact the solution is unique on account of the strictly monotonicity of $\mathbb{R}^N\ni x\to |x|^{p-2}x$. □

Consider the operator $V : D(V)\subseteq L^p \to L^{p'}$ defined by

$$V(u)(\cdot) = -a\left(u'(\cdot)\right)$$

for all $u\in D(V)=\left\{u\in C^1\left(T,\mathbb{R}^N\right) : a\left(u'(\cdot)\right)\in W^{1,p'},\ u(0)=u(b)=0\right\}$.

Proposition 3.2. *If hypotheses* $\mathbf{H}(a)$ *hold, then* $V : D(V)\subseteq L^p\to L^{p'}$ *is maximal monotone.*

Proof. Let $J : L^p\to L^{p'}$ *be* the continuous, strictly monotone (hence maximal monotone, too) coercive map, defined by

$$J(u)(\cdot) = |u(\cdot)|^{p-2}u(\cdot) \text{ for all } u\in L^p.$$

According to Proposition 3.1, we have

$$R(V+J) = L^{p'}. \tag{3.7}$$

We show that (3.7) implies the maximal monotonicity of $V(\cdot)$.

In what follows, let $(\cdot,\cdot)_{pp'}$ denote the duality brackets for the dual pairs $\left(L^{p'},L^{p}\right)$, that is,

$$(h,u)_{pp'}=\int_0^b (h(t),u(t))_{\mathbb{R}^N}\,dt \text{ for all } h\in L^{p'},\ \text{all } u\in L^{p}.$$

Evidently, $V(\cdot)$ is monotone. Suppose that for some $y\in L^p$ and some $v\in L^{p'}$ we have

$$(V(u)-v,u-y)_{pp'}\geq 0 \text{ for all } u\in D(V). \tag{3.8}$$

Exploiting the surjectivity of $V+J$ (see (3.7)), we can find $u_1\in D(V)$ such that

$$V(u_1)+J(u_1)=v+J(y).$$

We use this in (3.8) to obtain

$$(V(u_1)-V(u_1)-J(u_1)+J(y),u_1-y)_{pp'}\geq 0,$$

hence

$$(J(u_1)-J(y),u_1-y)_{pp'}\leq 0,$$

therefore $y=u_1\in D(V)$ and $v=V(u_1)$. This proves the maximality of $V(.)$. ☐

Now we consider the following approximation of problem (1.1):

$$\begin{cases} (a(u'(t)))'\in A_\lambda(u(t))+F(t,u(t),u'(t)) \text{ for a.a. } t\in T:=[0,b], \\ u(0)=u(b)=0,\ \lambda>0. \end{cases} \tag{3.9}$$

We prove the existence of solutions for problem (3.9).

Proposition 3.3. *If hypotheses* $\mathbf{H}(a)$, $\mathbf{H}(A)$, $\mathbf{H}(F)$ *hold, then problem* (3.9) *has a solution* $u_\lambda\in C^1\left(T,\mathbb{R}^N\right)$.

Proof. Let $\mathscr{A}_\lambda:L^p\to L^{p'}$ be the Nemitsky operator corresponding to $A_\lambda(\cdot)$, that is,

$$\mathscr{A}_\lambda(u)(\cdot)=A_\lambda(u(\cdot)) \text{ for all } u\in L^p.$$

This map is continuous, monotone, hence maximal monotone. Let $E_\lambda=V+J+\mathscr{A}_\lambda:D(V)\subseteq L^p\to L^{p'}$. Proposition 3.2 and Theorem 3.2.41 of Gasinski-Papageorgiou ([8], p. 328) imply that E_λ is maximal monotone. Also, for $u\in D(V)$ we have

$$\begin{aligned}(E_\lambda(u),u)_{pp'}&=(V(u),u)_{pp'}+(J(u),u)_{pp'}+(\mathscr{A}_\lambda(u),u)_{pp'}\\ &\geq C_0\left\|u'\right\|_p^p+\|u\|_p^p\end{aligned}$$

(see hypothesis $\mathbf{H}(a)(ii)$ and note that $\mathscr{A}_\lambda(0)=0$), hence E_λ is coercive.

But a maximal monotone, coercive map, is surjective (see Gasinski-Papageorgiou ([8], Corollary 3.2.31, p.319). Also E_λ is strictly monotone (since $J(\cdot)$ is), and so it is injective, therefore E_λ is a bijection.

Consider the map $E_\lambda^{-1}:L^{p'}\to D(V)\subseteq W_0^{1,p}$.

Claim 3: $E_\lambda^{-1}:L^{p'}\to W_0^{1,p}$ is completely continuous.

Let $g_n\xrightarrow{w}h$ in $L^{p'}$ and set $u_n=E_\lambda^{-1}(g_n)$ for all $n\in\mathbb{N}$. We have

$$E_\lambda(u_n)=g_n \text{ for all } n\in\mathbb{N},$$

and so

$$\begin{cases} -\left(a\left(u_n'(t)\right)\right)' + |u_n(t)|^{p-2}u_n(t) + A_\lambda(u_n(t)) = g_n(t) \text{ for a.a. } t\in T, \\ u_n(0) = u_n(b) = 0. \end{cases} \tag{3.10}$$

As before, we take the inner product with $u_n(t)$, integrate over $T=[0,b]$, use integration by parts and hypothesis $\mathbf{H}(a)(ii)$, and recall that for all $x\in\mathbb{R}^N$, $(A_\lambda(x),x)_{\mathbb{R}^N}\geq 0$ (since $A_\lambda(0)=0$). So, we obtain

$$C_0\left\|u_n'\right\|_p^p + \|u_n\|_p^p \leq \|g_n\|_{p'}\|u_n\|_p,$$

hence $\{u_n\}_{n\geq 1}\subseteq W_0^{1,p}$ is bounded, therefore $\{u_n\}_{n\geq 1}\subseteq C\left(T,\mathbb{R}^N\right)$ is relatively compact (recall that $W_0^{1,p}\hookrightarrow C\left(T,\mathbb{R}^N\right)$ compactly). Then from (3.10) and Proposition 2.1, we see that

$$\left\{\left(a\left(u_n'(\cdot)\right)\right)'\right\}_{n\geq 1}\subseteq L^{p'} \text{ is bounded.}$$

Also, from (3.10) we have

$$\left(a\left(u_n'(t)\right)\right)' = |u_n(t)|^{p-2}u_n(t) + A_\lambda(u_n(t)) - g_n(t) \text{ for a.a. } t\in T,$$

hence

$$a\left(u_n'(t)\right) = a\left(u_n'(0)\right) + \int_0^t\left[|u_n(s)|^{p-2}u_n(s) + A_\lambda(u_n(s)) - g_n(s)\right]ds$$

$$\text{for all } t\in T, \text{ all } n\in\mathbb{N}.$$

Set

$$\widetilde{k}(u_n)(t) := -|u_n(t)|^{p-2}u_n(t) - A_\lambda(u_n(t)) + g_n(t).$$

Then $\widetilde{k}(u_n)(\cdot)\in L^{p'}$ for all $n\in\mathbb{N}$ and we have

$$u_n'(t) = a^{-1}\left(a\left(u_n'(0)\right) - H\left(\widetilde{k}(u_n)\right)(t)\right), \text{ for all } t\in T.$$

Since $\int_0^b u_n'(t)\,dt = 0$ (by the Dirichlet boundary condition), we have

$$a\left(u_n'(0)\right) = \sigma\left(H\left(\widetilde{k}(u_n)\right)\right) \text{ for all } n\in\mathbb{N}$$

(see Manasevich-Mawhin [13], Proposition 2.2 (i) and the proof of Proposition 3.1).

Therefore

$$u_n'(t) = a^{-1}\left(\sigma\left(H\left(\widetilde{k}(u_n)\right)\right) - H\left(\left(\widetilde{k}(u_n)\right)(t)\right)\right), \text{ for all } t\in T. \tag{3.11}$$

Note that

$$\left\|H\left(\widetilde{k}(u_n)\right)\right\|_{C(T,\mathbb{R}^N)} \leq C_5 \text{ for some } C_5>0, \text{ all } n\in\mathbb{N}.$$

Also, we know that $\sigma: C\left(T,\mathbb{R}^N\right)\to\mathbb{R}^N$ is continuous and bounded (i.e., maps bounded sets to bounded sets). Similarly, recall that $N_1: C\left(T,\mathbb{R}^N\right)\to C\left(T,\mathbb{R}^N\right)$ defined by

$$N_1(u) = a^{-1}(u(\cdot)) \text{ for all } u\in C\left(T,\mathbb{R}^N\right)$$

is continuous and bounded. Therefore from (3.11) it follows that we can find $C_6>0$ such that

$$\left|u_n'(t)\right| \leq C_6 \text{ for all } n\in\mathbb{N}, \text{ all } t\in T,$$

hence

$$\left|a\left(u_n'(t)\right)\right| \leq C_7 \text{ for some } C_7>0, \text{all } n\in\mathbb{N}, \text{ all } t\in T$$

(recall that $a(\cdot)$ is continuous). So, finally we conclude that $\{a(u_n')\}_{n\geq 1}\subseteq W^{1,p'}$ is bounded, hence $\{a(u_n')\}_{n\geq 1}\subseteq C\left(T,\mathbb{R}^N\right)$ is relatively compact, and we derive that $\{u_n'\}_{n\geq 1}\subseteq C\left(T,\mathbb{R}^N\right)$ is relatively compact. Therefore we have proved that $\{u_n\}_{n\geq 1}\subseteq C^1\left(T,\mathbb{R}^N\right)$ is relatively compact. We may assume that

$$u_n\to u \text{ in } C^1\left(T,\mathbb{R}^N\right) \text{ as } n\to\infty.$$

For every $\psi\in C_0^1\left((0,b),\mathbb{R}^N\right)$, we have

$$\int_0^b \left(a\left(u_n'(t)\right),\psi'(t)\right)_{\mathbb{R}^N}dt+\int_0^b |u_n(t)|^{p-2}\left(u_n(t),\psi(t)\right)_{\mathbb{R}^N}dt$$
$$+\int_0^b \left(A_\lambda\left(u_n(t)\right),\psi(t)\right)_{\mathbb{R}^N}dt=\int_0^b \left(g_n(t),\psi(t)\right)_{\mathbb{R}^N}dt, \text{ for all } n\in\mathbb{N}.$$

Passing to the limit as $n\to+\infty$, we obtain

$$\int_0^b \left(a\left(u'(t)\right),\psi'(t)\right)_{\mathbb{R}^N}dt+\int_0^b |u(t)|^{p-2}\left(u(t),\psi(t)\right)_{\mathbb{R}^N}dt$$
$$+\int_0^b \left(A_\lambda\left(u(t)\right),\psi(t)\right)_{\mathbb{R}^N}dt=\int_0^b \left(g(t),\psi(t)\right)_{\mathbb{R}^N}dt.$$

Since $\psi\in C_0^1\left((0,b),\mathbb{R}^N\right)$ is arbitrary, it follows that

$$\begin{cases} -\left(a\left(u'(t)\right)\right)'+|u(t)|^{p-2}u(t)+A_\lambda\left(u(t)\right)=g(t) \text{ for a.a. } t\in T,\\ u(0)=u(b)=0,\end{cases}$$

hence $u=E_\lambda^{-1}(g)$. This proves Claim 3.

Next, let $N_3: W_0^{1,p}\to 2^{L^{p'}}$ be the multifunction defined by

$$N_3(u)=S^{p'}_{-F(\cdot,u(.),u'(.))}+J(u) \text{ for all } u\in W_0^{1,p}.$$

Claim 4: N_3 is $\mathscr{P}_{wkc}\left(L^{p'}\right)-$valued and is usc from $W_0^{1,p}$ into $L^{p'}$ with the w-topology (denoted hereafter by $L^{p'}_w$).

First we show the nonemptiness of the values of N_3. Hypotheses $\mathbf{H}(F)(i)$, (ii) do not imply that $F(t,x,y)$ is superpositionally measurable (see Hu-Papageorgiou [11], Example 7.2, p.227). So we cannot use directly in $F(\cdot,u(\cdot),u'(\cdot))$ the Yankov-von Neumann-Aumann selection theorem. To overcome this difficulty, we proceed as follows. Given $u\in W_0^{1,p}$ let $\{s_n\}_{n\geq 1}$, $\{r_n\}_{n\geq 1}$ be two sequences of $\mathbb{R}^N-$valued step functions such that $s_n(t)\to u(t)$ and $r_n(t)\to u'(t)$ for a.a. $t\in T$, as $n\to+\infty$ and $|s_n(t)|\leq|u(t)|$, $|r_n(t)|\leq|u'(t)|$, a.e. on T (note that $u,\ u'\in L^p$). See Dinculeanu ([4], p.99).

Then, for every $n\in\mathbb{N}$, hypothesis $\mathbf{H}(F)(i)$ implies that $t\to F(t,s_n(t),r_n(t))$ is Lebesgue measurable. So, for this multifunction we can use the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [11], p.158) and for each $n\in\mathbb{N}$, we obtain $f_n: T\to\mathbb{R}^N$, measurable functions such that

$$f_n(t)\in -F(t,s_n(t),r_n(t)) \text{ for almost all } t\in T.$$

On account of hypothesis $\mathbf{H}(F)(iii)$, we see that $\{f_n\}_{n\geq 1} \subseteq L^{p'}$ is bounded. So, we may assume that

$$f_n \xrightarrow{w} f \text{ in } L^{p'} \text{ as } n \to +\infty.$$

Invoking Proposition 3.9 of Hu-Papageorgiou ([11], p. 694), we obtain

$$f(t) \in \overline{conv} \limsup_{n\to\infty} \left(-F(t, s_n(t), r_n(t))\right) \subseteq -F\left(t, u(t), u'(t)\right) \text{ for a. a. } t \in T.$$

The last inclusion follows from hypothesis $\mathbf{H}(F)(ii)$. So, we see that

$$f \in S^{p'}_{-F(\cdot,u(\cdot),u'(\cdot))}.$$

Therefore we have established the nonemptiness of the values of $N_3(\cdot)$.

It is clear that $N_3(\cdot)$ has bounded, closed, convex values. Hence, $N_3(u) \in \mathscr{P}_{wkc}\left(L^{p'}\right)$ for all $u \in W_0^{1,p}$.

To show the upper semicontinuity of $N_3(\cdot)$, note that $N_3(\cdot)$ is bounded (that is maps bounded sets to bounded sets). Recall that bounded sets in $L^{p'}$ with the relative weak topology are metrizable. So, according to Proposition 2.23 of Hu-Papageorgiou ([11], p. 43), to prove the claimed upper semicontinuity of $N_3(\cdot)$ it suffices to show that $GrN_3 \subseteq W_0^{1,p} \times L_w^{p'}$ is sequentially closed (see also Section 2). To this end consider $\{(u_n, g_n)\}_{n\geq 1} \subseteq GrN_3$ such that

$$u_n \to u \text{ in } W_0^{1,p} \text{ and } g_n \xrightarrow{w} g \text{ in } L^{p'} \text{ as } n \to +\infty.$$

Using the compact embedding of $W_0^{1,p}$ into $C\left(T, \mathbb{R}^N\right)$, we have

$$u_n(t) \to u(t) \text{ for a.a. } t \in T, \text{ as } n \to \infty.$$

Also, we can say that

$$u_n'(t) \to u'(t) \text{ for a.a. } t \in T, \text{ as } n \to \infty.$$

Then, as before, we have

$$\begin{aligned} g(t) &\in \overline{conv} \limsup_{n\to\infty} \left(-F\left(t, u_n(t), u_n'(t)\right) + |u_n(t)|^{p-2} u_n(t)\right) \\ &\subseteq -F\left(t, u(t), u'(t)\right) + |u(t)|^{p-2} u(t) \text{ for a. a. } t \in T, \end{aligned}$$

hence $g \in N_3(u)$. This proves Claim 4.

We next consider the set

$$S = \left\{u \in W_0^{1,p} : u \in \beta E_\lambda^{-1} N_3(u),\ 0 < \beta < 1\right\}.$$

Claim 5: $S \subseteq W_0^{1,p}$ is bounded.

Let $u \in S$. We have

$$E_\lambda \left(\frac{1}{\beta} u\right) \in N_3(u),$$

hence

$$\begin{cases} -\left(a\left(\frac{1}{\beta} u'(t)\right)\right)' + \frac{1}{\beta} |u(t)|^{p-2} u(t) + A_\lambda \left(\frac{1}{\beta} u(t)\right) \\ \qquad\qquad = f(t) + |u(t)|^{p-2} u(t) \text{ for a.a. } t \in T \\ u(0) = u(b) = 0,\ f \in S^p_{-F(\cdot,u(\cdot),u'(\cdot))} \end{cases} \tag{3.12}$$

Hypothesis $\mathbf{H}(F)(iv)$ implies that given $\varepsilon > 0$, we can find $M_1 = M_1(\varepsilon) > 0$ such that for a.a. $t \in T$, all $|x| \geq M_1$, all $y \in \mathbb{R}^N$ and all $v \in -F(t, x, y)$, we have

$$(v, x)_{\mathbb{R}^N} \leq [\theta(t) + \varepsilon] |x|^p.$$

From hypothesis $\mathbf{H}(F)(iii)$ we see that for a. a. $t\in T$, all $|x|<M_1$, all $y\in\mathbb{R}^N$ and all $v\in -F(t,x,y)$, we have

$$|v|\leq \eta_{1,M_1}(t)+\eta_{2,M_1}(t)|y|^{p-1}$$

therefore

$$(v,x)_{\mathbb{R}^N}\leq M_1\eta_{1,M_1}(t)+M_1\eta_{2,M_1}(t)|y|^{p-1}.$$

Consequently, finally, for a. a. $t\in T$, all x, $y\in\mathbb{R}^N$ and all $v\in -F(t,x,y)$ we have

$$(v,x)_{\mathbb{R}^N}\leq(\theta(t)+\varepsilon)|x|^p+C_8|y|^{p-1}+\mu(t) \tag{3.13}$$

for some $C_8>0$ and $\mu\in L^1(T)_+$. Returning to (3.12), taking the inner product with $u(t)$, integrating over $T=[0,b]$, performing an integration by parts and using (3.13), we obtain

$$\frac{C_0}{\beta^{p-1}}\|u'\|_p^p+\left(\frac{1}{\beta^{p-1}}-1\right)\|u\|_p^p\leq\int_0^b(\theta(t)+\varepsilon)|u|^p\,dt+C_8\int_0^b|u'|^{p-1}\,dt+\|\mu\|_1.$$

Employing (2.2) and Holder's inequality, we arrive at

$$C_0\|u'\|_p^p-\int_0^b\theta(t)|u|^p\,dt-\frac{\varepsilon}{\widehat{\lambda}_1}\|u'\|_p^p\leq C_9\|u'\|_p^{p-1}+\|\mu\|_1$$

for some $C_9>0$. Therefore

$$\left(C_{10}-\frac{\varepsilon}{\widehat{\lambda}_1}\right)\|u'\|_p^p\leq C_9\|u'\|_p^{p-1}+\|\mu\|_1 \text{ for some } C_{10}>0 \tag{3.14}$$

(see hypothesis $\mathbf{H}(F)(iv)$ and Proposition 2.3). Choosing $\varepsilon\in\left(0,C_{10}\widehat{\lambda}_1\right)$, from (3.14) we have

$$\|u'\|_p^p\leq C_{11}\left(\|u'\|_p^{p-1}+1\right) \text{ for some } C_{11}>0. \tag{3.15}$$

From (3.15) and Poincare's inequality it follows that

$$\|u\|\leq C_{12} \text{ for some } C_{12}>0, \text{ all } u\in S,$$

and we conclude that $S\subseteq W_0^{1,p}$ is bounded. This proves Claim 5.

Now Claims 3, 4, and 5 permit the use of Proposition 2.2. So, we can find $u_\lambda\in D(V)$ such that

$$u_\lambda\in E_\lambda^{-1}N_3(u_\lambda).$$

Evidently $u_\lambda\in C^1\left(T,\mathbb{R}^N\right)$ and it is a solution of problem (3.9). □

Eventually we will let $\lambda\to 0^+$, to produce a solution of problem (1.1). To do this, we need an additional auxiliary result.

Let $1<r,\, r'<\infty$, $\frac{1}{r}+\frac{1}{r'}=1$ and let $\mathscr{A}:L^r\to 2^{L^{r'}}$ be the lifting (realization) of $A(\cdot)$ defined by

$$\mathscr{A}(u)=\left\{h\in L^{r'}:h(t)\in A(u(t)) \text{ for a.a. } t\in T\right\}$$

for all $u\in D(\mathscr{A})$, where

$$D(\mathscr{A})=\left\{y\in L^r:y(t)\in D(A) \text{ for a.a. } t\in T,\ S^{r'}_{A(u(\cdot))}\neq\varnothing\right\}.$$

Since $0\in D(\mathscr{A})$, it follows that $D(\mathscr{A})\neq\varnothing$.

The next Proposition is known for Hilbert spaces (see Brezis [3], p.25). However, our proof here is different and can be easily extended to Banach spaces.

Proposition 3.4. *If hypotheses* $\mathbf{H}(a)$ *hold, then* $\mathscr{A}: L^r \to 2^{L^{r'}}$ *is maximal monotone.*

Proof. Let $e: \mathbb{R}^N \to \mathbb{R}^N$ be defined by $e(x) = |x|^{r-2}x$. This map is continuous and strictly monotone (hence maximal monotone). The Nemitsky operator corresponding to $e(\cdot)$ is the map $J: L^r \to L^{r'}$ defined by

$$J(u)(\cdot) = |u(\cdot)|^{r-2} u(\cdot) \text{ for all } u \in L^r.$$

From the proof of Proposition 3.2, we know that to prove the maximality of $\mathscr{A}(\cdot)$ it suffices to show that

$$R(\mathscr{A}+J) = L^{r'}. \tag{3.16}$$

We know that $x \to (A+e)(x)$ is maximal monotone and coercive (recall that $0 \in A(0)$). Therefore $R(A+e) = \mathbb{R}^N$ (see Gasinski-Papageorgiou [8], Corollary 3.2.31, p. 319).

Let $h \in L^{r'}$ and consider the multifunction $L: T \to 2^{\mathbb{R}^N}$ defined by

$$L(t) = \left\{x \in \mathbb{R}^N : A(x) + e(x) \ni h(t)\right\}.$$

From the surjectivity of $(A+e)(\cdot)$, we see that $L(t) \neq \varnothing$ for a.a. $t \in T$. Consider the function $\eta : T \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^N$ defined by

$$\eta(t,x) = (x, h(t) - e(x)).$$

Then η is a Carathéodory function (that is, for all $x \in \mathbb{R}^N$, $t \to \eta(t,x)$ is measurable and for almost all $t \in T$, $x \to \eta(t,x)$ is continuous). Therefore $\eta(\cdot,\cdot)$ is jointly measurable (see Hu-Papageorgiou [11], Proposition 1.6, p. 142). We have

$$GrL = \left\{(t,x) \in T \times \mathbb{R}^N : \eta(t,x) \in GrA\right\} = \eta^{-1}(GrA).$$

Recall that $GrA \subseteq \mathbb{R}^N \times \mathbb{R}^N$ is closed (since A is maximal monotone). Then the joint measurability of η implies that

$$GrL = \eta^{-1}(GrA) \in \mathscr{L}_T \times \mathscr{B}\left(\mathbb{R}^N\right),$$

where $\mathscr{L}_T$ is the Lebesgue $\sigma-$field of T and $\mathscr{B}\left(\mathbb{R}^N\right)$ is the Borel $\sigma-$field of $\mathbb{R}^N$. We can apply Yankov-von Neumann-Aumann selection theorem and produce a measurable map $u: T \to \mathbb{R}^N$ such that $u(t) \in L(t)$ for a.a. $t \in T$. Then we have

$$h(t) \in A(u(t)) + e(u(t)) \text{ for a.a. } t \in T.$$

We take the inner product with $u(t) \in \mathbb{R}^N$ and obtain

$$|u(t)|^r \leq |h(t)|\,|u(t)| \text{ for a.a. } t \in T$$

(recall that $0 \in A(0)$), hence

$$|u(t)|^{r-1} \leq |h(t)| \text{ for a.a. } t \in T,$$

and it follows that $u \in L^r$. Therefore, finally we have $\mathscr{A}(u) + J(u) \ni h$ with $h \in L^{r'}$, and so, (3.16) holds, implying that $\mathscr{A}(\cdot)$ is maximal monotone. □

Remark 3.1. In fact the result is true if $\mathbb{R}^N$ is replaced by a reflexive Banach space X. The same proof with minor changes works in this more general case.

4. Existence Theorems

In this section we prove several existence theorems for problem (1.1). First we prove an existence theorem under slightly stronger hypotheses on the multivalued perturbation $F(t,x,y)$ ("convex" problem, since F is convex valued). Specifically, we assume:

$\mathbf{H}(F)_1$: $F : T\times\mathbb{R}^N\times\mathbb{R}^N \to \mathscr{P}_{kc}\left(\mathbb{R}^N\right)$ satisfies $\mathbf{H}(F)(i),(ii),(iv)$ as well as $\mathbf{H}(F)(iii)$ where the exponent $p-1$ is replaced by $q=\max\left\{1,\frac{p-1}{2}\right\}$ and $\eta_{1,k}\in L^2(T)_+$.

Theorem 4.1. *If hypotheses $\mathbf{H}(a),\mathbf{H}(A),\mathbf{H}(F)_1$ hold, then problem (1.1) has a solution $u\in C^1\left(T,\mathbb{R}^N\right)$.*

Proof. Let $\lambda_n\to 0^+$ and let $u_n=u_{\lambda_n}\in C^1\left(T,\mathbb{R}^N\right)$ be a solution of (3.9) with $\lambda=\lambda_n$, $n\in\mathbb{N}$ (see Proposition 3.3). We have

$$\left\{\begin{array}{l} -\left(a\left(u_n'(t)\right)\right)'=A_{\lambda_n}\left(u_n(t)\right)+f_n(t) \text{ for a.a. } t\in T,\\ u_n(0)=u_n(b)=0,\end{array}\right. \tag{4.1}$$

with $f_n\in S^{p'}_{F(\cdot,u_n(\cdot),u_n'(\cdot))}$ for all $n\in\mathbb{N}$. As before, acting on (4.1) with $u_n(t)$ we obtain

$$C_0\left\|u_n'\right\|_p^p\le\int_0^b\left(-f_n(t),u_n(t)\right)_{\mathbb{R}^N}dt \text{ for all } n\in\mathbb{N}. \tag{4.2}$$

From (3.13), we have

$$\left(-f_n(t),u_n(t)\right)_{\mathbb{R}^N}\le\left[\theta(t)+\varepsilon\right]\left|u_n(t)\right|^p+C_8\left|u_n'(t)\right|^{p-1}+\mu(t) \text{ for a.a. } t\in T.$$

Using this in (4.2), as in the proof of Proposition 3.3, and invoking Proposition 2.3, we deduce that

$$\left\{u_n\right\}_{n\ge1}\subseteq W_0^{1,p} \text{ is bounded.}$$

So, we may assume that

$$u_n\xrightarrow{w}u \text{ in } W_0^{1,p} \text{ and } u_n\to u \text{ in } C\left(T,\mathbb{R}^N\right). \tag{4.3}$$

We return to (4.1) and take the inner product with $A_{\lambda_n}u_n(t)$. Integrating over $T=[0,b]$, we have

$$\begin{aligned}&\int_0^b\left(-\left(a\left(u_n'(t)\right)\right)',A_{\lambda_n}\left(u_n(t)\right)\right)_{\mathbb{R}^N}dt+\left\|A_{\lambda_n}\left(u_n\right)\right\|_2^2 \\ &\le\int_0^b\left|f_n(t)\right|\left|A_{\lambda_n}\left(u_n(t)\right)\right|dt.\end{aligned} \tag{4.4}$$

Note that $A_{\lambda_n}(u_n) = \mathscr{A}_{\lambda_n}(u_n) \in C(T, \mathbb{R}^N)$. Performing an integration by parts in the first integral, we have

$$\int_0^b \left(-\left(a\left(u_n'(t)\right)\right)', A_{\lambda_n}(u_n(t)) \right)_{\mathbb{R}^N} dt$$

$$= -\left(\left(a\left(u_n'(b)\right)\right)', A_{\lambda_n}(u_n(b)) \right)_{\mathbb{R}^N} + \left(\left(a\left(u_n'(0)\right)\right)', A_{\lambda_n}(u_n(0)) \right)_{\mathbb{R}^N}$$

$$+ \int_0^b \left(a\left(u_n'(t)\right), \frac{d}{dt} A_{\lambda_n}(u_n(t)) \right)_{\mathbb{R}^N} dt$$

$$= \int_0^b \left(a\left(u_n'(t)\right), \frac{d}{dt} A_{\lambda_n}(u_n(t)) \right)_{\mathbb{R}^N} dt$$

(since $u_n(b) = u_n(0) = 0$ and $A_{\lambda_n}(0) = 0$).

Recall that for $n \in \mathbb{N}$, $A_{\lambda_n} : \mathbb{R}^N \to \mathbb{R}^N$ is Lipschitz continuous (see Proposition 2.1). So, by the Rademacher's theorem (see, for example Gasinski-Papageorgiou [8], Theorem 1.5.8, p. 56), we know that $A_{\lambda_n}(\cdot)$ is differentiable at every $x \in \mathbb{R}^N \backslash S_1$ with $|S_1|_N = 0$ (here $|.|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$). Then, for all $x \in \mathbb{R}^N \backslash S_1$, all $h \in \mathbb{R}^N$ and all $\tau > 0$, we have

$$\left(h, \frac{A_{\lambda_n}(x + \tau h) - A_{\lambda_n}(x)}{\tau} \right)_{\mathbb{R}^N} \geq 0 \text{ (since } A_{\lambda_n} \text{ is monotone)},$$

hence sending $\tau \to 0^+$ we get

$$\left(h, A_{\lambda_n}'(x) h\right)_{\mathbb{R}^N} \geq 0. \tag{4.5}$$

From the chain rule of Marcus-Mizel [14], we have

$$\frac{d}{dt} A_{\lambda_n}(u_n(t)) = A_{\lambda_n}'(u_n(t))\, u_n'(t) \text{ for a.a. } t \in T. \tag{4.6}$$

If $a(x) = k(x)x$ (cf. $\mathbf{H}(a)$), then

$$\int_0^b \left(-\left(a\left(u_n'(t)\right)\right)', A_{\lambda_n}(u_n(t)) \right)_{\mathbb{R}^N} dt \tag{4.7}$$

$$= \int_0^b \left(a\left(u_n'(t)\right), \frac{d}{dt} A_{\lambda_n}(u_n(t)) \right)_{\mathbb{R}^N} dt$$

$$= \int_0^b k\left(u_n'(t)\right) \left(u_n'(t), A_{\lambda_n}'(u_n(t))\, u_n'(t) \right)_{\mathbb{R}^N} dt$$

$$\geq 0 \text{ (see (4.6) and (4.5))}$$

Similarly if $a(x) = (k_m(x)\, x_m)_{m=1}^N$.

Let $M_1 = \sup \|u_n\|_{C(T, \mathbb{R}^N)}$. Then hypothesis $\mathbf{H}(F)_1\,(iii)$ implies that

$$|f_n(t)| \leq \eta_{1,M_1}(t) + \eta_{2,M_1}(t) \left|u_n'(t)\right|^q \text{ for a.a. } t \in T, \text{ all } n \in \mathbb{N} \tag{4.8}$$

with $\eta_{1,M_1} \in L^2(T)$, $\eta_{2,M_1} \in L^\infty(T)$. Since $u_n' \in C(T, \mathbb{R}^N)$ (recall that $u_n \in D(V)$ for all $n \in \mathbb{N}$) we see that $f_n \in L^2(T, \mathbb{R}^N)$ for all $n \in \mathbb{N}$. Moreover, by (4.3) and (4.8) it follows that $\{f_n\}_{n \geq 1}$ is bounded in

$L^2\left(T,\mathbb{R}^N\right)$. Then

$$\int_0^b |f_n(t)|\left|A_{\lambda_n}(u_n(t))\right| dt \leq \|f_n\|_2 \left\|\mathscr{A}_{\lambda_n}(u_n)\right\|_2 \leq M_2 \left\|\mathscr{A}_{\lambda_n}(u_n)\right\|_2 \tag{4.9}$$

for some $M_2 > 0$, all $n \in \mathbb{N}$. Returning to (4.4) and using (4.7) and (4.9), we have

$$\left\|\mathscr{A}_{\lambda_n}(u_n)\right\|_2^2 \leq M_2 \left\|\mathscr{A}_{\lambda_n}(u_n)\right\|_2,$$

hence

$$\left\{\mathscr{A}_{\lambda_n}(u_n)\right\}_{n\geq 1} \subseteq L^2 \subseteq L^{p'} \text{ is bounded (recall that } p' \leq 2 \leq p).$$

So, we may assume that

$$\mathscr{A}_{\lambda_n}(u_n) \xrightarrow{w} g \text{ in } L^2 \text{ and } L^{p'} \text{ as } n \to \infty. \tag{4.10}$$

As in the proof of Proposition 3.3 (see Claim 3), we obtain that $\{u_n\}_{n\geq 1} \subseteq C^1\left(T,\mathbb{R}^N\right)$ is relatively compact. We may assume that

$$u_n \to u \text{ in } C^1\left(T,\mathbb{R}^N\right) \text{ and so } a\left(u_n'(t)\right) \to a(u(t)) \text{ for all } t \in T. \tag{4.11}$$

Also, we have

$$f_n \xrightarrow{w} f \text{ in } L^{p'}\left(T,\mathbb{R}^N\right) \text{ with } f \in S^{p'}_{F(\cdot,u(\cdot),u'(.))}. \tag{4.12}$$

For every $\psi \in C_0^1\left((0,b),\mathbb{R}^N\right)$, we have

$$\int_0^b \left(-a\left(u_n'(t)\right), \psi'(t)\right)_{\mathbb{R}^N} dt = \int_0^b \left(A_{\lambda_n}(u_n(t)), \psi(t)\right)_{\mathbb{R}^N} dt + \int_0^b \left(f_n(t), \psi(t)\right)_{\mathbb{R}^N} dt$$

for all $n \in \mathbb{N}$. Passing to the limit as $n \to \infty$, we obtain

$$\int_0^b \left(-a\left(u'(t)\right)', \psi'(t)\right)_{\mathbb{R}^N} dt = \int_0^b \left(g(t), \psi(t)\right)_{\mathbb{R}^N} dt + \int_0^b \left(f(t), \psi(t)\right)_{\mathbb{R}^N} dt$$

(see (4.10) and (4.11)). Since $\psi \in C_0^1\left((0,b),\mathbb{R}^N\right)$ is arbitrary, it follows that

$$\left(a\left(u'(t)\right)\right)' = g(t) + f(t) \text{ for a.a. } t \in T, u(0) = u(b) = 0,$$

with $f \in S^{p'}_{F(\cdot,u(\cdot),u'(.))}$ (see (4.12)). Clearly $u(\cdot)$ will be a solution of (1.1) if we show that $g \in \mathscr{A}(u)$. This follows easily from (4.3), (4.10), Proposition 2.1 and Proposition 3.4 (applied to the special case $r = 2$). □

We also prove an existence theorem for problem (1.1) in the case where F is nonconvex valued ("nonconvex" problem). Now the hypotheses on the multivalued perturbation F are:

$\mathbf{H}(F)_2$: $F : T \times \mathbb{R}^N \times \mathbb{R}^N \to \mathscr{P}_k\left(\mathbb{R}^N\right)$ is a multifunction such that
- (i) $(t,x,y) \to F(t,x,y)$ is graph measurable;
- (ii) for a. a. $t \in T$, $(x,y) \to F(t,x,y)$ is lsc;
- (iii) same as hypothesis $\mathbf{H}(F)_1(iii)$;
- (iv) same as hypothesis $\mathbf{H}(F)(iv)$.

Theorem 4.2. *If hypotheses $\mathbf{H}(a)$, $\mathbf{H}(A)$, $\mathbf{H}(F)_2$ hold, then problem (1.1) has a solution $u \in C^1\left(T,\mathbb{R}^N\right)$.*

Proof. Let $N : W_0^{1,p} \to \mathscr{P}_f\left(L^{p'}\right)$ be the multifunction defined by

$$N(u) = S^{p'}_{F(\cdot,u(\cdot),u'(\cdot))} \text{ for all } u \in W_0^{1,p}.$$

We claim that $N(\cdot)$ is lsc. According to Proposition 2.26 of Hu-Papageorgiou ([11], p.45) it suffices to show that for every $v \in L^{p'}$, the function $u \to d_{p'}(v, N(u)) = \inf\left\{\|v-f\|_{p'} : f \in N(u)\right\}$ is upper semicontinuous. From Theorem 3.24 of Hu-Papageorgiou ([11], p.183), we have

$$d_{p'}(v, N(u)) = \left(\int_0^b \left[d\left(v(t), F\left(t, u(t), u'(t)\right)\right)\right]^{p'} dt\right)^{\frac{1}{p'}}. \tag{4.13}$$

We need to show that for every $\xi > 0$, the superlevel set

$$U_\xi = \left\{u \in W_0^{1,p} : d_{p'}(v, N(u)) \geq \xi\right\}$$

is closed. So, suppose that $\{u_n\}_{n\geq 1} \subseteq U_\xi$ and assume that $u_n \to u$ in $W_0^{1,p}$. Using (4.13) we have

$$\int_0^b \left[d\left(v(t), F\left(t, u_n(t), u_n'(t)\right)\right)\right]^{p'} dt \geq \xi^{p'} \text{ for all } n \in \mathbb{N}.$$

We may assume that

$$u_n(t) \to u(t) \text{ for all } t \in T \text{ and } u_n'(t) \to u'(t) \text{ for a. a. } t \in T. \tag{4.14}$$

From Fatou's lemma, we have

$$\int_0^b \limsup_{n\to\infty} \left[d\left(v(t), F\left(t, u_n(t), u_n'(t)\right)\right)\right]^{p'} dt \geq \xi^{p'}.$$

Hypothesis $\mathbf{H}(F)_2(ii)$ implies that $(x,y) \to d_{p'}(v(t), F(t,x,y))$ is upper semicontinuous. So,

$$\int_0^b \limsup_{n\to\infty} \left[d\left(v(t), F\left(t, u(t), u'(t)\right)\right)\right]^{p'} dt \geq \xi^{p'}$$

(see (4.14)) hence

$$d_{p'}(v, N(u)) \geq \xi,$$

that is, $u \in U_\xi$, and $N(\cdot)$ is lsc. Also $N(\cdot)$ has decomposable values. Therefore we can apply Theorem 8.7 of Hu-Papageorgiou ([11], p.245) and obtain a continuous map $d : W_0^{1,p} \to L^{p'}$ such that

$$d(u) \in N(u) \text{ for all } u \in W_0^{1,p}.$$

Then, as in the proof of Proposition 3.3, we show that for every $\lambda > 0$ the problem

$$\left(a\left(u'(t)\right)\right)' = A_\lambda(u(t)) + d(u)(t) \text{ for a.a. } t \in T, u(0) = u(b) = 0$$

admits a solution $u_\lambda \in C^1\left(T, \mathbb{R}^N\right)$. Passing to the limit as $\lambda \to 0^+$ and reasoning as in the proof of Theorem 4.1, we produce a solution $u \in C^1\left(T, \mathbb{R}^N\right)$ for the nonconvex problem. □

We can relax the condition on the map $a(\cdot)$ at the expense of strengthening the condition on $A(\cdot)$. The new hypotheses on $a(\cdot)$ and $A(\cdot)$ are:

$\mathbf{H}(a)'$: $a : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous map such that
(i) $a(\cdot)$ is strictly monotone;
(ii) $(a(x), x)_{\mathbb{R}^N} \geq C_0 |x|^p$ for all $x \in \mathbb{R}^N$, some $C_0 > 0$, and with $p \geq 2$.

$\mathbf{H}(A)'$: $A : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map such that $D(A) = \mathbb{R}^N$ and $0 \in A(0)$.

Remark 4.1. The hypothesis that $D(A) = \mathbb{R}^N$ precludes from considerations maps of the form $A = \partial i_C$ where i_C is the indicator function of a set $C \in \mathscr{P}_{fc}(\mathbb{R}^N)$. Such maps arise in problems with unilateral constraints (variational inequalities).

Theorem 4.3. *If hypotheses $\mathbf{H}(a)'$, $\mathbf{H}(A)'$, $\mathbf{H}(F)$ hold, then problem (1.1) has a solution $u \in C^1(T, \mathbb{R}^N)$.*

Proof. The particular structure of $a(\cdot)$ considered in hypothesis $\mathbf{H}(a)$ (namely that $a(x) = k(x)x$ or that $a(x) = (k_m(x) x_m)_{m=1}^N$ was only used in the proof of Theorem 4.1 in order to show the L^2-boundedness of $\{\mathscr{A}_{\lambda_n}(u_n)\}_{n\geq 1}$. So, Propositions 3.1, 3.2, 3.3 are still valid in the present setting. Moreover, if $\{u_n\}_{n\geq 1} \subseteq C^1(T, \mathbb{R}^N)$ are as in the proof of Theorem 4.1, from the first part of that proof we know that $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}$ is bounded, hence $\{u_n\}_{n\geq 1} \subseteq C(T, \mathbb{R}^N)$ is relatively compact (recall that $W_0^{1,p} \hookrightarrow C(T, \mathbb{R}^N)$ compactly). Therefore, we can find $M_3 > 0$ such that

$$\|u_n\|_{C(T,\mathbb{R}^N)} \leq M_3 \text{ for all } n \in \mathbb{N}.$$

Since $J_{\lambda_n}(0) = 0$ and $J_{\lambda_n}(\cdot)$ is nonexpansive (see Proposition 2.1), we have

$$\left\|J_{\lambda_n}(u_n(\cdot))\right\|_{C(T,\mathbb{R}^N)} \leq M_3 \text{ for all } n \in \mathbb{N}.$$

Then we have

$$A_{\lambda_n}(u_n(t)) \in A\left(J_{\lambda_n}(u_n(t))\right) \subseteq A\left(\overline{B}_{M_3}(0)\right), \tag{4.15}$$

where $\overline{B}_{M_3}(0) = \left\{v \in \mathbb{R}^N : |v| \leq M_3\right\}$. But, from Theorem 1.28 of Hu-Papageorgiou ([11], p. 308) we know that $A(\cdot)$ is usc with $\mathscr{P}_{kc}(\mathbb{R}^N)-$values. So, by Corollary 2.20 of Hu-Papageorgiou ([11], p. 42) we have that $A\left(\overline{B}_{M_3}(0)\right) \in \mathscr{P}_k(\mathbb{R}^N)$. Hence

$$\left|A_{\lambda_n}(u_n(t))\right| \leq M_4 \text{ for some } M_4 > 0, \text{ all } n \in \mathbb{N}, \text{ all } t \in T$$

(see (4.15)). Now the rest of the proof proceeds as the corresponding part of the proof of Theorem 4.1. □

We can derive a "nonconvex" version of this existence theorem, as well. The hypotheses on the multivalued perturbation are the following:

$\mathbf{H}(F)_2'$: $F : T \times \mathbb{R}^N \times \mathbb{R}^N \to \mathscr{P}_k(\mathbb{R}^N)$ is a multifunction satisfying $\mathbf{H}(F)_2$ $(i), (ii)$ and $\mathbf{H}(F)$ $(iii), (iv)$.

Theorem 4.4. *If hypotheses $\mathbf{H}(a)'$, $\mathbf{H}(A)'$, $\mathbf{H}(F)_2'$ hold, then problem (1.1) has a solution $u \in C^1(T, \mathbb{R}^N)$.*

Proof. We just combine the proofs of Theorems 4.2 and 4.3. □

5. An example

In this section, we present an example of a differential variational inequality (a problem with unilateral constraints) to which our general existence theory applies.

Let $\mathbb{R}^N_+$ be the usual positive cone of $\mathbb{R}^N$ (that is, $\mathbb{R}^N_+ = \left\{x = (x_m)_{m=1}^N \in \mathbb{R}^N : x_m \geq 0\right\}$. This is a closed, convex cone. Let $i_{\mathbb{R}^N_+}(\cdot)$ be the indicator function of $\mathbb{R}^N_+$, that is,

$$i_{\mathbb{R}^N_+}(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{R}^N_+, \\ +\infty & \text{if} \quad x \notin \mathbb{R}^N_+. \end{cases}$$

It is easily seen that $i_{\mathbb{R}^N_+}(\cdot) \in \Gamma_0\left(\mathbb{R}^N\right)$, the space of all proper, convex, lower semicontinuous functions on $\mathbb{R}^N$ (See Gasinski-Papageorgiou [8], Definition 4.21, p. 488). We set

$$A(x) = \partial i_{\mathbb{R}^N_+}(x) = N_{\mathbb{R}^N_+}(x) = \text{the normal cone to } \mathbb{R}^N_+ \text{ at } x,$$

where "∂" stands for the subdifferential in the sense of convex analysis. We know that for $x = (x_m)_{m=1}^N \in \mathbb{R}^N_+$ we have

$$N_{\mathbb{R}^N_+}(x) = \begin{cases} 0 & \text{if} \quad x_m > 0 \text{ for all } m = 1, \ldots, N, \\ -\mathbb{R}^N_+ \cap \{x\}^\perp & \text{if} \quad x_m = 0 \text{ for at least one } m = 1, \ldots, N. \end{cases}$$

So, $D(A) = \mathbb{R}^N_+$. Then problem (1.1) reduces to the following multivalued differential variational inequality

$$\begin{cases} (a(u'(t)))' \in F(t, u(t), u'(t)) \text{ a.e. on} \\ \qquad\qquad \{t \in T : u_m(t) > 0 \text{ for all } m = 1, \ldots, N\} \\ (a(u'(t)))' \in F(t, u(t), u'(t)) - h(t) \text{ a.e. on} \\ \qquad\qquad \{t \in T : u_m(t) = 0 \text{ for some } m = 1, \ldots, N\} \\ h(t) \in \mathbb{R}^N_+, (h(t), u(t))_{\mathbb{R}^N} = 0 \text{ for a.a. } t \in T, \\ \qquad u = (u_m)_{m=1}^N \in C^1\left(T, \mathbb{R}^N\right) \\ u_m(t) \geq 0 \text{ for all } t \in T, \text{ all } m = 1, \ldots, N, \\ u(0) = u(b) = 0. \end{cases} \tag{5.1}$$

If F satisfies hypotheses $\mathbf{H}(F)$ or $\mathbf{H}(F)_2$, then (5.1) admits a solution (see Theorems 4.1, 4.2).

If $F(t, x, y) = f(t, x, y)$ is single-valued, then (5.1) has the following more familiar form (recall that on $\mathbb{R}^N$, $x \leq y \Longleftrightarrow y - x \in \mathbb{R}^N_+$) :

$$\begin{cases} (a(u'(t)))' = f(t, u(t), u'(t)) \text{ a.e.} \\ \qquad\qquad \text{on } \{t \in T : u_m(t) > 0 \text{ for all } m = 1, \ldots, N\} \\ (a(u'(t)))' \leq f(t, u(t), u'(t)) \text{ a.e.} \\ \qquad\qquad \text{on } \{t \in T : u_m(t) = 0 \text{ for some } m = 1, \ldots, N\} \\ \left(f(t, u(t), u'(t)) - (a(u'(t)))', u(t)\right)_{\mathbb{R}^N} = 0 \text{ for a.a. } t \in T, \\ u = (u_m)_{m=1}^N \in C^1\left(T, \mathbb{R}^N\right), u(t) \geq 0 \text{ for all } t \in T, \\ u(0) = u(b) = 0. \end{cases}$$

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