DYNAMICS AND INTERPRETATION OF SOME INTEGRABLE SYSTEMS VIA MATRIX ORTHOGONAL POLYNOMIALS

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Abstract. In this work we characterize a high order Toda lattice in terms of a family of matrix polynomials orthogonal with respect to a complex matrix measure. In order to study the solution of this dynamical system we give explicit expressions for the Weyl function, generalized Markov function, and we also obtain, under some conditions, a representation of the vector of linear functionals associated with this system. We show that the orthogonality is embedded in these structure and governs the high order Toda lattice. We also present a Lax type theorem for the point spectrum of the Jacobi operator associated with a Toda type lattice.

1. Introduction

Consider the following infinite system of differential equations

$$
\begin{cases}
\dot{a}_n = c_n - c_{n-2} \\
\dot{b}_n = c_n a_{n+1} - c_{n-1} a_n + d_n - d_{n-2} \\
\dot{c}_n = c_n (b_{n+1} - b_n) + d_n a_{n+2} - d_{n-1} a_n \\
\dot{d}_n = d_n (b_{n+2} - b_n)
\end{cases}
, \quad n \in \mathbb{N}, \tag{1}
$$

where the dot, "$\cdot$", means the differentiation with respect to $t \in \mathbb{R}$ and where we assume that $a_0 = b_0 = c_0 = d_0 = 0$ and $c_1 = 0$.

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Particular cases of this kind of dynamical system appear in the literature in different contexts. For example, in [1] and [2], Bogoyavlenskii gave a classification of these dynamical systems which are a discrete generalization of a KdV equation and showed that such systems have interesting applications on Hamilton mechanics. On the other hand in the work [3] a particular case of a Bogoyavlenskii discrete dynamical system appears related with the study of spectral problems for higher order difference equations and it was studied using a method based on the analysis of the genetic sums formula for the moments of the associated operator. Also in [4] the authors studied another particular case of these dynamical systems investigating the spectral properties of the associated band operator. More recently, in [5], the authors propose to study systems of type (1) motivated by its bi-Hamiltonian structure, and the first two authors, studied the interpretation of some generalizations of the systems considered in the works mentioned before [6, 7, 8].

The system (1) can be written as matrix Sylvester equation known as a *Lax pair*,

$$\dot{J} = \big[J, J_-\big] := J\,J_- - J_-\,J\,,$$

where J and J_- are the operators which matrix representation is given respectively by

$$J = \begin{bmatrix} b_1 & a_2 & 1 & & & \\ c_1 & b_2 & a_3 & 1 & & \\ d_1 & c_2 & b_3 & a_4 & 1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \quad J_- = \begin{bmatrix} 0 & & & \\ c_1 & 0 & & \\ d_1 & c_2 & 0 & \\ & \ddots & \ddots & \ddots \end{bmatrix}. \tag{2}$$

Since, in general, the entries at the diagonals present a nonsymmetric operator, J, we cannot introduce the spectral measure and consider its evolution. However, since J is a bounded operator, then it is possible to define the *resolvent operator*, by

$$(zI - J)^{-1} = \sum_{n=0}^{\infty} \frac{J^n}{z^{n+1}}\,, \quad |z| > ||J||\,,$$

(see for example [9] and [10]) and, take the associated analytic function

$$R_J(z) = \sum_{n=0}^{\infty} \frac{e_0^{\mathsf{T}} J^n e_0}{z^{n+1}}\,, \quad |z| > ||J||\,, \tag{3}$$

where $e_0 = \begin{bmatrix} I_{2\times 2} & 0_{2\times 2} & \cdots \end{bmatrix}^{\mathsf{T}}$, as spectral data and then study its evolution. This approach for the complex orthogonality and for the nonsymmetric three diagonal matrix J was explored in the paper [11].

Now, if we denote by M_{ij} the 2×2 block matrices, of an infinite matrix M, formed by the entries of rows $2i-1$, $2i$ and columns $2j-1$, $2j$, the matrix J^n can be written in blocks notation as

$$J^n = \begin{bmatrix} J_{11}^n & J_{12}^n & \cdots \\ J_{11}^n & J_{11}^n & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} . \tag{4}$$

In this way, for each $n \in \mathbb{N}$, we have that the Weyl function (3) can also be written in the following form

$$R_J(z) = \sum_{n=0}^{\infty} \frac{J_{11}^n}{z^{n+1}} , \quad |z| > ||J|| . \tag{5}$$

As a consequence of the Lax pair representation for (1), we observe that the operator theory establishes a connection between these systems and the theory of approximation. In fact, with the Lax pair representation for (1), we can associate to this system the Weyl function of J, R_J. On the other hand, if $\{V_m\}_{m\in\mathbb{N}}$ is matrix sequence of orthogonal polynomials (cf. [12, 13]) defined by

$$x\, V_m(x) = A_{m+1}\, V_{m+1}(x) + B_m\, V_m(x) + C_m\, V_{m-1}(x) , \quad m \geq 1 , \tag{6}$$

with some specific initial conditions we will see that the Weyl function of J and the complex measure of orthogonality, given by the generalized Markov function, associated with the system of matrix orthogonal polynomials given above (6) are almost the same. Indeed, they are equal up to a normalizing constant.

Using the concepts of the matrix orthogonal polynomials theory presented in this work we show that if the vector orthogonality function is of exponential type we ensure that the coefficients of the matrix recurrence relation are a solution of the Toda type lattice and also under the representation of R_J a solution of the system (1) is obtained. These results are obtain due to the fact that (1) has a matrix interpretation in terms of the matrix coefficients A_m, B_m and C_m that appears in the recurrence relation (6). Moreover, these matrix coefficients contain the solution $\{a_n, b_n, c_n, d_n\}$, $n \in \mathbb{N}$ of the system (1) and, using the matrix orthogonality, we explicitly get the representations for them (cf. section 2).

The matrix interpretation is the main key to obtain the results we present. In this work we present the theory of matrix orthogonal polynomial with respect to a complex measure that assures that this interpretation is possible. These theory involves study of the so called direct and inverse spectral problems treated by one of the authors in [14] for the scalar case. Also Miranian in [15] study properties for the monic matrix orthogonal polynomials with respect to matrix measures that evolves in time with an exponential.

This work is organized as follows: In section 2, we present first a matrix interpretation of the dynamical system (1) that allows to use theory of matrix polynomials associated with complex measure of orthogonality to solve the dynamical system. The known theory of matrix orthogonality is presented afterwards. In section 3, we study the solution of the dynamical system (1). We show that the Weyl function associated to J play a main role in the solution of this problem. In the proof of the main theorem plays an important role certain orthogonal polynomials families that satisfy structure of relation of the type studied by Ismail and Durán in [16]. We also give a Lax type theorem for the Jacobi operator associated with the Toda type lattice. In section 4, we give explicit expressions for the Weyl function, and we also obtain a representation of the vector functionals associated with the system studied in section 3. Moreover, we find the expression for the generalized Markov function (or equivalently, a measure of orthogonality) that governs this Toda type system and rediscovering, as an application, the one treated, independently by, other authors like Durán and Grünbaum in [17] and Miranian in [15].

2. Connection with matrix orthogonality

To give an matrix interpretation of the dynamical system, we start considering, for $d_{n-1} \neq 0$, $n = 2, 3, \ldots$, the sequence of monic polynomials $\{p_n\}$ satisfying the five term recurrence relation

$$(7) \begin{cases} x^2 p_n = p_{n+2} + a_{n+2} p_{n+1} + b_{n+1} p_n + c_n p_{n-1} + d_{n-1} p_{n-2}, \; n \geq 2, \\ p_1(x) = x - a_1 \,, \quad a_1 \in \mathbb{R}, \; p_{-1}(x) = 0 \,, \; p_0(x) = 0 \,. \end{cases}$$

Notice that (7), with $\mathcal{B}_m = \begin{bmatrix} p_{2m} & p_{2m+1} \end{bmatrix}^\mathsf{T}$, can be written in matrix form as

$$(8) \qquad x^2 \, \mathcal{B}_m(x) = A_m \, \mathcal{B}_{m+1}(x) + B_m \, \mathcal{B}_m(x) + C_m \, \mathcal{B}_{m-1}(x) \,, \quad m \geq 1 \,,$$

where

$$A_m = \begin{bmatrix} 1 & 0 \\ a_{2m+3} & 1 \end{bmatrix} , \quad B_m = \begin{bmatrix} b_{2m+1} & a_{2m+2} \\ c_{2m+1} & b_{2m+2} \end{bmatrix} , \quad C_m = \begin{bmatrix} d_{2m-1} & c_{2m} \\ 0 & d_{2m} \end{bmatrix} ,$$

for $m \geq 1$, with $\mathcal{B}_{-1}(x) = 0_{2\times 1}$ and $\mathcal{B}_0(x) = \begin{bmatrix} 1 & x - a_1 \end{bmatrix}^\mathsf{T}$. It was proved in [18] that we can always write $\mathcal{B}_m$ in the form

$$\mathcal{B}_m(x) = V_m(x^2) \, \mathcal{P}_0(x) \,,$$

where V_m is a 2×2 matrix polynomial of degree m, $\mathcal{P}_0(x) = \begin{bmatrix} 1 & x \end{bmatrix}^{\mathsf{T}}$, and that the matrix sequence of polynomials, $\{V_m\}$, is defined by

$$x\,V_m(x) = A_{m+1}\,V_{m+1}(x) + B_m\,V_m(x) + C_m\,V_{m-1}(x)\,, \quad m \geq 1\,, \tag{9}$$

with $V_{-1} = 0_{2\times 2}$ and $V_0 = \begin{bmatrix} 1 & 0 \\ -a_1 & 1 \end{bmatrix}$. Since, the matrix operator J can also be written in a block tridiagonal matrix form

$$J = \begin{bmatrix} B_0 & A_0 & 0_{2\times 2} & \\ C_1 & B_1 & A_1 & \ddots \\ 0_{2\times 2} & C_2 & B_2 & \ddots \\ & \ddots & \ddots & \ddots \end{bmatrix}, \tag{10}$$

then it is related to the matrix sequence of polynomials $\{V_m\}$ through the recurrence relation (9). From now on this block matrix is said to be the 2×2 block Jacobi matrix associated with the above matrix polynomial sequences.

Notice that, in the present work, the polynomials p_n and V_m depend on $t \in \mathbb{R}$, as well as the coefficients a_n, b_n, c_n, d_n of the recurrence relations. For each t, $\{V_m\}$ forms a matrix sequence of orthogonal polynomials. For sake of simplicity in the following we suppress the t-dependence.

One of our goals is to study the solutions of (1) in terms of the operator J and its associated matrix polynomials V_m, but first we should establish some known results about vector and matrix orthogonality.

At this point it is worth mentioning that (1) has a matrix interpretation in terms of the matrix coefficients A_m, B_m and C_m that appears in the recurrence relation (8), i.e.

$$\begin{cases} \dot{A}_m = A_m\,D_{m+1} - D_m\,A_m\,, \\ \dot{B}_m = A_m\,C_{m+1} - C_m\,A_{m-1} + B_m\,D_m - D_m\,B_m\,, \\ \dot{C}_m = B_m\,C_m - C_m\,B_{m-1} + C_m\,D_{m-1} - D_m\,C_m\,, \end{cases} \tag{11}$$

$D_m = \left[\begin{smallmatrix} 0 & 0 \\ c_{2m+1} & 0 \end{smallmatrix}\right]$, $m = 0, 1 \ldots$. Also, these matrix coefficients contain the solution $\{a_n, b_n, c_n, d_n\}$, of the system (1) and, using the matrix orthogonality, we explicitly get the representations for them.

Now, to understand the kind of orthogonality that it will be used we need to remember the following results.

Let $\mathbb{P}$ be the linear space of polynomials with complex coefficients. Now, consider the space of vector of polynomials $\mathbb{P}^2 = \langle \mathcal{P}_j,\, j \in \mathbb{N} \rangle$, where $\mathcal{P}_j = x^{2j}\,\mathcal{P}_0$ with $\mathcal{P}_0 = \begin{bmatrix} 1 & x \end{bmatrix}^{\mathsf{T}}$, and the space $\mathcal{M}_{2\times 2}(\mathbb{C})$ of 2×2-matrices with complex entries. It is well known (see [18]) that there exist a *vector of linear functionals* $\mathcal{U} = \begin{bmatrix} u^1 & u^2 \end{bmatrix}^{\mathsf{T}}$ defined in $(\mathbb{P}^2)^*$, the linear space of vector linear functionals, here called *dual space*, acting in $\mathbb{P}^2$ over $\mathcal{M}_{2\times 2}(\mathbb{C})$ such that

$$\mathcal{U}(\mathcal{P}) := \left(\mathcal{U}.\mathcal{P}^{\mathsf{T}}\right)^{\mathsf{T}} = \begin{bmatrix} \langle u^1, p_1\rangle & \langle u^2, p_1\rangle \\ \langle u^1, p_2\rangle & \langle u^2, p_2\rangle \end{bmatrix} ,$$

where "." means the symbolic product of the vectors $\mathcal{U}$ and $\mathcal{P}^{\mathsf{T}}$, where $\mathcal{P}^{\mathsf{T}} = \begin{bmatrix} p_1 & p_2 \end{bmatrix}^{\mathsf{T}}$, $p_1, p_2 \in \mathbb{P}$. Notice that this definition is already known in a context of a vectorial interpretation of the multiple orthogonality (cf. [19]).

It is easy to verify that $\mathcal{U}$ is a linear, i.e., $\mathcal{U}$ satisfies $\mathcal{U}(A\,\mathcal{P} + B\,\mathcal{Q}) = A\mathcal{U}(\mathcal{P}) + B\mathcal{U}(\mathcal{Q})$, for A, B numerical matrices and $\mathcal{P}$, $\mathcal{Q}$ vector of polynomials and if $\widehat{A}(x) = \displaystyle\sum_{k=0}^{l} A_k\, x^k$ is a matrix polynomial we can define the *left multiplication of* $\mathcal{U}$ *by* $\widehat{A}$, denoted by $\widehat{A}\mathcal{U}$, as the vector of linear functionals such that

$$(\widehat{A}\mathcal{U})(\mathcal{P}) := (\widehat{A}\mathcal{U}.\mathcal{P}^{\mathsf{T}})^{\mathsf{T}} = \sum_{k=0}^{l} (x^k\,\mathcal{U})(\mathcal{P})\,(A_k)^{\mathsf{T}} .$$

The *Hankel matrices* associated with $\mathcal{U}$ are the matrices

$$U_m = \begin{bmatrix} \mathcal{U}_0 & \cdots & \mathcal{U}_m \\ \vdots & \ddots & \vdots \\ \mathcal{U}_m & \cdots & \mathcal{U}_{2m} \end{bmatrix} , \quad m \in \mathbb{N} ,$$

where $\mathcal{U}_j$ is the j-th moment associated with the vector of linear functionals $\mathcal{U}$, i.e., $\mathcal{U}_j = \mathcal{U}(x^{2j}\mathcal{P}_0)$. The vector of linear functionals $\mathcal{U}$ is said to be *quasi-definite* if all the leading principal submatrices of U_m , $m \in \mathbb{N}$, are non-singular.

A *vector sequence of polynomials* $\{\mathcal{B}_m\}$, with degree of $\mathcal{B}_m$ equal to m, is *left-orthogonal* with respect to the vector of linear functionals $\mathcal{U}$ if

$$(x^{2k}\,\mathcal{U})\,(\mathcal{B}_m) = \Delta_m \delta_{k,m} \,, \quad k = 0, 1, \ldots\,, \; m \,, \quad m \in \mathbb{N} \,, \tag{12}$$

where Δ_m is a non-singular 2×2 upper triangular matrix given by

$$\Delta_m = C_m \,\cdots\, C_1\,\Delta_0 \,, \quad m \geq 1 \,,$$

where Δ_0 is a 2×2 non-singular matrix and $\{C_m\}$ is a sequence of non-singular upper triangular matrices. Similarly, a *sequence of matrix polynomials* $\{G_m\}$, with degree of G_m equal to m, is *right-orthogonal* with respect to the vector of linear functionals $\mathcal{U}$ if it is *bi-orthogonal* with the vector sequence of polynomials $\{\mathcal{B}_m\}$ and to the vector of linear functionals $\mathcal{U}$, i.e., if

$$\left(G_n^{\mathsf{T}}(x^2)\right)\mathcal{U}(\mathcal{B}_m) = I_{2\times 2}\,\delta_{n,m}\,, \quad n, m \in \mathbb{N}\,.$$

In [18], necessary and sufficient conditions for the quasi-definiteness of $\mathcal{U}$, i.e., for the existence of a vector sequence of polynomials left-orthogonal with respect to the vector of linear functionals $\mathcal{U}$, were obtained. These sequences of polynomials satisfy non-symmetric three term recurrence relations. So, if $\{\mathcal{B}_m\}$ is a vector sequence of polynomials left-orthogonal with respect to $\mathcal{U}$ where $\mathcal{B}_m(x) = V_m(x^2)\mathcal{P}_0(x)$, with $\mathcal{P}_0(x) = \begin{bmatrix}1 & x\end{bmatrix}^{\mathsf{T}}$, then there exist sequences of numerical matrices $\{A_m\}$, $\{B_m\}$, and $\{C_m\}$, with A_m a non-singular lower triangular matrix and C_m a non-singular upper triangular matrix, such that $\{\mathcal{B}_m\}$ is defined by (8) with $\mathcal{B}_{-1}(x) = 0_{1\times 2}$ and $\mathcal{B}_0(x) = M\,\mathcal{P}_0(x)$, for a fixed matrix, M. Moreover, $\{V_m\}$ is defined by (9) with $V_{-1} = 0_{2\times 2}$ and $V_0 = M$, and $\{G_n\}$ is defined by

$$x\,G_n(x) = G_{n+1}(x)\,C_{n+1} + G_n(x)\,B_n + G_{n-1}(x)\,A_{n-1}\,, \quad n \geq 1\,,$$

with $G_{-1} = 0_{2\times 2}$ and $G_0 = \mathcal{U}\left(\mathcal{B}_0\right)$. These three term recurrence relations completely characterize each type of orthogonality, i.e., there exist a Favard type theorem for each of these cases. Moreover, the coefficients of the three term recurrence relation can be expressed in terms of a vector of linear functionals $\mathcal{U}$ or in terms of a matrix measure associated with $\{V_m\}$ or $\{G_n\}$ (cf. [18]).

Furthermore, *left and right vector orthogonality* is connected with *right and left matrix orthogonality* as will see bellow. So, if we consider the formal series

$$\frac{1}{z - x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{z^{n+1}}\,, \quad |x^2| < |z|\,,$$

the *generalized Markov matrix function*, $\mathcal{F}$, associated with $\mathcal{U}$ is defined by

$$\mathcal{F}(z) := \sum_{n=0}^{\infty} \frac{\left(x^{2n}\,\mathcal{U}\right)\left(\mathcal{P}_0(x)\right)}{z^{n+1}} = \begin{bmatrix} \displaystyle\sum_{n=0}^{\infty} \frac{\langle u^1, x^{2n}\rangle}{z^{n+1}} & \displaystyle\sum_{n=0}^{\infty} \frac{\langle u^2, x^{2n}\rangle}{z^{n+1}} \\ \displaystyle\sum_{n=0}^{\infty} \frac{\langle u^1, x^{2n+1}\rangle}{z^{n+1}} & \displaystyle\sum_{n=0}^{\infty} \frac{\langle u^2, x^{2n+1}\rangle}{z^{n+1}} \end{bmatrix},$$

with z such that $|x^2| < |z|$ for every $x \in \mathsf{L}$ where $\mathsf{L} = \cup_{j=1,2}\operatorname{supp} u^j$, and $\mathcal{P}_0(x) = \begin{bmatrix}1 & x\end{bmatrix}^{\mathsf{T}}$.

Theorem 2.1 (cf. [18]). *The matrix sequence $\{G_n\}$ and the vector sequence of polynomials $\{\mathcal{B}_m\}$ are bi-orthogonal with respect to $\mathcal{U}$ if,* *and only if, the sequence of matrix orthogonal polynomials* $\{G_n\}$ *and* $\{V_m\}$ *are bi-orthogonal with respect to* $\mathcal{F}$*, i.e.,*

$$\frac{1}{2\pi i}\int_C V_m(z)\mathcal{F}(z)G_n(z)dz = I_{2\times 2}\,\delta_{n,m}\,, \quad n,m\in\mathbb{N}\,, \tag{13}$$

where C *is a closed path in* $\{z\in\mathbb{C} : |z| > |x^2|, x\in \mathsf{L}\}$ *where is defined as before by* $\mathsf{L} = \cup_{j=1,2}\operatorname{supp} u^j$ *.*

The sequences of matrix polynomials $\{V_m\}$ and $\{G_m\}$ presented here are orthogonal with respect to the complex matrix measure of orthogonality $\mathcal{F}$. We note that $\mathcal{F}$ is a complex matrix measure of orthogonality (cf. [20]) which is not necessarily positive definite as in the orthornormal case considered in (cf. [21], [22]) and that can be determined by a Markov type theorem as the reader can see in [23]. Moreover, the matrix sequence of polynomials $\{V_m\}$ satisfy the three term recurrence relation (9), with

$$\begin{aligned} A_m &= \frac{1}{2\pi i}\int_C z\,V_m(z)\,\mathcal{F}(z)\,G_{m+1}(z)\,dz\,,\\ B_m &= \frac{1}{2\pi i}\int_C z\,V_m(z)\,\mathcal{F}(z)\,G_m(z)\,dz\,,\\ C_m &= \frac{1}{2\pi i}\int_C z\,V_m(z)\,\mathcal{F}(z)\,G_{m-1}(z)\,dz\,. \end{aligned}$$

For our purposes in this work we need the following definition:

Definition 2.2. Let $\mathcal{U}$ be a vector of linear functionals. We denote by $\widehat{\mathcal{U}}$, the *normalized vector of linear functionals* associated with $\mathcal{U}$, i.e.,

$$\widehat{\mathcal{U}} = ((\mathcal{U}(\mathcal{P}_0))^{-1})^{\mathsf{T}}\mathcal{U}\,, \text{ where } \quad \mathcal{P}_0(x) = \begin{bmatrix} 1 & x\end{bmatrix}^{\mathsf{T}}\,.$$

Furthermore, from this definition we have

$$\widehat{\mathcal{U}}(\mathcal{P}_0) = (((\mathcal{U}(\mathcal{P}_0))^{-1})^{\mathsf{T}}\mathcal{U})(\mathcal{P}_0) = \mathcal{U}(\mathcal{P}_0)(\mathcal{U}(\mathcal{P}_0))^{-1} = I_{2\times 2}.$$

From now on and in the next section we consider a normalized vector of linear functionals (we set $\mathcal{U} = \widehat{\mathcal{U}}$).

The next theorem shows how the generalized Markov function is directly related with the Weyl function associated with the block tridiagonal Jacobi matrix (10).

Theorem 2.3. *Let* $\mathcal{U}$ *be a normalized vector of linear functionals,* $\mathcal{F}$ *the generalized Markov function, and* R_J *the Weyl function associated with the Jacobi block matrix,* J*, defined by (10). Then, we have that*

$$R_J(z) = M\,\mathcal{F}(z)\,M^{-1}\,, \quad \textit{where} \quad M = \begin{bmatrix} 1 & 0\\ -a_1 & 1\end{bmatrix}\,.$$

Proof. Let $\{\mathcal{B}_m\}$ be a vector sequence of polynomials left-orthogonal with respect to $\mathcal{U}$. To determine the value of $e_0^{\mathsf{T}} J^n e_0$, $n \in \mathbb{N}$ we consider the following matricial identity

$$J \begin{bmatrix} \mathcal{B}_0(x) & \cdots & \mathcal{B}_m(x) & \cdots \end{bmatrix}^{\mathsf{T}} = x^2 \begin{bmatrix} \mathcal{B}_0(x) & \cdots & \mathcal{B}_m(x) & \cdots \end{bmatrix}^{\mathsf{T}},$$

from which we obtain for all $m \in \mathbb{N}$,

$$J^n \begin{bmatrix} \mathcal{B}_0(x) & \cdots & \mathcal{B}_m(x) & \cdots \end{bmatrix}^{\mathsf{T}} = x^{2n} \begin{bmatrix} \mathcal{B}_0(x) & \cdots & \mathcal{B}_m(x) & \cdots \end{bmatrix}^{\mathsf{T}}. \tag{14}$$

Hence, from the first equation of the relation (14) we have that

$$(e_0^{\mathsf{T}} J^n e_0)\, \mathcal{B}_0 + \cdots = x^{2n}\, \mathcal{B}_0 .$$

Applying the vector of linear functionals $\mathcal{U}$ to the last relation and considering the orthogonality conditions we have that

$$(e_0^{\mathsf{T}} J^n e_0)\, \mathcal{U}(\mathcal{B}_0) = (x^{2n}\, \mathcal{U})(\mathcal{B}_0)\,, \text{ i.e., } e_0^{\mathsf{T}} J^n e_0 = (x^{2n}\, \mathcal{U})(\mathcal{B}_0)(\mathcal{U}(\mathcal{B}_0))^{-1}.$$

But, from the initial conditions of the three term recurrence relation (8) we have $\mathcal{B}_0 = M\, \mathcal{P}_0$, then we obtain

$$e_0^{\mathsf{T}} J^n e_0 = M\, (x^{2n}\, \mathcal{U})(\mathcal{P}_0)(\mathcal{U}(\mathcal{P}_0))^{-1} M^{-1},$$

therefore

$$R_J(z) = M \sum_{n=0}^{\infty} \frac{(x^{2n}\, \mathcal{U})(\mathcal{P}_0)(\mathcal{U}(\mathcal{P}_0))^{-1}}{z^{n+1}} M^{-1}.$$

Since $\mathcal{U}$ is a normalized vector functional we get the desired representation for R_J. $\square$

3. Main Results

To establish the relation between the solutions of an integrable system and the vector of polynomials $\mathcal{B}_m$ we have to recall that in this case $\mathcal{U} = \mathcal{U}(t)$ depends on t and then, it is possible to define the derivative of $\mathcal{U}$ (cf. [24]) as usual:

$$\frac{d\mathcal{U}}{dt} : \mathbb{P}^2 \to \mathcal{M}_{2\times 2}(\mathbb{C})$$

such that, for each $\mathcal{P} \in \mathbb{P}^2$,

$$\frac{d\mathcal{U}}{dt}(\mathcal{P}) = \lim_{\Delta t \to 0} \frac{\mathcal{U}\{t + \Delta t\}(\mathcal{P}) - \mathcal{U}\{t\}(\mathcal{P})}{\Delta t}.$$

Obviously, the usual properties for this kind of operators are verified. In particular,

$$\frac{d}{dt}(\mathcal{U}(\mathcal{P})) = \frac{d\mathcal{U}}{dt}(\mathcal{P}) + \mathcal{U}(\dot{\mathcal{P}}),\ \forall\, \mathcal{P} \in \mathbb{P}^2. \tag{15}$$

Now, we establish the main result about the relation between the solutions of an integrable system of type (1) and the matrix polynomials, $\{V_m\}$, orthogonal with respect to the generalized Markov function, $\mathcal{F}$, associated with J by the previous theorem:

Theorem 3.1. *Assume that the sequence $\{a_n, b_n, c_n, d_n\}$, $n \in \mathbb{N}$, is uniformly bounded, i.e., for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$*

$$\exists\, K \in \mathbb{R}_+ :\ \max\{|a_n(t)|, |b_n(t)|, |c_n(t)|, |d_n(t)|\} \leq K \,.$$

Also, consider that $\dot{a}_1 = c_1$. Then, the following conditions are equivalent:

(a) *$\{a_n, b_n, c_n, d_n\}$, $n \in \mathbb{N}$, is a solution of (1), this is,*

$$\dot{J} = J J_- - J_- J \,. \tag{16}$$

(b) *For each $n \in \mathbb{N} \cup \{0\}$ we have that*

$$\frac{d}{dt} J_{11}^n = J_{11}^{n+1} - J_{11}^n J_{11} + J_{11}^n (J_-)_{11} - (J_-)_{11} J_{11}^n \,. \tag{17}$$

(c) *For $n \in \mathbb{N}$ we have that*

$$\dot{\mathcal{U}}_n = \mathcal{U}_{n+1} - \mathcal{U}_n \, \mathcal{U}_1 \,. \tag{18}$$

(d) *For all $z \in \mathbb{C}$ such that $|z| > ||J||$,*

$$\dot{\mathcal{F}}(z) = \mathcal{F}(z) \left(z I_{2\times 2} - \mathcal{U}_1 \right) - I_{2\times 2} \,. \tag{19}$$

(e) *For all $\mathcal{B} \in \mathbb{P}^2$ we have that*

$$\left(\frac{d}{dt} \mathcal{U} \right) (\mathcal{B}) = \mathcal{U}(x^2\, \mathcal{B}) - \mathcal{U}(\mathcal{B})\, \mathcal{U}_1 \,. \tag{20}$$

(f) *For all $m \in \mathbb{N} \cup \{0\}$, the polynomial $\mathcal{B}_m$ defined by (8) satisfies*

$$\dot{\mathcal{B}}_m(x) = -C_m\, \mathcal{B}_{m-1}(x) - D_m\, \mathcal{B}_m(x), \ \ \text{with} \ \ D_m = \begin{bmatrix} 0 & 0 \\ c_{2m+1} & 0 \end{bmatrix} . \tag{21}$$

(g) *The polynomial $\{V_m\}$ defined by $\mathcal{B}_m(x) = V_m(x^2)\, \mathcal{P}_0$ satisfies*

$$\dot{V}_m(x) = -C_m\, V_{m-1}(x) - D_m\, V_m(x) \,, \quad m \in \mathbb{N} \cup \{0\} \,, \tag{22}$$

with D_m given in (f).

Proof. We will prove this theorem according to the following scheme:

(a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (g) $\Rightarrow$ (a).

We start, proving that $(a) \Rightarrow (b)$. Since the sequence $\{a_n, b_n, c_n, d_n\}$, $n \in \mathbb{N}$ is a solution of (1) we have that $\dot{J} = \big[J, J_-\big]$. It is easy to prove by induction that

$$\frac{d}{dt}J^n = J^n J_- - J_- J^n \,.$$

In fact, for $n = 1$ the result is straightforward. Now, suppose that for $n = 2, \ldots, p$ the relation $\dfrac{d}{dt}J^p = J^p J_- - J_- J^p$ holds. Then, we just have to prove that the results is also valid for $n = p + 1$. Since,

$$\begin{aligned}\frac{d}{dt}J^{p+1} &= \frac{d}{dt}(J^p J) = \frac{d}{dt}(J^p)J + J^p\frac{d}{dt}(J)\\ &= (J^p J_- - J_- J^p)J + J^p(JJ_- - J_-J) = J^{p+1}J_- - J_-J^{p+1}\,,\end{aligned}$$

the result holds. In particular, we have that

$$\frac{d}{dt}J^n_{11} = (J^n J_-)_{11} - (J_- J^n)_{11}\,.$$

From (2) and (4),

$$\frac{d}{dt}J^n_{11} = J^n_{11}(J_-)_{11} + J^n_{12}(J_-)_{21} - (J_-)_{11}J^n_{11}\,.$$

But, we have that

$$J^{n+1}_{11} = J^n_{11}\, J_{11} + J^n_{12}\, J_{21}\,,$$

or equivalently, since $J_{21} = (J_-)_{21}$, that

$$J^n_{12}(J_-)_{21} = J^{n+1}_{11} - J^n_{11}\, J_{11}\,.$$

Therefore, we have that

$$\frac{d}{dt}J^n_{11} = J^{n+1}_{11} - J^n_{11}\, J_{11} + \big[J^n_{11}, (J_-)_{11}\big]\,.$$

To prove $(b) \Rightarrow (c)$ we have to read the differential equation (17) in terms of the moments. Regarding that we are working with normalized vector functionals and that $J^n_{11} = M(x^{2n}\mathcal{U})(\mathcal{P}_0)\, M^{-1}$ (cf. Theorem 2.3) the relation (17) becomes

$$\begin{aligned}\frac{d}{dt}(M(x^{2n}\mathcal{U})(\mathcal{P}_0)M^{-1}) &= M(x^{2(n+1)}\mathcal{U})(\mathcal{P}_0)M^{-1} - M(x^{2n}\mathcal{U})(\mathcal{P}_0)M^{-1}J_{11}\\ &\quad + M(x^{2n}\mathcal{U})(\mathcal{P}_0)M^{-1}(J_-)_{11} - (J_-)_{11}M(x^{2n}\mathcal{U})(\mathcal{P}_0)M^{-1}\,.\end{aligned}$$

Since the moment $\mathcal{U}_n = \mathcal{U}(\mathcal{P}_n) = (x^{2n}\mathcal{U})(\mathcal{P}_0)$ we have that

$$\begin{aligned}\frac{d}{dt}(M\mathcal{U}_n M^{-1}) &= M\,\mathcal{U}_{n+1}\, M^{-1} - M\,\mathcal{U}_n\, M^{-1}\, J_{11}\\ &\quad + M\,\mathcal{U}_n\, M^{-1}\,(J_-)_{11} - (J_-)_{11}\, M\,\mathcal{U}_n\, M^{-1}\,.\end{aligned} \tag{23}$$

Taking in consideration that

$$\dot{M} = \begin{bmatrix} 0 & 0 \\ -\dot{a}_1 & 0 \end{bmatrix}, \ \dot{\widehat{M^{-1}}} = \begin{bmatrix} 0 & 0 \\ \dot{a}_1 & 0 \end{bmatrix}, \ (J_-)_{11} = \begin{bmatrix} 0 & 0 \\ c_1 & 0 \end{bmatrix}, \ \dot{a}_1 = c_1 \,,$$

and the fact that (23) can be written like

$$\begin{aligned} \dot{\mathcal{U}}_n &= \mathcal{U}_{n+1} - M^{-1}\left(\dot{M} + (J_-)_{11} M\right)\mathcal{U}_n \\ &\qquad + \mathcal{U}_n\left(-M^{-1} J_{11}\, M - \dot{\widehat{M^{-1}}}\, M + M^{-1}\,(J_-)_{11}\right), \end{aligned}$$

we have $\dot{\mathcal{U}}_n = \mathcal{U}_{n+1} - \mathcal{U}_n\, M^{-1}\, J_{11} M$. Since $(x^{2n}\mathcal{U})(\mathcal{P}_0) = M^{-1}\, J_{11}^n\, M$, we have that the last relation is equivalent to (18).

Now, we prove that $(c) \Rightarrow (d)$ remember that $\mathcal{F}$ can be written

$$\mathcal{F}(z) = \sum_{n=0}^{\infty} \frac{\mathcal{U}_n}{z^{n+1}} \,.$$

Then, from (18),

$$\begin{aligned} \frac{d}{dt}\mathcal{F}(z) &= \sum_{n=0}^{\infty} \frac{\dot{\mathcal{U}}_n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{\mathcal{U}_{n+1}}{z^{n+1}} - \left(\sum_{n=0}^{\infty} \frac{\mathcal{U}_n}{z^{n+1}}\right)\mathcal{U}_1 \\ &= z\sum_{n=0}^{\infty} \frac{\mathcal{U}_{n+1}}{z^{n+2}} - \mathcal{F}(z)\mathcal{U}_1 = z\left(\mathcal{F}(z) - \frac{\mathcal{U}_0}{z}\right) - \mathcal{F}(z)\mathcal{U}_1 \\ &= \mathcal{F}(z)\left(zI_{2\times 2} - \mathcal{U}_1\right) - \mathcal{U}_0 \,, \end{aligned}$$

and as $\mathcal{U}_0 = I_{2\times 2}$ we get the desired equation for $\mathcal{F}$.

To prove that $(d) \Rightarrow (e)$ we are going to obtain the derivative of the vector functional $\mathcal{U}$ from (19). To do this, we use the linearity of $\mathcal{U}$ and the convergence of the series,

$$\mathcal{F}(z) = \sum_{n=0}^{\infty} \frac{\mathcal{U}\left(x^{2n}\mathcal{P}_0(x)\right)}{z^{n+1}} = \mathcal{U}_x\left(\frac{\mathcal{P}_0(x)}{z - x^2}\right) \ , \quad ||J|| < |z| \,. \tag{24}$$

For sake of simplicity here and in the next expressions we will use $\mathcal{U}_x = \mathcal{U}$. From (19) and (24),

$$\begin{aligned} \frac{d}{dt}\mathcal{U}\left(\frac{\mathcal{P}_0}{z - x^2}\right) &= \mathcal{U}\left(\frac{\mathcal{P}_0}{z - x^2}\right)\,\left(zI_{2\times 2} - \mathcal{U}_1\right) - \mathcal{U}_0 \\ &= \mathcal{U}\left(\left(1 + \frac{x^2}{z - x^2}\right)\mathcal{P}_0\right) - \mathcal{U}\left(\frac{\mathcal{P}_0}{z - x^2}\right)\mathcal{U}(x^2\mathcal{P}_0) - \mathcal{U}(\mathcal{P}_0) \\ &= \mathcal{U}\left(\frac{x^2}{z - x^2}\mathcal{P}_0\right) - \mathcal{U}\left(\frac{\mathcal{P}_0}{z - x^2}\right)\mathcal{U}(x^2\mathcal{P}_0) \,. \end{aligned}$$

Now, we define the auxiliary vector functionals $\mathcal{U}^1, \mathcal{U}^2 : \mathbb{P}^2 \to \mathcal{M}_{2\times 2}(\mathbb{C})$ such that:

$$\mathcal{U}^1 = \mathcal{U}(x^2\,\mathcal{B}) \quad \text{and} \quad \mathcal{U}^2 = \mathcal{U}(\mathcal{B})\,\mathcal{U}(x^2\mathcal{P}_0)\,,$$

for each $\mathcal{B} \in \mathbb{P}^2$. We remark that $\frac{1}{z-x^2}\,\mathcal{P}_0$ do not depend on $t \in \mathbb{R}$. In (24), denoting $\dot{\mathcal{U}} = \frac{d\mathcal{U}}{dt}$, we have that $\mathcal{U} = \mathcal{U}^1 - \mathcal{U}^2$ over $\frac{1}{z-x^2}\mathcal{P}_0$, being

$$\mathcal{U}\left(\frac{1}{z-x^2}\,\mathcal{P}_0\right) = \frac{1}{z}\mathcal{U}(\mathcal{P}_0) + \frac{1}{z^2}\mathcal{U}(x^2\mathcal{P}_0) + \cdots\,, \quad |z| > ||J||\,.$$

Hence, we have $\mathcal{U} = \mathcal{U}^1 - \mathcal{U}^2$ over $\mathbb{P}^2$, this is, we have (20).

For proving that $(e) \Rightarrow (f)$ we use the fact that $\dot{\mathcal{B}}_m$ can be written in terms of the sequence $\{\mathcal{B}_m\}$,

$$\dot{\mathcal{B}}_m = \sum_{j=0}^{m} \alpha_j^m \mathcal{B}_j\,. \tag{25}$$

For $m = 0, 1$, the above expression is

$$\dot{\mathcal{B}}_m(x) = \alpha_{m-1}^m\,\mathcal{B}_{m-1}(x) + \alpha_m^m\,\mathcal{B}_m(x)\,. \tag{26}$$

Let $m \geq 2$ be fixed. We are going to show that (26) holds, also for m. Due to the orthogonality conditions (12), i.e., $(x^{2k}\mathcal{U})\,(\mathcal{B}_m) = \Delta_m\delta_{k,m}$, $k = 0, \ldots, m-1$, $m \in \mathbb{N}$, from (25), we have that

$$\mathcal{U}(\dot{\mathcal{B}}_m) = \alpha_0^m\,\mathcal{U}(\mathcal{B}_0)\,.$$

In fact, using (15) and (20),

$$\begin{aligned} 0_{2\times 2} &= \frac{d}{dt}(\mathcal{U}(\mathcal{B}_m)) = \dot{\mathcal{U}}(\mathcal{B}_m) + \mathcal{U}(\dot{\mathcal{B}}_m) \\ &= \mathcal{U}(x^2\,\mathcal{B}_m) - \mathcal{U}(\mathcal{B}_m)\mathcal{U}(x^2\,\mathcal{P}_0) + \alpha_0^m\,\mathcal{U}(\mathcal{B}_0)\,. \end{aligned}$$

Thus, from orthogonality, we have $\alpha_0^m = 0_{2\times 2}$. We proceed by induction on m, assuming

$$\alpha_0^m = \alpha_1^m = \cdots = \alpha_{j-1}^m = 0_{2\times 2}\,,$$

for a fixed $j < m-1$. Using (25) and, again, (20) and the orthogonality conditions,

$$\begin{aligned} 0_{2\times 2} &= \frac{d}{dt}(\mathcal{U}(x^{2j}\,\mathcal{B}_m)) = \dot{\mathcal{U}}(x^{2j}\,\mathcal{B}_m) + \mathcal{U}(x^{2j}\,\dot{\mathcal{B}}_m) \\ &= \mathcal{U}(x^{2j+1}\,\mathcal{B}_m) - \mathcal{U}(x^{2j}\,\mathcal{B}_m)\mathcal{U}(x^2\,\mathcal{P}_0) + \alpha_j^m\,\mathcal{U}(x^{2j}\,\mathcal{B}_j) \\ &= \alpha_j^m\mathcal{U}(x^{2j}\,\mathcal{B}_j) = \alpha_j^m\Delta_j\,, \end{aligned}$$

where Δ_j is an invertible matrix. Thus, $\alpha_j^m = 0_{2\times 2}$ and (26) is verified for any $m \in \mathbb{N}$.

Our next purpose is to determine α_m^m and α_{m-1}^m. From, (26), we have that

$$\mathcal{U}(x^{2(m-1)}\dot{\mathcal{B}}_m) = \alpha_{m-1}^m \mathcal{U}(x^{2(m-1)}\mathcal{B}_{m-1}).$$

Then, because of (20) and the orthogonality conditions

$$\begin{aligned} 0_{2\times 2} &= \frac{d}{dt}(\mathcal{U}(x^{2(m-1)}\mathcal{B}_m)) \\ &= \mathcal{U}(x^{2m}\mathcal{B}_m) - \mathcal{U}(x^{2(m-1)}\mathcal{B}_m)\mathcal{U}(x^2\mathcal{P}_0) + \alpha_{m-1}^m \mathcal{U}(x^{2(m-1)}\mathcal{B}_{m-1})\,. \end{aligned}$$

Therefore $\alpha_{m-1}^m = -\Delta_m(\Delta_{m-1})^{-1} = -C_m$. On the other hand, writing

$$\mathcal{B}_m(x) = \sum_{j=0}^{m} \beta_j^m \mathcal{P}_j(x)\,, \tag{27}$$

and comparing the coefficient of x^{2m} and x^{2m+1} in both sides of (27), we obtain

$$\beta_j^m = \begin{bmatrix} 1 & 0 \\ \beta_m & 1 \end{bmatrix}.$$

Moreover, taking derivatives in (27) and comparing with (26), we get $\dot{\beta}_j^m = \alpha_m^m$, with

$$\alpha_m^m = \begin{bmatrix} 0 & 0 \\ \alpha_m & 0 \end{bmatrix},$$

where we need to determine α_m. From (26) and (20),

$$\begin{aligned} \alpha_m^m \mathcal{U}(x^{2m}\mathcal{B}_m) = \frac{d}{dt}\mathcal{U}(x^{2m}\mathcal{B}_m) - \mathcal{U}(x^{2(m+1)}\mathcal{B}_m) \\ + \mathcal{U}(x^{2m}\mathcal{B}_m)\mathcal{U}(x^2\mathcal{P}_0) + C_m\mathcal{U}(x^{2m}\mathcal{B}_{m-1})\,. \end{aligned}$$

Using the orthogonality conditions and (8)

$$\alpha_m^m \Delta_m = \left(\frac{d}{dt}(\Delta_m) - B_m\Delta_m + \Delta_m\mathcal{U}(x^2\mathcal{P}_0)\right).$$

Thus,

$$\alpha_m^m + B_m = \frac{d}{dt}(\Delta_m)(\Delta_m)^{-1} + \Delta_m(M^{-1}J_{11}M)(\Delta_m)^{-1}\,.$$

But, $\Delta_m = C_m C_{m-1}\cdots C_1\Delta_0$ with $\Delta_0 = M$ (see [18]). Then,

$$\begin{aligned} \alpha_m^m + B_m = {}& \frac{d}{dt}(C_mC_{m-1}\cdots C_1)(C_mC_{m-1}\cdots C_1)^{-1} \\ & + (C_mC_{m-1}\cdots C_1)(\dot{M}M^{-1} + J_{11})(C_mC_{m-1}\cdots C_1)^{-1}\,. \end{aligned} \tag{28}$$

The matrix $C_mC_{m-1}\cdots C_1$ is an upper triangular matrix. Moreover, because of $\dot{a}_1 = c_1$ also $\dot{M}M^{-1} + J_{11}$ is an upper triangular matrix and, then, the matrix in the left-side of (28) is upper triangular and, consequently $\alpha_m = c_{2m+1}$. Now, (g) is true as (22) is the interpretation of (21) for the polynomials $\{V_m\}$.

Finally, to prove that $(g) \Rightarrow (a)$ we have to take derivatives in (9),

$$\begin{aligned} x\dot{V}_m(x) = \dot{A}_m V_{m+1}(x) + A_m \dot{V}_{m+1}(x) + \dot{B}_m V_m(x) \\ + B_m \dot{V}_m(x) + \dot{C}_m V_{m-1}(x) + C_m \dot{V}_{m-1}(x), \ m \geq 1\,, \end{aligned}$$

Using (22) and taking into account (9) we get

$$\begin{aligned} &(A_m C_{m+1} - C_m A_{m-1} - D_m B_m + B_m D_m) V_m(x) \\ &\qquad + (B_m C_m - C_m B_{m-1} - D_m C_m + C_m D_{m-1}) V_{m-1}(x) \\ &+ (A_m D_{m+1} - D_m A_m) V_{m+1}(x) = \dot{A}_m V_{m+1}(x) + \dot{B}_m V_m(x) + \dot{C}_m V_{m-1}(x)\,. \end{aligned}$$

Hence, we arrive, for $m = 0, 1, \ldots$, to

$$\begin{cases} \dot{A}_m = A_m\, D_{m+1} - D_m\, A_m\,, \\ \dot{B}_m = A_m\, C_{m+1} - C_m\, A_{m-1} + B_m\, D_m - D_m\, B_m\,, \\ \dot{C}_m = B_m\, C_m - C_m\, B_{m-1} + C_m\, D_{m-1} - D_m\, C_m\,. \end{cases} \tag{29}$$

Taking into account that, with the above notation, $D_m = (J_-)_{m+1,m+1}$, we see that (29) is equivalent to (16). ☐

Remark 3.2. We shall notice that the equation (22) for the matrix polynomials, $\{V_m\}$, appears in the study of semi-classic families of matrix orthogonal polynomials done in [25] and [16].

Next, we state a Lax type theorem for the dynamical system (1) (see for instance [26]).

Theorem 3.3. *Let $\mathcal{V}$ be defined by*

$$\mathcal{V} = \begin{bmatrix} V_0(x;t) & \cdots & V_n(x;t) & \cdots \end{bmatrix}^{\mathsf{T}}\,,$$

and let λ be a spectral point of the Jacobi matrix $J(t)$, i.e.,

$$J(t)\,\mathcal{V}(\lambda(t)) = \lambda(t)\,\mathcal{V}(\lambda(t))\,.$$

Then $J(t)$ satisfy (16) *if, and only if,* $\dfrac{d}{dt}\,\lambda(t) = 0\,.$

Proof. If we apply the differential operator to the equation

$$J(t)\,\mathcal{V}(\lambda(t)) = \lambda(t)\,\mathcal{V}(\lambda(t))$$

we get that,

$$\dot{J}(t)\,\mathcal{V}(\lambda(t)) + J(t)\,\dot{\mathcal{V}}(\lambda(t)) = \dot{\lambda}(t)\,\mathcal{V}(\lambda) + \lambda(t)\,\dot{\mathcal{V}}(\lambda(t))\,.$$

Since by hypothesis J satisfy (16), we get

$$J(t)J_{-}(t)\mathcal{V}(\lambda(t)) - J_{-}(t)\lambda(t)\mathcal{V}(\lambda(t)) \\ + \big(J(t) - \lambda(t)\,\mathcal{I}\big)\dot{\mathcal{V}}(\lambda(t)) = \dot{\lambda}(t)\mathcal{V}(\lambda(t))\,,$$

and so

$$\big(J(t) - \lambda(t)\,\mathcal{I}\big)\big(J_{-}\,(t)\,\mathcal{V}(\lambda(t)) + \dot{\mathcal{V}}(\lambda(t))\big) = \dot{\lambda}(t)\,\mathcal{V}(\lambda(t))\,. \tag{30}$$

But, by theorem 3.1, (16) is equivalent to (22) which in matrix notation is read as

$$J_{-}(\lambda(t))\,\mathcal{V} + \dot{\mathcal{V}}(\lambda(t)) = \underline{\mathbf{0}}\,.$$

By (30) we have that $\dot{\lambda}(t) = 0$, as we intended to prove. □

4. Representations theorems

In this section we present a result that gives a explicit expression for the Weyl function, generalized Markov function, and we also present another result that gives, under some conditions, a representation of the vector of linear functionals associated with the system studied in the last section.

Considering that $e^{x^2t} = \sum_{k=0}^{+\infty} \frac{t^k}{k!}x^{2k}$ and given a vector of linear functionals $\mathcal{U}^0 : \mathcal{P} \to \mathcal{M}_{2\times 2}(\mathbb{R})$, which is the vector of functionals $\mathcal{U}$ for $t = 0$, we can always define a vector of linear functionals $e^{x^2t}\,\mathcal{U}^0 : \mathcal{P} \to \mathcal{M}_{2\times 2}(\mathbb{R})$ such as

$$(e^{x^2t}\,\mathcal{U}^0)(\mathcal{P}_j) = \left(\sum_{k=0}^{+\infty} \frac{t^k}{k!}x^{2k}\,\mathcal{U}^0\right)(x^{2j}\mathcal{P}_0) = \sum_{k=0}^{+\infty} \frac{t^k}{k!}\mathcal{U}^0(x^{2(j+k)}\,\mathcal{P}_0)\,.$$

Now we give a representation of $\mathcal{U}$ associated with the problem under discussion.

Theorem 4.1. *In the conditions of Theorem 3.1 assume that the vector of linear functionals $\mathcal{U}$ verifies $\mathcal{U}(\mathcal{P}) = (e^{x^2t}\,\mathcal{U}^0)(\mathcal{P})E$, for some $E \in \mathcal{M}_{2\times 2}(\mathbb{C})$. Then, $\{a_n, b_n, c_n, d_n\}$, $n \in \mathbb{N}$, is a solution of (1).*

Proof. Since $\mathcal{U}$ is a normalized vector functional necessarily the assumption $\mathcal{U}(\mathcal{P}) = (e^{x^2t}\mathcal{U}^0)(\mathcal{P})E$ implies

$$E = \big((e^{x^2t}\mathcal{U}^0)(\mathcal{P}_0)\big)^{-1}\,.$$

On the other hand, with this assumption, if we want to prove that $\{a_n, b_n, c_n, d_n\}$, $n \in \mathbb{N}$ is a solution of (1) it is sufficient to show that (18) holds. Now,

$$\frac{d}{dt}(e^{x^2 t}\mathcal{U}^0)(\mathcal{P}_0) = (e^{x^2 t}\mathcal{U}^0)(x^2\,\mathcal{P}_0)$$

and

$$\frac{dE}{dt} = -E\,\frac{d\big((e^{x^2 t}\mathcal{U}^0)(\mathcal{P}_0)\big)}{dt}\,E = -E\,\big((e^{x^2 t}\mathcal{U}^0)(x^2\mathcal{P}_0)\big)\,E = -E\,\mathcal{U}(x^2\mathcal{P}_0)$$

if we take derivatives in $\mathcal{U}(x^{2k}\,\mathcal{P}_0) = (e^{x^2 t}\mathcal{U}^0)(x^{2k}\,\mathcal{P}_0)E$, we arrive to (18). $\square$

Theorem 4.2. *Assume that the sequence $\{a_n, b_n, c_n, d_n\}$, $n \in \mathbb{N}$, is uniformly bounded, i.e., for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$*

$$\exists\, K \in \mathbb{R}_+ :\ \max\{|a_n(t)|, |b_n(t)|, |c_n(t)|, |d_n(t)|\} \le K\,.$$

Then, $\{a_n, b_n, c_n, d_n\}$, $n \in \mathbb{N}$, is a solution of (1) if, and only if, the Weyl function satisfy

$$\dot{R}_J(z) = R_J(z)(zI_{2\times 2} - J_{11}) - I_{2\times 2} + \left[R_J(z), (J_-)_{11}\right], \tag{31}$$

for all $z \in \mathbb{C}$ such that $|z| > ||J||$.
Moreover, the Weyl function is given by

$$R_J(z) = e^{zt} M\, T(t,z)(N(t))^{-1}\,, \tag{32}$$

where

$$N(t) = \begin{bmatrix} e^{\int_0^t b_1 ds} & e^{\int_0^t b_1 ds} \int_0^t a_2\, e^{\int_0^s (b_2 - b_1) dr} ds \\ 0 & e^{\int_0^t b_2 ds} \end{bmatrix},$$

$$T(t,z) = -\int_0^t e^{-zs} M^{-1} N(s) ds + M_0^{-1} R_0(z)\,,$$

and M_0 and $R_0(z)$ are, respectively, M and $R_J(z)$ for $t = 0$.

Proof. The first part of the proof of this result is trivial. From theorem 3.1 we know that if $\{a_n, b_n, c_n, d_n\}$, $n \in \mathbb{N}$, is a solution of (1) it is equivalent to say that (17) is verified. Starting by (17), to obtain the relation (31) for R_J it is sufficient to take derivatives in (5) and to substitute $\dot{J}_{11}^n$ in

$$\dot{R}_J(z) = \sum_{n=0}^{\infty} \frac{\dot{J}_{11}^n}{z^{n+1}}\,, \quad |z| > ||J||\,.$$

by its equivalent condition (17).

Reciprocally, if (31) holds, using the fact that $R_J(z) = M\mathcal{F}(z)M^{-1}$ and the paragraph (d) of theorem 3.1, we get $\{a_n, b_n, c_n, d_n\}$, $n \in \mathbb{N}$, is a solution of (1).

Moreover, it is easy to see that M, $T(t,z)$ and $N(t)$ are, respectively, the solutions of the following Cauchy problems:

$$\begin{cases} \dot{X} = -(J_-)_{11}X\,, \\ X(0) = M_0\,, \end{cases} \quad \begin{cases} \dot{X} = -e^{-zt}M^{-1}N(t)\,, \\ X(0) = M_0^{-1}R_0(z)\,, \end{cases} \quad \begin{cases} \dot{X} = (J_{11} - (J_-)_{11})X, \\ X(0) = I_{2\times 2}\,. \end{cases}$$

Taking derivatives in the right-hand side of (32), and checking the initial conditions, we prove that R_J is a solution of the following Cauchy problem:

$$\begin{cases} \dot{X} = X(zI_{2\times 2} - J_{11}) - I_{2\times 2} + \big[X, (J_-)_{11}\big]\,, \\ X(0) = R_0(z)\,. \end{cases} \tag{33}$$

From [24], we know that (33) has a unique solution. On the other hand, from (31) we have that R_J is a solution of (33). Then, we arrive to (32). □

Theorem 4.3. *Assume that the sequence* $\{a_n, b_n, c_n, d_n\}$, $n \in \mathbb{N}$, *is uniformly bounded, i.e., for all* $n \in \mathbb{N}$ *and* $t \in \mathbb{R}$

$$\exists\, K \in \mathbb{R}_+ :\ \max\{|a_n(t)|, |b_n(t)|, |c_n(t)|, |d_n(t)|\} \leq K\,.$$

Then $\{a_n, b_n, c_n, d_n\}$, $n \in \mathbb{N}$, *is a solution of (1) if, and only if, the generalized Markov function is given by*

$$\mathcal{F}(z) = e^{-zt}\Big(\int e^{zt}\, S(t)\, dt\Big)\, S^{-1}(t)$$

where $S(t)$ *is such that* $\dot{S}(t) = -S(t)\,\mathcal{U}_1$, *for all* $z \in \mathbb{C}$ *with* $|z| > ||J||$.

Proof. Similar to the previous one. □

From this theorem we seen that the orthogonal matrix measures (cf. theorem 2.1) that govern our high order Toda lattice are of type studied in [17], i.e.

$$\mathcal{F}_0(z) = z^{\alpha}\, e^{-z}\, e^{A\,z}\, z^{B}\, C\,. \tag{34}$$

When $\mathcal{U}_1$ do not depends on t, the Markov function that governs our Toda type system (1) with initial data associated with $\mathcal{F}_0(z)$ given by (34) is of type $\mathcal{F}(z) = e^{z\,t}\,\mathcal{F}_0(z)$, which is the one studied in [15].

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